Cross Ratios, Anosov Representations and the Energy Functional on Teichmüller Space

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Résumé

Nous étudions deux classes de représentations linéaires d'un groupe de surface : les représentations de Hitchin et les représentations symplectiques maximales. En reliant ces représentations à des birapports, nous montrons qu'elles sont *déplaçantes*, c'est à dire que leurs longueurs de translation sont grossièrement contrôlées par celles du graphe de Cayley. Ceci nous permet de montrer que le groupe modulaire agit proprement sur l'espace de ces représentations et que la fonctionnelle énergie associée à une telle représentation est propre. Nous en déduisons alors l'existence de surfaces minimales dans les quotients d'espaces symétriques associés et en tirons deux conséquences : un résultat de rigidité pour les représentations symplectiques et un résultat partiel concernant la description de la composante de Hitchin en termes purement holomorphes.

Abstract

We study two classes of linear representations of a surface group: Hitchin and maximal symplectic representations. We relate them to cross ratios and thus deduce that they are *displacing* which means that their translation lengths are roughly controlled by the translations lengths on the Cayley graph. As a consequence, we show that the mapping class group acts properly on the space of representations and that the energy functional associated to such a representation is proper. This implies the existence of minimal surfaces in the quotient of the associated symmetric spaces, a fact which leads to two consequences: a rigidity result for maximal symplectic representations and a partial result concerning a purely holomorphic description of the Hichin component.

1 Introduction

Let S be a closed connected oriented surface of genus greater than one. Monodromies of hyperbolic structures on S define a distinguished class of homo-

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morphisms from the fundamental group $\pi_1(S)$ into $\mathrm{PSL}(2,\mathbb{R})$. In this paper we study two generalisations of these surface group representations, one in which we replace $\mathrm{PSL}(2,\mathbb{R})$ by $\mathrm{PSL}(n,\mathbb{R})$ and one in which we generalise $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{PSp}(2,\mathbb{R})$ to $\mathrm{PSp}(2n,\mathbb{R})$.

The first generalisation uses the irreducible representation of $\mathrm{PSL}(2,\mathbb{R})$ in $\mathrm{PSL}(n,\mathbb{R})$. A *Fuchsian representation* from $\pi_1(S)$ into $\mathrm{PSL}(n,\mathbb{R})$ is a representation which decomposes as the product of a faithful cocompact representation from $\pi_1(S)$ to $\mathrm{PSL}(2,\mathbb{R})$ and the irreducible representation from $\mathrm{PSL}(2,\mathbb{R})$ to $\mathrm{PSL}(n,\mathbb{R})$ (see [30] and Section 4.1). The representations we study, called *Hitchin representations*, are those which may be *deformed* into a Fuchsian representation. In [26], Hitchin studies the moduli space of reductive (*i.e.* whose Zariski closure is a reductive group) Hitchin representations. For a combinatorial point of view on a related subject, see Fock and Goncharov in [16].

The second generalisation exploits the fact that the homogeneous space M associated to $\mathrm{PSp}(2n,\mathbb{R})$ is Hermitian symmetric and thus carries an invariant symplectic form ω (see Section 4.2.1 for details). Given a representation ρ from $\pi_1(S)$ to $\mathrm{PSp}(2n,\mathbb{R})$, if f is any ρ-equivariant map from the universal cover of S to M, then $f^*\omega$ is invariant under the action of $\pi_1(S)$. The following number

$$\tau(\rho) = \frac{n}{2\pi}\int_S f^*\omega$$

is then an integer independent of the choice of f. This number, called the *Toledo invariant* of ρ, remains constant under continuous deformations of the representation and satisfies a generalised Milnor-Wood Inequality (see [43])

$$|\tau(\rho)| \leqslant n|\chi(S)|.$$

By definition, a *maximal symplectic representation* is one for which the Toledo invariant attains the upper bound in this inequality. The notion of maximality and a suitable version of the Milnor-Wood Inequality extend to all Hermitian symmetric spaces. W. Goldman shows in [18, 19] that maximal representations in $\mathrm{PSL}(2,\mathbb{R})$ are precisely monodromies of hyperbolic structures. In the general case, these maximal representations have been extensively studied by Bradlow, García-Prada, Gothen, Mundet i Riera (as well as Xia in a specific example) ([3, 5, 4, 17, 22, 45]) using Higgs bundle techniques on one hand and Burger, Iozzi and Wienhard ([6, 9, 8, 44]) using bounded cohomology techniques on the other hand.

Both type of representations – Hitchin representations and maximal symplectic representations – can be thought of as generalisations of the $\mathrm{PSL}(2,\mathbb{R})$-representations which arise from monodromies of hyperbolic structures and hence as generalising Teichmüller-Thurston theory.

The maximal symplectic representations and the Hitchin representations are known to share several fundamental properties, including:

- They are Anosov as defined in [30]. For Hitchin representations this is proved in [30], for maximal representations this is shown in [7] by Burger, Iozzi, Wienhard and the author.

- The Zariski closure of the images are reductive. For Hitchin representations, see Proposition 4.1.5. For maximal representations, this is proved by Burger, Iozzi and Wienhard in [8].

- They are discrete: see [30] for Hitchin representations and the proof by Burger, Iozzi and Wienhard in [8] for maximal representations.

The results presented in this paper extend this list of common features by showing that both types of representation have the property that we call *displacing*. More precisely, let Γ be a finitely generated subgroup of the isometry group of a metric space X: we say that Γ is *displacing* if, given a finite generating set $\mathcal{G}$ of Γ, there exist positive constants A and B such that for all elements γ of Γ

$$\inf_{x\in X} d(x,\gamma(x)) \geqslant A \inf_{\eta\in\Gamma} \|\eta\gamma\eta^{-1}\|_{\mathcal{G}} - B.$$

where $\|\gamma\|_{\mathcal{G}}$ is the word length of γ with respect to $\mathcal{G}$. It is easy to check that this definition is independent of the generating set $\mathcal{G}$. Note that cocompact groups are always displacing, as are convex-cocompact groups whenever X is *Hadamard* (*i.e.* complete, nonpositively curved and simply connected). If ρ is a representation from a finitely generated group Γ with values in a connected semi-simple real Lie group G without compact factor and with trivial centre, then ρ is *displacing* if the group $\rho(\Gamma)$ is displacing as a group of isometries of the associated symmetric space.

We now briefly summarise results of Delzant, Guichard, Mozes and the author in [13] which compare this notion to the fact that orbit maps are quasiisometries. While the two notions turn out to be equivalent for surface groups and more generally hyperbolic groups, they are not equivalent for every group: there are known examples which have displacing representations whose orbit maps are not quasiisometries and also nondisplacing representations for which the orbit maps are quasiisometries.

The starting point of this article is the following result.

Theorem 1.0.1 *Hitchin and maximal symplectic representations are displacing.*

It has already been observed by Burger, Iozzi, Wienhard and the author in [7] that the orbit maps are quasiisometries for maximal symplectic representations. Here we prove the theorem by relating Hitchin representations and maximal symplectic representations to cross ratios (cf Theorems 4.1.6 and 4.2.4).

The two main applications of this result are that

- the mapping class group acts properly on certain moduli spaces, and

- the energy functional on Teichmüller space is proper.

Let us be more specific. Let $\mathrm{Hom}_H(\pi_1(S),\mathrm{PSL}(n,\mathbb{R}))$ be the space of Hitchin homomorphisms and

$$\mathrm{Rep}_H(\pi_1(S),\mathrm{PSL}(n,\mathbb{R})) = \mathrm{Hom}_H(\pi_1(S),\mathrm{PSL}(n,\mathbb{R}))/\mathrm{PSL}(n,\mathbb{R}),$$

where the action of $\mathrm{PSL}(n, \mathbb{R})$ is by conjugation.

Similarly, let $\mathrm{Hom}_T(\pi_1(S), \mathrm{PSp}(2n, \mathbb{R}))$ be the space of maximal symplectic homomorphisms and

$$\mathrm{Rep}_T(\pi_1(S), \mathrm{PSp}(2n, \mathbb{R})) = \mathrm{Hom}_T(\pi_1(S), \mathrm{PSp}(2n, \mathbb{R}))/\mathrm{PSp}(2n, \mathbb{R}).$$

Both spaces Hom_H and Hom_T are unions of connected components, each of which is a component of the corresponding space of reductive homomorphisms. Moreover, since they consist of reductive representations, the quotient spaces Rep_H and Rep_T are Hausdorff (hence locally compact). This last fact follows from the identification due to Hitchin [25] of reductive representations with polystable Higgs bundle, although more direct proofs could be obtained. The *mapping class group* $\mathcal{M}(S)$ – that is the group of outer automorphisms of $\pi_1(S)$ which may be represented by orientation preserving diffeomorphisms – acts by precomposition on these spaces. The following result will be an immediate consequence of Theorem 1.0.1.

Theorem 1.0.2 *The mapping class group $\mathcal{M}(S)$ acts properly by precomposition on the spaces $\mathrm{Rep}_H(\pi_1(S), \mathrm{PSL}(n, \mathbb{R}))$ and $\mathrm{Rep}_T(\pi_1(S), \mathrm{PSp}(2n, \mathbb{R}))$.*

In the case of $\mathrm{PSL}(3, \mathbb{R})$, W. Goldman proved in [20] that the mapping class group acts properly on the moduli space of convex $\mathbb{RP}^2$ structures. Moreover, together with Choi in [10], he identified this moduli space with the Hitchin component.

Our second main application concerns the energy functional. We first recall briefly the general framework and refer to Paragraph 5 for precise definitions. Let ρ be a representation from $\pi_1(S)$ to a connected semi-simple real Lie group G without compact factor and with trivial centre. Let M be the symmetric space associated to G and let M_ρ be the flat M-bundle over S defined by the representation ρ. Let $\Gamma(S, M_\rho)$ be the space of smooth sections of M_ρ. If J is a complex structure on S and f an element of $\Gamma(S, M_\rho)$, we define

$$\mathrm{Energy}_J(f) = \int_S \langle df \wedge df \circ J \rangle.$$

Then, the *energy functional* e_ρ, associated to the representation ρ, is the map from the space of all complex structures on S to the real numbers defined by

$$e_\rho(J) = \inf\{\mathrm{Energy}_J(f) \mid f \in \Gamma(S, M_\rho)\}.$$

The value of this function depends only on the isotopy class of the complex structure J, and hence defines a function on Teichmüller space. Denoted by e_ρ and also called the *energy functional*, this function is smooth on Teichmüller space (*cf* Paragraph 5.2). We shall prove the following result

Theorem 1.0.3 *If ρ is a Hitchin representation or a maximal symplectic representation, then the energy functional e_ρ is a proper function on Teichmüller space.*

It is classical that critical points of the energy functional are related to minimal surfaces. Indeed, using Gulliver's definition of a branched immersion ([24]), we obtain the following consequence

Corollary 1.0.4 *Let ρ be a Hitchin or maximal symplectic representation. Then there exists a minimal branched immersion from S into $M/\rho(\pi_1(S))$ which represents ρ at the level of homotopy groups.*

This corollary will lead to two applications that we shall explain in the next section.

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2 Application of the main results and outline of the paper

2.1 The minimal area and the Toledo invariant

We restrict ourselves in this paragraph to representations of surface groups with values in $\mathrm{PSp}(2n, \mathbb{R})$.

Definition 2.1.1 [Minimal area] *The* minimal area *of a representation ρ from $\pi_1(S)$ to $\mathrm{PSp}(2n, \mathbb{R})$ is*

$$\mathrm{MinArea}(\rho) = \inf\{e_\rho(J) \mid J \in \mathcal{T}(S)\}.$$

Definition 2.1.2 [Diagonal representation] *A homomorphism ρ from $\pi_1(S)$ with values in $\mathrm{PSp}(2n, \mathbb{R})$ is* diagonal *if it factors as $\rho = \varphi \circ \delta \circ \sigma$ where σ is a cocompact homomorphism of $\pi_1(S)$ into $\mathrm{PSL}(2, \mathbb{R})$, δ is the diagonal mapping from $\mathrm{PSL}(2, \mathbb{R})$ into*

$$\prod_{i=1}^{i=n} \mathrm{PSL}(2, \mathbb{R})$$

and φ is an embedding of this product in $\mathrm{PSp}(2n, \mathbb{R})$ corresponding to a decomposition of $\mathbb{R}^{2n}$ in a direct sum of 2-dimensional orthogonal symplectic vector spaces.

Note that the set of diagonal homomorphisms is invariant under conjugation.

Theorem 2.1.3 *For every representation ρ, we have*

$$\frac{n}{2\pi}\mathrm{MinArea}(\rho) \geqslant |\tau(\rho)|,$$

where $\tau(\rho)$ is the Toledo invariant of ρ. If furthermore ρ is maximal and

$$\frac{n}{2\pi}\mathrm{MinArea}(\rho) = \tau(\rho),$$

then ρ is a diagonal representation.

As a corollary of the proof, we have the following result

Corollary 2.1.4 *Let ρ be a maximal representation. Assume that there exists a holomorphic equivariant map f from S to the associated symmetric space of* $\mathrm{PSp}(2n,\mathbb{R})$. *Then ρ is diagonal and f is totally geodesic.*

2.2 The Hitchin map is surjective

In his article [26], N. Hitchin gives explicit parametrisations of the Hitchin components $\mathrm{Rep}_H(\pi_1(S), \mathrm{PSL}(n,\mathbb{R}))$. Namely, given a choice of a complex structure J over S, he produces a homeomorphism

$$H_J : \mathcal{Q}(2,J) \oplus \ldots \oplus \mathcal{Q}(n,J) \to \mathrm{Rep}_H(\pi_1(S), \mathrm{PSL}(n,\mathbb{R})),$$

where $\mathcal{Q}(p,J)$ denotes the space of holomorphic p-differentials on the Riemann surface (S,J). The first step in the construction of this map uses results from K. Corlette's seminal paper [11] – see also [15, 27] – to identify conjugacy classes of representations with harmonic mappings to symmetric spaces. The second step is to associate holomorphic differentials to a harmonic mapping with values in a symmetric space. This is accomplished by means of a construction very similar to the construction of characteristic classes in Chern-Weil theory – see Paragraph 8.1.3.

However, in this construction, the homeomorphism H_J depends on the choice of the complex structure J. In particular, this choice breaks the mapping class group symmetry. The construction thus does not give any information on the topological nature of the quotient of $\mathrm{Rep}_H(\pi_1(S), \mathrm{PSL}(n,\mathbb{R}))$ by the mapping class group.

We now explain a construction which is equivariant with respect to the action of the mapping class group and which conjecturally leads to a complex analytic description of the quotient. Let $\mathcal{E}^{(n)}$ be the vector bundle over Teichmüller space whose fibre above the (isotopy class of the) complex structure J is

$$\mathcal{E}^{(n)}_J = \mathcal{Q}(3,J) \oplus \ldots \oplus \mathcal{Q}(n,J).$$

The dimension of the total space of $\mathcal{E}^{(n)}$ is the same as that of the Hitchin component

$$\mathrm{Rep}_H(\pi_1(S), \mathrm{PSL}(n,\mathbb{R}))$$

since the dimension of the "missing" quadratic differentials in $\mathcal{E}^{(n)}_J$ accounts for the dimension of Teichmüller space. The *Hitchin map* [1] is then the map from

[1] We are aware that this terminology is awkward since this Hitchin map is some kind of an inverse of what is usually called the Hitchin fibration.

$\mathcal{E}^{(n)}$ to $\mathrm{Rep}_H(\pi_1(S), \mathrm{PSL}(n,\mathbb{R}))$ defined by

$$(J, \omega) \to H_J(0, \omega).$$

It follows from Hitchin's construction that this map is equivariant with respect to the mapping class group action. We prove

Theorem 2.2.1 *The Hitchin map is surjective.*

Our strategy is to identify $\mathcal{E}^{(n)}$ with the moduli space of equivariant minimal surfaces in the associated symmetric space and then, by tracking a critical point of the energy, to prove that there exists an equivariant minimal surface for every representation.

We conjectured in [30] that the Hitchin map is a homeomorphism. This would be a consequence of the following:

Conjecture 2.2.2 *If ρ is a Hitchin representation, then there exists a non degenerate minimum of e_ρ.*

This conjecture is well known to be true for $n = 2$. For $n = 3$, it is proved in [31] by relating real projective structures, affine spheres and Blaschke metrics (as in [28, 31] or in [34]). By our previous discussion, Conjecture 2.2.2 would imply the following consequence which sheds light on the action of the mapping class group $\mathcal{M}(S)$ on the Hitchin components:

Conjecture 2.2.3 *The quotient $\mathrm{Rep}_H(\pi_1(S), \mathrm{PSL}(n,\mathbb{R}))/\mathcal{M}(S)$ is homeomorphic to the total space of the vector bundle E – in the orbifold sense – over the Riemann moduli space, whose fibre at a point J is*

$$E_J = \mathcal{Q}(3, J) \oplus \ldots \oplus \mathcal{Q}(n, J).$$

Again, by the previous discussion this result is true for $n = 2$ and $n = 3$.

2.3 Outline of the paper

- 3. Cross ratios and the boundary at infinity. We recall the basic definitions (cross ratios, periods), explain how cross ratios are related to flows and finally show how this relation helps to control the growth of the periods (Proposition 3.3.1).

- 4. Representations and cross ratios. We explain that Hitchin and maximal symplectic representations are reductive, how they generate curves in Grassmannian spaces and how they relate to cross ratios.

- 5. The energy functional and the minimal area. We recall basic results about existence of equivariant harmonic mappings in symmetric spaces as well as classical definitions and results concerning minimal surfaces, energy and Teichmüller space.

- 6. Displacing representations and the energy functional. We introduce the notion of displacing representations. Using the relation with cross ratios, we show that the Hitchin and maximal symplectic representations are displacing. We then prove the main properties of displacing representations and obtain Theorems 1.0.2 and 1.0.3. As a consequence, we deduce the existence of equivariant minimal surfaces for our two main examples.

- 7. The Toledo invariant and the minimal area. We prove Theorem 2.1.3.

- 8. The Hitchin map. We prove Theorem 2.2.1.

3 Cross ratios and the boundary at infinity

3.1 The boundary at infinity

Let $\partial_\infty \pi_1(S)$ be the boundary at infinity of $\pi_1(S)$. We recall that $\partial_\infty \pi_1(S)$ is a circle with a Hölder structure and is equipped with an action of $\pi_1(S)$ by Hölder homeomorphisms. Up to equivariant Hölder homeomorphisms, this action can be characterised by the following two properties

- every orbit is dense,
- every nontrivial element of $\pi_1(S)$ has exactly two fixed points: one attractive and one repulsive.

If one fixes an uniformisation of the universal cover of the surface equipped with a complex structure, then $\partial_\infty \pi_1(S)$ can be identified with the real projective line $\mathbb{RP}^1$ considered as the boundary of the Poincaré disk model.

3.2 Cross ratios

Let X be a metric space equipped with an action of a group Γ by Hölder homeomorphisms. Let

$$X^{4*} = \{(x, y, z, t) \in X^4 \mid x \neq t \text{ and } y \neq z\}.$$

We equip X^{4*} with the diagonal action of Γ. In the sequel, our main examples are $X = \partial_\infty \pi_1(S)$ equipped with the natural action of $\pi_1(S)$ by Hölder homeomorphisms, or variants of that.

Definition 3.2.1 [Cross ratio] *A* cross ratio *on X is a real valued Γ-invariant Hölder function $\mathbb{B}$ on X^{4*} which satisfies the following rules*

$$\begin{aligned}
\mathbb{B}(x, y, z, t) &= \mathbb{B}(z, t, x, y), && (1)\\
\mathbb{B}(x, y, z, t) &= 0 \quad \Leftrightarrow x = y \quad \text{or } z = t, && (2)\\
\mathbb{B}(x, y, z, t) &= 1 \ \Leftrightarrow x = z \quad \text{or } y = t, && (3)\\
\mathbb{B}(x, y, z, t) &= \mathbb{B}(x, y, z, w)\mathbb{B}(x, w, z, t), && (4)\\
\mathbb{B}(x, y, z, t) &= \mathbb{B}(x, y, w, t)\mathbb{B}(w, y, z, t). && (5)
\end{aligned}$$

The classical cross ratio $\mathbf{b}$ on $\mathbb{RP}^1$, defined in an affine chart by

$$\mathbf{b}(x,y,z,t) = \frac{(x-y)(z-t)}{(x-t)(y-z)},$$

is an example of a cross ratio with respect to the action of $\mathrm{PSL}(2,\mathbb{R})$.

Definition 3.2.2 [Period] *Let $\mathbb{B}$ be a cross ratio on $\partial_\infty\pi_1(S)$ and γ be a nontrivial element in $\pi_1(S)$. The* period $\ell_{\mathbb{B}}(\gamma)$ *is*

$$\ell_{\mathbb{B}}(\gamma) = \log|\mathbb{B}(\gamma^-, \gamma y, \gamma^+, y)|, \tag{6}$$

where γ^+ and γ^- are respectively the attracting and repelling fixed points of γ on $\partial_\infty\pi_1(S)$ and y is any element of $\partial_\infty\pi_1(S)$ different from γ^+ and γ^-.

Relation (4) and the invariance under the action of γ imply that $\ell_{\mathbb{B}}(\gamma)$ does not depend on y. Moreover, by Equation (1), $\ell_{\mathbb{B}}(\gamma) = \ell_{\mathbb{B}}(\gamma^{-1})$.

For more information and examples on a related notion see the work of Otal and Ledrappier in [37, 33]. For other applications to representations of surface groups, see [29, 32].

3.3 Periods and lengths

The next proposition compares periods with length of geodesics.

Proposition 3.3.1 *We fix a hyperbolic metric on S. For every nontrivial γ in $\pi_1(S)$, let $\lambda(\gamma)$ be the length of the closed geodesic associated to γ for this hyperbolic metric. Let $\mathbb{B}$ be a cross ratio. Then there exists a positive constant A, depending only on the cross ratio and the choice of the hyperbolic metric, such that or all nontrivial element γ in $\pi_1(S)$*

$$\frac{1}{A}\lambda(\gamma) \leqslant \ell_{\mathbb{B}}(\gamma) \leqslant A\lambda(\gamma).$$

The idea of the proof is to define *compatible flows* on the space of oriented triples of pairwise distinct points of $\partial_\infty\pi_1(S)$ and study their periodic orbits.

3.3.1 Compatible flows on the space of oriented triples

Recall first that the orientation on S induces an orientation on $\partial_\infty\pi_1(S)$.

Definition 3.3.2 [Oriented triples] *We denote by $\partial_\infty\pi_1(S)^{3+}$ the* space of oriented triples of pairwise distinct points *of $\partial_\infty\pi_1(S)$.*

The quotient $\partial_\infty\pi_1(S)^{3+}/\pi_1(S)$ is compact and homeomorphic to the unitary tangent bundle of the surface S for any auxiliary metric.

Definition 3.3.3 [Compatible flow] *A continuous flow $\{\phi_t\}_{t\in\mathbb{R}}$ on $\partial_\infty\pi_1(S)^{3+}$ is* compatible *if*

- *every homeomorphism ϕ_t is $\pi_1(S)$-equivariant,*
- *every homeomorphism ϕ_t acts without fixed points,*
- *for every real number t, for every triple (x, z, y) in $\partial_\infty\pi_1(S)^{3+}$, there exists u in $\partial_\infty\pi_1(S)$ such that $\phi_t(x, z, y) = (x, u, y)$.*

If we identify $\partial_\infty\pi_1(S)^{3+}$ with the unit tangent bundle of S for some hyperbolic metric, then a compatible flow is nothing but a reparametrisation of the geodesic flow.

Definition 3.3.4 [Period for a flow] *For notational convenience, given a compatible flow $\{\phi_t\}_{t\in\mathbb{R}}$, we define for every real number t the map*

$$\hat{\phi}_t : \partial_\infty\pi_1(S)^{3+} \to \partial_\infty\pi_1(S),$$

by the condition that $(x, \hat{\phi}_t(x, z, y), y) = \phi_t(x, z, y)$.

If γ is a nontrivial element of $\pi_1(S)$, we then define the period *of γ with respect to $\{\phi_t\}_{t\in\mathbb{R}}$ to be the least positive number $\ell_\phi(\gamma)$ such that*

$$\hat{\phi}_{\ell_\phi(\gamma)}(\gamma^-, y, \gamma^+) = \gamma(y).$$

We first prove

Proposition 3.3.5 *Let $\{\phi_t\}_{t\in\mathbb{R}}$ and $\{\psi_t\}_{t\in\mathbb{R}}$ be two compatible flows. Then there exists a constant K such that for all nontrivial element γ of $\pi_1(S)$*

$$\ell_\psi(\gamma) \leqslant K\ell_\phi(\gamma).$$

Proof: There exists a positive continuous $\pi_1(S)$-invariant positive function Θ defined on $\partial_\infty\pi_1(S)^{3+}$ such that

$$\hat{\psi}_1(x, y, z) = \hat{\phi}_{\Theta(x,y,z)}(x, y, z).$$

Since $\partial_\infty\pi_1(S)^{3+}/\pi_1(S)$ is compact, it follows that there exists a constant A such that for all oriented triple (x, y, z)

$$0 < \Theta(x, y, z) \leqslant A.$$

Then we have

$$\ell_\psi(\gamma) \leqslant A\ell_\phi(\gamma) + A.$$

By compactness of $\partial_\infty\pi_1(S)^{3+}/\pi_1(S)$, there exists a constant l such that for all nontrivial element γ of $\pi_1(S)$, we have

$$\ell_\phi(\gamma) \geqslant l > 0.$$

Therefore it follows that

$$\ell_\psi(\gamma) \leqslant (A + A/l)\ell_\phi(\gamma).$$

Q.e.d.

3.3.2 Proof of Proposition 3.3.1

We show that every cross ratio is associated to a compatible flow and thereby conclude the proof of Proposition 3.3.1.

Proposition 3.3.6 *Let $\mathbb{B}$ be a cross ratio. Let x and y be two distinct elements of $\partial_\infty\pi_1(S)$. Let I be one of the connected components of $\partial_\infty\pi_1(S)\setminus\{x,y\}$. Let z be an element of I. Then the map from I to $\mathbb{R}$ given by*

$$\varphi \ : \ t \to \log(\mathbb{B}(x,t,y,z)),$$

is a homeomorphism.

Proof: By the definition of cross ratio, if (x,s,t,y) is cyclically oriented, then $\mathbb{B}(x,s,y,t)$ is greater than 1. In particular $\mathbb{B}(x,t,y,z)$ is positive if z and t belong to I and thus φ is well defined.

We now prove that φ is injective. Suppose that $\varphi(s)=\varphi(t)$. This implies that

$$\mathbb{B}(x,t,y,s)=\frac{\mathbb{B}(x,t,y,z)}{\mathbb{B}(x,s,y,z)}=e^{\varphi(t)-\varphi(s)}=1.$$

Hence $s=t$ by the definition of a cross ratio. The same proof shows that φ is increasing. It follows that $\varphi(I)$ is an interval $]\alpha,\beta[$.

We now prove that $\beta=+\infty$. Assume on the contrary that β is finite. By definition

$$\lim_{t\to y}\log(\mathbb{B}(x,t,y,z))=\beta.$$

Choose an auxiliary compatible flow $\{\psi_t\}_{t\in\mathbb{R}}$ on $\partial_\infty\pi_1(S)^{3+}$. Since $(x,t,\hat{\psi}_1(x,t,y),y)$ is cyclically oriented, we have

$$\lim_{t\to y}\hat{\psi}_1(x,t,y)=y.$$

Hence,

$$\lim_{t\to y}\mathbb{B}(x,t,y,\hat{\psi}_1(x,t,y))=\lim_{t\to y}\left(\frac{\mathbb{B}(x,t,y,z)}{\mathbb{B}(x,\hat{\psi}_1(x,t,y),y,z)}\right)=1. \tag{7}$$

Now choose a sequence $\{t_n\}_{n\in\mathbb{N}}$ converging to y. Since the quotient $\partial_\infty\pi_1(S)^{3+}/\pi_1(S)$ is compact, there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of elements in $\pi_1(S)$ and an oriented triple (X,Y,T) of pairwise distinct elements in $\partial_\infty\pi_1(S)$ such that

$$\lim_{n\to\infty}(\gamma_n(x),\gamma_n(y),\gamma_n(t_n))=(X,Y,T)\in\partial_\infty\pi_1(S)^{3+}\ .$$

If follows from Assertion (7) that

$$\begin{aligned}
\mathbb{B}(X,T,Y,\hat{\psi}_1(X,T,Y)) &= \lim_{n\to\infty}\mathbb{B}(\gamma_n(x),\gamma_n(t_n),\gamma_n(y),\hat{\psi}_1(\gamma_n(x),\gamma_n(t_n),\gamma_n(y)) \\
&= \lim_{n\to\infty}\mathbb{B}(\gamma_n(x),\gamma_n(t_n),\gamma_n(y),\gamma_n(\hat{\psi}_1(x,t_n,y)) \\
&= \lim_{n\to\infty}\mathbb{B}(x,t_n,y,\hat{\psi}_1(x,t_n,y)) \\
&= 1.
\end{aligned}$$

This yields the conclusion that $T = \hat{\psi}_1(X, T, Y)$, which contradicts the fact that ψ_1 has no fixed points. A similar argument yields $\alpha = -\infty$. Therefore φ is a homeomorphism. Q.E.D.

Proposition 3.3.7 *There exists a compatible flow $\{\phi_t\}_{t\in\mathbb{R}}$ on $\partial_\infty \pi_1(S)^{3+}$ such that* $\log(\mathbb{B}(x, z, y, \hat{\phi}_t(x, z, y))) = t$.

PROOF: By Proposition 3.3.6, the family of homeomorphisms $\{\phi_t\}_{t\in\mathbb{R}}$ is well defined. The multiplicative cocycle relation

$$\mathbb{B}(x, y, z, t).\mathbb{B}(x, t, z, u) = \mathbb{B}(x, y, z, u),$$

implies that $\{\phi_t\}_{t\in\mathbb{R}}$ is a one-parameter group. Q.E.D.

Proposition 3.3.1 is now a consequence of Propositions 3.3.7 and 3.3.5.

4 Representations and cross ratios

We shall in the sequel distinguish between *homomorphism* from a group to an other and *representation* which we consider as a class of homomorphism up to conjugation. Whenever a property is defined for a homomorphism, this definition will be extended to representation whenever the property is invariant by conjugation.

We explain that our two favourite classes of representations from $\pi_1(S)$ are associated to cross ratios whose periods can be computed from holonomy. We also recall that these representations are reductive.

4.1 Hitchin representations

Definition 4.1.1 [DEFORMATION] *Let ρ_0 and ρ_1 be representations of $\pi_1(S)$ with values in a topological group G. A* deformation *of ρ_0 into ρ_1 is a family of representations $\{\rho_t\}_{t\in[0,1]}$ such that for every element γ of $\pi_1(S)$, the map $t \to \rho_t(\gamma)$ is continuous.*

Definition 4.1.2 [HITCHIN HOMOMORPHISMS] *Following [30], a* Fuchsian *homomorphism from $\pi_1(S)$ to* $\mathrm{PSL}(n, \mathbb{R})$ *is a homomorphism ρ which factors as $\rho = \iota\circ\rho_0$, where ρ_0 is a convex-cocompact injective homomorphism with values in* $\mathrm{PSL}(2, \mathbb{R})$ *and ι is an irreducible homomorphism from* $\mathrm{PSL}(2, \mathbb{R})$ *to* $\mathrm{PSL}(n, \mathbb{R})$.

A Hitchin homomorphism *is a homomorphism that may be deformed into a Fuchsian homomorphism.*

Definition 4.1.3 [REDUCTIVE HOMOMORPHISM] *A homomorphism is* reductive *if the Zariski closure of its image is a reductive group.*

Later we will show that every Hitchin homomorphism is reductive. In [26], Hitchin studies the moduli space of reductive Hitchin representations.

In [30], we showed the following result.

Theorem 4.1.4 *Let ρ be a reductive Hitchin representation. Let γ be a non-trivial element of $\pi_1(S)$. Then $\rho(\gamma)$ is $\mathbb{R}$-split.*

4.1.1 Reductivity

Recall that a subgroup of (or a homomorphism with values in) $\mathrm{PSL}(n, \mathbb{R})$ is *irreducible* if it does not preserve any proper subspace of $\mathbb{R}^n$. We show the following result.

Proposition 4.1.5 *Every Hitchin representation is irreducible.*

Proof: In [30], Lemma 10.1, using elementary observations on Higgs bundle, we show that every *reductive* Hitchin representation is irreducible. We now explain that every Hitchin representation is reductive, hence irreducible.

Obviously the set of irreducible homomorphisms is open since its complement is closed. Hence, to conclude the proof, it suffices to show that the set of irreducible Hitchin homomorphisms is closed.

Let ρ be a limit of Hitchin homomorphisms. Let G be the Zariski closure of $\rho(\pi_1(S))$. Let N be the nilradical of G. Let $R = G/N$ be the reductive part of G – *i.e* the Levi component. We identify R with a subgroup of G so that $G = N \rtimes R$. Let π be the projection from G to R.

We first prove that $\pi \circ \rho$ is also a limit of reductive homomorphisms. Indeed, there exists an element h in the centraliser of R such that for all u in N,

$$\lim_{n \to \infty} h^{-n} u h^n = 1.$$

It follows that

$$\pi \circ \rho = \lim_{n \to \infty} h^{-n} \rho h^n.$$

In particular, $\pi \circ \rho$ is also a limit of Hitchin homomorphisms and hence a Hitchin homomorphism itself. By construction, $\pi \circ \rho$ is reductive and hence irreducible.

We now prove by contradiction that N is trivial. Assume the contrary. Then, since N is unipotent, the set of vectors fixed by N is a proper subspace of $\mathbb{R}^n$. This set is fixed by R and hence by $\pi \circ \rho$, from which it follows that $\pi \circ \rho$ is not irreducible. Hence we obtain a contradiction.

We have just shown that N is trivial. By definition ρ is reductive, and hence irreducible. Q.e.d.

I owe this argument to F. Paulin.

4.1.2 Cross ratios

In [29], we showed how to associate a cross ratio to every Hitchin representation. More precisely, we showed the following:

Theorem 4.1.6 *Let ρ be a Hitchin representation. Then there exists a cross ratio $\mathbb{B}$ on $\partial_\infty\pi_1(S)^{4*}$ such that for every nontrivial element γ of $\pi_1(S)$, the period of γ is given by*

$$\ell_{\mathbb{B}}(\gamma) = \log\left(\left|\frac{\lambda_{\max}(\rho(\gamma))}{\lambda_{\min}(\rho(\gamma))}\right|\right) . \tag{8}$$

Here $\lambda_{\max}(\rho(\gamma))$ and $\lambda_{\min}(\rho(\gamma))$ are respectively the eigenvalues of of $\rho(\gamma)$ with the maximum and minimum modulus.

Using O. Guichard's work [23], we also proved a converse of this statement (see [29]).

4.2 Symplectic Anosov structures

In [7], together with Burger, Iozzi and Weinhard, we studied maximal representations from surface groups to $\mathrm{PSp}(2n, \mathbb{R})$. We first recall some definitions and notation from [8].

4.2.1 The symplectic structure

In this section, we normalise the symplectic form on the associated symmetric space M and construct the Toledo invariant.

We denote by $\mathfrak{g}$ the Lie algebra of $G = \mathrm{PSp}(2n, \mathbb{R})$ and we identify $\mathfrak{g}$ with the Lie algebra of Killing vector fields on M. For every m in M, let $\mathfrak{k}_m$ be the Lie algebra of the stabiliser K_m of m in G. We finally identify T_mM with the orthogonal of $\mathfrak{k}_m$ in $\mathfrak{g}$ with respect to the Killing form.

We choose a continuous map ∂_θ from M to $\mathfrak{g}$ such that, for every m in M, $\partial_\theta(m)$ is a generator of the centre of $\mathfrak{k}_m$ verifying

$$\exp(s\partial_\theta(m)) = 1 \Leftrightarrow s \in 2\pi\mathbb{Z}.$$

Observe that ∂_θ is well defined up to sign. The complex structure on TM is then given by the following map from TM to itself

$$J : A \mapsto [\partial_\theta, A].$$

We normalise the Killing form $\langle\ ,\rangle$ so that $\|\partial_\theta\| = 1$.

Definition 4.2.1 [Canonical complex structure] *The* canonical symplectic structure *ω of M is given, for all tangent vectors X and Y, by*

$$\omega(X, Y) = \langle[X, Y], \partial_\theta\rangle.$$

4.2.2 The Toledo invariant

The homomorphism

$$\det : K_m \to \mathbb{T},$$

is a degree n map when restricted to the centre of K_m. It follows that $n.\omega$ is the curvature of a line bundle L.

The following inequality is fundamental.

$$|\tau(\rho)| \leqslant n|\chi(S)|.$$

For $n = 1$, this is the Milnor-Wood Inequality [36]. This specific inequality for the symplectic group is due to V. Turaev [43]. It has been extended to other Hermitian symmetric spaces (see in particular [14, 41, 42, 9, 8]). We shall restrict ourselves to $\mathrm{PSp}(2n,\mathbb{R})$ although the discussion extends to the general case as well.

Definition 4.2.2 [Maximal representation] *A* maximal representation *is a representation, say ρ, for which*

$$|\tau(\rho)| = n|\chi(S)|.$$

4.2.3 Reductivity

Using bounded cohomology techniques, Burger, Iozzi and Wienhard prove in [8]:

Theorem 4.2.3 *Every maximal representation is reductive.*

4.2.4 Cross ratios and maximal representations: the main result

For every A in $\mathrm{PSp}(2n,\mathbb{R})$, let $\{\lambda_i\}_{1\leqslant i\leqslant 2n}$ be the eigenvalues (with multiplicities) of A ordered so that

$$|\lambda_1| \leqslant |\lambda_2| \ldots \leqslant |\lambda_{2n}|.$$

We define

$$c(A) = \prod_{i=n+1}^{i=2n} |\lambda_i|.$$

The main result of this paragraph is the following.

Theorem 4.2.4 *Let ρ be a maximal symplectic representation. Then there exists a cross ratio $\mathbb{B}$ on $\partial_\infty\pi_1(S)$ such that*

$$\ell_{\mathbb{B}}(\rho(\gamma)) = 2\log c(\rho(\gamma)).$$

We prove this theorem in Paragraph 4.2.8.

4.2.5 Positivity

Let $\mathcal{L}(E)$ be the Grassmannian of Lagrangian spaces in a vector space E equipped with a symplectic form ω.

Definition 4.2.5 *A triple of Lagrangian spaces (F, G, L) is* positive *if*

- *$F \oplus L = E$ and*
- *$\omega(u_f, u_l)$ is positive for every pair of vectors (u_f, u_l) in $F \times L$ such that $u_f + u_l$ belongs to G,*

Definition 4.2.6 [POSITIVE CURVE] *An oriented curve ξ from the circle $\mathbb{T}$ to $\mathcal{L}(E)$ is* positive *if $(\xi(x), \xi(y), \xi(z))$ is positive for every oriented triple of pairwise distinct points (x, y, z) in $\mathbb{T}$.*

With Burger, Iozzi and Wienhard, we proved the following result (see [7])

Theorem 4.2.7 *Let E be a symplectic vector space. Let ρ be a maximal symplectic homomorphism from $\pi_1(S)$ to $PSp(E)$. Then there exists a positive Hölder ρ-equivariant curve ξ from $\partial_\infty \pi_1(S)$ to $\mathcal{L}(E)$.*

Furthermore, $\xi(\gamma^+)$ (respectively $\xi(\gamma^-)$) is generated by the eigenvectors of $\rho(\gamma)$ corresponding to the eigenvalues of absolute value greater than 1 (resp. smaller than 1).

4.2.6 The cross ratio of four Lagrangian spaces

Let (L_1, L_2, L_3, L_4) be a quadruple of Lagrangian spaces in a symplectic space of dimension $2n$. We suppose that L_4 is transverse to L_1 and that L_2 is transverse to L_3. Let l^1, l^2, l^3 and l^4 be bases of L_1, L_2, L_3 and L_4 respectively. For every pair $(a, b) \in \{1, 2, 3, 4\}^2$, we consider the $n \times n$ matrix

$$A_{l^a, l^b} = (\omega(l^a_i, l^b_j)).$$

We observe that for every endomorphism g of L^a whose matrix in the basis l^a is G then

$$A_{g(l^a), l^b} = G.A_{l^a, l^b}. \tag{9}$$

Similarly

$$A_{l^a, l^b} = -A^t_{l^b, l^a}.$$

We now define

$$B(\ell_1, \ell_2, \ell_3, \ell_4) = \frac{\det(A_{l^1, l^2}). \det(A_{l^3, l^4})}{\det(A_{l^1, l^4}). \det(A_{l^3, l^2})}.$$

By Assertion (9), $B(\ell_1, \ell_2, \ell_3, \ell_4)$ depends only on (L_1, L_2, L_3, L_4). Hence we can define

$$B(L_1, L_2, L_3, L_4) = B(\ell_1, \ell_2, \ell_3, \ell_4).$$

The following proposition follows easily from the definition

Proposition 4.2.8 *We have*

$$B(L_1, L_2, L_3, L_4) B(L_1, L_4, L_3, L_5) = B(L_1, L_2, L_3, L_5), \tag{10}$$

$$B(L_1, L_2, L_3, L_4) = B(L_2, L_1, L_4, L_3). \tag{11}$$

Finally, if (L, U, V) are generic,

$$B(L, U, L, V) = 1, \tag{12}$$

$$B(L, L, U, V) = 0. \tag{13}$$

4.2.7 Cross ratios and positivity

Proposition 4.2.9 *If the triples of Lagrangian spaces* (E, F_1, G) *and* (E, F_2, G) *are positive, then*

$$B(E, F_1, G, F_2) > 0.$$

If moreover the triple (F_1, F_2, G) *is positive, then*

$$B(E, F_1, G, F_2) > 1.$$

Proof: We assume that (E, F_1, G) and (E, F_2, G) are positive. Let p be the projection onto G along E. Let q_i be the quadratic form on F_i defined by

$$q_i(u) = \omega(p(u), u).$$

Since (E, F_i, G) is positive, it follows that q_i is positive definite. By simultaneous orthogonalisation, we can choose an basis f^i of F_i, which is orthogonal for q_i, such that $p(f^1) = p(f^2) = g$. Let

$$e^i = (1 - p)(f^i) = f^i - g$$

be the corresponding bases of E. We then have

$$A_{e^i, f^j} = A_{e^i, g} = -A_{g, f^i},$$

and also

$$\omega(e_i^j, g_k) = q(f_i^j, f_k^j) = \omega(g_i, f_k^j).$$

Furthermore let

$$\lambda_i = \frac{q(f_i^2, f_i^2)}{q(f_i^1, f_i^1)},$$

then

$$e_i^2 = \lambda_i e_i^1.$$

It follows that

$$\begin{aligned} B(E, F_1, G, F_2) &= \frac{\det(A_{e^1, f^1}).\det(A_{g, f^2})}{\det(A_{e^1, f^2}).\det(A_{g, f^1})} \\ &= \frac{\det(A_{e^1, g}).\det(-A_{e^2, g})}{\det(A_{e^1, g}).\det(-A_{e^1, g})} \\ &= \prod_i \lambda_i > 0. \end{aligned}$$

Finally, assume that (F_1, F_2, G) is positive. Let π be the projection onto G along F_1. Recall that

$$f_i^2 = e_i^2 + g_i = \lambda_i e_i^1 + g_i = \lambda_i f_i^1 + (1 - \lambda_i) g_i.$$

It follows that

$$\begin{aligned}\omega\big(\pi(f_i^2),(1-\pi)(f_i^2)\big) &= \omega\big((1-\lambda_i)g_i,\lambda_i f_i^1\big)\\ &= (1-\lambda_i)\lambda_i\omega(g_i,f_i^1)\\ &= \lambda_i(\lambda_i-1)q_1(f_i^1,f_i^1).\end{aligned}$$

Hence the positivity of (F_1, F_2, G) implies that $\lambda_i > 1$ and therefore

$$B(E,F_1,G,F_2)=\prod_i \lambda_i > 1.$$

Q.E.D.

Finally, we have the following:

Proposition 4.2.10 *Let S be a symplectic automorphism preserving two transverse Lagrangian spaces E and F. Then, for every Lagrangian space G, we have*

$$B(E,G,F,S(G))=\frac{\det(S|_E)}{\det(S|_F)}=\det(S|_E)^2.$$

Proof: Let S be a symplectic transformation. Let e be a basis of a space K invariant by S. We have

$$\det(A_{e,S(l)})=\det(A_{S^{-1}e,l})=\frac{\det(A_{e,l})}{\det(S|_K)}\ ,$$

from which the formula follows. Q.E.D.

4.2.8 Cross ratios and maximal representations

We now prove Theorem 4.2.4. Let ρ be a maximal symplectic homomorphism. Let ξ be the positive map from $\partial_\infty\pi_1(S)$ to $\mathcal{L}(\mathbb{R}^{2n})$ associated to ρ by Theorem 4.2.7. By Proposition 4.2.9 the following formula (using the notation of Paragraph 4.2.6) defines a cross ratio on $\partial_\infty\pi_1(S)$

$$\mathbb{B}(x,y,z,t)=B(\xi(x),\xi(y),\xi(z),\xi(t)).$$

Furthermore, by Proposition 4.2.10, we have

$$\ell_{\mathbb{B}}(\gamma)=2\log\det\big(\rho(\gamma)|_{\xi(\gamma^+)}\big).$$

The Theorem follows from this.

5 The energy functional and the minimal area

Let M be a Hadamard manifold– *i.e* complete, nonpositively curved and simply connected. Let

$$\rho:\pi_1(S)\longrightarrow \mathrm{Iso}(M)$$

be a representation from $\pi_1(S)$ to the group of isometries of M and let M_ρ be the associated M bundle over S. We denote by $\mathcal{F}_\rho$ the space of ρ-equivariant smooth mappings from the universal cover $\widetilde{S}$ of S to M:

$$\mathcal{F}_\rho = \{f : \widetilde{S} \to M \mid f \circ \gamma = \rho(\gamma) \circ f\}.$$

The space $\mathcal{F}_\rho$ is canonically identified with the space $\Gamma(S, M_\rho)$ of smooth sections of M_ρ.

5.1 The energy of a map and the energy functional

Let $\langle , \rangle$ be the metric on M. Let f be an element of $\mathcal{F}_\rho$. Let J be a complex structure on S lifted to $\widetilde{S}$. Let u and v be tangent vectors in $\widetilde{S}$, the following expression defines an exterior differential 2-form on $\widetilde{S}$

$$\langle df \wedge df \circ J\rangle(u, v) := \frac{1}{2}\left(\langle Tf(v), Tf(Ju)\rangle - \langle Tf(u), Tf(Jv)\rangle\right).$$

Notice that $\langle df \wedge df \circ J\rangle$ is $\pi_1(S)$-invariant, hence defines an exterior differential 2-form on S also denoted by $\langle df \wedge df \circ J\rangle$.

Definition 5.1.1 [Energy of a map] *The* energy *of f with respect to J is the following real number*

$$\mathrm{Energy}(J, f) = \int_S \langle df \wedge df \circ J\rangle.$$

The definition above is slightly nonstandard and restricted to dimension 2. We use it in order to emphasise the conformal invariance of the energy. Observe that for any diffeomorphism ϕ of S isotopic to the identity and lifted to a diffeomorphism Φ of $\widetilde{S}$ we have

$$\mathrm{Energy}(J, f) = \mathrm{Energy}(\Phi^* J, f \circ \Phi). \tag{14}$$

Definition 5.1.2 [Energy functional on Teichmüller space] *Let ρ be a representation of $\pi_1(S)$ in $\mathrm{Iso}(M)$. The* energy functional *associated to ρ is the real valued function on the Teichmüller space $\mathcal{T}(S)$ of S defined by*

$$e_\rho : \ J \to e_\rho(J) := \inf\{\mathrm{Energy}(J, f) \mid f \in \mathcal{F}_\rho\}.$$

In the definition above, we have used that $e_\rho(J)$ only depends on the isotopy class of J which is a consequence of Equation (14).

5.2 Harmonic mappings

By definition, a harmonic mapping is a critical point of the energy. Whenever ρ is reductive, the existence of a ρ-equivariant harmonic mapping is guaranteed by Corlette's Theorem in the context of symmetric spaces [11]. Note that [27] gives an alternative simpler proof which works in the general context of Hadamard manifolds:

Theorem 5.2.1 [CORLETTE]. *Let ρ be a reductive representation from $\pi_1(S)$ into a connected semi-simple real Lie group G without compact factor and with trivial centre. Let M be the associated symmetric space. Then there exists a ρ-equivariant harmonic mapping f from S to M. Furthermore this mapping is unique up to an isometry of M and minimises the energy.*

The definition of the energy extends to maps from higher dimensional manifolds – although in that case it is not anymore a conformal invariant – and the above statement holds in this general context.

As a byproduct of the proof of this theorem, together with a simple application of the implicit function Theorem, the energy functional is a smooth function on Teichmüller space. Combining Corlette's Theorem with Propositions 4.2.3, 4.1.5, we deduce the following result.

Proposition 5.2.2 *Let ρ be a Hitchin or maximal symplectic representation. Let J be a complex structure on S. Then there exists a unique (up to isometries) ρ-equivariant harmonic mapping $f_{\rho,J}$. Moreover*

$$\mathrm{Energy}(J, f_{\rho,J}) = e_\rho(J).$$

5.3 Minimal area

Let f be an element of $\mathcal{F}_\rho$ and let $R(f)$ be the open set of points x in S for which $T_x f$ is injective. We observe that the induced bilinear form $f^*(g_M)$ is invariant under $\pi_1(S)$ and defines a metric on $R(f)$. We define the *area* of f to be the area of $R(f)$ with respect to this metric, *i.e.*

$$\mathrm{Area}(f) = \mathrm{area}_{f^*(g_M)}(R(f)).$$

We recall that

$$\mathrm{Area}(f) \leqslant \mathrm{Energy}(J, f), \tag{15}$$

with equality if and only if f is conformal with respect to J. Finally one can find a sequence of complex structure $\{J_n\}_{n\in\mathbb{N}}$ on S so that

$$\mathrm{Area}(f) = \lim_{n\to\infty} \mathrm{Energy}(J_n, f), \tag{16}$$

Definition 5.3.1 [MINIMAL AREA] *The* minimal area of ρ *is*

$$\mathrm{MinArea}(\rho) = \inf\{e_\rho(J) \mid J \in \mathcal{T}(S)\}.$$

It follows from Assertions (15) and (16) that

$$\mathrm{MinArea}(\rho) = \inf\{\mathrm{Area}(f) \mid f \in \mathcal{F}_\rho\}.$$

We also recall the classical results of Sacks–Uhlenbeck [39][38] and Schoen–Yau [40]:

Theorem 5.3.2 [Sacks–Uhlenbeck], [Schoen–Yau] *Let J be a point in Teichmüller space which is a critical point of the energy. Let f_J be a mapping such that*

$$\mathrm{Energy}(J, f) = \mathrm{MinArea}(\rho).$$

Then f is harmonic and conformal.

6 Displacing representations and the energy functional

In Definition 6.1.2, we introduce the notion of displacing homomorphism. In Theorem 6.1.3, we prove that Hitchin and maximal symplectic representations are displacing and finally show in Theorem 6.2.1 that the energy functional associated to a displacing representation is a proper map.

6.1 Displacing representations

Definition 6.1.1 [Displacement function] *Let γ be an isometry of a metric space M. The* displacement *of γ is*

$$d(\gamma) = \inf_{x \in M} d(x, \gamma(x)).$$

If M is the Cayley graph of a group Γ with set of generators $\mathcal{G}$ and word length $\| \ \|_{\mathcal{G}}$, then

$$d(\gamma) = \inf_{\eta \in \Gamma} \|\eta\gamma\eta^{-1}\|_{\mathcal{G}}.$$

The displacement function is explicit in the case of $M = \mathrm{PSL}(n, \mathbb{R})/\mathrm{SO}(n, \mathbb{R})$. Let A be an element in $\mathrm{PSL}(n, \mathbb{R})$. Let $\{\lambda_i\}_{1 \leqslant i \leqslant n}$ be the eigenvalues of A, then up to a multiplicative constant that depends on the normalisation of the metric on M

$$d(A) = \sqrt{\sum_{i=0}^{n} \big(\log|\lambda_i|\big)^2}. \tag{17}$$

Definition 6.1.2 [Displacing homomorphism]. *Let M be a metric space. A homomorphism ρ from a finitely generated group Γ to $\mathrm{Iso}(M)$ is* displacing *if for every finite generating set $\mathcal{G}$ of Γ there exist positive constants A and B such that such that for every γ in Γ*

$$d(\rho(\gamma)) \geqslant A. \inf_{\eta \in \Gamma} \|\eta\gamma\eta^{-1}\|_{\mathcal{G}} - B.$$

where $\|\gamma\|_{\mathcal{G}}$ is the word length of the element γ of Γ.

In other words, the displacement of a displacing representation is roughly controlled by the displacement in the Cayley graph. Alternatively, in the case

$\Gamma = \pi_1(S)$, the representation ρ is displacing if for every hyperbolic metric g on S, there exist positive constants A and B such that for every nontrivial element γ in $\pi_1(S)$

$$d(\rho(\gamma)) \geqslant A.\lambda_g(\gamma) - B.$$

Here $\lambda_g(\gamma)$ is the length of the closed geodesic, with respect to g, representing the nontrivial element γ.

6.1.1 Examples of displacing representations

Representations in $\mathrm{Iso}(M)$ are displacing for cocompact groups. The same holds for convex-cocompact groups whenever M is Hadamard. The use of cross ratios allow us to identify other examples of displacing representations.

Theorem 6.1.3 *Every Hitchin representation is displacing. Every maximal symplectic representation is displacing.*

Proof: Let ρ be a Hitchin representation. Let $\mathbb{B}$ be the cross ratio associated to ρ by Theorem 4.1.6. Let $\{\lambda_i\}_{1\leqslant i\leqslant n}$ be the eigenvalues of the element $\rho(\gamma)$. We order the eigenvalues so that $|\lambda_j| \geqslant |\lambda_i|$ if $j > i$. Combining Equations (8) and (17), we obtain

$$\begin{aligned} d(\rho(\gamma)) &= \sqrt{\sum_{i=1}^{i=n} (\log|\lambda_i|)^2} \\ &\geqslant \frac{1}{\sqrt{n}} \sum_{i=1}^{i=n} |\log|\lambda_i|| \\ &\geqslant \frac{1}{\sqrt{n}} \left(|\log|\lambda_n|| + |\log|\lambda_1||\right). \end{aligned}$$

Since $|\lambda_n| \geqslant 1 \geqslant |\lambda_1|$, we get

$$d(\rho(\gamma)) \geqslant \frac{1}{\sqrt{n}} \log\frac{|\lambda_n|}{|\lambda_1|} = \frac{1}{\sqrt{n}}\ell_{\mathbb{B}}(\gamma).$$

By Theorem 3.3.1, there exist positive constants A and B such that

$$\ell_{\mathbb{B}}(\gamma) \geqslant A.\lambda(\gamma) - B,$$

where $\lambda(\gamma)$ is the length of the geodesic associated to γ in some auxiliary hyperbolic metric. It follows that every Hitchin representation is displacing.

For maximal symplectic representations, we have a very similar argument. Let ρ be such a representation and let $\mathbb{B}$ be the cross ratio associated to it by Proposition 4.2.4. Since the injection i from $\mathrm{PSp}(2n,\mathbb{R})$ into $\mathrm{PSL}(2n,\mathbb{R})$ gives rise to a totally geodesic embedding of the corresponding symmetric spaces, it suffices to show that $i \circ \rho$ is displacing. Let $\{\lambda_i\}_{1 \leqslant i \leqslant 2n}$ be the eigenvalues, with multiplicities, of the element $\rho(\gamma)$. We order them so that

$$|\lambda_1| \leqslant |\lambda_2| \ldots \leqslant |\lambda_{2n}|.$$

We observe that $\lambda_i \lambda_{2n+1-i} = 1$ from which it follows that for i less than n, $|\lambda_i| \leqslant 1 \leqslant |\lambda_{2n+1-i}|$. Then, we have

$$\begin{aligned}
d(\rho(\gamma)) &= \sqrt{\sum_{i=1}^{i=2n} (\log |\lambda_i|)^2} \\
&\geqslant \frac{1}{\sqrt{n}} \sum_{i=1}^{i=2n} |\log |\lambda_i|| \\
&\geqslant \frac{1}{n} \log \left| \frac{\prod_{i=n+1}^{i=2n} \lambda_i}{\prod_{i=1}^{i=n} \lambda_i} \right| \\
&\geqslant \frac{1}{\sqrt{n}} \log \left(\prod_{i=n+1}^{i=2n} |\lambda_i|^2 \right) = \frac{1}{\sqrt{n}} \ell_{\mathbb{B}}(\gamma).
\end{aligned}$$

The result follows from this last inequality. Q.E.D.

6.2 Energy of displacing representations

The main result of this section, proved in Paragraph 6.2.2, is the following.

Theorem 6.2.1 *Let ρ be a displacing representation from $\pi_1(S)$ to the isometry group of a Hadamard manifold. Then the energy functional e_ρ, defined from $\mathcal{T}(S)$ to $\mathbb{R}$, is proper.*

In particular, we recover as a corollary the result of W. Goldman and R. Wentworth [21] that the energy functional is proper for convex-cocompact representations.

Corollary 6.2.2 *Let ρ be a Hitchin or maximal representation. Then there exists a complex structure J_0 on S and a J_0-conformal harmonic ρ-equivariant mapping f defined from the universal cover of S to the corresponding symmetric space such that*

$$\text{Area}(f) = \text{Energy}(J_0, f) = \text{MinArea}(\rho).$$

PROOF: By Proposition 6.1.3 and the previous theorem, there exists a complex structure J_0 on S which achieves the minimum of the energy functional. By Proposition 5.2.2, there exists a ρ-harmonic mapping f, such that

$$\text{Energy}(J_0, f) = e_\rho(J_0) = \text{MinArea}(\rho).$$

We conclude the proof by applying Theorem 5.3.2 of Sachs–Uhlenbeck and Schoen–Yau. Q.E.D.

6.2.1 The intersection is proper

Let g and g_0 be two hyperbolic metrics on S. Let US and U_0S be the associated unit tangent bundles, with geodesic flows $\{\phi_t\}_{t\in\mathbb{R}}$ and $\{\phi_t^0\}_{t\in\mathbb{R}}$ generated by X and X_0 respectively. Let μ and μ_0 be the corresponding Liouville measures normalised to be probability measures. We know that these two geodesic flows are orbit conjugate. In other words, there exist a homeomorphism F from US to US_0 and a positive continuous function $\psi_{(g,g_0)}$ on US such that F is differentiable along X and $DF(\psi_{(g,g_0)}X) = X_0$.

Definition 6.2.3 [INTERSECTION] *The* intersection *of g and g_0 is*

$$\operatorname{inter}(g, g_0) = \int_{US} \psi_{(g,g_0)} d\mu.$$

The following proposition is a classical result. For the sake of completeness since we could not find a good reference for it, we include a sketchy proof. For a less down to earth point of view and for extra information on intersections, we refer to Francis Bonahon's original article ([1]) or to Curt McMullen's notes ([35]).

Proposition 6.2.4 *Fixing g_0, the function $g \mapsto \operatorname{inter}(g, g_0)$ is a proper map from $\mathcal{T}(S)$ to $\mathbb{R}$.*

PROOF: We denote by $[\gamma]$ the free homotopy class of a closed curve γ. By definition, the intersection of two homotopy classes of closed curves c_1 and c_2 in the compact surface S is

$$\operatorname{inter}(c_1, c_2) = \inf\{\sharp(\gamma_1 \cap \gamma_2) \mid [\gamma_i] = c_i\}.$$

If γ_1 and γ_2 are geodesics with distinct support for a negatively curved metric g, then

$$\operatorname{inter}([\gamma_1], [\gamma_2]) = \sharp(\gamma_1 \cap \gamma_2).$$

We denote by $\mathcal{G}(S)$ the set of closed geodesics in S. Let η be a simple closed curve. Using a tubular neighbourhood of η, we see that there exists a constant $C(\eta, g)$, depending only on the isotopy class of η and the metric g, such that for every closed geodesic γ we have

$$\operatorname{inter}([\eta], [\gamma]) \leqslant C(\eta, g)\lambda_g([\gamma]). \tag{18}$$

Now let g be a hyperbolic metric and let

$$\mathcal{G}_L = \{\gamma \in \mathcal{G}(S) \mid \lambda_g(\gamma) \leqslant L\}.$$

According to the equirepartition of closed geodesics due to R. Bowen in [2], for every continuous function f on US the following formula relates the integral of f with respect to the Liouville measure for g with its average along closed geodesics

$$\int_{US} f d\mu_g = \lim_{L\to\infty} \left(\frac{1}{\sharp(\mathcal{G}_L)} \sum_{\gamma\in\mathcal{G}_L} \frac{\int_\gamma f dt}{\lambda_g(\gamma)} \right). \tag{19}$$

Hence, if g_0 is another hyperbolic metric

$$\text{inter}(g, g_0) = \lim_{L\to\infty} \left(\frac{1}{\sharp(\mathcal{G}_L)} \sum_{\gamma\in\mathcal{G}_L} \frac{\lambda_{g_0}(\gamma)}{\lambda_g(\gamma)} \right).$$

Furthermore, if η is a simple closed geodesic, then we also have

$$\lambda_g(\eta) = \lim_{L\to\infty} \left(\frac{1}{\sharp(\mathcal{G}_L)} \sum_{\gamma\in\mathcal{G}_L} \frac{\text{inter}(\eta, \gamma)}{\lambda_g(\gamma)} \right). \tag{20}$$

Combining Equation (20) and Inequality (18), we obtain that for every simple closed geodesic η,

$$\lambda_g(\eta) \leqslant C(\eta, g_0)\text{inter}(g, g_0).$$

But we can find a finite set of simple closed curves $\mathcal{A}$ such that the function

$$\lambda_{\mathcal{A}} : g \to \sum_{\eta\in\mathcal{A}} \inf_{c\in[\eta]} \lambda_g(c)$$

is proper on Teichmüller space. The statement thus follows from the following inequality

$$\lambda_J(g) \leqslant \text{inter}(g, g_0) \sum_{\eta\in\mathcal{A}} C(\eta, g_0).$$

Q.E.D.

6.2.2 The energy functional is proper

We now prove Theorem 6.2.1 by adapting a beautiful argument of C. Croke and A. Fathi ([12]). We use the notation of the previous paragraph. Let ρ be a displacing representation. Let J be a complex structure on S and let g be the associated hyperbolic metric whose area form is $d\sigma$ and Liouville measure – normalised to be a probability measure – is $d\mu$. Let g_0 be a fixed hyperbolic metric. We now prove that there exists a constant K dependent on g_0 but independent of J such that

$$e_\rho(J) \geqslant K(\text{inter}(g, g_0))^2.$$

Let f an element of $\mathcal{F}_\rho$. We consider the function from US to $\mathbb{R}$

$$h : u \to \|Tf(u)\|.$$

From the definition of the energy, we have

$$\text{Energy}(J, f) = \frac{1}{2}\int_S \text{trace}(Tf^*Tf)d\sigma = 2\pi|\chi(S)| \int_{US} h^2 d\mu.$$

By the Cauchy-Schwarz inequality

$$\text{Energy}(J, f) \geqslant 2\pi|\chi(S)| \left(\int_{US} h \, d\mu \right)^2.$$

Let γ be a closed orbit of the geodesic flow of g and let $\lambda_g(\gamma)$ be the length of γ with respect to g. We denote also by γ the corresponding conjugacy class in the fundamental group. Let c be the curve from the interval $[0, \lambda_g(\gamma)]$ to M defined by

$$c:\ t \to f(\widetilde{\gamma}(t)),$$

where $\widetilde{\gamma}$ is a lift of γ in $\widetilde{S}$. Then

$$\begin{aligned}
\int_\gamma h\ dt &= \text{length}(c) \\
&\geqslant d(c(0), \rho(\gamma)(c(0)) \\
&\geqslant A\lambda_{g_0}(\gamma) - B = A\int_\gamma \psi_{(g,g_0)} dt - B.
\end{aligned}$$

From Equation (19), it follows that for every f

$$\begin{aligned}
\sqrt{\frac{\text{Energy}(J,f)}{2\pi|\chi(S)|}} &\geqslant \int_{US} h\ \ d\mu = \lim_{L\to\infty}\left(\frac{1}{\sharp(\mathcal{G}_L)}\sum_{\gamma\in\mathcal{G}_L}\frac{\int_\gamma h dt}{\lambda_g(\gamma)}\right) \\
&\geqslant A\lim_{L\to\infty}\left(\frac{1}{\sharp(\mathcal{G}_L)}\sum_{\gamma\in\mathcal{G}_L}\frac{\int_\gamma \psi_{(g,g_0)} dt}{\lambda_g(\gamma)}\right) \\
&\quad -B\lim_{n\to\infty}\left(\frac{1}{\sharp(\mathcal{G}_L)}\sum_{\gamma\in\mathcal{G}_L}\frac{1}{\lambda_g(\gamma)}\right) \\
&\geqslant A\int_{US}\psi_{(g,g_0)} d\mu = A\ \text{inter}(g, g_0).
\end{aligned}$$

Hence

$$e_\rho(J) \geqslant 2\pi|\chi(S)|A^2(\text{inter}(g, g_0))^2.$$

Finally, by Proposition 6.2.4, the function $g \mapsto \text{inter}(g, g_0)$ is proper. Hence, the energy functional e_ρ is proper.

6.3 Mapping class group and displacing representations

6.3.1 Point set topology

We recall some elementary point set topology. Let X be a topological space. We define an equivalence relation on X as follows: we say that $x \sim y$, if for any continuous function f from X to a Hausdorff topological space, we have $f(x) = f(y)$. We denote by $X^\sharp$ the quotient $X/\sim$. Observe that

Proposition 6.3.1 *The space $X^\sharp$ is Hausdorff. Moreover, we have a morphism from the group of homeomorphisms of X to the group of homeomorphisms of $X^\sharp$. Finally, every continuous map from X to a Hausdorff space factors through $X^\sharp$.*

PROOF: Let us denote by $[x]$ the equivalence class of a point x in X. For any continuous function f from X to a Hausdorff space, the preimage of any point is a union of equivalence classes. This implies the last statement of the assertion. Moreover, by definition, if $[x] \neq [y]$, there exists a continuous function f with values in a Hausdorff space so that $f(x) \neq f(y)$. Then the preimage of disjoint neighbourhoods of $f(x)$ and $f(y)$ are disjoint neighbourhoods of $[x]$ and $[y]$. Q.E.D.

Let G be a topological group acting continuously on a topological space M.

Definition 6.3.2 *The action of G on M is* proper *if the map from $G \times M$ to $M \times M$ defined by $(g, m) \to (gm, m)$ is proper, or equivalently if, for any compact K in M, the set*

$$G_K = \{g \in G \mid gK \cap K \neq \emptyset\}$$

is a compact of G.

This definition immediately implies the following result.

Proposition 6.3.3 *Assume that the topological group G acts continuously on the topological spaces M and N. Assume that the action on N is proper. Assume that there exists a continuous G-equivariant map from M to N. Then the action of G on M is proper.*

This notion of a proper action is mainly interesting under hypotheses on M whenever one is interested in the quotient space: for instance the quotient of a locally compact Hausdorff space by a proper action is Hausdorff. Moreover in the sequel, we shall only consider the case of continuous actions on Hausdorff spaces.

6.3.2 Proper actions of the mapping class group

Let $\mathrm{Hom}_{\mathrm{disp}}(\pi_1(S), \mathrm{Iso}(M))$ be the space of displacing homomorphisms from $\pi_1(S)$ to $\mathrm{Iso}(M)$. Let us define

$$\mathrm{Rep}^{\sharp}_{\mathrm{disp}}(\pi_1(S), \mathrm{Iso}(M)) = [\mathrm{Hom}_{\mathrm{disp}}(\pi_1(S), \mathrm{Iso}(M)) / \mathrm{Iso}(M)]^{\sharp}$$

Observe that $\mathcal{M}(S)$ acts continuously on $\mathrm{Rep}^{\sharp}(\pi_1(S), \mathrm{Iso}(M))$.

Proposition 6.3.4 *The mapping class group $\mathcal{M}(S)$ acts properly on the space $\mathrm{Rep}^{\sharp}_{\mathrm{disp}}(\pi_1(S), \mathrm{Iso}(M))$.*

PROOF: Let $\mathbb{R}^{\pi_1(S)}$ be the space of maps from $\pi_1(S)$ to $\mathbb{R}$. We equip $\mathbb{R}^{\pi_1(S)}$ with the product topology. We fix a hyperbolic metric on S and for every γ in $\pi_1(S)$ we denote by $\lambda(\gamma)$ the length of the closed geodesic associated to γ. Let

$$\mathbb{R}^{\pi_1(S)}_{\mathrm{hyp}} = \{\ell \in \mathbb{R}^{\pi_1(S)} \mid \exists A, B > 0, \quad \forall \gamma \in \pi_1(S), \quad \ell(\gamma) \geqslant A\lambda(\gamma) - B.\}$$

For every divergent sequence $\{g_n\}_{n\in\mathbb{N}}$ of elements of $\mathcal{M}(S)$, there exists an element γ in $\pi_1(S)$ such that after extracting a subsequence

$$\lim_{n\to\infty} \lambda(g_n(\gamma)) = \infty.$$

It follows that $\mathcal{M}(S)$ acts properly on $\mathbb{R}^{\pi_1(S)}_{\mathrm{hyp}}$. Let

$$\mathfrak{S} : \left\{ \begin{array}{rcl} \mathrm{Hom}_{\mathrm{disp}}(\pi_1(S), \mathrm{Iso}(M)) & \rightarrow & \mathbb{R}^{\pi_1(S)}, \\ \rho & \mapsto & \{d(\rho(\gamma))\}_{\gamma\in\pi_1(S)}. \end{array}\right.$$

Observe that $\mathfrak{S}$ is a continuous map, invariant under conjugation and equivariant with respect to the action of the mapping class group. Therefore $\mathfrak{S}$ yields the existence of a continuous $\mathcal{M}(S)$-equivariant map from $\mathrm{Rep}^{\sharp}_{\mathrm{disp}}(\pi_1(S), \mathrm{Iso}(M))$ to $\mathbb{R}^{\pi_1(S)}_{\mathrm{hyp}}$. Thus, by Proposition 6.3.3, the properness of the action of $\mathcal{M}(S)$ on $\mathbb{R}^{\pi_1(S)}_{\mathrm{hyp}}$ implies the properness of the action of $\mathcal{M}(S)$ on $\mathrm{Rep}^{\sharp}_{\mathrm{disp}}(\pi_1(S), \mathrm{Iso}(M))$. Q.E.D.

REMARK: The Hausdorff space $\mathrm{Rep}^{\sharp}_{\mathrm{disp}}(\pi_1(S), \mathrm{Iso}(M))$ is locally compact when M is a symmetric space: indeed $\mathrm{Hom}(\pi_1(S), \mathrm{Iso}(M))$ is locally compact. We do not know however under which conditions on M such a result holds.

Our main goal is the following result.

Corollary 6.3.5 *The mapping class group $\mathcal{M}(S)$ acts properly on the spaces $\mathrm{Rep}_H(\pi_1(S), \mathrm{PSL}(n, \mathbb{R}))$ and $\mathrm{Rep}_T(\pi_1(S), \mathrm{PSp}(2n, \mathbb{R}))$. Moreover the quotient spaces are Hausdorff.*

PROOF: We write the proof only in the case of $\mathrm{PSL}(n, \mathbb{R})$, the other case being similar. Let M be the symmetric space of $\mathrm{PSL}(n, \mathbb{R})$ and observe that $\mathrm{PSL}(n, \mathbb{R})$ is a subgroup of $\mathrm{Iso}(M)$. Let

$$\mathrm{Rep}^{\sharp}_H(\pi_1(S), \mathrm{PSL}(n, \mathbb{R})) = [\mathrm{Hom}_H((\pi_1(S), \mathrm{PSL}(n, \mathbb{R}))/\mathrm{PSL}(n, \mathbb{R})]^{\sharp}.$$

From Theorem 6.1.3, it follows that $\mathrm{Hom}_H(\pi_1(S), \mathrm{PSL}(n, \mathbb{R}))$ consists only of displacing homomorphisms. Since the induced map from $\mathrm{Rep}^{\sharp}_H(\pi_1(S), \mathrm{PSL}(n, \mathbb{R}))$ to $\mathrm{Rep}^{\sharp}_{\mathrm{disp}}(\pi_1(S), \mathrm{Iso}(M))$ is an $\mathcal{M}(S)$-equivariant continuous map, it follows from the previous proposition that the action of $\mathcal{M}(S)$ on $\mathrm{Rep}^{\sharp}_H(\pi_1(S), \mathrm{PSL}(n, \mathbb{R}))$ is proper.

We already know that $\mathrm{Rep}_H(\pi_1(S), \mathrm{PSL}(n, \mathbb{R}))$ is Hausdorff, this implies by the construction before Proposition 6.3.1 that

$$\mathrm{Rep}_H(\pi_1(S), \mathrm{PSL}(n, \mathbb{R})) = \mathrm{Rep}^{\sharp}_H(\pi_1(S), \mathrm{PSL}(n, \mathbb{R})).$$

Hence the first part of the statement follows. The last part follows from the local compactness of the spaces under consideration. Q.E.D.

7 The Toledo invariant and the minimal area

We now restrict ourselves to the case of maximal symplectic representations.

Theorem 7.0.6 *Let ρ be a representation of $\pi_1(S)$ in $\mathrm{PSp}(2n,\mathbb{R})$. Then*

$$\frac{n}{2\pi}\mathrm{MinArea}(\rho) \geqslant |\tau(\rho)|.$$

Moreover, if ρ is maximal and the inequality above is an equality, then ρ is diagonal. In this situation, there exists a unique minimal equivariant immersion, realised by a totally geodesic embedding, of $\widetilde{S}$ in the symmetric space of $\mathrm{PSp}(2n,\mathbb{R})$.

We begin with a lemma which uses the identification of tangent vectors to elements of the Lie algebra of Killing vector fields as in the beginning of Paragraph 4.2.1.

Lemma 7.0.7 *Let M be an irreducible Hermitian symmetric space of noncompact type. Let ω be its symplectic form. Let (u,v) be an orthonormal pair of tangent vectors at a point m of M. Let κ be the sectional curvature of the plane generated by (u,v), then*

$$|\omega(u,v)| \leqslant \sqrt{-\kappa}. \tag{21}$$

Moreover the equality occurs exactly whenever the Lie bracket $[u,v]$ generates the centre of the Lie algebra of the stabiliser of m in the isometry group of M.

Proof: In the notation of Paragraph 4.2.1, the symplectic structure is given by

$$\omega(u,v) = \langle [u,v], \partial_\theta\rangle \leqslant \sqrt{-\langle [u,v],[u,v]\rangle}.$$

The curvature tensor satisfies $R(u,v)w = [[u,v],w]$. The result then follows by

$$\sqrt{-\langle [u,v],[u,v]\rangle} \leqslant \sqrt{-\kappa}.$$

Q.e.d.

We can now prove Theorem 7.0.6.

Proof: The first point is immediate. Let ω be the Kähler form on the associated symmetric space X. Let (u,v) be an orthonormal system in T_xX. Then, $\omega(u,v) \leqslant 1$ with equality if and only if the plane generated by (u,v) is complex. It follows that for every ρ-equivariant mapping f, we have

$$\frac{n}{2\pi}\mathrm{Area}(f) \geqslant \frac{n}{2\pi}\int_S f^*(\omega) = \tau(\rho). \tag{22}$$

Moreover, if the equality in (22) holds for an immersion f, then there exists an invariant complex structure on $\tilde{S}$ for which f is a holomorphic map.

Assume now that ρ is maximal. According to Theorem 6.2.2, there exist a complex structure J on S and a J-conformal harmonic mapping f such that

$$\mathrm{Area}(f) = \mathrm{MinArea}(\rho).$$

Let f be such a conformal harmonic mapping. We know by Proposition 2.4 and Example (3) of the article by Gulliver, Osserman and Royden [24] that f is a branched minimal immersion. Let $x_1, \ldots, x_n$ be the branch points of order $k_1, \ldots, k_n$ respectively. Let $\hat{S} = S \setminus \{x_1, \ldots, x_n\}$. Denote by κ the curvature of the metric $f^* g_X$ on $\hat{S}$. Note that

$$\frac{1}{2\pi} \int_{\hat{S}} \kappa d\mu - \sum_i (k_i - 1) = \chi(S).$$

Let κ_f be the sectional curvature of the 2-plane $Tf(T\hat{S})$ and let B be the second fundamental form of f. By the Gauss equation

$$\kappa = \kappa_f - \|B\| \leqslant \kappa_f.$$

Finally, assume that $\tau(\rho) = \frac{n}{2\pi}\mathrm{MinArea}(\rho)$. Let μ be the measure of area of the metric $f^* g_X$. We have by Inequality (21)

$$\begin{aligned} 2\pi|\tau(\rho)| &= n\left|\int_{\hat{S}} f^*\omega\right| \\ &\leqslant n\int_{\hat{S}} \sqrt{-\kappa_f} d\mu \\ &\leqslant n\sqrt{\mathrm{Area}(f)\int_{\hat{S}} -\kappa_f d\mu} \\ &\leqslant \sqrt{2n\pi\tau(\rho)}\sqrt{\int_{\hat{S}} -\kappa_f d\mu}. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{1}{n}|\tau(\rho)| &\leqslant -\frac{1}{2\pi}\int_{\hat{S}} \kappa_f d\mu \\ &\leqslant -\frac{1}{2\pi}\int_{\hat{S}} \kappa d\mu - \frac{1}{2\pi}\int_{\hat{S}} \|B\| d\mu \\ &\leqslant -\chi(S) - \sum_i (k_i - 1) - \frac{1}{2\pi}\int_{\hat{S}} \|B\| d\mu \\ &\leqslant \frac{1}{n}|\tau(\rho)| - \sum_i (k_i - 1) - \frac{1}{2\pi}\int_{\hat{S}} \|B\| d\mu. \end{aligned}$$

As a first consequence, we see that $k_i = 1$ for all i. In other words, f is an immersion. Moreover B vanishes everywhere. This means that f is totally geodesic. It follows from the equality case in Lemma 7.0.7 that f is associated to an embedding of $\mathrm{PSL}(2,\mathbb{R})$ in $\mathrm{PSp}(2n)$ whose Lie algebra contains the centre of the Lie algebra of the maximal compact subgroup. Hence ρ is diagonal. Q.E.D.

Corollary 7.0.8 *Let ρ be a maximal representation. Assume that there exists a holomorphic equivariant map f from S to the associated symmetric space of* $\mathrm{PSp}(2n,\mathbb{R})$. *Then ρ is diagonal and f is totally geodesic.*

Proof: If f is holomorphic then f is minimal and $\mathrm{Area}(f) = \int_S f^*\omega$. By Inequality (22), it follows that $\mathrm{MinArea}(\rho) = \tau(\rho)$. Hence by the previous theorem, ρ is diagonal and f is totally geodesic. Q.E.D.

8 The Hitchin map

8.1 Representations and holomorphic differentials

We first recall that representations from $\pi_1(S)$ to a semi-simple connected real Lie group G, without compact factor and with trivial centre, give rise to holomorphic differentials by a construction quite similar to the basic construction in Chern-Weil theory. The construction that we now describe associates to every reductive representation from $\pi_1(S)$ to G, to every complex structure J on S and to every $\mathrm{Ad}(G)$-invariant polynomial q of degree n on the Lie algebra of G, a holomorphic n-ic differential on (S, J).

By Corlette's Theorem ([11] and [27] for an alternative simpler proof), there exists a ρ-equivariant harmonic mapping f from S to the symmetric space M associated to G. Moreover this mapping is unique up to an isometry of M. We define

$$\mathrm{Hom}_{\mathrm{red}}(\pi_1(S), G)$$

to be the space of reductive homomorphisms from $\pi_1(S)$ to G and

$$\mathrm{Rep}_{\mathrm{red}}(\pi_1(S), G) = \mathrm{Hom}_{\mathrm{red}}(\pi_1(S), G)/G.$$

8.1.1 Harmonic maps on surfaces

We begin with a standard observation on harmonic maps on surfaces. Let S be a Riemann surface whose complex structure is denoted by J. Let f be a smooth map from S to a smooth manifold M. Let

$$\Omega^1(S, f^*TM)$$

be the space of one-forms on S with values in the pull back vector bundle f^*TM and let

$$\Omega^1_{\mathbb{C}}(S, f^*TM \otimes_{\mathbb{R}} \mathbb{C})$$

be the space of complex linear one-forms on S with values in the complexified vector bundle $f^*TM_{\mathbb{C}} = f^*TM \otimes_{\mathbb{R}} \mathbb{C}$. For every one-form ω on S with values in

f^*TM, we denote by $\omega_{\mathbb{C}}$ its complexification. This complexification is defined for every tangent vector u in S by

$$\omega_{\mathbb{C}}(u) = \omega(u) - i\omega(Ju).$$

The map $\omega \to \omega_{\mathbb{C}}$ is a linear map from $\Omega^1(S, f^*TM)$ to $\Omega^1_{\mathbb{C}}(S, f^*TM_{\mathbb{C}})$.

Definition 8.1.1 [HOLOMORPHIC ONE-FORM] *An element β of $\Omega^1_{\mathbb{C}}(S, f^*TM_{\mathbb{C}})$ is* holomorphic *if*

$$\nabla_{Ju}\beta = i\nabla_u\beta.$$

We consider the tangent map Tf of f as a one-form on S with values in the pullback bundle by f of TM. Then, we have the following classical observation.

Proposition 8.1.2 *The map f is harmonic if and only if $Tf_{\mathbb{C}}$ is holomorphic.*

PROOF: Indeed, f is harmonic if and only if for every X

$$\nabla_X Tf(X) + \nabla_{JX} Tf(JX) = 0.$$

Since $\nabla_X Tf(Y)$ is symmetric in X and Y, the condition above is equivalent to

$$\nabla_X Tf(Y) + \nabla_{JX} Tf(JY) = 0,$$

for all X and Y in TS. This turns out to be equivalent to

$$\nabla_{JX} Tf(Y) - \nabla_X Tf(JY) = 0,$$

for all X and Y in TS. On the other hand, by definition

$$\begin{aligned}(\nabla_{JX} Tf_{\mathbb{C}} - i\nabla_X Tf_{\mathbb{C}})(Y) &= (\nabla_{JX} Tf(Y) - \nabla_X Tf(JY)) \\ &- i(\nabla_X Tf(Y) + \nabla_{JX} Tf(JY)).\end{aligned}$$

The statement follows from these remarks. Q.E.D.

8.1.2 Holomorphic differentials

Let M be a Riemannian manifold. Let p be a parallel section of $(T^*M)^{\otimes k}$. We denote by $p_{\mathbb{C}}$ the parallel section of $(T^*M_{\mathbb{C}})^{\otimes k}$ characterised by

$$p_{\mathbb{C}}\big|_{(TM)^{\otimes k}} = p.$$

Let f be a map from a Riemann surface S to M. Then we have the following easy observation.

Proposition 8.1.3 *Let β be an element in $\Omega^1_{\mathbb{C}}(S, f^*TM_{\mathbb{C}})$. Suppose that β is holomorphic. Then $p_{\mathbb{C}}(\beta, \beta, \ldots, \beta)$ is a holomorphic differential of degree k.*

As a specific example of this construction, we can take $p = g$, the Riemannian metric on M. If f is harmonic by Propositions 8.1.1 and 8.1.3

$$H(f) := g_{\mathbb{C}}(Tf_{\mathbb{C}}, Tf_{\mathbb{C}})$$

is a quadratic differential, which is called the *Hopf differential* of f. Observe that we have the following result.

Proposition 8.1.4 *Let f be a harmonic map from a surface S to M. Then the Hopf differential of f vanishes if and only if f is minimal.*

PROOF: Indeed, the quadratic differential $g_{\mathbb{C}}(Tf_{\mathbb{C}}, Tf_{\mathbb{C}})$ vanishes if and only if f is conformal and hence minimal. Q.E.D.

8.1.3 Commutative Chern-Weil Theory

When M is the symmetric space associated to $G = \mathrm{Iso}(M)$, then every $\mathrm{Ad}(G)$-invariant symmetric k-multilinear form P on the Lie algebra of G gives rise naturally to a parallel k-tensor field function $\check{P}$ on M. Indeed such a P gives naturally rise to a G-invariant tensor field $\check{P}$ on M. But on a symmetric space, any tensor field invariant under isometries is parallel.

Let J be a complex structure on S and P be a symmetric $\mathrm{Ad}(G)$-invariant multilinear form P of degree k on $\mathcal{G}$. Let $\mathcal{Q}(k, J)$ be the space of holomorphic k-differentials on S equipped with the complex structure J. Combining the above constructions, we define a map $F_{P,J}$ from $\mathrm{Rep}_{\mathrm{red}}(\pi_1(S), G)$ to $\mathcal{Q}(k, J)$ by

$$\rho \mapsto F_{P,J}(\rho) := \check{P}_{\mathbb{C}}(Tf_{\mathbb{C}}, \ldots, Tf_{\mathbb{C}}),$$

where f is a ρ-equivariant harmonic mapping from S to M given by Corlette's Theorem. The uniqueness part of Corlette's result shows that $F_{P,J}$ is well defined.

When $G = \mathrm{PSL}(n, \mathbb{R})$, let σ_k be the symmetric polynomial of degree k seen as a homogeneous function of degree k on the Lie algebra $\mathfrak{g}$ of G. There exists a unique k-multilinear symmetric $\mathrm{Ad}(G)$-invariant form p_k on $\mathfrak{g}$ so that

$$p_k(A, \ldots, A) = \sigma_k(A).$$

Notice that up to a multiplicative constant F_{p_2} is the metric on M. We define the map

$$\xi_J = \bigoplus_{k=2}^{k} F_{p_k, J}.$$

We can now state Hitchin's Theorem [26].

Theorem 8.1.5 [HITCHIN] *The map ξ_J is a homeomorphism from the space of Hitchin representations $\mathrm{Rep}_H(\pi_1(S), \mathrm{PSL}(n, \mathbb{R}))$ to $\mathcal{Q}(2, J) \oplus \ldots \oplus \mathcal{Q}(n, J)$.*

As in the introduction, we define the *Hitchin map*

$$H\left\{\begin{array}{ccl}\mathcal{E}^{(n)} & \to & \mathrm{Rep}_H(\pi_1(S),\mathrm{PSL}(n,\mathbb{R})),\\ (J,\omega) & \mapsto & \xi_J^{-1}(\omega).\end{array}\right.$$

This map is equivariant with respect to the mapping class group action. The following result is now immediate.

Theorem 8.1.6 *The Hitchin map is surjective.*

PROOF: Let ρ be a Hitchin representation. By Corollary 6.2.2, there exists a complex structure J on S and a ρ equivariant conformal harmonic mapping f with respect to J. It follows by Proposition 8.1.4 that the quadratic differential $F_{p_2,J}$ vanishes. This shows that the Hitchin map is surjective. Q.E.D.

8.1.4 The normal bundle to the space of Fuchsian representations

We conclude with a partial result. The energy functional associated to a faithful cocompact representation ρ with values in $\mathrm{PSL}(2,\mathbb{R})$ is the same – up to a multiplicative constant only depending on n – as the energy functional $e_{\check{\rho}}$ associated to the Fuchsian representation $\check{\rho} = \iota \circ \rho$ with values in $\mathrm{PSL}(n,\mathbb{R})$. Hence the energy functional $e_{\check{\rho}}$ has a unique strict minimum. Therefore the same holds for representations which are closed to being Fuchsian. It follows that the Hitchin map is a diffeomorphism from a small neighbourhood of the zero section onto its image. This implies that the normal bundle of the space of Fuchsian representations in the Hitchin component can be identified – equivariantly with respect to the action of the mapping class group – with $\mathcal{E}^{(n)}$.

9 Comments and extensions

We conclude this article with two comments

1. The theory of Hitchin representations extends to all real split groups. Similarly, the theory of maximal representations extends to all isometry groups of Hermitian symmetric spaces. It is quite natural to conjecture that the constructions of this article extend to these more general cases. The fact that these representations are (at least conjecturally) Anosov representations is certainly meaningful from this point of view. However, one cannot expect all the results here to extend to all Anosov representations since one can construct Anosov representations which are not reductive. It also remains a puzzle to understand the algebraic conditions under which Anosov representations are associated to cross ratios; this is a crucial argument in our paper.

2. For maximal symplectic representations in other components than Hitchin's, the map from the space of equivariant minimal surfaces to the space of representation is surjective for the same reason that apply to the Hitchin

component. However, in that case, the structure of a generic fibre is mysterious.

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