Anosov Flows, Surface Groups and Curves in Projective Space

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1 Introduction

In his beautiful paper [13], N. Hitchin studies the connected components of the space

$$Rep(\pi_1(S), PSL(n,\mathbb{R})) = Hom^{red}(\pi_1(S), PSL(n,\mathbb{R}))/PSL(n,\mathbb{R}),$$

of reducible representations of the fundamental group of a compact surface S into $PSL(n,\mathbb{R})$. By reducible, we mean representations which are as sum of irreducible representations. Using Higgs bundle techniques, he proves two remarkable results. The first deals with the number of components of this space.

Theorem 1.1 [HITCHIN] *If $n > 2$, the space $Rep(\pi_1(S), PSL(n,\mathbb{R}))$ has three connected components when n is odd, and six when n is even*

Note that in in [8], W. Goldman gives a complete description of these connected components in the case of finite covers of $PSL(2,\mathbb{R})$. In the case of $PSL(2,\mathbb{R})$, two homeomorphic components, called Teichmüller spaces, play a central role. These two components are well known to be homeomorphic to a ball of dimension $6g-6$.

N. Hitchin generalises this situation to $PSL(n,\mathbb{R})$. Indeed, one of these components when n is odd, and two when n is even, has a very simple topology. Let us define an *n-Fuchsian* representation to be a representation ρ which can be written as $\rho = \iota \circ \rho_0$, where ρ_0 is a cocompact representation with values in $PSL(2,\mathbb{R})$ and ι is the irreducible representation of $PSL(2,\mathbb{R})$ in $PSL(n,\mathbb{R})$, We denote by $Rep_H(\pi_1(S), PSL(n,\mathbb{R}))$ a connected component that contains Fuchsian representations, and call it a *Hitchin component*. In fact, when n is odd there is one Hitchin component, and when n is even two isomorphic ones. N. Hitchin proves in [13]

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Theorem 1.2 [Hitchin] *Each component $Rep_H(\pi_1(S), PSL(n, \mathbb{R}))$ is homeomorphic to a ball of dimension $\chi(S)(1-n^2)$.*

This last result actually extends to the case of adjoint groups of real split forms. Although Hitchin's proof gives an explicit parametrisation of this component, the construction by itself sheds no light on the geometry underlying these representations. Using Higgs bundle techniques, one can prove that the representations in Hitchin component are irreducible (Lemma 10.1), but it seems hard to detect by these means whether these representations are faithful, discrete, as to understand whether the group of outer automorphisms of $\pi_1(S)$ acts properly on this specific component.

Nevertheless, the geometric significance of this component is well known in dimensions 2 and 3. For $n = 2$, it is the Teichmüller component, corresponding to holonomies of hyperbolic structures on S. For $n = 3$, S. Choi and W. Goldman proves in [3]

Theorem 1.3 [Choi-Goldman] *For $n = 3$, Hitchin component consists of holonomies of convex real projective structures on S. That is, for every representation ρ in $Rep_H(\pi_1(S), PSL(3, \mathbb{R}))$, there exists an open set Ω in $\mathbb{P}(\mathbb{R}^3)$ such that $\Omega/\rho(\pi_1(S))$ is homeomorphic to S.*

As a consequence of this result, when $n = 3$, a representation ρ in Hitchin component preserves a C^1-convex curve in $\mathbb{P}(\mathbb{R}^3)$, namely the boundary of the open set Ω obtained by the previous theorem.

Our first result generalises this last situation. Let us introduce a definition. A curve ξ with values in $\mathbb{P}(\mathbb{R}^n)$ is said to be *hyperconvex* if for any distinct points $(x_1, \ldots, x_n)$ the following sum is direct

$$\xi(x_1) + \ldots + \xi(x_n).$$

Furthermore, we say a hyperconvex curve is a *Frenet curve*, if there exists a family of maps $(\xi^1, \xi^2, \ldots, \xi^{n-1})$ with $\xi^p \subset \xi^{p+1}$, called the *osculating flag of* ξ, such that

- $\xi = \xi^1$ and ξ^p takes values in the Grassmannian of p-planes,
- if $(n_1, \ldots, n_l)$ are positive integers such that $\sum_{i=1}^{i=l} n_i \leq n$, if $(x_1, \ldots, x_l)$ are distinct points, then the following sum is direct

$$\xi^{n_1}(x_1) + \ldots + \xi^{n_l}(x_l); \tag{1}$$

- finally, for every x, let $p = n_1 + \ldots + n_l$, then

$$\lim_{(y_1, \ldots, y_l)\ \xrightarrow[y_i \text{ all distinct}]{} x} \left(\bigoplus_{i=1}^{i=l} \xi^{n_i}(y_i)\right) = \xi^p(x). \tag{2}$$

We observe that for a Frenet hyperconvex curve, ξ^1 completely determines ξ^p. Also, if ξ^1 is C^∞, then $\xi^p(x)$ is generated by the derivatives at x of ξ^1 up to order $p-1$. However, in general, a Frenet hyperconvex curve has no reason to be C^∞ although its image is obviously a C^1-submanifold. Our main result is the following

Theorem 1.4 *For every representation ρ in a Hitchin component, there exists a ρ-equivariant hyperconvex Frenet curve from $\partial_\infty \pi_1(S)$ to $\mathbb{P}(\mathbb{R}^n)$.*

Note that the Veronese embedding from $\mathbb{P}(\mathbb{R}^2)$ to $\mathbb{P}(\mathbb{R}^n)$ is a $SL(2,\mathbb{R})$-equivariant hyperconvex Frenet curve; it corresponds to Fuchsian representations. Our main theorem therefore says that a curve similar to the Veronese embedding persists under possibly large deformation of the representation.

We recall that we say that a representation ρ of Γ with values in a semi-simple Lie group G is *purely loxodromic* if, for every γ in Γ different from the identity, $\rho(\gamma)$ is conjugate to an element in the interior of the Weyl chamber. For $G = PSL(n,\mathbb{R})$, this just means that the eigenvalues of $\rho(\gamma)$ are real with multiplicity 1. For $G = PSL(2,C)$, we recover the classical definition. As a consequence of the techniques involved in the proof, we also obtain

Theorem 1.5 *Every representation in Hitchin component is discrete, faithful and purely loxodromic.*

This theorem is a generalisation of a classical result for Teichmüller Space. It also bears some ressemblance to a recent, beautiful result of M. Burger, A. Iozzi and A. Wienhard [2], announced while the second draft of this paper was completed. They proves in particular that surface group representations with maximal Toledo invariant also have discrete images. Although the methods and the target groups are different (so that the results have non empty intersection, although neither contains the other), it appears that dynamical ideas quite similar to those appearing in this paper can be applied to their situation, improving some geometrical aspects. It is also quite surprising that two classes of groups, namely isometry groups of Hermitian symmetric spaces on the one hand, and $SL(n,\mathbb{R})$ (and quite plausibly all real split forms) on the other hand, have some common features, not shared for instance with $PSL(2,\mathbb{C})$.

We will also prove in paper [18] that the mapping class group acts properly on the Hitchin component

We will also state converses and refinements of these results in Section 4. We will now describe more precisely the structure of this paper, before proceeding to discussions and conjectures.

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1.1 Summary

We describe briefly the content of the main sections of this article.

- 2: **Geometric Anosov flows.** We introduce in this section the notion of *Anosov structure* which are "geometric structure" related to flows. Our main aim in this paper is to describe the representations in Hitchin component in terms of holonomies of such structures. As a preliminary, we will show how such holonomies form an open set in the space of representations.

- 3: **Quasi-Fuchsian and Anosov representations.** This geometric structure is specified to the case of study rank 1 subgroups of semi-simple Lie groups, and more specifically the irreducible copy of $PSL(2,\mathbb{R})$ in $PSL(n,\mathbb{R})$. We introduce *quasi-Fuchsian representations* as deformations of Fuchsian representations. In a similar way as for classical quasi-Fuchsian representations in $PSL(2,\mathbb{C})$, limit curves appear of taking values in the flag manifold instead of $\mathbb{CP}^1$ as in the "classical". The properties of these curves will play a central role in the sequel.

- 4: **Statement of the main results.** With all the basic notions in hand, we can state the main results of this paper: first, that the limit curve of a quasi-Fuchsian representation is built from a hyperconvex curve and, second, that every representation in Hitchin component is quasi-Fuchsian. We also state converse results.

- 5: **Hyperconvex curves.** In this section, we study more specifically hyperconvex curves and prove in general that they admit "left" and "right" osculating flags. This section is independent of the rest of this paper.

- 6: **Preserving hyperconvex curves.** We prove that a representation preserving a hyperconvex curve is the holonomy of an Anosov structure.

- 7: **Curves and Anosov representations.** This is the core of the the article: we prove in Corollary 7.2 that the limit curve of certain Anosov representations is the osculating flag of a Frenet hyperconvex curve .

- 8: **Anosov representations, 3-hyperconvexity and Property (H)** In the core of the proof of the previous result, certain properties to be satisfied by limit curves were introduced. We study here their relations with quasi-Fuchsian representations.

- 9: **Closedness.** We show that the set of quasi-Fuchsian representations is closed in the space of all representations. This permits us to conclude the proofs of our main results.

- 10: **Appendix**: some lemmas.

1.2 Further discussions and conjectures

This section is rather programmatic containing various announcements and precise conjectures. This section should be skipped by a reader interested in concrete results. It is a rather random collection of remarks aimed at suggesting how many aspects of Teichmüller theory considered as a dictionary between various fields of mathematics should extend to the study of Hitchin components.

1.2.1 Crossratios: $n = \infty$

In a subsequent article, currently under preparation [17], we explain the relation between Hitchin components and crossratios on $\pi_1(S)$. We define a crossratio on $\pi_1(S)$ is a real Hölder function b defined on $(\partial_\infty \pi_1(S))^4 \setminus \{(x, y, z, t)/x = w, z = y\}$ satisfying the following rules

$$\begin{aligned} b(x, y, z, t) &= \frac{b(x, y, z, w)}{b(x, w, z, t)}, \\ b(x, y, z, t) &= b(x, t, z, y)^{-1}, \\ b(x, y, z, t) &= b(z, t, x, y). \end{aligned}$$

As an example of crossratio, one has the classical projective crossratio and the crossratio associated by J.-P. Otal to a negatively curved metric on S [22]. These have been extensively studied U. Hamenstädt in [11] (Note however our definition includes more general crossratios than those defined by Otal and that some of their results are not true in our generality).

For a complete description of various aspects of crossratios, one is advised to read F. Ledrappier's presentation [19]. Associated to a crossratio are numbers called *periods*. If γ is an element of $\pi_1(S)$, let γ^+ (resp. γ^-) be the attracting (resp. repelling) fixed point of γ on $\partial_\infty(\pi_1(S))$. We define the period $l(\gamma)$ of γ by

$$\forall y \in \partial_\infty(\pi_1(S)), \quad l(\gamma) = \log |b(\gamma^+, \gamma^-, y, \gamma y)|.$$

It turns out that a crossratio is completely determined by its set of periods which in the case of Otal's crossratio is just the collection of lengths of the corresponding closed orbits.

The main result of our article [17] explains that there exists a correspondence between representations in a Hitchin component and crossratios satisfying some functional relations, one for each n, which are completely explicit but technical to state. Under this correspondence, the period of γ is equal to

$$\log(\frac{\lambda_{max}(\rho(\gamma))}{\lambda_{min}(\rho(\gamma))}),$$

where ρ is the corresponding representation and $\lambda_{max}(A)$ (resp. $\lambda_{min}(A)$) is the largest (resp. smallest) real eigenvalue of the matrix A. According to this result, each component $Rep_H(\pi_1(S), PSL(n, \mathbb{R}))$ embeds in the space of all crossratios, which may be considered as a candidate for $Rep_H(\pi_1(S), PSL(\infty, \mathbb{R}))$. This is a rather mysterious picture, but is has the advantage of (almost) describing Hitchin component as a space of objects, crossratios, that may be thought as "geometric structures" on the surface.

Following discussions with Nigel Hitchin, we have found that this picture is coherent with a conjectural picture of his. Namely, he suggested to consider the group $SL(\infty, \mathbb{R})$ as the group of symplectic diffeomorphisms of $\mathcal{G} = \mathbb{RP}^1 \times \mathbb{RP}^1 \setminus \Delta$. On our dynamical side, a crossratio defines a measure, equivalent to the Lebesgue measure, on $\mathcal{G}$. For instance, the choice of a negatively curved metric defines a symplectic form on the space of geodesics by symplectic reduction, and this space is identified with $\mathcal{G}$ via the identification of S^1 with the boundary at infinity. This measure is trivially invariant under the action of $\pi_1(S)$. It follows that after the conjugation by a homeomorphism sending the measure associated to the crossratio to the "standard measure" on $\mathcal{G}$, we obtain a representation of $\pi_1(S)$ in the group of symplectic homeomorphisms of $\mathcal{G}$. It is striking that these two pictures coming from different areas of mathematics agree.

1.2.2 Universal Hitchin components: $g = \infty$

One observes that Theorem 1.4 allows us to let the genus g of the surface tends to infinity an thus provides an extension of the theory of the universal Teichmüller space. Indeed, we may consider the space $\mathcal{T}(n)$ of all Frenet hyperconvex curves in $\mathbb{P}(\mathbb{R}^n)$, this is a natural candidate for the *universal Hitchin component*, generalising the group of quasi-symmetric homeomorphisms when $n = 2$. Here are some natural quastions: how do the various components sit in this space? Does it have a Kähler geometry?

1.2.3 Frenet curves and integrable systems:

We hope to relate this subject to that of integrable systems. We strongly encourage the reader to consult G. Segal very clear exposition [26]. Hyperconvex Frenet curve maybe constructed (at least locally) is through differential equations. Namely, we consider an n^{th}-order linear differential operator - a *Hill operator* - of the following form

$$L(f) = f^{(n)} + a_2 f^{(n-2)} + a_3 f^{(n-3)} + \dots a_n. \tag{3}$$

If $(f_i, \dots, f_n)$ are n independent solutions of the equation $L(f) = 0$, the projective coordinates given by

$$[f_1, \dots, f_n]$$

define locally a hyperconvex Frenet curve. A different choice of f_i yields the same curve up to a projective transformation. Since the curves in Theorem 1.4 have low regularity (they are usually only C^1), they cannot be related to smooth regular operators like the one in Formula (3). However it would be interesting to know whether they can described by some operator in a weak sense.

This question is motivated by the following fact: the space of Hill operators is naturally a symplectic manifold and its Poisson algebra relates to the so-called $W(n)$-algebras, where $W(2)$ is the Virasoro Algebra (*cf* [26])

Apparently, physicists tend to believe that a Teichmüller theory should hold for those $W(n)$-algebras for which Hitchin components would play the role of Teichmüller spaces. Honestly, I have never understood in the papers that allude to this question what they really expect as a link between $W(n)$-algebras and Hitchin components. Apparently, the goal is rather to obtain Hitchin component as a "double quotient" of $W(n)$-algebras analogously to Kontsevich's result [14] for Virasoro algebra, rather than to copy the relation of Virasoro algebra with the universal Teichmüller space outlined in our previous discussion.

However, Theorem 1.4 provides at least a relation between $W(n)$-algebras and Hitchin component which may well be consistent with the expected picture. Moreover, the fact that we still have a candidate for a companion to $W(\infty)$ as discussed in the Paragraph 1.2.1, seems appealing.

1.2.4 Holomorphic differentials and the link with Hitchin theory

In order to prove his theorem, N. Hitchin gives explicit parametrisations of Hitchin components. Namely, given a choice of a complex structure J over a given compact surface S, he identifies the component $Rep_H(\pi_1(S), PSL(n,\mathbb{R}))$ with the vector space

$$\mathcal{Q}(2,J) \oplus \ldots \oplus \mathcal{Q}(n,J),$$

where $\mathcal{Q}(p,J)$ denotes the space of holomorphic p-differentials on the Riemann surface (S,J). The main idea in the proof is first to identify representations with harmonic mappings as in K. Corlette's seminal paper [4], (see also [5], [15]), second to use the fact a harmonic mapping f taking values in a symmetric space gives rise to holomorphic differentials $q_2(f),\ldots$ in manner similar to that in which a connection gives rise to differential forms in Chern-Weil theory.

Can one improve this parametrisation, and in particular eliminate the dependance on the choice of a complex structure and thus obtain a parametrisation by holomorphic objects invariant under the mapping class group? Here is a suggestion: a rather standard check shows that the quadratic differential part $q_2(f)$ vanishes exactly when f is minimal. We may now ask whether, fixing the representation ρ, we can choose in a unique manner a complex structure on S such that the associated harmonic is actually minimal. Another way to state this question is the following conjecture which I have discussed many times with Bill Goldman

Conjecture 1.6 *Let ρ be a representation in Hitchin component. For every complex structure j in Teichmüller space $\mathcal{T}$, let $e(j)$ be the energy of the corresponding harmonic mapping. Then e has a unique minimum.*

This conjecture is well known to be true for $n = 2$. For $n = 3$, one can prove it using ideas linking real projective structures, affine spheres, Blaschke metric as in J. Loftin paper [20] or in the preprint [16]; in order to complete the circle of ideas contained in these papers, one has just to realise that, for an affine sphere S, the Blashke metric, seen as a map from S to $SL(3,\mathbb{R})/SO(3)$, is minimal. For general n, one can at least show that e is proper [18].

If the last conjecture is true, then following our previous discussion, we would obtain the following result, which helps to understand the action of the mapping class group $\mathcal{M}(S)$ on Hitchin components

Conjecture 1.7 *The quotient $Rep_H(\pi_1(S), PSL(n,\mathbb{R}))/\mathcal{M}(S)$ is homeomorphic to the total space of the vector bundle E over the Riemann moduli space, whose fibre at a point J is*

$$E_J = \mathcal{Q}(3,J) \oplus \ldots \oplus \mathcal{Q}(n,J).$$

Again, by the previous discussion this result is true for $n = 2$ and $n = 3$. The fact that the energy is proper would say that the map we can define using Hitchin's identfication (described in the beginning of this paragraph) from E to $Rep_H(\pi_1(S), PSL(n,\mathbb{R}))/\mathcal{M}(S)$ is at least surjective.

1.2.5 Compactification

W. Thurston (*cf* [6]) gives a compactification of Teichmüller space, which has been extended in many ways. More specifically in [23], A. Parreau gives a compactification of the set of discrete representations in $SL(n,\mathbb{R})$. Since all representations in Hitchin component are discrete, her work provides a compactification of $Rep_H(\pi_1(S), PSL(n,\mathbb{R}))$. It would be interesting to relate this compactification to Theorem 1.4.

1.2.6 Further extensions and questions

This article only deals with the group $PSL(n,\mathbb{R})$, although Hitchin Theorem 1.2 actually extends to adjoint groups of all real split forms. It is rather tempting to conjecture that at least Theorem 1.5 extends to this general context. This is trivial in cases like $PSO(n, n+1)$ when the corresponding component is a subset of the component for $PSL(n,\mathbb{R})$.

Another natural extension is to consider surfaces with marked points, the holonomy around marked points being forced to preserve a full flag. To my knowledge, even the case $n = 3$ is not known, although Hitchin's version has been studied in [1].

Note however that in their remarkable paper [7], Volodya Fock and Sacha Goncharov provide a construction and a combinatorial description of a "Teichmüller space" for surfaces with punctures or boundary, as well as coordinates and Poisson structures. Actually their picture extends to the case of real split groups. For the moment, although it appears quite plausible, it is clear that their Teichmüller space is indeed is connected component.

2 Geometric Anosov Flows

Our starting point is to obtain representations in $Rep(\pi_1(S), PSL(n,\mathbb{R}))$ as holonomies of "geometric structures" associated to flows. For these new geometric structures, we prove the Proposition 2.1 which is an analog in our context of the Thurston-Lok Holonomy Theorem [21][10], which states that a deformation of the holonomy representation of a compact manifold can be obtained through a deformation of the structure.

We then describe an example arising from the consideration of a rank 1 subgroup of a semisimple group, thus making sense of the notion of *quasi-Fuchsian representations*. We finally concentrate on the case which is the subject of this paper, associated to the irreducible copy of $PSL(2,\mathbb{R})$ in $PSL(n,\mathbb{R})$.

As a motivation for our notion of geometric structure, we begin with the following remark. When one defines a (G,X)-geometric structure on a manifold M as an atlas modelled on X with transition maps in G, one requires that M and X have the same dimension and that the charts are homeomorphisms (or at least submersions in the case of transverse structures to foliations), although this is not formally necessary. However, if X is allowed to have a larger dimension, the corresponding "geometric structure" would not be rigid enough and would be too vague to have a useful meaning. Nevertheless, the presence of a flow and a subsequent hyperbolic hypothesis will allow us to enlarge the definition in this direction, and still obtain "rigid" geometric structures.

Before proceeding to the definition, we recall that of a *contracting (or dilating) bundle* over a dynamical system.

Let X be a topological space equipped with a flow ϕ_t. Let E be a topological vector bundle over X such that the action of ϕ_t lifts to an action of a flow ψ_t by bundle automorphisms. Let us assume that E is equipped with a metric g. The bundle E is *contracting* (resp. *dilating*), if there exist positive constants A and B, such that for every u in E, for every t such that $t > 0$ (resp. $t < 0$)

$$\|\psi_t(u)\| \leq Ae^{-B|t|}\|u\|.$$

It is useful and classical to remark that if X is compact, then

1. the metric g plays no role, and
2. the parametrisation of the flow plays no role either, that is if we change the parametrisation of the flow the bundle will remain contracting for this new flow.

Therefore the property of being contracting or dilating over a compact topological space X depends only on the orbit lamination $\mathcal{L}$, the bundle E and the "parallel transport" on E along leaves of $\mathcal{L}$.

2.0.7 (M,G)-Anosov structure

Let M be a manifold equipped with a pair of continuous foliations $\mathcal{E}^{\pm}$, whose tangential distributions are $E^{\pm}$, such that

$$TM = E^+ \oplus E^-.$$

Let G be a Lie group of diffeomorphisms preserving these foliations.

Let V be a manifold equipped with an Anosov flow ψ_t. Let $\mathcal{L}$ be the orbit foliation. Let $\tilde{V}$ be a Galois covering of V with covering group Γ.

We shall say that V is *A-modelled* on M ("A" standing for Anosov), if there exists a representation ρ of Γ in G, called the *holonomy representation*, a continuous map F from $\tilde{V}$ to M, called the *developing map*, satisying the following properties

- Γ-*equivariance*:
$$\forall \gamma \in \Gamma,\ F \circ \gamma = \rho(\gamma) \circ F,$$

- *Flow invariance*:
$$F \circ \psi_t(x) = F(x),$$

- *Hyperbolicity*: We consider the induced bundle $F^{\pm} = F^* E^{\pm}$. By flow invariance, these bundles are equipped with a parallel transport along the orbit of ψ_t (induced for instance by the pull back of any connection on $E^{\pm}$). By Γ-equivariance this parallel transport is invariant under Γ. Our final hypothesis is that the corresponding lift of the action of ψ_t on F^+ (resp. on F^-) is contracting (resp. dilating).

We also say that $(V, \mathcal{L})$ admits an (M, G)-*Anosov structure*.

2.0.8 Remarks

1. In general, in the examples we shall study, the continuous map F will have a very low regularity. It will only be Hölder.

2. As we shall see in the proof of Proposition 2.1, it will turn out that the notion of being A-modelled is fairly rigid. In other words, if we fix the holonomy representation, the only allowed infinitesimal transformations of F are translations by ψ_t.

3. This notion m ay be linked to another very classical one. We first consider the associated M-bundle over V generated by ρ, that is, $M_\rho = (M \times \tilde{V})/\Gamma$ where the action is the diagonal one. By construction, we have now a Γ-invariant flow φ_t on $M \times \tilde{V}$ given by $\varphi_t(m, v) = (m, \psi_t(v))$. This flow gives rise to a flow ϕ_t on N_ρ lifting ψ_t. We now observe that F gives rise to a flow equivariant section of N_ρ which we call σ_F. Our hyperbolicity condition now just means that $\sigma_F(V)$ is a hyperbolic subset of M_ρ with respect to ϕ_t.

From this last observation and the stability of hyperbolic sets, we obtain the following Proposition

Proposition 2.1 *Let M be a manifold equipped with a pair of foliations as described above. Let G be the group of diffeomorphisms preserving these foliations. Let V be a compact manifold equipped with an Anosov flow ψ_t. Let $\tilde{V}$ be a Galois covering with covering group Γ. Let $\mathcal{O}$ be the subset of all homomorphisms ρ from Γ to G which are holonomy representations of (M, G)-Anosov structures. Then $\mathcal{O}$ is open.*

PROOF: We use the notations of the previous paragraph. We first have to prove that $\sigma_F(V)$ is an isolated hyperbolic set of N_ρ. That is, we have to find an *isolating neighbourhood* U characterised by the property that

$$\sigma_F(V) = \bigcap_{n\in\mathbb{Z}} \phi^n(U).$$

We recall that M has a local product structure given by the two foliations $\mathcal{E}^\pm$.

Let us denote by π the fibration $M_\rho \to V$ described above. We fix for every x in V a complete Riemannian metric g_x on $\pi^{-1}(x) \approx M$, which we may assume depends continuously on x. If $\pi(y) = x$, we consider $d_y^\pm$ the associated distance on the leaves $\mathcal{E}_y^\pm$ through y of the foliations $\mathcal{E}^\pm$. We denote by $B_y^\pm(\epsilon)$ the ball of radius ϵ on $\mathcal{E}^\pm$ centred at y.

Since M has a local product structure, for every y, we may find a real positive number ϵ, such that

- for every x in $B_y^+(\epsilon)$, for every t in $B_y^-(\epsilon)$, the leaves $\mathcal{E}_x^-$ and $\mathcal{E}_z^+$ have a unique intersection $G_y(x,z)$ in the ball of centre y and radius 10ϵ, and
- G_y is a differentiable embedding.

We set

$$U_y(\epsilon) = G_y\big(B_y^+(\epsilon) \times B_y^-(\epsilon)\big).$$

Since $\sigma_F(V)$ is compact, we can find ϵ that satisfies the above conditions for all y in $\sigma_F(V)$. We now consider the set

$$U(\epsilon) = \bigcup_{y\in\sigma_F(V)} U_y(\epsilon),$$

which is a neighbourhood of $\sigma_F(V)$.

For ϵ small enough, since $\sigma_F(V)$ is a hyperbolic set, there exists positive constants A and B, such that

$$\forall z, w \in B_y^\pm(\epsilon), \forall t > 0. \;\; d_y(\phi_{\pm t}(z), \phi_{\pm t}(w)) \leq d_y(z,w)Ae^{-Bt},$$

This last condition implies that U is an isolating neighbourhood.

By Theorem 7.4 of C. Robinson's book [25], which is stated for diffeomorphisms but for which the discussion in the following page permits us to extend the proof to flows, we deduce that $\sigma_F(V)$ is stable. In our case, this implies that after a small perturbation $\hat\rho$ of ρ, there exists a hyperbolic set W of $N_{\hat\rho}$ and a homeomorphism h from $\sigma_F(V)$ to W close to the identity which conjugates the flows up to a small time change.

We now prove that there exists a section $\hat\sigma$ such that $W = \hat\sigma(V)$. Indeed, $H = \pi \circ h \circ \sigma_F$ is a mapping from V to V, C^0-close to the identity which conjugates the flows up to a small time change. Since the flow of ψ_t on V is Anosov, we deduce that H is an homeomorphism. It follows that $\pi : W \to V$ is also a homeomorphism. Hence, W is the image of a section $\hat\sigma$.

Finally, we know that W is a hyperbolic set and recalling that the tangent spaces to the foliations $\mathcal{E}^{\pm}$ are invariant under the flow, it follows that these tangent spaces remain contracting and dilating bundles after a small perturbation. Q.E.D.

3 Quasi-Fuchsian and Anosov representations

We now give concrete examples of the situation described above.

3.1 Rank 1 subgroups and geometric Anosov structures

Let G be a semi-simple group and be $\mathcal{G}$ the associated Lie algebra. Let H be a connected rank 1 semi-simple subgroup of G. We consider the unit tangent bundle of the symmetric space associated to H with its geodesic flow. We are going to describe geometric Anosov structures carried by this flow.

We introduce some notations.

- Let A be the real split Cartan subgroup of H and $Z(A)$ the centraliser of A in G. Let $Z_0(A)$ be the connected component of $Z(A)$ containing the identity.
- We denote $U = Z_0(A)$. Let $M = G/U$. We observe that G acts on the left on M.
- Let $U \cap H = W \times A$, where the Lie algebra of W is orthogonal to A.
- We recall that the right action of A on H/W is identified with the geodesic flow of the unit tangent bundle of the symmetric space of H. Let $\mathcal{L}$ be the orbit foliation of this flow.
- Let P^+ (resp. P^-) be the parabolic subgroup whose Lie algebra is generated by the eigenvectors of non negative (resp. nonpositive) eigenvalues of $ad(A)$. Note that M is an open set in $G/P^+ \times G/P^-$. Let $\mathcal{E}^{\pm}$ be the pair of foliations coming from this product structure on M.

We are interested in (M, G)-Anosov structures, which we abusively call (H, G)-*Anosov structures*.

3.1.1 Fuchsian representations.

Our initial result is the following

Proposition 3.1 *Let Γ be a torsion free discrete subgroup of H. Let $V = \Gamma\backslash H/W$. Then $(V, \mathcal{L})$ admits a canonical (H, G)-Anosov structure. The developing map is the identification of H/W with the left orbit in H of the identity class in M. The corresponding holonomy representation is the injection of Γ in G through H. We call such a representation an* (H, G)-Fuchsian representation.

PROOF: We let H act by the left on $M = G/U$. Let m_0 be the class of the identity in M. We define F from H/W to M by

$$F(g) = gm_0.$$

We consider E the pulled back vector bundle on H defined by

$$E = F^*TM.$$

We also consider the bundles $E^{\pm}$ arising from the product structure on M. We wish to prove that the right A action on $E^{\pm}$ is contracting/dilating. We observe that H acts by the left on E by an action that lifts the standard left action of H on H/W. We denote by g_* the linear map from E_{m_0} to E_{gm_0} associated to the action of an element g of H.

We recall that W is compact and that $Wm_0 = m_0$. We may now choose a metric q_{m_0} on E_{m_0} invariant under the action of W. We now equip the bundle E with the metric q defined by $q_{gm_0}(u, u) = (g^*q_{m_0})(u, u) = q_{m_0}(g_*^{-1}(u), g_*^{-1}(u))$. This is a well defined metric.

Note that this metric is invariant under the left action and hence under the action of any discrete subgroup. Furthermore, every left H-invariant metric on E arises from this construction.

We finally have a right action of A on M commuting with the left H action. This action of A preserves globally the orbit $F(H/W)$. Whenever H/W is identified with the unit tangent bundle of the symmetric space of H, the corresponding action of A on H/W is the geodesic flow. We therefore obtain a right action of A on E. If a is an element of A and q a left H invariant metric, then $\tilde{q} = a^*q$ is also a H-invariant metric which is completely determined by q. By construction, we have $\tilde{q}_{m_0} = Ad(a)q_{m_0}$, it thus follows that the action of A on $E^{\pm}$ is contracting/dilating. Q.E.D.

3.1.2 Anosov, quasi-Fuchsian representations and limit curves.

We now assume that Γ is a cocompact lattice. We define an (H, G)-*Anosov representation* of Γ in G as the holonomy of an (H, G)-Anosov structure on $\Gamma\backslash H/W$ with its geodesic flow. We define an (H, G)-*quasi-Fuchsian representation* in G as a representation in the connected component of Fuchsian representations of the set of (H, G)-Anosov representations: therefore, an (H, G)-quasi-Fuchsian representation is a representation that can be deformed through Anosov representations to a Fuchsian one. By Proposition 2.1, the set of (H, G)-Anosov representations is open. One may check that the $(PSL(2, \mathbb{R}), PSL(2, \mathbb{C}))$-quasi-Fuchsian representations coincides with quasi-Fuchsian representations in the classical sense. We recall that, in the classical case, a quasi-Fuchsian representation preserves a quasi circle on $\mathbb{C}P^1$. We now describe the non-classical counterpart of this result.

Proposition 3.2 *Let Γ be a cocompact lattice in H. Let ρ be an (H, G)-Anosov representation of Γ in G. Let $\partial_\infty\Gamma$ be the boundary at infinity of Γ. Then, there* *exist Hölder ρ-equivariant mappings $\xi^{\pm}$ from $\partial_\infty\Gamma$ to $G/P^{\pm}$ called the* positive and negative limit curves of ρ. *Moreover*

- *if $x \neq y$, then $\xi^+(x)$ and $\xi^-(y)$ are opposite parabolics, and*
- *if γ^+ is an attractive fixed point of γ in $\partial_\infty\Gamma$, then $\xi^{\pm}(\gamma^+)$ is an attractive fixed point of $\rho(\gamma)$ in $G/P^{\pm}$.*

Proof: By definition of an Anosov structure, the stable and unstable manifolds of ϕ_t (along $F(\tilde{V})$) are the right and left orbit foliations by P^+ and P^-. Furthermore, these foliations are well known to be Hölder (see Theorem 19.1.6. of [12])

We therefore have ρ equivariant Hölder maps from $\tilde{V}$ to G/P^+ and G/P^- which are constant along the central stable (resp. unstable) foliations of the geodesic flow of H/W. Since the space of these central stable leaves is identified with $\partial_\infty\Gamma$, the result now follows and the final two statements are immediate.

Q.e.d.

3.2 Irreducible copy of $PSL(2,\mathbb{R})$ in $PSL(n,\mathbb{R})$.

From now on we concentrate on the following example. First, we consider $V = US$, the unit tangent bundle of a compact hyperbolic surface, and $\tilde{V}$ the unit tangent bundle of the universal cover of S. We consider the lamination $\mathcal{L}$ given by the orbit foliation of the geodesic flow. We also consider $\mathcal{F}^{\pm}$ the central stable and unstable foliations of the geodesic flow. It is well known that this data depends only on the fundamental group $\pi_1(S)$ of the surface. Indeed we may describe this data the following way. Let

$$\Delta_3 = \{(x_1, x_2, x_3) \in (\partial_\infty\pi_1(S))^3 / \exists i \neq j,\ x_i = x_j\}.$$

Let us choose an arbitrary orientation on $\partial_\infty\pi_1(S)$ and let $\partial_\infty\pi_1(S)^{3+}$ be the space of positively ordered triples. Then the following identification holds

$$\begin{array}{rcl} \tilde{V} & = & \partial_\infty\pi_1(S)^{3+} \setminus \Delta_3 \\ V & = & (\partial_\infty\pi_1(S)^{3+} \setminus \Delta_3)/\pi_1(S). \end{array}$$

Furthermore, for every $x = (x_+, x_-, x_0)$ in US, the leaf $\mathcal{L}_x$ of $\mathcal{L}$ through x in $\tilde{V}$ is

$$\mathcal{L}_x = \{(y_+, y_-, y_0)/y_+ = x_+, y_- = x_-\}.$$

Similarly

$$\mathcal{F}^{\pm}_x = \{y_+, y_-, y_0)/x_{\pm} = y_{\pm}\}.$$

We model these flows on a specific situation, namely we consider

- $G = PSL(n,\mathbb{R})$, and
- H the image of the irreducible representations of $PSL(2,\mathbb{R})$.

In order to simplify our notation we shall speak of *n-quasi-Fuchsian representations* (resp. n-Anosov structures), or just quasi-Fuchsian representations when there is no ambiguity, instead of $(H, PSL(n,\mathbb{R}))$-quasi-Fuchsian representations.

3.2.1 Description of the model

In the case in question, A lies in the interior of the Weyl Chamber and $U = Z_0(A)$ is nothing other than the full Cartan subgroup of G, that is the subgroup of diagonal matrices in a given basis. It is useful to think of $M = G/U$ as an open set in $Flag \times Flag$, where $Flag$ is the space of flags.

We recall that a point of M is a family of n lines $\mathbb{L} = \{L_i\}_{i\in\{1,\ldots,n\}}$ in a direct sum.

3.2.2 A vector bundle description of n-Anosov representations

We immediately have

Proposition 3.3 *Let ρ be an n-Anosov representation of $\pi_1(S)$ in $PSL(n,\mathbb{R})$ which can be lifted to $SL(n,\mathbb{R})$. Let E be the associated $\mathbb{R}^n$ bundle over $V = US$ with its flat connection ∇. Then E splits as the sum of n continuous line bundles V_i which are parallel along the leaves of $\mathcal{L}$. Moreover, let $E^+ = (E_i^+)$ (resp. $E^- = (E_i^-)$) be the corresponding positive and negative flag bundles,*

$$\begin{aligned} E_i^+ &= \bigoplus_{j=1}^{j=i} V_j \\ E_i^- &= \bigoplus_{j=n-i-1}^{j=n} V_j. \end{aligned}$$

The subbundle E_i^+ (resp. E_i^-) is parallel along $\mathcal{F}^+$ (resp. $\mathcal{F}^-$). Finally, if we lift the action of $\mathcal{L}$ by the connection, this action is contracting on $V_i^* \otimes V_j$ for $i > j$.

Furthermore, if we lift the vector bundle E over $\tilde{V}$ and identify this bundle with the trivial bundle $\mathbb{R}^n \times \tilde{V}$ by the flat connection, we have the following identification with the positive and negative limit curves of ρ

$$E^{\pm}_{(x_+,x_0,x_-)} = \xi^{\pm}(x_{\pm}). \tag{4}$$

Conversely, the holonomy of such a connection is an n-Anosov representation.

Proof: It suffices to remark that $\mathbb{L} = (V_1,\ldots,V_n)$ is a section of the associated $M = PSL(n,\mathbb{R})/U$ bundle and the tangent spaces to the associated foliations are

$$\begin{aligned} E^+ &= \bigoplus_{i>j} (V_i^* \otimes V_j), \\ E^- &= \bigoplus_{i<j} (V_i^* \otimes V_j). \end{aligned}$$

Q.E.D.

3.2.3 Faithfulness and discreteness

We recall that an element of a semi-simple Lie group is *purely loxodromic* if it is conjugate to an element in the interior of the Weyl chamber. In the case of $PSL(n,\mathbb{R})$, this just means that it is real split with eigenvalues of multiplicity 1.

Although purely loxodromic elements are referred by some as being*strictly hyperbolic*, we feel this latter terminology may be confusing from the dynamical systems point of view: purely loxodromic elements may well have 1 as an eigenvalue whereas this is not felt to be compatible with strict hyperbolicity for a dynamicist.

We then have

Proposition 3.4 *Let ρ be an n-Anosov representation. Then, for each γ in $\pi_1(S)$ different to the identity, $\rho(\gamma)$ is purely loxodromic. In particular ρ is faithful. Furthermore if ρ is n-quasi-Fuchsian, it is irreducible and discrete.*

PROOF: An element is purely loxodromic if it has an attractive fixed point in the space of flags. Consequently the first assertion of the proposition follows from Proposition 3.2. Next, ρ is obviously faithful since loxodromic elements are non trivial. Irreducibility follows from Lemma 10.1 and discreteness from Lemma 10.4 which are both proved in an independent appendix. Q.E.D.

3.2.4 Basic properties of limit curves and 2-hyperconvexity.

If ρ is an n-Anosov representation, by Proposition 3.2 we obtain two Hölder mappings ξ^+ and ξ^- from $\partial_\infty\pi_1(S)$ into the corresponding homogeneous spaces G/P^+ and G/P^-, which in our case are both identified with the space of flags. Since, for every attracting point γ^+ in $\partial_\infty\pi_1(S)$ of some element γ in $\pi_1(S)$, $\xi^\pm$ is an attracting point of $\rho(\gamma)$ in the space of flags and such an attracting point is unique for a loxodromic element, we conclude that $\xi^+(\gamma^+) = \xi^-(\gamma^+)$ and hence, by density of the fixed points, that $\xi^+ = \xi^-$.

From now on, we will write

$$\xi^\pm = \xi = (\xi^1, \xi^2, \ldots, \xi^{n-1}).$$

Here, ξ^i takes values in the Grassmannian of i-planes in $E = \mathbb{R}^n$. By definition, we have

$$\forall x \in \partial_\infty\pi_1(S),\ \xi^i(x) \subset \xi^{i+1}(x).$$

The curve ξ will be called the *limit curve* of ρ. We remark that, for $x \neq y$, $\xi(x)$ and $\xi(y)$ are transverse flags since they correspond to opposite parabolics (*cf.* Proposition 3.2). Hence, we have the following property which we shall call *2-hyperconvexity*,

$$\forall x, y \in \partial_\infty\pi_1(S), x \neq y \implies \xi^p(x) \oplus \xi^{n-p}(y) = E. \tag{5}$$

Indeed the curve ξ cannot be completely arbitrary; it must have some properties. For instance, in the $(PSL(2,\mathbb{R}), PSL(2,\mathbb{C}))$ situation, such curves are quasi-circles. It also follows from S. Choi and W. Goldman's work that in the $(PSL(2,\mathbb{R}), PSL(3,\mathbb{R}))$ case, the curve ξ^1 is C^1 and bounds a convex set [3].

4 Statement of the main results

We state now our main theorem concerning the properties of the curve ξ, which generalises S.-Y. Choi and W. Goldman's result.

4.1 Quasi-Fuchsian representations, limit curves and Hitchin components

Our main Theorem is a slight refinement of Theorem 1.4

Theorem 4.1 *Let ρ be a representation in a Hitchin component. Then ρ is quasi-Fuchsian. Furthermore, let*

$$\xi = (\xi^1, \xi^2, \ldots, \xi^{n-1})$$

be its limit curve. Then ξ^1 is a hyperconvex Frenet curve, and ξ is its osculating flag. Furthermore, for any triple of distinct points (x, y, z) of $\partial_\infty \pi_1(S)$ the following sum is direct,

$$\big(\xi^{k+1}(y) + \xi^{n-k-2}(x)\big) + (\xi^{k+1}(z) \cap \xi^{n-k}(x)) = E. \tag{6}$$

We recall that by definition, ξ is the *osculating flag* of the hyperconvex Frenet curve ξ^1 if:

- for $(x_1, \ldots, x_p)$ be pairwise distinct points of $\partial_\infty \pi_1(S)$, p be an integer, $(n_1, \ldots, n_l)$ be positive integers such that

  $$l = \sum_{i=1}^{i=p} n_i \leq n$$

  , the following sum is direct

  $$\sum_{i=1}^{i=l} \xi^{n_i}(x_i), \tag{7}$$

- and, for every $x \in \partial_\infty \pi_1(S)$,

  $$\lim_{(y_1,\ldots,y_p)\to x, y_i \text{all distinct}} \Big(\bigoplus_{i=1}^{i=p} \xi^{n_i}(y_i)\Big) = \xi^l(x). \tag{8}$$

As a consequence, ξ is completely determined by ξ^1, and ξ^1 is a C^1 curve. Theorem 4.1 together with Proposition 3.4 give rise to Theorem 1.5. Theorem 4.1 is proved in Paragraph 9.1.1.

4.2 Converse results

It turns out that the curve ξ^1 contains all the information needed to reconstruct the geometric structure.

Theorem 4.2 *Let ρ be a representation of $\pi_1(S)$ in $SL(E)$. Let ξ^1 be a ρ-equivariant continuous map from $\partial_\infty\pi_1(S)$ to $\mathbb{P}(E)$. Assume that for all distinct points $(x_1, \ldots, x_n)$, we have the following direct sum*

$$\xi^1(x_1) + \ldots + \xi^1(x_n) = E.$$

Then ρ is an n-Anosov representation and ξ^1 is the projection in $\mathbb{P}(E)$ of the limit curve ξ of ρ. Finally, ξ^1 is a hyperconvex Frenet curve and ξ is its osculating flag.

This theorem is proved in Paragraph 6.2. Unfortunately, we cannot prove that every n-Anosov representation is quasi-Fuchsian. To our present knowledge, the set of n-Anosov representations may well not be connected. However, Olivier Guichard recently proved the following result, which was conjectured in an earlier version of the present paper and which gives a complete geometric characterisation of Hitchin components [9]

Theorem 4.3 [Guichard] *Let ρ be a representation of $\pi_1(S)$ in $SL(E)$. Let ξ^1 be a ρ-equivariant continuous map from $\partial_\infty\pi_1(S)$ to $\mathbb{P}(E)$. Suppose that for all distinct points $(x_1, \ldots, x_n)$, we have the following direct sum*

$$\xi^1(x_1) + \ldots + \xi^1(x_n) = E.$$

Then the representation ρ is in Hitchin component.

These two theorems provide a converse result to Theorem 4.1.

5 Hyperconvex curves

5.1 Definition and notations

Let ξ be a map from an interval J to $\mathbb{P}(E)$. We say that ξ is *hyperconvex*, if for all n-uples of distinct points $(x_1, \ldots, x_n)$, we have

$$\xi(x_1) + \ldots + \xi(x_n) = E.$$

If $p \leq n$, and if $X = (x_1, \ldots, x_p)$ is a p-uple of distinct points, we will write

$$\xi^{(p)}(X) = \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_p).$$

We also say that $X < x$ if for all i, $x_i < x$. We write $X \to x^+$ as a shorthand for $X \to x, X < x$. We use a similar convention for $X \to x^-$.

We shall actually need a refinement of the notion of hyperconvexity in order to take into account *non continuous* maps. Let Ω be an orientation on E. Let ξ

be a, not necessarily continuous, map from an interval J to $\mathbb{P}(E)$. We say that ξ is $*$-*hyperconvex* if the following sum is direct,

$$\xi^{(p)}(X) = \xi(x_1) + \ldots + \xi(x_p),$$

and if, there exists a map $\hat{\xi}$, the *lift of* ξ, with values in $E \setminus \{0\}$ such that the following holds

1. for all y in J, $\hat{\xi}(y) \in \xi(y)$, and
2. for every l n-uple of distinct increasing points $X = (x_1, \ldots, x_n)$, we have

   $$\Omega(\hat{\xi}(x_1), \ldots, \hat{\xi}(x_n)) \geq 0.$$

   Note that this last inequality and the first condition actually imply that

   $$\Omega(\hat{\xi}(x_1), \ldots, \hat{\xi}(x_n)) > 0. \tag{9}$$

   The existence of this "coherent" lift should be understood in the following way: the map ξ, though not being continuous, preserves some ordering.

It is also obvious that such a lift exists whenever ξ is continuous. It follows that every hyperconvex curve defined on a (contractible) interval is in particular $*$-hyperconvex.

5.2 Left and right osculating flags

The main result of this section is the following

Lemma 5.1 *Let ξ be an $*$-hyperconvex map from J to $\mathbb{P}(E)$. Assume that the sequence $\{X_m\}_{m\in\mathbb{N}}$ (resp. $\{Y_m\}_{m\in\mathbb{N}}$) converges to $(x_1, \ldots, x_p)$ (resp. $(y_1, \ldots, y_{n-p})$) with*

$$x_1 \leq \ldots \leq x_p < y_1 \leq y_2 \ldots \leq y_{n-p}.$$

Assume also that $\{\xi^{(p)}(X_m)\}_{m\in\mathbb{N}}$ (resp. $\{\xi^{(n-p)}(Y_m)\}_{m\in\mathbb{N}}$) converges to F (resp. G).

Then

$$F \oplus G = E. \tag{10}$$

Furthermore, for every p, with $n \geq p \geq 1$, there exist maps ξ^p_+ and ξ^p_- from J to $Gr(p, E)$ such that

$$\begin{aligned} \lim_{X \to x^\pm} \xi^{(p)}(X) &= \xi^p_\pm(x), && (11)\\ \lim_{(z,y) \overset{z\neq y}{\to} x^\pm} (\xi(z) \oplus \xi^p_\pm(y))) &= \xi^{p+1}_\pm(x), && (12)\\ \xi^p_\pm(x) &\subset \xi^{p+1}_\pm(x). && (13) \end{aligned}$$

Finally, if $\xi_+^p = \xi_-^p$, then both maps are continuous and

$$\lim_{X\to x} \xi^{(p)}(X) \quad = \quad \xi_{\pm}^p(x), \tag{14}$$

In particular, if $\xi_+^1 = \xi_-^1$, then both maps are equal to ξ and the latter is continuous.

We shall begin with some preliminaries concerning *increasing maps* before proving the lemma.

5.3 Increasing maps

We clarify some properties of *increasing maps*. Let p an arbitrary integer. Let I be an oriented interval. We define

$$I^{(p)} = \{(x_1, \ldots, x_p) \in I^p / x_i \leq x_{i+1}\}.$$

We define partial orderings on $I^{(p)}$ by

$$\begin{aligned}(x_1, \ldots, x_p) &\leq (y_1, \ldots, y_p) \text{ iff } \forall i,\ x_i \leq y_i, \\ (x_1, \ldots, x_p) &< (y_1, \ldots, y_p) \text{ iff } \forall i,\ x_i < y_i,\end{aligned}$$

We also define

$$\hat{I}^{(p)} = \{(x_1, \ldots, x_p) \in I^p / x_i < x_{i+1}\}.$$

Let f be a map from $\hat{I}^{(p)}$ to $\mathbb{R}$. We say that f is *increasing* if for every $(x_1, \ldots, x_p)$ in $\hat{I}^{(p)}$ and for every j,

$$\begin{aligned} x_{j-1} < z < y < x_{j+1} \quad &\Longrightarrow \\ f(x_0, \ldots, x_{j-1}, z, x_{j+1}, \ldots, x_p) \quad &\leq \quad f(x_0, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_p).\end{aligned}$$

Note that this immediately implies

$$X \leq Y \implies f(X) \leq f(Y).$$

For f an increasing map and X in $I^{(p)}$, we define

$$\begin{aligned} f^+(X) &= \inf_{Y>X} f(Y) \\ f^-(X) &= \sup_{Y<X} f(Y).\end{aligned}$$

The next proposition summarises the properties that we shall need in the sequel. All these properties are immediate.

Proposition 5.2 *Suppose that f is increasing. Then*

$$
\begin{aligned}
f^-(X) &= \lim_{Y \overset{Y \leq X}{\to} X} (f(Y)) && (15)\\
f^+(X) &= \lim_{Y \overset{Y \geq X}{\to} X} (f(Y)) && (16)\\
f^-(X) &\leq f^+(X) && (17)\\
f^+(X) &\leq f^-(Y), \text{if } X < Y, && (18)\\
f^-(X) &\geq f^+(Y), \text{if } X > Y. && (19)
\end{aligned}
$$

Finally, suppose that $f^+ = f^-$ are everywhere equal. Then they are both continuous and

$$\lim_{Y \to X} f(Y) = f^+(X).$$

5.4 $*$-Hyperconvex curves and increasing maps

We begin with the following observation which follows at once from hyperconvexity. Let $(y_1, \ldots, y_{n-1}, w_1, w_2)$ be distinct points of the one-dimensional manifold J. We denote $Y = (y_1, \ldots, y_{n-1})$ and $W = (w_1, w_2)$. Let ξ be a $*$-hyperconvex curve from J to $\mathbb{P}(E)$. Then

$$\dim\left(\xi^{(n-1)}(Y) \cap \xi^{(2)}(W)\right) = 1. \tag{20}$$

Let I an interval contained in $J \setminus \{w_1, w_2\}$. We now consider the map f_W

$$\begin{cases} \hat{I}^{(n-1)} & \rightarrow \quad \mathbb{P}\big(\xi^{(2)}(W)\big) \setminus \{\xi(w_1)\} \\ Y = (y_1, \ldots y_{n-1}) & \mapsto \quad \xi^{(n-1)}(Y) \cap \xi^{(2)}(W). \end{cases}.$$

We observe that by Assertion (20), and since

$$\xi(y_1) \oplus \ldots \oplus \xi(y_{n-1}) \oplus \xi(w_1) = E,$$

this map is well defined. We now prove

Proposition 5.3 *Assume that the lift $\hat{\xi}$ of ξ is well defined on J. Then, for a suitable choice of orientations on I and on $\mathbb{P}\big(\xi^{(2}(W)\big) \setminus \{\xi(w_1)\}$, the map f_W is increasing.*

Proof: Let $u_1 = \hat{\xi}(w_1)$. If $y_1 < y_2 < \ldots < y_{n-1}$ are in I, by Inequality 9, we have

$$\Omega(u_1, \hat{\xi}(y_1), \ldots, \hat{\xi}(y_{n-1})) > 0.$$

We choose the orientation on $\xi^{(2)}(W)$ given by the form

$$\omega(w, t) = \Omega(\hat{\xi}(y_1), \ldots, \hat{\xi}(y_{n-2}), w, t).$$

By Inequality (9), we observe this orientation is independent of the choice of $(y_1, \ldots, y_{n-2})$ in I, provided that

$$y_1 < \ldots < y_{n-2}.$$

This choice gives an ordering on $\mathbb{P}\big(\xi^{(2}(W)\big) \setminus \{\xi(w_1)\}$ in the following way. For every L in $\mathbb{P}\big(\xi^{(2}(W)\big) \setminus \{\xi(w_1)\}$, we choose $x(L)$ in L such that $\omega(u_1, x(L)) > 0$. Next, we say that $L < L'$ if

$$\omega(x(L), x(L')) > 0.$$

We may now prove that the map f_W is increasing. Let

$$Q = \xi(y_1) \oplus \ldots \xi(y_{n-2}).$$

Let z be such that

$$y_1 < y_2 < \ldots < y_{j-1} < z < y_j < \ldots < y_{n-2}.$$

Let

$$L_z = f_W(y_1, \ldots, y_{j-1}, z, y_j, \ldots, y_{n-2}).$$

Let $\hat{x}(z)$ in L_z be such that

$$\hat{x}(z) = (-1)^{n-j-1}\hat{\xi}(z) + w(z), \quad w(z) \in Q.$$

Then $\omega(u_1, \hat{x}(z)) > 0$. Assume now that

$$y_1 < \ldots < y_{j-1} < z < t < y_j < \ldots < y_{n-2}.$$

Then we have,

$$\begin{aligned}
\omega(\hat{x}(z), \hat{x}(t)) &= \Omega(\hat{\xi}(y_1), \ldots, \hat{\xi}(y_{n-2}), \hat{x}(z), \hat{x}(t)) \\
&= \Omega(\hat{\xi}(y_1), \ldots, \hat{\xi}(y_{j-1}), \hat{\xi}(z), \hat{\xi}(t), \hat{\xi}(y_j), \ldots, \hat{\xi}(y_{n-2})) \\
&> 0.
\end{aligned}$$

Consequently,

$$f_W(y_1, \ldots, y_{j-1}, z, y_j, \ldots, y_{n-2}) < f_W(y_1, \ldots, y_{j-1}, t, y_j, \ldots, y_{n-2}).$$

Q.E.D.

5.5 Proof of Lemma 5.1

5.5.1 First step: Assertion (11)

PROOF: We use the notations of the previous paragraph. Let p be an integer less than n. Let x be a point in J, I a small neighbourhood of x and $Z = (y_1, \ldots, y_{n-p-1}, w_1, w_2)$ some tuple of cyclically oriented points in $J \setminus \{x\}$. Let us denote $Y = (y_1, \ldots, y_{n-p-1})$, $W = (w_1, w_2)$. By Propositions 5.2 and 5.3, there exist maps $F^{\pm}_{p,Y,W}$ from I to $\mathbb{P}(\xi^{(2)}(W))$ such that

$$\lim_{X \mapsto x^{\pm}} \Big((\xi^{(p)}(X) \oplus \xi^{(n-p-1)}(Y)) \cap \xi^{(2)}(W) \Big) = F^{\pm}_{p,Y,W}(x).$$

Using the fact that the choice of Z is arbitrary, we now show that there exist maps $\xi^p_\pm$ verifying Assertion (11) characterised by

$$\left(\xi^p_\pm(x) \oplus \xi^{(n-p-1)}(Y)\right) \cap \xi^{(2)}(W) = F^\pm_{p,Y,W}(x).$$

We justify this last point in detail.

This is achieved in two steps. First, we fix Y. Let U be an interval of $J\pi_1(S) \setminus I \cup Y$. We consider the subspace

$$H^\pm_Y(x) = \sum_{W\in U^{(2)}} F^\pm_{p,Y,W}(x).$$

Note that,

$$\sum_{W\in U^{(2)}} \Big(\left(\xi^{(p)}(X) \oplus \xi^{(n-p-1)}(Y)\right) \cap \xi^{(2)}(W)\Big) \subset \xi^{(p)}(X) \oplus \xi^{(n-p-1)}(Y).$$

Hence,

$$\dim\Big(\sum_{W\in U^{(2)}} \left(\xi^{(p)}(X) \oplus \xi^{(n-p-1)}(Y)\right) \cap \xi^{(2)}(W)\Big) \leq n-1. \tag{21}$$

We deduce that $\dim(H^\pm_Y) \leq n-1$. We now prove that $\dim(H^\pm_Y) = n-1$ and

$$\lim_{X\mapsto x^\pm} (\xi^{(p)}(X) \oplus \xi^{(n-p-1)}(Y)) = H^\pm_Y(x). \tag{22}$$

Let $\{X_n\}_{n\in\mathbb{N}}$ be a subsequence converging to (say) x^+, such that

$$P_n = \xi^{(p)}(X_n) \oplus \xi^{(n-p-1)}(Y),$$

converges to some hyperplane H. By hyperconvexity, we choose w_1 in U such that

$$H \oplus \xi(w_1) = E.$$

By hyperconvexity again, we choose $(w_2, \ldots, w_n)$ in U such that

$$\xi(w_1) \oplus \ldots \xi(w_n) = E.$$

Let $W_i = (w_1, w_i)$. We then observe that

$$H = \bigoplus_i (H \cap \xi^{(2)}(W_i)).$$

Since

$$H \cap \xi^{(2)}(W_i) = F^+_{p,W_i,Y}(x).$$

It follows that $H \subset H^\pm_Y$, hence $\dim(H^\pm_Y) \geq n-1$. Combining this with Inequality (21), we obtain that $H = H^\pm_Y$, and Assertion (22) now follows.

Our next step uses a similar approach. We consider an interval U not containing x, and we define

$$\xi_{\pm}^{p}(x) = \bigcap_{Y \in U^{(n-p-1)}} H_{p,Y}^{\pm}(x).$$

Since

$$\xi^{(p)}(X) \subset \bigcap_{Y \in U^{(n-p-1)}} (\xi^{(p)}(X) \oplus \xi^{(n-p-1)}(Y)),$$

we obtain

$$\dim \xi_{\pm}^{p}(x) \geq p. \tag{23}$$

We now prove Assertion (11) and that $\dim \xi_{\pm}^{p}(x) \leq p$. As before, let again $\{X_n\}_{n\in\mathbb{N}}$ be a subsequence converging to (say) x^+, such that $\xi^{(p)}(X_n) \oplus \xi^{(n-p-1)}(Y)$ converges to some p-plane P. By hyperconvexity, we may now choose $(y_1, \ldots, y_{n-p})$ in U such that

$$P \oplus \xi(y_1) \oplus \ldots \oplus \xi(y_{n-p}) = E.$$

We denote $Y_i = (\ldots, y_j, \ldots)_{j \neq i}$, and we observe that

$$P = \bigcap_i (P \oplus \xi^{(n-p-1)}(Y_i)).$$

In particular, since

$$P \oplus \xi^{(n-p-1)}(Y_i) = H_{p,Y_i}(x),$$

we obtain that $\xi_{\pm}^{p}(x) \subset P$. Consequently by Inequality (23) $\xi_{\pm}^{p}(x) = P$ and Assertion (11) now follows. Q.E.D.

5.5.2 Second step: completion of the proof of Lemma 5.1

Proof: Assume that $\{X_m\}_{m\in\mathbb{N}}$ (resp. $\{Y_m\}_{m\in\mathbb{N}}$) converges to $(x_1, \ldots, x_p)$ (resp. $(y_1, \ldots, y_{n-p})$) with

$$x_1 \leq \ldots \leq x_p < y_1 \leq y_2 \ldots \leq y_{n-p}.$$

Assume that $\{\xi^{(p)}(X_m)\}_{m\in\mathbb{N}}$ (resp. $\{\xi^{(n-p)}(Y_m)\}_{m\in\mathbb{N}}$) converges to F (resp. G). We aim to show that

$$F \oplus G = E. \tag{24}$$

We assume the contrary. We thus consider the smallest integer m for which there exist integers p and q, such that $p + q = m$, satisfying the following property: there exist sequences $\{X_m\}_{m\in\mathbb{N}}$ (resp. $\{Y_m\}_{m\in\mathbb{N}}$) converging to $(x_1, \ldots, x_p)$ (resp. $(y_1, \ldots, y_{n-p})$) with

$$x_1 \leq \ldots \leq x_p < y_1 \leq y_2 \ldots \leq y_{n-p}.$$

such that

- $\{\xi^{(p)}(X_m)\}_{m\in\mathbb{N}}$ (resp. $\{\xi^{(q)}(Y_m)\}_{m\in\mathbb{N}}$) converges to P (resp. Q), and
- $P\cap Q\neq\{0\}$.

Let $H=P+Q$. For m large enough, we denote $X_m=(x_1^m,\ldots,x_p^m)$ and $Y_m=(y_1^m,\ldots,y_q^m)$, with $x_i^m<x_{i+1}^m<y_j^m<y_{j+1}^m$. We introduce

$$X_m^-=(x_1^m,\ldots,x_{p-1}^m),\quad Y_m^-=(y_2^m,\ldots,y_q^m).$$

After extracting a subsequence, we may assume that $\{\xi^{(p-1)}(X_m^-)\}_{m\in\mathbb{N}}$ and $\{\xi^{(q-1)}(Y_m^-)\}_{m\in\mathbb{N}}$ converge respectively to P^- and Q^-. By the minimality of $m=p+q$, we obtain

$$P^-\oplus Q=P\oplus Q^-=P+Q=H. \tag{25}$$

By hyperconvexity, we may choose a collection $Z=(z_1,\ldots,z_{n-p-q})$ of points in U, and pair of points $W=(w_1,w_2)$ not in U such that the following sums are direct

$$\begin{aligned} H+\xi^{(n-p-q)}(Z)+\xi(w_1) &= E, & (26)\\ P^-+Q^-+\xi^{(n-p-q)}(Z)+\xi^{(2)}(W) &= E. & (27)\end{aligned}$$

Using the notations of Paragraph 5.4, we now consider the family of maps g_m defined by

$$g_m(t)=f_W(X_m^-,t,Y_m^-,Z).$$

By Proposition 5.3, all these maps are increasing. Using (25) and (26), we obtain that

$$\lim_{m\to\infty}(g_m(x_p^m))=\lim_{m\to\infty}(g_m(y_1^m))=(H\oplus\xi^{(n-p-q)}(Z))\cap\xi^{(2)}(W):=D.$$

We recall that

$$\lim_{m\to\infty}x_p^m=x_p,\quad \lim_{m\to\infty}y_1^m=y_1.$$

Since all the maps g_m are increasing, it follows that for all t in the interval I joining x_p and y_1, we have

$$\lim_{m\to\infty}(g_m(t))=D.$$

On the other hand by (27), for all t in I, we have

$$\begin{aligned} W_m(t) &:= \xi^{(p-1)}(X_m^-)\oplus\xi^{(q-1)}(Y_m^-)\oplus\xi(t)\oplus\xi^{(n-p-q)}(Z)\\ &= \xi^{(p-1)}(X_m^-)\oplus\xi^{(q-1)}(Y_m^-)\oplus\xi^{(n-p-q)}(Z)\oplus g_m(t).\end{aligned}$$

It thus follows that for all t in I,

$$\xi(t)\subset\lim_{n\to\infty}W_m(t)=P^-\oplus Q^-\oplus\xi^{(m-n-2)}(Z)\oplus D\subsetneq E.$$

This last assertion contradicts the hypothesis of hyperconvexity, hence concludes the proof of Assertion (10).

Assertions (10) and (11) imply trivially Assertion (13). Finally we observe that the final assertion concerning the case where $\xi_+^p=\xi_-^p$ is a consequence of the last statement of Proposition and the result now follows. 5.2.Q.E.D.

6 Preserving hyperconvex curves

We prove the following converse of Proposition 3.3.

Theorem 6.1 *Let ρ be a representation of $\pi_1(S)$ in $SL(E)$. Let ξ^1 be a ρ-equivariant $*$-hyperconvex map from $\partial_\infty\pi_1(S)$ to $\mathbb{P}(E)$. Suppose that for all integers p less than n there exist , ρ-equivariant maps $\xi^p_\pm$ from $\partial_\infty\pi_1(S)$ to $Gr(p,E)$ such that*

$$\lim_{y\to x^\pm}(\xi^1(y)\oplus\xi^p_\pm(x)) \quad = \quad \xi^{p+1}_\pm(x). \tag{28}$$

Suppose moreover that if

- *$\{x_m\}_{m\in\mathbb{N}}$ (resp. $\{y_m\}_{m\in\mathbb{N}}$) converges to x, (resp. to y), with $x\neq y$,*
- *$p+q\leq n$ and $Z=(z_1,\ldots,z_{n-p+q})$ are $n-p-q$ points pairwise distinct and different from x and y,*
- *$\{\xi^p_+(x_m)\}_{m\in\mathbb{N}}$ (resp. $\{\xi^p_-(x_m)\}_{m\in\mathbb{N}}$, $\{\xi^q_+(y_m)\}_{m\in\mathbb{N}}$) converges to P^+ (resp. to P^-, Q),*

then

$$P^\pm\oplus Q\oplus\xi^1(z_1)\oplus\ldots\oplus\xi^1(z_{n-p-q}) \quad = \quad E, \tag{29}$$

As a conclusion, then

- *ρ is n-Anosov,*
- *$\xi^p_+=\xi^p_-$,*
- *$(\xi^1,\xi^2_-,\ldots,\xi^{n-1}_-)$ is the limit curve of ρ.*

The following corollary is immediate

Corollary 6.2 *Let ρ be a representation of $\pi_1(S)$ in $SL(E)$. Let*

$$\xi=(\xi^1,\ldots,\xi^{n-1})$$

be a ρ-equivariant continuous map from $\partial_\infty\pi_1(S)$ to $Flag(E)$. Assume that ξ^1 is hyperconvex and that

$$\begin{aligned}\forall x,y\in\partial_\infty\pi_1(S),\forall p,\ x\neq y \quad &\Longrightarrow \quad \xi^p(x)\oplus\xi^{n-p}(y)=E,\\ \lim_{y\to x}(\xi^1(y)\oplus\xi^p(x)) \quad &= \quad \xi^{p+1}(x).\end{aligned}$$

Then ρ is n-Anosov and ξ is the limit curve of ρ.

In Paragraph 6.2, we obtain Theorem 4.2 as a corollary to these results.

6.1 Proof of Theorem 6.1

PROOF: Let us choose an orientation on $\partial_\infty \pi_1(S)$. Let

$$M = \{(x, y, w) \in \partial_\infty \pi_1(S)^3, \text{ distinct and cyclically oriented }\}.$$

Note that $\pi_1(S)$ acts properly on M, in such a way that the quotient is compact and homeomorphic to the unit tangent bundle of the surface S. We write the generic element x of M as $x = (x_+, x_0, x_-)$. Consider on M the lamination whose leaves are

$$\mathcal{L}_{x_+,x_-} = \{(x_+, w, x_-) \in M / w \in \partial_\infty \pi_1(S)\}.$$

This lamination is $\pi_1(S)$ equivariant and its quotient is identified with the lamination by leaves of the geodesic flow. Consider also the following 2-dimensional laminations whose leaves are:

$$\begin{aligned} \mathcal{F}^+_{x_+} &= \{(x_+, w, y) \in M/w, y \in \partial_\infty \pi_1(S)\} \\ \mathcal{F}^-_{x_-} &= \{(y, w, x_-) \in M/w, y \in \partial_\infty \pi_1(S)\}. \end{aligned}$$

Now consider the E-associated bundle on $M/\pi_1(S)$ to ρ, also denoted abusively by E. Consider the subbundles E_i^+ and E_i^- of E given by

$$\begin{aligned} E^+_{i\ (x_+,x_0,x_-)} &= \xi^i_+(x_+) \\ E^-_{i\ (x_+,x_0,x_-)} &= \xi^i_-(x_-). \end{aligned}$$

Note these bundles are not *a priori* continuous. The bundle E_i^+ (resp. E_i^-) is parallel along the leaves of $\mathcal{F}^+$ (resp. $\mathcal{F}^-$). Let $V^i = E_i^+ \cap E^-_{n-i+1}$. This 1-dimensional subbundle of E is well defined (by Hypothesis (29)), which is parallel along the leaves of $\mathcal{L}$. We observe that $E^+_{i-1} \oplus E^-_{n-i}$ is a subspace supplementary to V^i is . We denote by α_i a 1-form whose kernel is this supplementary subspace.

We first define a metric on $((V^i)^* \otimes V^{i+1})_w$. Write $w = (x_+, x_0, x_-)$. Let u be a nonzero element of V^i, $z(w)$ a nonzero element of $\xi^1(x_0)$. The metric on $((V^i)^* \otimes V^{i+1})_w$ is given by

$$\|\phi\|_w = \Big|\frac{\langle \alpha_{i+1} | \phi(u)\rangle \langle \alpha_i | z(w)\rangle}{\langle \alpha_{i+1} | z(w)\rangle \langle \alpha_i | u\rangle}\Big|.$$

This metric is well defined since by Hypothesis (29) the following sum is direct

$$\xi_+^{j-1}(x_+) + \xi_-^{n-j}(x_-) + \xi^1(x_0),$$

and in particular for all j,

$$\langle \alpha_j | z(w)\rangle \neq 0.$$

This metric is obviously independent of the choice of u and $z(w)$. However, for the moment it is not obvious that this metric is continuous (although, one may easily show that it is bounded).

We introduce a definition. We say a $(V_i)^* \otimes V_{i+1}$ is a *weakly contracting bundle* for a metric if whenvever σ is a parallel section of $(V_i)^* \otimes V_{i+1}$ along a leaf of $\mathcal{L}$, then

$$\begin{aligned}\lim_{x_0 \to x_+} \|\sigma\|_{(x_+,x_0,x_-)} &= 0\\ \lim_{x_0 \to x_-} \|\sigma\|_{(x_+,x_0,x_-)} &= \infty.\end{aligned}$$

We will now prove that $(V_i)^* \otimes V_{i+1}$ are weakly contracting bundles for this metric. We observe that a parallel section σ of $(V_i)^* \otimes V_{i+1}$ along a leaf of $\mathcal{L}$ corresponds to a fixed element ϕ in $(V_i)^* \otimes V_{i+1}$. We have

$$\frac{\|\phi\|_w}{\|\phi\|_t} = \Big|\frac{\langle \alpha_{i+1}|z(t)\rangle\langle\alpha_i|z(w)\rangle}{\langle \alpha_{i+1}|z(w)\rangle\langle\alpha_i|z(t)\rangle}\Big|.$$

Let $w = (x_+, x_0, x_-)$ and imagine that x_0 converges to x_+. By Hypothesis (28), we now may choose $z'(w) = \alpha + \beta$ in

$$\xi^1(x_0) \oplus E^+_{i-1} = \xi^1(x_0) \oplus \xi^{i-1}_+(x_+),$$

with $0 \neq \alpha \in \xi^1(x_0)$ and $\beta \in \xi^{i-1}_+(x_+)$ such that $z'(w)$ converges to a nonzero vector $u \in \xi^i_+(x_+) \cap \xi^{n-i+1}_-(x_-)$ when x_0 converges to x_+. We observe that

$$\frac{\langle \alpha_{i+1}|z(t)\rangle\langle\alpha_i|z(w)\rangle}{\langle \alpha_i|z(t)\rangle\langle\alpha_{i+1}|z(w)\rangle} = \frac{\langle \alpha_{i+1}|z(t)\rangle\langle\alpha_i|z'(w)\rangle}{\langle \alpha_i|z(t)\rangle\langle\alpha_{i+1}|z'(w)\rangle}.$$

To conclude, we remark that

$$\lim_{x_0 \to x_+} \frac{\langle \alpha_i|z'(w)\rangle}{\langle\alpha_{i+1}|z'(w)\rangle} = \frac{\langle\alpha_i|u\rangle}{\langle\alpha_{i+1}|u\rangle} = \infty. \tag{30}$$

A similar reasoning when x_0 tends to x_-, implies that the bundles are weakly contracting.

We may now show that $\xi^p_+ = \xi^p_-$. First, by Hypothesis (29), $\xi^p_-(x_+)$ is the graph of a homomorphism ψ in $Hom(E^p_+, E^p_-) = (E^p_+)^* \otimes E^p_-$. Since $(E^p_+)^* \otimes E^p_-$ is a weakly contracting bundle and ψ is parallel, in order to prove that $\xi^p_+ = \xi^p_-$, it suffices to show $\|\psi\|$ is uniformly bounded. Let $\{x_m\}_{m\in\mathbb{N}}$ be a sequence of points in M such that $\{\|\psi\|_{x_m}\}_{m\in\mathbb{N}}$ tends to $+\infty$. Since $\pi_1(S)$ acts cocompactly on M and everything is invariant under $\pi_1(S)$, we can as well assume that $\{x_m\}_{m\in\mathbb{N}}$ converges to y in M. We may extract a subsequence such that

- for all i, $\{(V_i)_{x_m}\}_{m\in\mathbb{N}}$ converges to W_i in E_y.
- $\{(E^i_\pm)_{x_m}\}_{m\in\mathbb{N}}$ converges to $F^i_\pm$,
- $\{\xi^p_-(x_m)\}_{m\in\mathbb{N}}$ converges to Q .

By Hypothesis (29), Q is the graph of a map ϕ from F^p_+ to F^p_-. In particular, $\|\phi\|$ is bounded. The contradiction now follows, since

$$\lim_{m\to\infty}(\|\psi\|_{x_m}) = \|\phi\| \neq \infty.$$

Now that we know that $\xi_+^p = \xi_-^p$, we may conclude by Proposition 5.1 that both maps are continuous. The metric we have defined previously is therefore continuous. Repeating the argument above for the weakly contracting property, the limits that we obtain are now uniform. Given that, we obtain that the bundles are contracting. We perform now this step in more details. To prove that the bundles $(V_i)^* \otimes V_{i+1}$ are contracting, we need to show that there exists a constant $t_0 > 0$, such that if Ψ_t is the lift of the geodesic flow ψ_t on M, then for every vector σ in $(V_i)^* \otimes V_{i+1}$, we have

$$\forall t > t_0,\ \|\Psi_t(\sigma)\| \leq \frac{1}{2}\|\sigma|.$$

We shall prove this by contradiction. Suppose the contrary. There exists thus a sequence of points $\{w_n\}_{n\in\mathbb{N}}$ of $M/\pi_1(S)$ and a sequence $\{t_n\}_{n\in\mathbb{N}}$ converging to $+\infty$, a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ in $((V_i)^* \otimes V_{i+1})_{w_n}$ such that, for all n

$$\frac{\|\Psi_{t_n}(\sigma_n)\|}{\|\sigma_n\|} \geq \frac{1}{2}. \tag{31}$$

We may now lift the sequence w_n to M. Since the action of $\pi_1(S)$ is cocompact, we may assume the sequence converges to $w_0 = (x_+^0, x_0^0, x_-^0)$. We define

$$\begin{aligned} w_n &= (x_+^n, x_0^n, x_-^n), \\ \phi_{t_n}(w_n) &= (x_+^n, y_0^n, x_-^n). \end{aligned}$$

By assumption

$$\begin{aligned} \lim_{n\to\infty}(x_+^n) &= \lim_{n\to\infty}(y_0^n) = x_+^0, \\ \lim_{n\to\infty}(x_0^n) &= x_0^0, \\ \lim_{n\to\infty}(x_-^n) &= x_-^0 \end{aligned}$$

For all t, let $z(t)$ be a nonzero element in $\xi^1(t)$. We have

$$\Big|\frac{\langle\alpha_{i+1}|z(x_0^n)\rangle\langle\alpha_i|z(y_0^n)\rangle}{\langle\alpha_{i+1}|z(y_0^n)\rangle\langle\alpha_i|z(x_0^n)\rangle}\Big| = \frac{\|\Psi_{t_n}(\sigma_n)\|}{\|\sigma_n\|} \geq \frac{1}{2}. \tag{32}$$

Given that

$$\lim_{n\to\infty}(\xi^1(y_0^n) \oplus \xi^i(x_+^n)) = \xi^{i+1}(x_+^0),$$

we obtain as above

$$\lim_{n\to\infty}\Big(\frac{\langle\alpha_{i+1}|z(x_0^n)\rangle\langle\alpha_i|z(y_0^n)\rangle}{\langle\alpha_{i+1}|z(y_0^n)\rangle\langle\alpha_i|z(x_0^n)\rangle}\Big) = 0.$$

We thus obtain a contradiction. It follows that the bundles are indeed contracting. The conclusion now follows by Proposition 3.3. Q.E.D.

6.2 Proof of Theorem 4.2

Before proceeding to the proof, we shall prove a lemma of independent interest that will be used in the sequel.

6.2.1 Direct sums and limits

Our first lemma is the following:

Lemma 6.3 *Let ξ be the limit curve of an Anosov representation. Suppose that for all distinct points $(x_1, \ldots, x_q)$ in $\partial_\infty \pi_1(S)$, and integers $(n_1, \ldots, n_q)$ with $k = \sum n_i \leq n$, the following sum is direct*

$$\xi^{n_1}(x_1) + \ldots\ldots \xi^{n_q}(x_q).$$

Suppose moreover that

$$\lim_{(x_0, x_1, \ldots, x_l) \to x} \left(\xi^{n_1}(x_1) \oplus \ldots\ldots \xi^{n_q}(x_q)\right) \quad = \quad \xi^k(x). \tag{33}$$

Then, for all y distinct from $(x_1, \ldots, x_q)$, the following sum is direct

$$\xi^{n-k}(y) + \xi^{n_1}(x_1) + \ldots\ldots \xi^{n_q}(x_q).$$

Proof: If $(y, x_1, \ldots, x_q)$ is a collection of $q+1$ distinct points of $\partial_\infty \pi_1(S)$, it is a classical result that there exist two distinct points, t and z, and a sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ of elements of $\pi_1(S)$, such that

$$\begin{aligned} \forall i \leq q, \ \lim_n(\gamma_n(x_i)) &= t, \\ \lim_n(\gamma_n(y)) &= z. \end{aligned}$$

Now by 2-hyperconvexity,

$$\xi^k(z) \oplus \xi^{n-k}(t) = E.$$

By Hypothesis (33), for m sufficiently large, we have

$$\xi^{n-k}(\gamma_m(y)) \oplus \xi^{n_1}(\gamma_m(x_1)) \oplus \ldots \oplus \xi^{n_q}(\gamma_m(x_q)) = E.$$

Since

$$\xi^s(\gamma(w)) = \rho(\gamma)\xi^s(w),$$

The result now follows. Q.e.d.

6.2.2 Proof of Theorem 4.2

From Lemma 5.1 and Theorem 6.1, we deduce immediately that ρ is n-Anosov. Furthermore, ξ^1 is the projection in $\mathbb{P}(E)$ of the limit curve ξ of ρ, and we have, for every $x \in \partial_\infty \pi_1(S)$,

$$\lim_{(y_1, \ldots, y_l) \to x, y_i \text{all distinct}} \left(\bigoplus_{i=1}^{i=l} \xi^1(y_i)\right) = \xi^l(x). \tag{34}$$

To conclude the proof of Theorem 4.2, it suffices to verify Assertions (1) and (2) of the definition of Frenet hyperconvex curve.

We first prove Assertion (1). Let $(x_1, \ldots, x_p)$ be pairwise distinct points of $\partial_\infty \pi_1(S)$. Let p be an integer. Let $(n_1, \ldots, n_p)$ be positive integers such that

$$k = \sum_{i=1}^{i=p} n_i \leq n.$$

We want to prove that the following sum is direct

$$\xi^{n_i}(x_i) + \ldots + \xi^{n_p}(x_p). \tag{35}$$

We prove this result by induction on p. By 2-hyperconvexity, it is true for $p = 2$. Suppose that it is true for $p = q - 1$. By Assertion (34), we obtain that

$$\lim_{(x_1, \ldots, x_{q-1}) \to x} (\xi^{n_1}(x_1) \oplus \ldots \oplus \xi^{n_{q-1}}(x_{q-1})) = \xi^{k-n_q}(x). \tag{36}$$

The induction follows from Lemma 6.3. Finally, Assertion (2) is an immediate consequence of Assertions (34) and (1). Q.E.D.

7 Curves and Anosov representations.

7.1 Definitions

We introduce some definitions.

7.1.1 (p, l)-direct.

Let (p, l) be integers such that $p + l \leq n$. We say the limit curve ξ (or the corresponding representation ρ) is (p, l)-*direct* if for all distinct $(y, x_0, \ldots, x_l)$ the following sum is direct

$$\xi^{n-p-l}(y) + \xi^p(x_0) + \xi^1(x_1) + \ldots + \xi^1(x_l).$$

Note that to say that the representation is $(1, n-1)$-direct is equivalent to say that ξ^1 is hyperconvex.

7.1.2 (p, l)-convergent.

Let (p, l) be integers such that $p + l \leq n$. We say that the limit curve ξ (or the corresponding representation ρ) is (p, l)-*convergent* if firstly for all distinct points $(x_0, \ldots, x_l)$ in $\partial_\infty \pi_1(S)$ the following sum is direct

$$\xi^p(x_0) + \xi^1(x_1) + \ldots + \xi^1(x_l),$$

and, secondly

$$\lim_{(x_0, x_1, \ldots, x_l) \to x} \left(\xi^p(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_l)\right) = \xi^{p+l}(x).$$

7.1.3 3-hyperconvexity

We say that the limit curve (or the corresponding representation) is *3-hyperconvex*, if, for $k+p+l \leq n$ and distinct (x, y, z), the following sum is direct

$$\xi^k(x) + \xi^p(y) + \xi^l(z).$$

7.1.4 Property (H)

We say that the limit curve ξ (or the corresponding representation) satisfies *Property (H)*, if for every triple of distinct points x, y and z and integer k, we have

$$\xi^{k+1}(y) \oplus (\xi^{k+1}(z) \cap \xi^{n-k}(x)) \oplus \xi^{n-k-2}(x) \quad = \quad E.$$

7.2 Main Lemma

As a main step in the proof of Theorem 4.1, we shall prove the following lemma.

Lemma 7.1 *Let ξ be the limit curve of an Anosov representation. Suppose that*

- *ξ is 3-hyperconvex,*
- *ξ satisfies Property (H).*

Then, ξ is (k, l)-convergent for all integers with $k + l \leq n$. In particular, ξ^1 is hyperconvex.

We state a corollary that follows at once from Theorem 4.2 and gives the form in which this lemma will be used in the sequel. Lemma 7.1

Corollary 7.2 *Let ξ be the limit curve of an Anosov representation. Assume*

- *it is 3-hyperconvex,*
- *and satisfies Property (H).*

Then ξ^1 is a hyperconvex Frenet curve, and ξ is its osculating flag.

We begin with an observation that is a consequence of Lemma 6.3:

Lemma 7.3 *Let ξ be the limit curve of an Anosov representation. Assume the representation is (p, l)-convergent, then it is (p, l)-direct*

7.3 Bundles

In order to prove Lemma 7.1, we will require the vector bundle description of Proposition 3.3 along with some general preliminary results.

7.3.1 Hyperconvex rank 2 vector bundle

Here we describe the framework in which we will work. We begin by defining various conventions. Let M be a manifold. For any vector bundle F over M, we denote by F_x the fibre at a point x of a vector bundle F, for any foliation $\mathcal{L}$ we denote by $\mathcal{L}_x$ the leaf passing through x.

We are first interested in actions on compact manifolds of flows which preserve one dimensional foliations. Namely

- let ϕ_t be a flow of homeomorphisms of a compact topological manifold M.
- let $\mathcal{F}$ be a 1-dimensional foliation of M with no compact leaves. We suppose that this foliation is invariant under the flow of ϕ_t.

This is for instance satisfied when ϕ_t is an Anosov flow and $\mathcal{F}^+$ is the stable (or unstable) foliation of ϕ_t.

We are interested in bundles over M and lifting of such actions to actions on these bundles. We shall say a vector bundle E over M admits a *flag action* if it satisfies the following conditions:

- E is a vector bundle of rank 2 equipped with a parallel transport along the leaves of $\mathcal{F}$;
- the action of ϕ_t lifts to an action ψ_t by bundle automorphisms on E which preserves the parallel transport;
- E admits a direct sum decomposition into continuous oriented subbundles of rank 1
$$E = W^1 \oplus W^2.$$
Moreover this decomposition is invariant under ψ_t.
- W^2 is parallel along leaves of $\mathcal{F}$.
- *Contraction assumption.* We assume that $(W^1)^* \otimes W^2$ is a contracting vector bundle for ψ_t. It follows that if we take a 1-dimensional vector space L in $W^2_x \oplus W^1_x$ different to W^2_x, then
$$\lim_{t\to\infty} d(\psi_t(L), W^1_{\phi_t(x)}) = 0.$$

We now introduce some notations. Suppose that x and y are on the same leaf of $\mathcal{F}$. We denote by $W^1_{x,y}$ the vector subspace of F^i_x which is the parallel transport of W^1_y along the leaf containing x and y. We say that the bundle E is *hyperconvex* if and only if, for all distinct z and y in the leaf $\mathcal{F}_x$

$$W^1_{x,z} \oplus W^1_{x,y} \quad = \quad E_x. \tag{37}$$

Our main lemma is the following

Lemma 7.4 *Suppose that the rank 2 vector bundle E equipped with a flag action is hyperconvex. Then the map J_x*

$$\left\{ \begin{array}{ccc} \mathcal{F}_x & \to & \mathbb{P}(E_x) \setminus \{W_x^2\} \\ y & \mapsto & W_{x,y}^1 \end{array} \right.$$

is a homeomorphism. Moreover, for every x in M

$$\lim_{y \to \infty,\ y \in \mathcal{F}_x} (W_{xy}^1) = W_x^2.$$

We begin by describing how this result fits with our framework. Then we prove a preliminary lemma which permits us to conclude.

7.3.2 Hyperconvex bundles and Anosov representations

Rank 2 vector bundles with a flag action arise naturally from Anosov representations. Indeed using the notations of Proposition 3.3, we will show that the bundles E_k^+/E_{k-2}^+ are of this type. More precisely, let E be the vector bundle associated to an n-Anosov representation with its flat connection as in Proposition 3.3. Let $F_k = E_k^+/E_{k-2}^+$. We observe that F is equipped with a flat connection along $\mathcal{F}^+$. It is trivial that $W^2 = E_{k-1}^+/E_{k-2}^+$ is parallel for this connection. We may thus identify F_k with $E_{n-k+2}^- \cap E_k^+$. In this interpretation, we have

$$W^2 = E_{n-k+2}^- \cap E_{k-1}^+ = V^{k-1}.$$

Let

$$W^1 = E_{n-k+1}^- \cap E_k^+ = V^k.$$

We then have the following

Proposition 7.5 *Let E be the vector bundle associated to an n-Anosov representation by Proposition 3.3. Then F_k with the structure described above is a rank 2 vector bundle equipped with a flag action. Furthermore, using Identification (4) of Proposition 3.3, we have*

$$W_{(x,x_0,w),(x,x_0,y)}^1 = (\xi^{n-k+1}(y) \oplus \xi^{k-2}(x)) \cap \xi^{n-k+2}(w) \cap \xi^k(x). \tag{38}$$

Finally, the representation satisfies Property (H) if and only if the bundles F_k are hyperconvex.

Proof: By definition, the bundle F_k (with its flat connection) along the leaf of $\mathcal{F}^+$ passing through (x, x_0, w) is identified with the trivial bundle whose fibre is

$$\xi^k(x)/\xi^{k-2}(x).$$

We identify $\xi^k(x) \cap \xi^{n-k+2}(w)$ with this fibre using the projection along $\xi^{k-2}(x)$. We then get

$$W_{(x,x_0,w),(x,x_0,y)}^1 = ((\xi^{n-k+1}(y) \cap \xi^k(x)) \oplus \xi^{k-2}(x)) \cap \xi^{n-k+2}(w).$$

This in turn implies Identification (38). Therefore hyperconvexity is equivalent to having that for $y \neq t$

$$\begin{array}{rcl} ((\xi^{n-k+1}(y) \cap \xi^k(x)) \oplus \xi^{k-2}(x)) & \cap & \xi^{n-k+2}(w) \\ \oplus((\xi^{n-k+1}(t) \cap \xi^k(x)) \oplus \xi^{k-2}(x)) & \cap & \xi^{n-k+2}(w) \\ = \xi^k(x) & \cap & \xi^{n-k+2}(w). \end{array} \tag{39}$$

If we add $\xi^{k-2}(x)$ to both sides of Equality (39), since $\xi^{k-2}(x) \oplus \xi^{n-k+2}(w) = E$ by 2-hyperconvexity, we obtain

$$(\xi^{n-k+1}(y) \cap \xi^k(x)) + \xi^{k-2}(x) + (\xi^{n-k+1}(t) \cap \xi^k(x)) \quad = \quad \xi^k(x). \tag{40}$$

Adding $\xi^{n-k}(y)$ to both sides of Equality (40) yields

$$\xi^{n-k+1}(y) + \xi^{k-2}(x) + (\xi^{n-k+1}(t) \cap \xi^k(x)) \quad = \quad E. \tag{41}$$

For dimensional reasons, the above sum is direct. We thus obtain the following equality which is nothing else than Property (H),

$$\xi^{n-k+1}(y) \oplus \xi^{k-2}(x) \oplus (\xi^{n-k+1}(t) \cap \xi^k(x)) = E.$$

Conversely, suppose that Property (H) is satisfied. This implies that taking the intersection with $\xi^k(x)$ yields

$$(\xi^{n-k+1}(y) \cap \xi^k(x)) \oplus \xi^{k-2}(x) \oplus (\xi^{n-k+1}(t) \cap \xi^k(x)) = \xi^k(x).$$

Let us denote by π the projection on $B = \xi^{n-k-2}(w)$ along $A = \xi^{k-2}(x)$. Let $L(m)$ be the line $\xi^{n-k+1}(m) \cap \xi^k(x)$. The last assertion can be restated as

$$L(y) \oplus L(t) \oplus A = \xi^k(x).$$

Applying π yields

$$\pi(L(t)) \oplus \pi(L(y)) = \pi(\xi^k(x)) = \xi^k(x) \cap \xi^{n-k+2}(w).$$

This is precisely Assertion (39), and the result follows. Q.E.D.

7.3.3 Invariant subbundle

We begin by making a definition which will be used in the following proof. A subbundle L of E is said to be *invariant* if

$$\psi_t(L_x) = L_{\phi_t(x)}.$$

Note that no regularity is assumed on L.

Lemma 7.6 *Let L be an invariant subbundle of rank 1 of $E = W^1 \oplus W^2$ equipped with a flag action. Suppose that, for some auxiliary (continuous) metric d on the bundle $\mathbb{P}(E)$, we have*

$$\exists \epsilon,\ \forall x \in M,\ d(L_x, W^1_x) > \epsilon > 0.$$

Then

$$\forall x \in M,\ L_x = W^2_x.$$

We stress again that no regularity is assumed on L.

PROOF: This is an immediate consequence of our contraction Assumption (7). Indeed, if the lemma is not true, for some x, $L_x \neq W^2_x$. Consequently, our contraction Assumption (7) implies that for some large positive s

$$d(L_{\phi_s(x)}, W^1_x) = d(\psi_s(L_x), W^1_x) < \frac{\epsilon}{2}.$$

Q.E.D.

7.3.4 Proof of Lemma 7.4

PROOF: By assumption, we know that for $y \neq z$, we have $W^1_{x,y} \oplus W^1_{x,z} = E_x$. Therefore the continuous maps J_x are injective. Since $\mathbb{P}(E_x)$ is of dimension 1, the following limits exist (after a choice of orientation on $\mathcal{F}$)

$$\begin{aligned}\lim_{y\to+\infty} J_x(y) &= J^+_x \\ \lim_{y\to-\infty} J_x(y) &= J^-_x\end{aligned}$$

We observe moreover that the bundles $J^{\pm}$ are flow invariant, although not *a priori* continuous. In order to conclude, we merely have to show that

$$J^+_x = J^-_x = W^2_x.$$

We may assume that $\mathcal{F}$ is the orbit lamination of a flow θ_t. Let us introduce the continuous bundles $L^{\pm}_x = W^1_{x,\theta_{\pm1}(x)}$. We observe that for all x, Assumption (37) implies that $L^{\pm}_x \neq W^1_x$. Moreover, since all the maps J_x are monotone, J^+_x and W^1_x are not in the same connected component of $\mathcal{P}(E_x) \setminus \{L^-_x, L^+_x\}$ (and the same holds for J^-_x). It follows that there exists $\epsilon > 0$ for which

$$d(J^{\pm}_x, W^1_x) \geq d(L^{\pm}_x, W^1_x) \geq \epsilon.$$

Finally, Lemma 7.6 implies that for all x, $J^{\pm}_x = W^2_x$. Q.E.D.

7.4 Property (H) and hyperconvex bundles

We recall that an n-Anosov representation, with limit curve ξ, satisfies *Property (H)*, if, for every triple of distinct points x, y and z, and every integer k, we have

$$\xi^{k+1}(y) \oplus (\xi^{k+1}(z) \cap \xi^{n-k}(x)) \oplus \xi^{n-k-2}(x) \quad = \quad E.$$

We explain now various ways to verify this property, and use its relation with hyperconvex bundles to obtain Proposition 7.7 which is the main result of this section.

7.4.1 The main property

Let x and z be two distinct points of $\partial_\infty \pi_1(S)$. Let

$$G_{p,x,z} = \xi^{n-p+2}(z) \cap \xi^p(x).$$

We consider the map $\mathcal{Y}_{p,x,z}$ defined by

$$\begin{cases} \partial_\infty \pi_1(S) \setminus \{x\} & \rightarrow \quad \mathbb{P}(G_{p,x,z}) \setminus \{\xi^{p-1}(x) \cap \xi^{n-p+2}(z)\} \\ \qquad\qquad y & \mapsto \quad (\xi^{n-p+1}(y) \oplus \xi^{p-2}(x)) \cap G_{p,x,z}. \end{cases}$$

Proposition 7.8 explains that this application is well defined. Our main result in this paragraph is the following Proposition

Proposition 7.7 *Suppose that the representation satisfies Property (H). Then, the map $\mathcal{Y}_{p,x,z}$ from $\partial_\infty \pi_1(S) \setminus \{x\}$ to $\mathbb{P}(G_{p,x,z}) \setminus \{\xi^{p-1}(x) \cap \xi^{n-p+2}(z)\}$ is surjective.*

7.4.2 Construction of the map $\mathcal{Y}$

We use the notation of the previous paragraph

Proposition 7.8 *We have*

$$\begin{aligned} \dim(\mathcal{Y}_{p,x,z}(y)) &= 1, & (42)\\ \mathcal{Y}_{p,x,z}(y) \oplus \left(\xi^{p-1}(x) \cap \xi^{n-p+2}(z)\right) &= G_{p,x,z}. & (43) \end{aligned}$$

Furthermore, Property (H) is equivalent to the following assertion

$$\mathcal{Y}_{p,x,z}(t) \neq \mathcal{Y}_{p,x,z}(y), . \qquad (44)$$

Proof: From Identification (38), we obtain that

$$W^1_{(x,x_0,w),(x,x_0,y)} = \mathcal{Y}_{k,x,w}(y).$$

All the above results follow from Paragraph 7.3.2. Q.e.d.

7.4.3 Proof of Proposition 7.7

Proof: From Proposition 7.5, we know that, if the representation satisfies Property (H), the bundles $F_k = E^k/E^{k-2}$ are hyperconvex. Moreover by Identification (38), we get that

$$W^1_{(x,x_0,w),(x,x_0,y)} = \mathcal{Y}_{k,x,w}(y).$$

Proposition now 7.7 follows from Lemma 7.4. Q.e.d.

7.5 Proof of Lemma 7.1

7.5.1 Preliminary facts: case $l = 1$.

Let ξ be the limit curve of the Anosov representation ρ. Let (z, x, x_0, x_1, y) be distinct points of $\partial_\infty \pi_1(S)$. Write $X = (z, x, x_0, x_1)$. We introduce

$$\begin{aligned} U_{y,x_0} &= (\xi^{n-k}(y) \oplus \xi^{k-1}(x_0) \cap \xi^{n-k+1}(z)), \\ Z_{x_0,x_1} &= (\xi^{k-1}(x_0) \oplus \xi^1(x_1)) \cap \xi^{n-k+1}(z). \end{aligned}$$

We first observe that due to 2-hyperconvexity (*cf.* Assertion (5)) the sums in the definition of Z_{x_0,x_1} and U_{y,x_0} are indeed direct. We now prove

Lemma 7.9 *If ξ is a limit curve of a 3-hyperconvex quasi-Fuchsian representation, then*

$$\begin{aligned} \dim(Z_{x_0,x_1}) &= 1 \\ \dim(U_{y,x_0}) &= n - k. \end{aligned}$$

Moreover,

$$Z_{x_0,x_1} \oplus U_{y,x_0} = \xi^{n-k+1}(z).$$

Proof: Let

$$C_y = \xi^{n-k}(y) \oplus \xi^{k-1}(x_0).$$

Let π be the projection along $B = \xi^{k-1}(x_0)$ onto $A = \xi^{n-k+1}(z)$. We observe that 2-hyperconvexity implies that $A \oplus B = E$. We recall that

$$\pi(W) = (W + B) \cap A.$$

In particular

$$\begin{aligned} \pi(C_y) &= C_y \cap A = U_{y,x_0} \\ \pi(\xi^1(x_1)) &= Z_{x_0,x_1}. \end{aligned}$$

We begin by computing the dimensions of Z_{x_0,x_1} and U_{y,x_0}. We first observe that

$$\dim Z_{x_0,x_1} = \dim(\pi(\xi^1(x_1)) \leq 1.$$

By 2-hyperconvexity, the following sum is direct

$$\xi^1(x_1) + \underbrace{\xi^{k-1}(x_0)}_{B}.$$

Hence

$$\xi^1(x_1) \not\subset B,$$

and

$$\dim(Z_{x_0,x_1}) = 1.$$

We now consider U_{y,x_0}. We know that

$$\dim(C_y) = n-1.$$

Next, since the curve is assumed to be 3-hyperconvex, we have

$$\underbrace{\xi^{n-k}(z)}_{\subset A} \oplus \underbrace{\xi^{k-1}(x_0) \oplus \xi^1(y)}_{\subset C_y} = E.$$

Hence

$$A + C_y = E,$$

and

$$A \not\subset C_y.$$

Hence,

$$\dim(U_{y,x_0}) = \dim(\pi(C_y)) = \dim(C_y \cap A) = \dim(A) - 1 = n-k.$$

Finally, since the curve is 3-hyperconvex,

$$\underbrace{\left(\xi^{n-k}(y) \oplus \xi^{k-1}(x_0)\right)}_{C_y} \oplus \xi^1(x_1) \quad = \quad E. \tag{45}$$

Applying π to both sides of Formula (45) yields

$$Z_{x_0,x_1} + U_{y,x_0} = \xi^{n-k+1}(z).$$

The result now follows. Q.E.D.

7.5.2 Proof of Lemma 7.1: case $l = 1$

We concentrate on the case $l = 1$ of Lemma 7.1. We prove

Lemma 7.10 *Suppose that the limit curve ξ of the Anosov representation ρ is*

- *3-hyperconvex, and*
- *satisfies Property (H).*

Then it is $(k, 1)$-convergent for all k.

PROOF: We prove this Lemma by induction on $n - k$ and use the following induction hypothesis

$$\lim_{(x_0,x_1)\to x} \left(\xi^k(x_0) \oplus \xi^1(x_1)\right) \quad = \quad \xi^{k+1}(x). \tag{46}$$

Due to 2-hyperconvexity, this result is true for $k = n - 1$. We aim to prove

$$\lim_{(x_0,x_1)\to x} \left(\xi^{k-1}(x_0) \oplus \xi^1(x_1)\right) \quad = \quad \xi^k(x). \tag{47}$$

We recall that, by 2-hyperconvexity, for $z \neq x$,

$$\xi^{k-1}(x) \oplus \xi^{n-k+1}(z) = E. \tag{48}$$

As in the previous paragraph, we introduce

$$\begin{aligned}
Z_{x_0,x_1} &= (\xi^{k-1}(x_0) \oplus \xi^1(x_1)) \cap \xi^{n-k+1}(z),\\
U_{y,x_0} &= (\xi^{n-k}(y) \oplus \xi^{k-1}(x_0)) \cap \xi^{n-k+1}(z),\\
B &= \xi^{n-k+1}(z),\\
G_{x,z} &= B \cap \xi^{k+1}(x).
\end{aligned}$$

Using the notations of the previous paragraph, since $\xi^{k-1}(x_0)$ converges to $\xi^{k-1}(x)$, by Assertion (48) we first observe Assertion (47) follows from

$$\lim_{(x_0,x_1)\to x} Z_{x_0,x_1} = \xi^{n-k+1}(z) \cap \xi^k(x).$$

We thus aim now to prove this last assertion. We shall do this by "trapping" Z_{x_0,x_1} using U_{y,x_0}.

Let V be a connected neighbourhood of x homeomorphic to the interval. We choose an orientation on V. We say that (x_0,x_1) tends to x_+, (resp. x_-) if $x_0 > x_1$ (resp. $x_0 < x_1$). Let $\alpha \in \{+,-\}$. Let Λ_x^α be the set of accumulation points of Z_{x_0,x_1} when (x_0,x_1) tends to x_α. We note that Λ_x^α is a connected subset of $\mathbb{P}(B)$. Note that,

$$Z_{x_0,x_1} \subset (\xi^k(x_0) \oplus \xi^1(x_1)) \cap \xi^{n-k+1}(z).$$

By Hypothesis (46), it follows that the set Λ_x^α is actually a subset of the projective space $\mathbb{P}(G_{x,z})$. Therefore Λ_x^α is a closed interval of the 1-dimensional manifold $\mathbb{P}(G_{x,z})$.

We now choose an auxiliary metric $\langle,\rangle$ on B, a unit vector z_{x_0,x_1} depending continuously on (x_0,x_1) in Z_{x_0,x_1}, a normal vector u_{y,x_0} to U_{y,x_0}, also depending continuously on (x_0,x_1). By Lemma 7.9, we have, after possibly replacing u by $-u$), for all y, for $X = (x_0,x_1)$ close to x,

$$\langle z_{x_0,x_1}, u_{y,x_0}\rangle > 0.$$

We now consider $\hat{\Lambda}_x^\alpha$ the set of accumulation points of z_{x_0,x_1} as (x_0,x_1) goes to α.

We now observe , that for all y, for all w in $\hat{\Lambda}_x^\alpha$, we have

$$\langle w, u_{y,x}\rangle \geq 0.$$

Hence, for all y,t, $\hat{\Lambda}_x^\alpha$ is contained in the closure of one of the connected components of

$$G_{x,z} \setminus \big((U_{y,x} \cap G_{x,z}) \cup (U_{t,x} \cap G_{x,z})\big).$$

However, by Definition 7.4.1

$$U_{y,x} \cap G_{x,z} = \mathcal{Y}_{k+1,x,z}(y),$$

It thus follows that Λ^α_x is contained in the closure of one of the connected components of

$$\mathbb{P}(G_x) \setminus \{\mathcal{Y}_{k+1,x,z}(y), \mathcal{Y}_{k+1,x,z}(t)\}.$$

By Proposition 7.7, we know that the map $\mathcal{Y}_{k+1,x,z}$ from $\partial_\infty\pi_1(S) \setminus \{x\}$ to $\mathbb{P}(G_{k,x,z}) \setminus \{\xi^k(x) \cap \xi^{n-k+1}(z)\}$ is surjective. We thus obtain

$$\Lambda^\alpha_x = \{\xi^k(x) \cap \xi^{n-k+1}(z)\}.$$

Since this is true for every end α, we the result now follows. Q.E.D.

7.5.3 Main Lemma: case $l > 1$.

The main Lemma 7.1 will follow by an induction process proved in Paragraph 7.5.6 from a combination of Lemma 7.10 and the following statement

Lemma 7.11 *Let ξ be the limit curve of a quasi-Fuchsian representation. Let k and l be integers such that $k + l \leq n$ and $l > 2$. We suppose moreover that the curve is*

1. $(k, l-1)$*-convergent,*
2. (k, l)*-convergent, and*
3. $(k-1, l-1)$*-convergent,*

Then the curve is $(k-1, l)$-convergent.

7.5.4 Preliminary facts: case $l > 1$

We now suppose that the limit curve ξ satisfies the hypothesis of Lemma 7.11. That is we suppose that the curve is

1. $(k, l-1)$-convergent,
2. (k, l)-convergent,
3. $(k-1, l-1)$-convergent,

Note that since the curve is (k, l)-direct by Hypothesis (2) and Lemma 7.3, the following sum is direct for all m with $m \leq k$,

$$\xi^m(x_0) + \xi^1(x_1) + \ldots + \xi^1(x_l)). \tag{49}$$

Let z be a point of $\partial_\infty\pi_1(S)$. Let $Y = (y_0, y_1, \ldots, y_{l-1}$ be a l-uple of cyclically ordered points of $\partial_\infty\pi_1(S) \setminus \{z\}$. We define

$$C(z, Y) \quad = \quad \xi^{n-k-l}(z) \oplus \xi^k(y_0) \oplus \bigoplus_{i=1}^{i=l-1} \xi^1(x_i).$$

The sum in the definition of $C(z, Y)$ is direct due to Hypothesis 7.11.(1) and Lemma 7.3. We also require various some choices of orientation. Let $I =$

$\partial_\infty \pi_1(S) \setminus \{z\}$. For all p, for all w in I, we choose an orientation on $\xi^p(w)$ depending continuously on w. For all k, we choose an arbitrary orientation on $\xi^k(z)$. It follows that there exists a family of 1-forms $\alpha(z, Y)$, depending continuously on Y, such that

$$C(z, Y) \quad = \quad ker(\alpha(z, Y)).$$

Now, let $X = (z, x_0, x_1, \ldots, x_l)$ be a cyclically oriented $l+2$-uple of distinct points of $\partial_\infty \pi_1(S)$. Let

$$\begin{aligned} X^+ &= (x_0, \ldots, x_{l-1}) \\ X^- &= (x_0, \ldots, x_{l-2}, x_l). \end{aligned}$$

We introduce

$$\begin{aligned} U_X^+ &= C(z, X^+) \cap \xi^{n-k-l+2}(z), \\ U_X^- &= C(z, X^-) \cap \xi^{n-k-l+2}(z), \\ Z_X &= (\xi^{k-1}(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_l)) \cap \xi^{n-k-l+2}(z). \end{aligned}$$

By Assertion (49), the sum in the definition of Z_X is indeed direct.

We prove

Proposition 7.12 *If ξ is the limit curve of a quasi-Fuchsian representation which satisfies the hypothesis of Lemma 7.11, then*

$$\begin{aligned} \dim(Z_X) &= 1 \\ \dim(U_X^\pm) &= n-k-l+1. \end{aligned}$$

Moreover,

$$Z_X \oplus U_X^+ \quad = \quad Z_X \oplus U_X^- = \xi^{n-k-l+2}(z), \tag{50}$$

and considering orientations, we obtain

$$[\alpha(z, X^+)]|_{Z_X} = -[\alpha(z, X^-)]|_{Z_X}. \tag{51}$$

Proof: Let

$$\begin{aligned} C^+ &= C(z, X^+) = \xi^{n-k-l}(z) \oplus \xi^k(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_{l-2}) \oplus \xi^1(x_{l-1}), \\ B^+ &= \xi^{k-1}(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_{l-2}) \oplus \xi^1(x_{l-1}), \\ B^- &= \xi^{k-1}(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_{l-2}) \oplus \xi^1(x_l), \\ C^- &= C(z, X^-) = \xi^{n-k-l}(z) \oplus \xi^k(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_{l-2}) \oplus \xi^1(x_l). \end{aligned}$$

Let $\pi^\pm$ be the projection along $B^\pm$ onto $A = \xi^{n-k-l+2}(z)$. By Lemma 7.3 and Hypothesis 7.11.(3), we obtain that $A \oplus B^\pm = E$. We recall that

$$\pi^\pm(W) = (W + B^\pm) \cap A.$$

In particular

$$\begin{aligned}\pi^{\pm}(C^{\pm}) = C^{\pm}\cap A &= U_X^{\pm}\\ \pi^{+}(\xi^1(x_l)) &= Z_X,\\ \pi^{-}(\xi^1(x_{l-1})) &= Z_X.\end{aligned}$$

We first compute the dimensions of Z_X and U_X^+. We observe that

$$\dim Z_X = \dim(\pi^+(\xi^1(x_l)) \leq 1.$$

However, the following sum is direct (*cf* Hypothesis 7.11.(2))

$$\xi^k(x_0) + \xi^1(x_1) + \ldots + \xi^1(x_{l-1}) + \ \xi^1(x_l).$$

It thus follows that the following sum is also direct

$$\xi^1(x_l) + \ \underbrace{\xi^{k-1}(x_0) + \xi^1(x_1) + \ldots + \xi^1(x_{l-1})}_{B^+}.$$

Hence

$$\xi^1(x_l) \not\subset B^+,$$

and

$$\dim(Z_X) = 1.$$

We now consider U_X^+, the proof for U_X^- being similar by symmetry. First, we know that

$$\dim(C^+) = n-1.$$

Hypothesis 7.11.(3) and Lemma 7.3 yield

$$\underbrace{\xi^{n-k-l+2}(z)}_{A}\oplus \ \underbrace{\xi^{k-1}(x_0)\oplus\xi^1(x_1)\oplus\ldots\oplus\xi^1(x_{l-1})}_{\subset C^+} = E.$$

Hence

$$A + C^+ = E,$$

and

$$A \not\subset C^+.$$

It thus follows that

$$\dim(U_X^+) = \dim(\pi^+(C^+)) = \dim(C^+\cap A) = \dim(A) - 1 = n-k-l+1.$$

Finally, by Hypothesis 7.11.(2) and Lemma 7.3, we have

$$\underbrace{\left(\xi^{n-k-l}(z)\oplus\xi^k(x_0)\oplus\xi^1(x_1)\oplus\ldots\oplus\xi^1(x_{l-1})\right)}_{C^+}\oplus\xi^1(x_l) \quad = \quad E. \qquad (52)$$

Applying π^+ to both sides of Formula (52) yields

$$Z_X + U_X^+ = \xi^{n-k-l+2}(z).$$

The same holds for U_X^-. It finally remains to verify that the orientations on $Z_X \oplus U_X^+$ and $Z_X \oplus U_X^-$ are opposite. We denote by $\overline{V}$ the opposite of the oriented vector space V.

Since $(z, x_0, \ldots, x_l)$ are distinct and cyclically oriented, there exists an arc $t \mapsto w_t$ joining x_l to x_{l-1} such that

$$\forall t, w_t \notin \{z, x_0, \ldots, x_{l-2}\}.$$

Let

$$B_t = \xi^{k-1}(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_{l-2}) \oplus \xi^1(w_t).$$

We observe that, as before, B_t satisfies

$$B_t \oplus \xi^{n-k-l+2}(z) = E. \tag{53}$$

We choose an orientation on E such that with respect to this orientation

$$E = C^+ \oplus \xi^1(x_l) = \ldots \oplus \xi^1(x_{l-1}) \oplus \xi^1(x_l).$$

We recall that B_t is oriented. We now choose an orientation on $\xi^{n-k-l+2}(z)$ which is compatible with Equation (53). It follows that, considering $\xi^{n-k-l+2}(z)$ as an oriented space, we have

$$U_X^+ \oplus Z_X = \xi^{n-k-l+2}(z).$$

Conversely, since

$$E = \overline{C^- \oplus \xi^1(x_{l-1}))} = \overline{\ldots \oplus \xi^1(x_l) \oplus \xi^1(x_{l-1})}.$$

We obtain

$$\overline{U_X^+ \oplus Z_X} = \xi^{n-k-l+2}(z).$$

Q.E.D.

7.5.5 Proof of Lemma 7.11

We first state the following elementary lemma.

Lemma 7.13 *Let E be a vector space, Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of oriented lines converging to an oriented line L_∞. Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of oriented hyperplanes converging to an oriented hyperplane P_∞. Suppose that the following sums are direct with opposite orientations*

$$L_n \oplus P_{2n} = \overline{L_n \oplus P_{2n+1}}.$$

Then $L_\infty \subset P_\infty$.

Proof: We choose an auxiliary metric g on E. Let u_n be the positive unit vector in L_n. Let v_n be the normal unit vector to P_n. By the hypothesis, we obtain

$$g(u_n, v_{2n}).g(u_n, v_{2n+1}) < 0.$$

Therefore, by passing to the limit, we find that $g(u_\infty, v_\infty) = 0$. Q.E.D.

We now proceed to the main proof. We shall always assume that

$$(z, x_0, x_1, \ldots, x_l)$$

are distinct and positively cyclically oriented. As before, we choose an orientation on $\xi^p(w)$ depending continuously on w in $\partial_\infty \pi_1(S) \setminus \{z\}$. We suppose the following hypotheses

$$\lim_{(x_0,\ldots,x_l)\to x} \left(\xi^k(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_l)\right) = \xi^{k+l}(x), \tag{54}$$

$$\lim_{(x_0,\ldots,x_l)\to x} \left(\xi^k(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_{l-1})\right) = \xi^{k+l-1}(x), \tag{55}$$

$$\lim_{(x_0,\ldots,x_{l-2})\to x} \left(\xi^{k-1}(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_{l-1})\right) = \xi^{k+l-2}(x). \tag{56}$$

We may actually assume that the limit in Assertion (55) is a limit as oriented vector spaces. We aim to prove that

$$\lim_{(x_0,\ldots,x_l)\to x} \left(\xi^{k-1}(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_l)\right) = \xi^{k+l-1}(x). \tag{57}$$

We use the notations and results of the preceding paragraph.

By Hypothesis (56), Assertion (57) reduces to

$$\lim_{(x_0,\ldots,x_l)\to x} Z_X = \xi^{n-k-l+2}(z) \cap \xi^{k+l-1}(x).$$

Our aim is to now prove this last assertion. We shall do this by "trapping" Z_X using $U_X^\pm$.

Let Λ_x be the set of accumulation points of Z_X when $(x_0, \ldots, x_l)$ tends to x. We note that Λ_x is a subset of $\mathbb{P}(\xi^{n-k-l+2}(z))$. We recall that

$$Z_X \subset \xi^k(x_0) \oplus \xi^1(x_1) \oplus \ldots \oplus \xi^1(x_l).$$

By Hypothesis (54), the set Λ_x is actually a subset of the projective space $\mathbb{P}(W)$, where $W = \xi^{n-k-l+2}(z) \cap \xi^{k+l}(x)$.

Finally, by Hypothesis (54)

$$\lim_{X\to x} (U_X^\pm) = \underbrace{(\xi^{n-k-l}(z) \oplus \xi^{k+l-1}(x)) \cap \xi^{n-k-l+2}(z)}_{D}.$$

We observe that by 2-hyperconvexity $\xi^{n-k-l+2}(z) + \xi^{k+1}(x) = E$. It follows that D is indeed a hyperplane of $\xi^{n-k-l+2}(z)$. We may choose the orientation on D such the limit is to be considered as a limit oriented vector spaces. By Equation (51) and Lemma 7.13, we obtain

$$\Lambda_x \subset \mathbb{P}(D).$$

The result now follows since

$$\begin{aligned} D \cap W &= \left(\xi^{n-k-l}(z) \oplus \xi^{k+l-1}(x)\right) \cap \xi^{n-k-l+2}(z) \cap \xi^{k+l}(x) \\ &= \xi^{k+l-1}(x) \cap \xi^{n-k-l+2}(z). \end{aligned}$$

Q.E.D.

7.5.6 Final induction

PROOF: It remains to prove our main Lemma 7.1, using Lemma 7.10 and Lemma 7.11. We do this by induction. We say that the limit curve is l-*superconvergent*, if it is (k,l)-convergent for all k.

By Lemma 7.10, the limit curve if 1-superconvergent. We assume by induction that the curve is $l-1$-superconvergent. By Lemma 7.11 and an easy induction, in order to prove that the curve is l-superconvergent, it suffices to show that it is $(n-l,l)$-convergent. However the curve is $(n-l,l)$-convergent precisely when the following sum is direct

$$\xi^{n-l}(x_0) + \xi^1(x_1) + \ldots + \xi^1(x_l) = E.$$

Finally the fact that this sum is direct follows from the fact that the curve is $(1,l-1)$-convergent, and $(1,l-1)$-direct by Lemma 7.3. All these conditions are guaranteed by the induction hypothesis. Q.E.D.

8 Anosov representations, Property (H) and 3-hyperconvexity

We now clarify some relations between Property (H), 3-hyperconvexity, and Anosov representations. We first say that a representation is S*-irreducible* if its restriction to all finite index subgroups is irreducible. By Lemma 10.1, every representation in a Hitchin component is S-irreducible. We denote

- $\mathcal{A}$ (resp. $\mathcal{QF}$) the space of n-Anosov S-irreducible (resp. quasi-Fuchsian representation) representations,
- $\mathcal{A}_H$ (resp. $\mathcal{QF}_H$) the space of S-irreducible Anosov (resp. quasi-Fuchsian) representations satisfying Property (H)
- $\mathcal{A}_3$ (resp. $\mathcal{QF}_3$) the set of S-irreducible Anosov (resp. quasi-Fuchsian) representations which are 3-hyperconvex.

The results of this section are summarised in the following proposition.

Proposition 8.1 *$\mathcal{A}_3$ is open in $\mathcal{A}$. $\mathcal{A}_H$ is a connected subset of $\mathcal{A}$. Furthermore $\mathcal{QF}_H = \mathcal{QF}$, and every Fuchsian representation is 3-hyperconvex.*

The proof of Proposition 8.1 proceeds as follows: we prove in the following paragraph that $\mathcal{A}_H$ and $\mathcal{A}_3$ are open in $\mathcal{A}$, and in the subsequent one we prove that $\mathcal{A}_H$ is closed in $\mathcal{A}$; finally we prove that every Fuchsian representation is 3-hyperconvex and satisfies Property (H). This will conclude the proof of Proposition 8.1.

8.1 Open

We first make the following obervation.

Proposition 8.2 *The sets $\mathcal{A}_H$ and $\mathcal{A}_3$ are open in $\mathcal{A}$.*

PROOF: This follows at once from the fact that $(\partial_\infty\pi_1(S)^3\backslash\Delta)/\pi_1(S)$ is compact and that the conditions defining 3-hyperconvexity and Property (H) are open in the corresponding product of flag manifolds. Q.E.D.

8.2 Closed

The aim of this paragraph is to prove the following assertion.

Proposition 8.3 *The set $\mathcal{A}_H$ is closed in $\mathcal{A}$.*

PROOF: We consider ρ an S-irreducible Anosov representation limit of representations in $\mathcal{A}_H$. Let ξ be the associated curve, and $\mathcal{Y} = \mathcal{Y}_{k,x,z}$ the associated map defined as in Paragraph 7.4.1. By Proposition 7.8, it suffices to prove that $\mathcal{Y}$ is injective. Since $\mathcal{Y}$ is a limit of continuous injective maps of a 1-dimensional manifold into another, $\mathcal{Y}$ is monotone.

Therefore, if $\mathcal{Y}$ fails to be injective, there is an open set U in $\partial_\infty\pi_1(S)$ on which it is constant. We will prove that this would lead to a contradiction. Indeed we obtain the following property which contradicts Lemma 10.2.

Assertion. There exists some $(n-k-1)$-plane A, such that

$$\forall y \in U,\ \dim \xi^{k+1}(y) \cap A \geq 1. \tag{58}$$

We first observe that

$$\begin{aligned}
\mathcal{Y}(y) &= (\xi^{k+1}(y) \oplus \xi^{n-k-2}(z)) \cap G_{k,x,z} \\
&= (\xi^{k+1}(y) \oplus \xi^{n-k-2}(z)) \cap \xi^{n-k}(z) \cap \xi^{k+2}(x) \\
&= ((\xi^{k+1}(y) \cap \xi^{n-k}(z)) \oplus \xi^{n-k-2}(z)) \cap \xi^{k+2}(x).
\end{aligned}$$

Since $\xi^{k+2}(x)$ is supplementary to $\xi^{n-k-2}(z)$, we have

$$\mathcal{Y}(y) = \mathcal{Y}(t) \implies P(y) = P(t),$$

where

$$P(y) = (\xi^{k+1}(y) \cap \xi^{n-k}(z)) \oplus \xi^{n-k-2}(z).$$

Note that $P(y)$ has dimension $n-k-1$. We thus obtain Assertion (58). Q.E.D.

8.3 Back to Fuchsian representations

We now prove that $\mathcal{QF}_H$ is not empty. To be precise

Lemma 8.4 *Every Fuchsian representation satisfies Property (H).*

PROOF: We first observe that for every pair of distinct points x and z the following sum is direct

$$\xi^{k+1}(z) + \xi^{n-k-2}(x).$$

Consequently, the following is also direct,

$$(\xi^{k+1}(z) \cap \xi^{n-k}(x)) + \xi^{n-k-2}(x) = P(z,x).$$

It follows that if a Fuchsian representation does not satisfy Property (H), then there exists a triple of distinct points (x, y, z) such that

$$\dim(\xi^{k+1}(y) \cap P(z,x)) > 0.$$

In the case of a Fuchsian representation, the limit curve is the Veronese embedding and is equivariant under the whole action of $SL(2,\mathbb{R})$. Since $SL(2,\mathbb{R})$ acts transitively on the set of triples of distinct points, we find that there exists an $n-k-1$- plane P (namely $P(z,x)$ for some x and z) a such that for every y,

$$\dim(\xi^{k+1}(y) \cap P) > 0.$$

It follows that there exist a $k+1$-plane Q such that for every A in $SL(2,\mathbb{R})$,

$$\dim(\rho(A)Q \cap P) > 0.$$

This last assertion contradicts Proposition 10.3. Q.E.D.

Actually, one could prove the previous proposition by an explicit computation. Indeed, we may identify $\partial_\infty \pi_1(S) \setminus \{x\}$ with $\mathbb{RP}^1 \setminus \{\infty\} = \mathbb{R}$, The irreducible representation of $SL(2,\mathbb{R})$ of dimension n is then the representation on homogeneous polynomials of degree $n-1$ in variables t and s. We find that

$$\xi^k(x) = \{P(s,t)/\exists Q \text{ such that } = (s+tx)^{n-k}Q(s,t)\}.$$

It is an exercise (left to the reader) to prove the previous proposition using this construction.

A similar reasoning permits us to obtain

Proposition 8.5 *Every Fuchsian representation is 3-hyperconvex.*

9 Closedness

In this section, we aim to prove the following result

Lemma 9.1 *The set*

$$\tilde{\mathcal{A}} = \mathcal{A}_3 \cap \mathcal{A}_H$$

of 3-hyperconvex Anosov representations satisfying Property (H) is closed in the space of S-irreducible representations.

This lemma will be deduced from Lemma 9.2. We first show that we may obtain Theorem 4.1 as a corollary to this result.

9.1 Proof of Theorems 4.1

9.1.1 Theorem 4.1

Proof: We merely have to place the previous statements in the correct order. First, we know by Lemma 10.1 that every representation in a Hitchin component is S-irreducible. By Proposition 8.1, $\mathcal{QF}_H \cap \mathcal{QF}_3$ is open in $\mathcal{QF}$, and thus in the Hitchin component by Lemma 2.1. Using Proposition 8.1 a second time, we find that $\mathcal{QF}_H \cap \mathcal{QF}_3$ is non empty since it contains all Fuchsian representations. By Lemma 9.1 it is closed. Hence $\mathcal{QF}_H \cap \mathcal{QF}_3$ coincides with the Hitchin component. Let ρ be a representation in this component. Let

$$\xi = (\xi^1, \xi^2, \ldots, \xi^{n-1})$$

be its limit curve. By Corollary 7.2, we know that if $\rho \in \mathcal{QF}_H \cap \mathcal{QF}_3 = Rep_H(\pi_1(S), PSL(n,\mathbb{R}))$, ξ^1 is a hyperconvex Frenet curve and ξ is its osculating flag. The result now follows Q.e.d.

9.2 Convergence of limit curves

We aim to prove the following lemma,

Lemma 9.2 *Let $\{\rho_m\}_{m\in\mathbb{N}}$ be a sequence of Anosov representations satisfying Property (H) converging to an S-irreducible representation ρ. Let $\xi_m = (\xi_m^1, \ldots, \xi_m^{n-1})$ be the limit curve of ρ_m. Then there exists*

- *a sequence of homeomorphisms ϕ_m of S^1 with $\partial_\infty \pi_1(S)$,*
- *a monotone map π from S^1 to itself,*
- *an injective map $\hat{\xi}^1$ from S^1 to $\mathbb{P}(E)$, and*
- *an injective left continuous orientation preserving map ϕ_0 from $\partial_\infty \pi_1(S)$ to S^1*

such that

- *after extracting a subsequence, the sequence of mappings $\{\xi^1_m \circ \phi_m\}_{m\in\mathbb{N}}$ converges to $\hat{\xi}^1 \circ \pi$, and*
- *the map $\hat{\xi}^1 \circ \phi_0$ is ρ-equivariant and $*$-hyperconvex.*

We first show how this Lemma implies Lemma 9.1. Indeed, by Theorem 4.2 the limit representation is Anosov. Moreover by Proposition 8.1, it satisfies Property (H). The proof of the lemma itself requires several steps which we now briefly describe using the notations and the hypothesis of this Lemma.

1. *Convergence of the images (Proposition 9.3)*: there exists a sequence of homeomorphisms ϕ_m of S^1 with $\partial_\infty\pi_1(S)$ such that $\{\xi^1_m \circ \phi_m\}_{m\in\mathbb{N}}$ converges to a rectifiable curve ξ^1
2. *Preliminary facts*: we prove various lemmas of independent interest concerning rectifiable curves invariant under group actions.
3. *The limit and the boundary at infinity*: this is the core of the proof. In particular Lemma 9.2 is a consequence of Proposition 9.6. We essentially prove that $\xi^1 = \hat{\xi}^1 \circ \pi$ where $\hat{\xi}^1$ is $*$-hyperconvex and ρ-equivariant, and π is monotone from S^1 to $\partial_\infty\pi_1(S)$.

From now on, we use the notation of the Lemma. That is, we consider

- a sequence $\{\rho_m\}_{m\in\mathbb{N}}$ of 3-hyperconvex Anosov representations satisfying Property (H) converging to a S-irreducible representation ρ, and
- $\xi_m = (\xi^1_m, \ldots, \xi^{n-1}_m)$, the limit curve of ρ_m.

9.3 Convergence of the images

Proposition 9.3 *After passing to a subsequence, there exists a sequence of homeomorphisms ϕ_m of S^1 with $\partial_\infty\pi_1(S)$such that $\{\xi^1_m \circ \phi_m\}_{m\in\mathbb{N}}$ converges to a rectifiable curve ξ^1.*

PROOF: By Arzela-Ascoli theorem, it suffices to show that there exists a constant B such that ξ^1_m is rectifiable with length bounded above by B. Moreover, if c is a curve in $\mathbb{P}(E)$, then

$$\mathrm{length}(c) \leq \int_{P(E^*)} \sharp(c \cap P)d\mu(P).$$

In our case, by Lemma 7.1, ξ^1_m is hyperconvex, and we thus obtain

$$\mathrm{length}(\xi^1_m) \leq (dim(E) - 1)\mu(P(E^*)).$$

The result now follows. Q.E.D.

9.4 Preliminary facts

9.4.1 Wormlike

Let Z be a subset of $\mathbb{P}(E)$. We define $\langle Z\rangle$ to be the vector subspace generated by all the elements of Z:

$$\langle Z\rangle = \sum_{u\in Z} u.$$

Finally, suppose that Γ acts on S^1. Let ρ be a faithful representation of Γ in $SL(E)$. Let ξ be a ρ-equivariant injective map from S^1 to $\mathbb{P}(E)$. Let

$$\Gamma_{\mathbb{R}} = \{\gamma \in \Gamma, \gamma \neq id, \rho(\gamma) \text{ is diagonalisable over } \mathbb{R}\}.$$

For every γ in $\Gamma_{\mathbb{R}}$, let $Fix(\gamma)$ (resp. $Fix^+(\gamma)$) be the set of (resp. attractive) fixed points of $\rho(\gamma)$ in $\mathbb{P}(E)$. We define

$$\Lambda_{\xi,\rho,\Gamma} = \{a \in S^1/\exists\gamma \in \Gamma_{\mathbb{R}}, \xi(a) \in Fix^+(\gamma)\}.$$

We prove the following lemma.

Lemma 9.4 . *Let Γ be a group acting on S^1 by orientation preserving homeomorphisms. Let ρ be an S-irreducible representation of $\pi_1(S)$ in $SL(E)$. Suppose that ξ is a ρ-equivariant rectifiable injective map of finite length from S^1 to $\mathbb{P}(E)$. Then*

- *(i) $\xi(S^1)$ is not included in a finite union of proper vector subspaces of E,*
- *(ii) For every γ in $\Gamma_{\mathbb{R}}$, $S^1 \setminus Fix(\gamma)$, has finitely many connected components.*
- *(iii) For every γ in $\Gamma_{\mathbb{R}}$, there exists a unique γ^+ in S^1 such that $\xi(\gamma^+) \in Fix^+(\gamma)$. In particular, $\Lambda_{\xi,\rho,\Gamma}$ is not empty if $\Gamma_{\mathbb{R}}$ is not empty, and*
- *(iv) if U is a neighbourhood of a point c^+ in $\Lambda_{\xi,\rho,\Gamma}$, then*

  $$\xi(U) \not\subset P_0 \cup P_1,$$

  for any proper vector subspaces P_0 and P_1 of E. In particular, $\langle\xi(U)\rangle = E$.

PROOF: Statement (i) is a consequence of the fact that the connected component of the identity of the Zariski closure of $\rho(\Gamma)$ is irreducible. Indeed, let $E_1, \ldots, E_p$ be proper vector subspaces such that

$$\xi(S^1) \subset E_1 \cup \ldots \cup E_p.$$

We may assume that $< \xi(S^1) \cap E_i >= E_i$. It follows that for every γ in Γ, we have

$$\gamma(E_i) \subset E_1 \cup \ldots \cup E_p.$$

The same property holds for γ in the Zariski closure H of $\rho(\gamma)$. Let E_k be such that $\dim(E_k) = \sup_i(\dim(E_i))$. Then, for every element g in H close to the identity, $g(E_k) = E_k$. It follows that the identity component of H preserves E_k, and is thus not irreducible. We thus obtain the contradiction.

Let us now describe the action on $\mathbb{P}(E)$ of an element f of $SL(E)$ diagonalisable over $\mathbb{R}$. We state the following elementary facts leaving the proofs to the reader.

(a) The stable manifold W of a fixed point z of f in $\mathbb{P}(E)$ is described in the following way. There exists a vector subspace $\tilde{W}$ of E such that W is an open set in $\mathbb{P}(\tilde{W})$. Furthermore W is open in $\mathbb{P}(E)$, if and only if z is an attractive fixed point.

(b) Every closed invariant set of f contains a fixed point.

(c) If x is such that $f^n(x)$ converges to a and $f^{-n}(x)$ converges to b with $a \neq b$, when n goes to infinity, then a and b belong to different connected components of the space of fixed points of f.

(d) f has at most one attractive fixed point.

We now prove (ii). Let $I =]\alpha, \beta[$ be a connected component of $S^1 \setminus Fix(\gamma)$. Since γ is orientation preserving I is fixed by γ. Furthermore, by (c), $\xi(\alpha)$ and $\xi(\beta)$ belong to different connected components of the space of fixed points of $\rho(\gamma)$. Let $\mathcal{W}$ be the set of connected components of $Fix(\rho(\gamma))$. We find that

$$\text{length}(\xi(I)) \geq \epsilon_0 = \inf_{A,B\in\mathcal{W}, A\neq B} d(A,B).$$

Since by hypothesis, $\xi(S^1)$ has finite length, we obtain (ii).

We now proceed to (iii). Suppose that $Fix^+(\gamma)$ is empty. Note that every connected component $]\alpha, \beta[$ of $S^1 \setminus Fix(\gamma)$ is mapped to the stable manifold of $\xi(\alpha)$ and the unstable manifold of $\xi(\beta)$ (after a suitable choice of orientation). Therefore by (a), if $Fix^+(\gamma)$ is empty, then $\xi(I)$ lies in a proper subspace of E. Since $\rho(\gamma) \neq id$, $\xi(Fix(\gamma))$ lies in a finite union of proper vector subspaces. By (ii), it follows that $\xi(S^1)$ lies in a finite union of proper vector subspaces of E. We thus obtain a contradiction to (i). Uniqueness follows from (d) and the injectivity of ξ.

We shall now prove (iv) by similar arguments. Let c^+ be the point which is mapped to the attractive fixed point of $\rho(\gamma)$, with $\gamma \in \Gamma_{\mathbb{R}}$. Write the finite decomposition into connected components as follows

$$S^1 \setminus Fix(\gamma) = \bigsqcup_i V_i.$$

By convention, we assume that V_1 and V_2 have c^+ in their closure. Let $i \geq 3$. The closure of V_i contains an element which is mapped by ξ to a fixed point c which is neither attractive nor repulsive. The sets V_i lie in the stable (or unstable) manifold of c. The same holds for $\xi(V_i)$. By (a), it follows that $\xi(V_i)$ lies in a proper subspace E_i of E, for $i \geq 3$.

We now assume that there exists a neighbourhood U of c^+, and two proper vector subspaces P_0 and P_1 of E, such that

$$\xi(U) \subset P_0 \cup P_1.$$

We choose U small enough so that $U \subset \gamma^{-1}(U)$. Then, for each i, if Q_i is a limit of $\rho(\gamma^{n_q})(P_i)$ for some subsequence n_q, we obtain

$$\bigcup_{n\in\mathbb{N}} \xi(\gamma^{-n}(U)) \subset Q_0 \cup Q_1.$$

However,

$$\bigcup_{n\in\mathbb{N}} \gamma^{-n}(U) = \overline{V_1 \cup V_2}.$$

It follows that $\xi(S^1)$ lies in the union of E_i for $i \geq 3$, $Fix(\rho(\gamma))$ and $Q_0 \cup Q_1$, and we thus obtain a contradiction by (i). Q.E.D.

9.4.2 Weak worm

We now prove a weak version of the preceding Lemma

Lemma 9.5 *Let ξ be a rectifiable map from S^1 to $\mathbb{P}(E)$ parametrised by arc length. Let ρ be a representation of $\pi_1(S)$ in $SL(E)$. Suppose that ρ is S-irreducible. Suppose also that $\xi(S^1)$ is $\rho(\pi_1(S))$-invariant. Let x, y be two distinct points of S^1, then one of the connected components I of $S^1 \setminus \{x, y\}$ satisfies $\langle \xi(I) \rangle = E$.*

The main point here is that ξ is not assumed to be injective. If it were, it would be a homeomorphism, and we would be able to deduce an action of $\pi_1(S)$ on S^1 such that ξ is ρ-equivariant. The result would then follow by Lemma 9.4.

PROOF: We suppose the contrary, in which case both connected components I_0 and I_1 of $S^1 \setminus \{x, y\}$ satisfy

$$\langle \xi(I_i) \rangle = P_i \subsetneq E.$$

Consequently $\xi(S^1) \subset P_0 \cup P_1$. It thus follows by a reasoning similar to that used in (i) of the previous lemma, which equally well applies in this more general situation, that ρ is not S-irreducible. We thus obtain a contradiction and the result follows. Q.E.D.

9.5 The limit and the boundary at infinity

By Proposition 9.3, we may assume that the sequence of curves $\{\xi^1_m\}_{m\in\mathbb{N}}$, parametrised by the arc-length, converges. In particular, let ξ^1 be the limit. *A priori*, by using the arc-length parametrisation, we have lost control over the action of $\pi_1(S)$. We merely know that $\xi^1(S^1)$ is globally invariant by $\rho(\pi_1(S))$. Our aim is now to show that this action is semi-conjugate to the action of $\pi_1(S)$ on $\partial_\infty \pi_1(S)$.

We begin by replacing ξ^1 by its arc-length parametrisation $\hat{\xi}^1$ so that we have $\xi^1 = \hat{\xi}^1 \circ \pi$ with π monotone.

We aim to prove

Proposition 9.6 *There is an injective left continuous map preserving the orientation φ_0 from $\partial_\infty \pi_1(S)$ to S^1, such that $\hat{\xi}^1 \circ \varphi_0$ is ρ equivariant and $*$-hyperconvex.*

Note that Proposition 9.6 implies Lemma 9.2. The proof may be divided in the following steps, we prove

1. We first prove that $\hat{\xi}^1$, when restricted to a certain (non empty) subset Λ, is "hyperconvex". This is Proposition 9.5.1;
2. Second, we prove Lemma 9.9 which, combined with the propositions of the previous section, implies Proposition 9.6.

9.5.1 Λ-Hyperconvexity

We prove the following related two propositions

Proposition 9.7 *The map $\hat{\xi}^1$ is injective.*

As a consequence, it is a homeomorphism onto its image, and we deduce that there exists an action of $\pi_1(S)$ on S^1 by homeomorphisms such that $\hat{\xi}^1$ is ρ-equivariant. Let Γ_0 the normal subgroup of index 2 of orientation preserving elements of $\pi_1(S)$. Let

$$\Lambda = \Lambda_{\xi,\rho,\Gamma_0}.$$

We observe that Λ is $\pi_1(S)$ invariant. We shall also prove.

Proposition 9.8 *For any n-tuple $(x_1, \ldots, x_n)$ of distinct points of the closed set $\overline{\Lambda}$, the following sum is direct*

$$\sum_{i=1}^{i=n} \hat{\xi}^1(x_i).$$

Proof: We write $\tilde{\xi}^1_m = \xi^1_m \circ \phi_m$, so that

$$\lim_{m\to\infty} \tilde{\xi}^1_m = \xi^1 = \hat{\xi}^1 \circ \pi.$$

We now prove the propositions. We split the proof in two parts, which use very similar ideas.

Injectivity: proof of Proposition 9.5.1. First we aim to prove that the map is injective. We suppose the contrary? Let $y \neq z$ be such that $\hat{\xi}^1(y) = \hat{\xi}^1(z)$. By Lemma 9.5, since ρ is S-irreducible, one of the connected component J of $\partial_\infty \pi_1(S) \setminus \{y, z\}$ is such that $\langle \xi(J) \rangle = E$.

We may therefore find n points $(x_1, \ldots, x_n)$ in J such that the following sums are direct

$$\begin{array}{rcl} \hat{\xi}^1(x_1) + \ldots + \hat{\xi}^1(x_n) & = & E, \\ \forall i, \quad \sum_{i \neq j} \hat{\xi}^1(x_j) + \hat{\xi}^1(y) & = & E \end{array} \tag{59}$$

Let I be an interval containing y and z and none of the x_i. For any point $t \in \{y, z, x_1, \ldots, x_n\}$, we denote by $\dot{t}$ a point such that $\pi(\dot{t}) = t$. For any distinct integers i, j, we write

$$W_{ij} = (\dot{x}_i, \dot{x}_j), \quad Y_{ij} = (\ldots, \dot{x}_l, \ldots)_{l \notin \{i,j\}}.$$

As in Section 5.4, we may study the maps F_{ij}^m defined by

$$\left\{ \begin{array}{ccl} I & \rightarrow & \mathbb{P}\big(\tilde{\xi}_m^{(2)}(W_{ij})\big) \setminus \{\tilde{\xi}_m^1(\dot{x}_i)\} \\ t & \mapsto & \big(\tilde{\xi}_m^{(n-2)}(Y_{ij}) \oplus \tilde{\xi}_m^1(t)\big) \cap \tilde{\xi}_m^{(2)}(W_{ij}). \end{array} \right. .$$

By Assertion (59), we obtain

$$\lim_{m \to \infty} (F_{ij}^m(\dot{y})) = \big(\bigoplus_{k \neq i,j} \hat{\xi}^1(x_k) \oplus \hat{\xi}^1(y)\big) \cap \big(\hat{\xi}^1(x_i) \oplus \hat{\xi}^1(x_j)\big) = \lim_{m \to \infty} (F_{ij}^m(\dot{z})).$$

By Proposition 5.3, all the maps F_{ij}^m are monotone. It follows that for all t in $[\dot{y}, \dot{z}]$, we have

$$\lim_{m \to \infty} (F_{ij}^m(\dot{y})) = \lim_{m \to \infty} (F_{ij}^m(t)).$$

However, for t in a neighbourhood of y, the following sums are direct

$$\forall i, \quad \sum_{i \neq j} \hat{\xi}^1(x_j) + \xi^1(t) \quad = \quad E,$$

This implies that

$$\lim_{m \to \infty} (F_{ij}^m(t)) = \big(\bigoplus_{k \neq i,j} \hat{\xi}^1(x_k) \oplus \hat{\xi}^1(\pi(t))\big) \cap \big(\hat{\xi}^1(x_i) \oplus \hat{\xi}^1(x_j)\big)$$

Hence, for all t in a non empty open set, we have

$$\hat{\xi}^1(t) = \hat{\xi}^1(y).$$

However, since $\hat{\xi}^1$ is parametrised by arc length and, hence, not locally constant, this is impossible.

Λ-*Hyperconvexity: proof of Proposition 9.8.* In this proof, we only use the following property of the elements of Λ: If U is a neighbourhood of an element of Λ, $\xi(U)$ is not included in a union of two proper subspaces (cf. Lemma 9.4(iv)). This property is then true for any element in the closure $\overline{\Lambda}$ of Λ.

Let p be the smallest integer less than n (when it exists) such that there exist p cyclically oriented points $(x_1, \ldots, x_{p-2}, y, z)$ with $x_i, y, z, \in \overline{\Lambda}$ such that the following sum is not direct

$$H = \sum_{i=1}^{i=p-2} \hat{\xi}^1(x_i) + \hat{\xi}^1(y) + \hat{\xi}^1(z).$$

By Proposition , $p \geq 3$. We observe that, by minimality of p, the following sums are direct and equal

$$\sum_{i=1}^{i=p-2} \hat{\xi}^1(x_i) + \hat{\xi}^1(y) = \sum_{i=1}^{i=p-2} \hat{\xi}^1(x_i) + \hat{\xi}^1(z) = H. \tag{60}$$

By our initial observation, we may choose $(x_{p-1}, \ldots, x_n)$ in an arbitrarily small neighbourhood J of x_1 in $\overline{\Lambda}$ such that the following sums are direct

$$\begin{aligned} \hat{\xi}^1(x_1) + \ldots + \hat{\xi}^1(x_n) &= E, \\ \forall i \geq p-1, \quad \sum_{j \neq i} \hat{\xi}^1(x_j) + \hat{\xi}^1(y) &= E \\ \forall i \geq p-1, \quad \sum_{j \neq i} \hat{\xi}^1(x_j) + \hat{\xi}^1(z) &= E. \end{aligned} \tag{61}$$

Let I be an interval containing y and z and none of the x_i. As in the previous proof, for any point $t \in \{x_1, \ldots, x_n, y, z\}$, we denote by $\dot{t}$ a point such that $\pi(\dot{t}) = t$. For any distinct integers i, j, let uslet us write

$$W_{ij} = (\dot{x}_i, \dot{x}_j), \quad Y_{ij} = (\ldots, \dot{x}_l, \ldots)_{l \notin \{i,j\}}.$$

As in Section 5.4, we study the maps F_{ij}^m defined for $i, j \geq p-1$, by

$$\left\{ \begin{array}{ccc} I & \rightarrow & \mathbb{P}\big(\tilde{\xi}_m^{(2)}(W_{ij})\big) \setminus \{\tilde{\xi}_m^1(\dot{x}_i)\} \\ t & \mapsto & \big(\tilde{\xi}_m^{(n-2)}(Y_{ij}) \oplus \tilde{\xi}_m^1(t)\big) \cap \tilde{\xi}_m^{(2)}(W_{ij}). \end{array} \right. .$$

By Assertions (61) and (60), for all $i, j \geq p-1$, we obtain

$$\lim_{m \to \infty} (F_{ij}^m(\dot{y})) = \big(\bigoplus_{k \neq i,j} \hat{\xi}^1(x_k) \oplus \hat{\xi}^1(y)\big) \cap \big(\hat{\xi}^1(x_i) \oplus \hat{\xi}^1(x_j)\big) = \lim_{m \to \infty} (F_{ij}^m(\dot{z})).$$

By Proposition 5.3, all the maps F_{ij}^m are monotone. It follows that for all t in $[\dot{y}, \dot{z}]$, we have

$$\lim_{m \to \infty} (F_{ij}^m(\dot{y})) = \lim_{m \to \infty} (F_{ij}^m(t)).$$

Moreover for t in a neighbourhood of y, the following sums are all direct

$$\forall i, \quad \sum_{i \neq j} \hat{\xi}^1(x_j) + \xi^1(t) \quad = \quad E,$$

This implies that

$$\lim_{m\to\infty}(F_{ij}^m(t)) = \Big(\bigoplus_{k\neq i,j}\hat{\xi}^1(x_k)\oplus\hat{\xi}^1(\pi(t))\Big)\cap\big(\hat{\xi}^1(x_i)\oplus\hat{\xi}^1(x_j)\big).$$

Hence, for all t in a right neighbourhood U^+ of y, we have

$$\bigoplus_{i=1}^{i=p-2}\hat{\xi}^1(x_i)\oplus\hat{\xi}^1(t) = \bigoplus_{i=1}^{i=p-2}\hat{\xi}^1(x_i)\oplus\hat{\xi}^1(y).$$

Consequently, there exists a proper subspace H^+ of E, such that

$$\forall t\in U^+,\quad \hat{\xi}^1(t)\subset H^+.$$

By symmetry (using cyclic permutation of $(x_1,\ldots,y,z)$), we find that there exist a left neighbourhood U^- of y, and a proper subspace H^- of E such that

$$\forall t\in U^-,\quad \hat{\xi}^1(t)\subset H^-.$$

We thus obtain a neighbourhood U of y and two proper subspaces H^+ and H^- of E such that

$$\forall t\in U,\quad \hat{\xi}^1(t)\subset H^-\cup H^+.$$

This contradicts our initial observation, and the result follows. Q.E.D.

9.5.2 Conjugating to the action on the boundary at infinity

We use the framework described in Proposition 9.3. That is we consider

1. a sequence $\{\rho_m\}_{m\in\mathbb{M}}$ of representations of $\pi_1(S)$ in $PSL(E)$, converging to ρ, such that for all non trivial γ in $\pi_1(S)$, $\rho_m(\gamma)$ is purely loxodromic
2. a sequence of maps $\{\xi_m^1\}_{m\in\mathbb{N}}$ from $\partial_\infty\pi_1(S)$ to $\mathbb{P}(E)$, such that each ξ_m^1 is ρ_m-equivariant,
3. a sequence of homeomorphisms ϕ_m of S^1 into $\partial_\infty\pi_1(S)$ such that $\{\xi_m^1\circ\phi_m\}_{m\in\mathbb{N}}$ converges to $\hat{\xi}^1\circ\pi$ where $\hat{\xi}^1$ is an embedding and π is a monotone map from S^1 to $\partial_\infty\pi_1(S)$.

We say in this situation that $(\hat{\xi}^1,\rho)$ is a *good limit.* We observe that in this case, there exists an action ρ of $\pi_1(S)$ on S^1 such that $\hat{\xi}^1$ is ρ-equivariant.

The following lemma completes the proof of Proposition 9.6.

Lemma 9.9 *Let $\pi_1(S)$ be a surface group. Let ρ be a representation of $\pi_1(S)$ in $SL(E)$. Suppose that*

- *the restriction of ρ to every finite index subgroup is irreducible, and*
- *there exists a map $\hat{\xi}^1$ from S^1 to $\mathbb{P}(E)$ such that $(\hat{\xi}^1,\rho)$ is a good limit (in particular every element of $\rho(\pi_1(S))$ has real eigenvalues).*

Then the induced action of $\pi_1(S)$ on S^1 for which $\hat{\xi}^1$ is ρ-equivariant is topologically semi-conjugate to the action of $\pi_1(S)$ on $\partial_\infty\pi_1(S)$ in the following sense: there exists an orientation preserving, left continuous, $\pi_1(S)$-equivariant map φ_0 from $\partial_\infty\pi_1(S)$ to S^1.

Finally, $\hat{\xi}^1 \circ \varphi_0$ is $*$-hyperconvex.

PROOF: We first observe that $\Gamma_{\mathbb{R}}$ is not empty. Indeed, the connected component of the Zariski closure of $\rho(\pi_1(S))$ is irreducible. It therefore contains a non trivial diagonalisable element. Hence so does $\rho(\pi_1(S))$. However, since this element only has real eigenvalues, it follows that $\Gamma_{\mathbb{R}}$ is not empty. Let now Γ^0 be the subgroup of finite index of orientation preserving elements of $\pi_1(S)$ acting on S^1. As before $\Gamma^0_{\mathbb{R}}$ is not empty and invariant by conjugation. As in Paragraph 9.4.1, let.

$$\Lambda = \Lambda_{\xi,\rho,\Gamma^0} \subset S^1.$$

By Lemma 9.4(*iii*) , the set Λ is not empty and is invariant under the action of $\pi_1(S)$.

Let γ be an element of $\Gamma^0_{\mathbb{R}}$, we define

- γ^+ to be such that $\xi(\gamma^+)$ is the attractive fixed point of $\rho(\gamma)$. The point γ^+ is well defined by Lemma 9.4.(*iii*). Let

$$\Lambda = \{\gamma^+, \gamma \in \Gamma^0_{\mathbb{R}}\},$$

- γ_0^+ to be the attractive fixed point of $\rho_0(\gamma)$, and

$$\Lambda_0 = \{\gamma_0^+, \gamma \in \Gamma^0_{\mathbb{R}}\},$$

We know that

$$\gamma_0^+ = \lambda_0^+ \implies \exists p, q \neq 0, \lambda^p = \gamma^q \implies \gamma^+ = \lambda^+.$$

We therefore have a well defined map φ_0, possibly not injective, defined from Λ_0 to Λ. We observe that by the minimality of the action of $\rho_0(\Gamma)$, Λ_0 is dense

We now prove that φ_0 preserves the cyclic ordering. We use our hypothesis concerning the construction of $\hat{\xi}^1$, in particular that $\hat{\xi}^1$ is a good limit. We begin with the following observations. By construction, $\xi^1_m(\gamma_0^+)$ is an attractive fixed point of $\rho_m(\gamma)$ (*cf* Hypotheses (1) and (2)), hence

$$\lim_{m\to\infty}(\xi^1_m(\gamma_0^+)) = \hat{\xi}^1(\gamma^+) \tag{62}$$

We now extract a subsequence such that $\{\tilde{\gamma}^+_m\}_{m\in\mathbb{N}} = \{\phi_m^{-1}(\gamma_0^+)\}_{m\in\mathbb{N}}$ converges to a point $\tilde{\gamma}^+$. By Hypothesis (3),

$$\lim_{m\to\infty}(\xi^1_m(\phi_m(\tilde{\gamma}^+_m))) = \hat{\xi}^1 \circ \pi(\tilde{\gamma}^+) \tag{63}$$

Combining Assertions (63) and (62), and using the injectivity of $\hat{\xi}^1$ we find that $\pi(\tilde{\gamma}^+) = \gamma^+$. It follows that φ_0 preserves the orientation. Indeed, let

γ, λ and δ be three elements of Γ such that $(\gamma_0^+, \lambda_0^+, \delta_0^+)$ are cyclically oriented. Then so are $(\phi_n(\gamma_0^+), \phi_n(\lambda_0^+), \phi_n(\delta_0^+))$, hence $(\tilde{\gamma}_0^+, \tilde{\lambda}_0^+, \tilde{\delta}_0^+)$ and finally $(\pi(\tilde{\gamma}_0^+), \pi(\tilde{\lambda}_0^+), \pi(\tilde{\delta}_0^+)) = (\gamma^+, \lambda^+, \delta^+)$. Since, by minimality, Λ_0 is dense and since φ_0 preserves the orientation, we may extend it by left continuity to a Γ equivariant orientation preserving map from $\partial_\infty \pi_1(S)$ to S^1. We finally prove that φ_0 is injective. Let

$$U = \{x \in S^1/\varphi_0 \text{ is constant on a neighbourhood of } x\}.$$

We show that $U = \emptyset$. We can observe that U is an open $\rho_0(\pi_1(S))$-invariant strict subset of $\partial_\infty \pi_1(S)$. By minimality of the action of $\pi_1(S)$ on $\partial_\infty \pi_1(S)$ we conclude that $U = \emptyset$. This implies that φ_0 is strictly monotone, and thus injective.

It now remains to prove that $\hat{\xi}^1 \circ \varphi_0$ is $*$-hyperconvex. First, we observe that $\varphi_0(\partial_\infty \pi_1(S)) = \overline{\Lambda}$. By Proposition 9.5.1, for n distinct points $(x_1, \ldots, x_n)$ of $\overline{\Lambda}$, the following sum is direct

$$\sum_i \hat{\xi}^1(x_i).$$

This implies the first condition of $*$-hyperconvexity. The second condition on $*$-hyperconvexity is a closed condition and thus follows from the fact that $\xi^1 \pi$ is the limit of $\{\xi^1_m \circ \phi_m\}$ which is a sequence of hyperconvex maps. Q.E.D.

10 Appendix: some lemmas

We prove various lemmas that are used several times throughout this paper. This appendix is entirely self contained. The following lemma is trivial for experts in the theory of Higgs field experts and is certainly well known.

Lemma 10.1 *If ρ belongs to a Hitchin component, then the connected component of the Zariski closure of $\rho(\pi_1(S))$ is irreducible, or equivalently the restriction of ρ to every finite index subgroup is irreducible.*

Proof: We recall the relevant part of Hitchin construction. We consider a surface S, its canonical bundle K and the holomorphic vector bundle

$$E = K^{-n\otimes} \oplus K^{(2-n)\otimes} \oplus \ldots\ldots K^{n\otimes}.$$

We consider the Higgs field ϕ which is a section of $End(E)$ associated to a companion matrix.

Every representation in a Hitchin component arises from such a Higgs field. By Lemma 1.2 in [27], the parallel sections of the endomorphism bundle are exactly those holomorphic sections which commute with the Higgs field. Let A be such a section. Since A is holomorphic, the first row of its matrix in the decomposition of E vanishes. It is now easy to verify that a matrix whose first row vanishes and that commutes with a companion matrix is zero.

We have thus proved that the endomorphism bundle has no parallel sections, and hence that the representation is irreducible.

Since the restriction to a finite index subgroup comes from a representation in a Hitchin component on the corresponding finite cover, the second part of the lemma follows. Q.E.D.

The following result is used several times.

Lemma 10.2 *Let Γ be a surface group. Let ρ be a representation of $\pi_1(S)$ in $SL(n,\mathbb{R})$. Suppose that there exists a continuous ρ-equivariant map ξ^{k+1} from $\partial_\infty\pi_1(S)$ to the Grassmannian of $k+1$-planes in $\mathbb{R}^n$. Suppose also that there exists a $(n-k)$-plane A, and a non empty open set U in $\partial_\infty\pi_1(S)$ such that*

$$\forall y \in U,\ \dim(\xi^{k+1}(y) \cap A) \geq 1. \tag{64}$$

Then the restriction of ρ to a finite index subgroup is not irreducible, or, equivalently, the connected component of the identity of the Zariski closure of $\rho(\pi_1(S))$ is not irreducible.

PROOF: *First step.* We begin by showing that there exists some $(n-k)$-plane B, such that

$$\forall y \in \partial_\infty\pi_1(S),\ \dim(\xi^k(y) \cap B) \geq 1. \tag{65}$$

First, there exists a smaller open subset O of U and an element $\gamma \in \pi_1(S)$ such that

$$\begin{aligned} \gamma^i(O) &\subset \gamma^{i+1}(O) \\ O_\infty &= \bigcup_{i\in\mathbb{N}} \gamma^i(O) \text{ is dense in } \partial_\infty\pi_1(S). \end{aligned}$$

Next, we observe that, for $B^i = \rho(\gamma)^{-i}(A)$, we have

$$\forall y \in \gamma^i(O),\ \dim(\xi^k(y) \cap B^i) \geq 1.$$

We now extract from $\{B^i\}_{i\in\mathbb{N}}$ a subsequence which converges to a $(n-k)$-plane B. It follows that

$$\forall y \in O_\infty,\ \dim(\xi^k(y) \cap B) \geq 1.$$

Hence, since O_∞ is dense, we obtain Assertion (65). The result now follows from the following proposition applied to G the Zariski closure of $\rho(\pi_1(S))$. Q.E.D.

Proposition 10.3 *Let G be an algebraic subgroup of $SL(n,\mathbb{R})$. If there exist a k-plane C, and an $(n-k)$-plane B such that*

$$\forall g \in G,\ \dim(g(C) \cap B) \geq 1. \tag{66}$$

Then the connected component of the identity of G is not irreducible.

PROOF: Let G, C, B be as above. We observe that $B + C$ is a proper subspace, and $B \cap C$ is not reduced to $\{0\}$. Let g_0 be an element of G such that

$$\forall g \in G,\ p := \dim(g_0(C) \cap B) \leq \dim(g(C) \cap B).$$

Now, let $D = g_0(C)$, and let F be a codimension $p-1$ vector subspace of B such that,

$$\dim(D \cap F) = 1.$$

Note that

$$\forall g \in G,\ \dim(g(D) \cap F) \geq 1. \tag{67}$$

Let $\mathcal{G}$ be the Lie algebra of G.

We prove the following assertion which contradicts the irreducibility of the connected component of the identity of G,

$$\forall \alpha \in \mathcal{G},\ \alpha(D \cap F) \subset D + F. \tag{68}$$

Indeed, let $(e_0, \ldots, e_k)$ be a basis of D, and let $(u_1, \ldots, u_l)$ be a basis of F such that (u_l) is a basis of $D \cap F$. We observe that

$$e_0 \wedge \ldots \wedge e_k \wedge u_1 \wedge \ldots \wedge u_{l-1} \neq 0.$$

Let h be an element of $\mathcal{G}$. By Assertion (67), we obtain

$$e_0 \wedge \ldots \wedge e_k \wedge e^{t\alpha}(u_1 \wedge \ldots \wedge u_l) = 0.$$

We now take the first order term in t of the above series. Since

$$e_0 \wedge \ldots \wedge e_k \wedge u_l = 0,$$

we have,

$$e_0 \wedge \ldots \wedge e_k \wedge u_1 \wedge \ldots \wedge u_{l-1} \wedge \alpha(u_l) = 0.$$

This implies Assertion (68), and the result now follows.Q.E.D.

Finally, the following lemma is of independent interest

Lemma 10.4 *Let Γ be a subgroup of $SL(n, \mathbb{R})$ whose elements are all real split. Suppose that every finite index subgroup of Γ is irreducible. Then Γ is discrete.*

PROOF: Let G be the Zariski closure of Γ. From the irreducibility hypothesis, it follows G is semi-simple. Suppose that Γ is not discrete. Since Γ is Zariski dense, it is a classical result that its closure (for the usual topology) contains one of the non trivial factors H of G. However the closure of Γ consists of elements whose elements have only real eigenvalues. This implies that the maximal compact subgroup of H is reduced to the identity which never happens for a simple Lie group. we thus obtain a contradiction and the result follows. Q.E.D.

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