Cross Ratios and Identities for Higher Teichmüller-Thurston Theory

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Abstract

We generalise the McShane-Mirzakhani identities from hyperbolic geometry to arbitrary cross ratios. We define and study Hitchin representations of open surface groups to $\mathrm{PSL}(n, \mathbb{R})$. We associate to these representations cross ratios and then give explicit expressions for our generalised identities in terms of (a suitable choice of) Fock–Goncharov coordinates.

1 Introduction

In [18], the second author establishes an identity for lengths of simple closed geodesics on punctured hyperbolic surfaces. The results holds for surfaces with multiple cusps, however, in order to simplify the exposition, we consider in this section only the case where Σ is a complete connected orientable finite area hyperbolic surface with a single cusp. The length $\ell(C)$ of a homotopy class C of closed curves is defined to be the infimum of the set of lengths of curves freely homotopic to C with respect to the hyperbolic metric; this naturally extends to families $\{C_i\}_i$ by setting $\ell(\{C_i\}_i) = \sum_i \ell(C_i)$. We denote by $\mathcal{P}$ the set of embedded pants (with marked boundary) in Σ up to isotopy, such that the first boundary component of the pair of pants is the cusp.

With this notation, McShane's identity is

$$1 = \sum_{P \in \mathcal{P}} \frac{1}{e^{\frac{\ell(\partial P)}{2}} + 1}. \tag{1}$$

In [19], using McShane's method, M. Mirzakhani extends this identity to hyperbolic surfaces with totally geodesic boundary. Let Σ be a complete finite area connected hyperbolic surface with a single totally geodesic boundary component $\partial\Sigma$. Mirzakhani's identity is

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$$\ell(\partial\Sigma) = \sum_{P\in\mathcal{P}} \log\left(\frac{e^{\frac{\ell(\partial P)}{2}} + e^{\ell(\partial\Sigma)}}{e^{\frac{\ell(\partial P)}{2}} + 1}\right), \qquad (2)$$

where $\mathcal{P}$ is the set of embedded pants (with marked boundary) up to homotopy, such that the first boundary component of the pair of pants is $\partial\Sigma$.

The purpose of the paper is twofold. Firstly, we show that the identity above has a natural formulation in terms of (generalised) cross ratios. Then, using this formulation, we study identities arising from the cross ratios constructed by the first author in [15] for representations from fundamental groups of surfaces to $\mathrm{PSL}(n,\mathbb{R})$.

In order to avoid lengthy hypotheses, we shall use the following conventions throughout this article: unless otherwise stated, all surfaces are assumed to be compact, connected and oriented, all closed surfaces have genus at least 2, all surfaces with non empty boundary are of finite type and their double have genus at least 2.

We now give a brief overview of the main ideas.

Our first aim in this article is to show that the identities above have a more general interpretation. Let Σ be a closed surface (of genus at least 2 accordinng to our convention). The boundary at infinity $\partial_\infty\pi_1(\Sigma)$ of the fundamental group $\pi_1(\Sigma)$ is a one dimensional compact connected Hölder manifold – hence Hölder homeomorphic to the circle $\mathbb{T}$ – equipped with an action of $\pi_1(\Sigma)$ by Hölder homeomorphisms (see [9, 10, 11, 20]).

A *cross ratio* on $\partial_\infty\pi_1(\Sigma)$ is a Hölder function $\mathbf{B}$ defined on

$$\partial_\infty\pi_1(\Sigma)^{4*} = \{(x,y,z,t) \in \partial_\infty\pi_1(\Sigma)^4 \mid x \neq t \text{ and } y \neq z\},$$

invariant under the diagonal action of $\pi_1(\Sigma)$, and which satisfies some algebraic rules. Roughly speaking, these rules encode two conditions which constitute a normalisation, and two multiplicative cocycle identities which hold for some of the variables:

$$\begin{array}{rrcl}
\text{NORMALISATION:} & \mathbf{B}(x,y,z,t) & = & 0 \quad \Leftrightarrow x = y\,, \text{ or } z = t, \\
\text{NORMALISATION:} & \mathbf{B}(x,y,z,t) & = & 1 \quad \Leftrightarrow x = z\,, \text{ or } y = t, \\
\text{COCYCLE IDENTITY:} & \mathbf{B}(x,y,z,t) & = & \mathbf{B}(x,y,z,w)\mathbf{B}(x,w,z,t), \\
\text{COCYCLE IDENTITY:} & \mathbf{B}(x,y,z,t) & = & \mathbf{B}(x,y,w,t)\mathbf{B}(w,y,z,t).
\end{array}$$

These relations imply an essential symetry

$$\text{SYMMETRY:} \quad \mathbf{B}(x,y,z,t) \quad = \quad \mathbf{B}(z,t,x,y).$$

The *period* of a nontrivial element γ of $\pi_1(\Sigma)$ with respect to $\mathbf{B}$ is the following real number

$$\log|\mathbf{B}(\gamma^-,\gamma y,\gamma^+,y)| := \ell_{\mathbf{B}}(\gamma),$$

where γ^+ (respectively γ^-) is the attracting (respectively repelling) fixed point of γ on $\partial_\infty \pi_1(\Sigma)$ and y is any element of $\partial_\infty \pi_1(\Sigma) \setminus \{\gamma^+, \gamma^-\}$; as the notation suggests, one checks immediately that the period is independent of y (see Paragraph 2.2.2). In Section 2.1 we extend the above definitions to cover open surfaces of finite type.

These definitions are closely related to those given by Otal in [21, 22], those discussed from various perspectives by Ledrappier in [16] and those of Bourdon in [5] in the context of CAT(-1)-spaces.

The archetype of a cross ratio arises in hyperbolic geometry in the following way. A complete hyperbolic metric on Σ gives rise to an identification of $\partial_\infty \pi_1(\Sigma)$ with the real projective line. Thus, the projective cross ratio on the projective line gives rise to a cross ratio on $\partial_\infty \pi_1(\Sigma)$ – called a *hyperbolic cross ratio*. The period of γ is just the hyperbolic length of the closed geodesic associated to γ. We recall that the projective cross ratio $\mathbf{b}$ on the projective line – which we shall refer to in the sequel as the *classical cross ratio* – is given in projective coordinates by

$$\mathbf{b}(x, y, z, t) = \frac{(x - y)(z - t)}{(x - t)(z - y)},$$

so that $\mathbf{b}(x, 0, 1, \infty) = x$.

More generally, as was observed in [15] and as described in the following paragraphs, cross ratios are associated to certain representations from $\pi_1(\Sigma)$ to $\mathrm{PSL}(n, \mathbb{R})$ for a closed surface Σ.

Let P be the – oriented – sphere minus three points. We choose once and for all a *marking* of P, that is three elements α_0,β_0, and γ_0 of $\pi_1(P)$ represented by loops freely homotopic to the oriented boundary loops and such that $\alpha_0\gamma_0\beta_0 = 1$. An *isotopy class of pair of pants* in Σ is an isotopy class of embeddings of P in Σ such that the images α, β and γ of α_0, β_0 and γ_0 are non trivial. The triple (α, β, γ) is well defined up to conjugacy by elements of $\pi_1(\Sigma)$ and satisfies $\alpha\gamma\beta = 1$. Henceforth, we refer to α as the *first* boundary component of the pair of pants.

Given a cross ratio $\mathbf{B}$ on $\partial_\infty \pi_1(\Sigma)$, we define the value of the *pant gap function* $G_{\mathbf{B}}$ associated to $\mathbf{B}$ of an isotopy class of pair of pants P represented by (α, β, γ) to be

$$G_{\mathbf{B}}(P) = \log |\mathbf{B}(\alpha^+, \gamma^-, \alpha^-, \beta^+)|.$$

Note that the first boundary component α plays a special role. We shall prove

Theorem 1.0.1 *Let Σ be compact surface with exactly one boundary component representing a non trivial element α in $\pi_1(\Sigma)$. Let $\mathbf{B}$ be a cross ratio on $\partial_\infty \pi_1(\Sigma)$. Let $\mathcal{P}$ be the space of isotopy classes of pairs of pants in Σ whose first boundary component is α, then*

$$\ell_{\mathbf{B}}(\alpha) = \sum_{P \in \mathcal{P}} G_{\mathbf{B}}(P).$$

Moreover, after suitably extending the notion of cross ratio (see Subsection 2.1), the theorem generalises to more general open surfaces of finite type. It also generalises "at a cusp" so that we recover Formula (1). The complete results – Theorem 4.1.2 and Theorem 4.2.4 – are proved in Section 4.

Examining more closely the case of hyperbolic cross ratios, we observe that the pant gap function for P can be computed in terms of the lengths of just the boundary components of the pair of pants P. Indeed recall that every hyperbolic pair of pants with totally geodesic boundary is determined up to isometry by the length of its three boundary components and hence every geometric function is a function of these three parameters. In Section 5, using Thurston's *shear coordinates*, described by Bonahon in [3], and elementary manipulations involving the classical cross ratio – as opposed to hyperbolic trigonometry in the original proofs – we recover Mirzakhani-McShane formulae (1) and (2) for the pant gap function.

We now discuss briefly additional ideas that allow our approach to extend to higher dimensions. For a closed surface Σ, the first author gives in [15] an interpretation of *Hitchin representations* – which fill a connected component of the space of representations from $\pi_1(\Sigma)$ to $\mathrm{PSL}(n,\mathbb{R})$ – as the space of rank n cross ratios on $\partial_\infty\pi_1(\Sigma)$ (see precise Definition in [14]). The second aim of this article – accomplished in Section 9 – is to extend these results to open surfaces.

Let us be more precise. An element in $\mathrm{PSL}(n,\mathbb{R})$ is *purely loxodromic* if it is real split and has simple eigenvalues, or in other words all its eigenvalues are real with mutilplicity 1. Let Σ be a compact surface possibly with boundary. A representation from $\pi_1(\Sigma)$ to $\mathrm{PSL}(n,\mathbb{R})$ is *Fuchsian* if it factors as a discrete faithful representation without parabolics into $\mathrm{PSL}(2,\mathbb{R})$ composed with the irreducible representation from $\mathrm{PSL}(2,\mathbb{R})$ to $\mathrm{PSL}(n,\mathbb{R})$. A representation from $\pi_1(\Sigma)$ to $\mathrm{PSL}(n,\mathbb{R})$ is *Hitchin* if the boundary components have purely loxodromic images under the representation, and if the representation can be deformed into a Fuchsian representation so that the images of the boundary components stay purely loxodromic.

In Section 9, we obtain the following generalisation of results in [15]

Theorem 1.0.2 *Let S be a compact connected orientable surface with non empty boundary whose double has genus at least 2. Let ρ be a Hitchin representation of $\pi_1(\Sigma)$. Then the image under ρ of every non trivial element of the fundamental group is purely loxodromic and, moreover, there exists a cross ratio $\mathbf{B}$ on $\partial_\infty\pi_1(S)$ whose periods satisfy*

$$\forall\gamma\in\pi_1(S),\quad \ell_{\mathbf{B}}(\gamma)=\log\left(\left|\frac{\lambda_{\max}(\rho(\gamma))}{\lambda_{\min}(\rho(\gamma))}\right|\right),$$

where $\lambda_{\max}(\rho(\gamma))$ and $\lambda_{\min}(\rho(\gamma))$ are the eigenvalues of respectively maximum and minimum absolute values of the element $\rho(\gamma)$.

Furthermore, let Σ be an incompressible surface embedded in S. Then ρ restricted to $\pi_1(\Sigma)$ is a Hitchin representation.

Theorem 9.0.3 is a more detailed statement of this result. Its proof involves a doubling construction described in Theorem 9.2.4 and which also yields

Corollary 1.0.3 *Let Σ be a compact surface with boundary. Let ρ be a Hitchin representation of $\pi_1(\Sigma)$. Then there exists a* closed *surface S containing Σ and a Hitchin representation $\widehat{\rho}$ of $\pi_1(S)$ such that ρ is the restriction of $\widehat{\rho}$.*

In a series of articles [14, 13, 15], the first author shows that Hitchin representations for closed surfaces are discrete and faithful, that every non trivial element is purely loxodromic and that the mapping class group acts properly on the moduli space of Hitchin representations. A consequence of the above corollary is that these concepts and results for closed surfaces carry over for surfaces with boundary.

For a cross ratio associated to a representation ρ from $\pi_1(\Sigma)$ to $\mathrm{PSL}(n, \mathbb{R})$, the gap function of a pair of pants associated to the triple (α, β, γ) of elements of $\pi_1(\Sigma)$ depends only on the triple $(\rho(\alpha), \rho(\beta), \rho(\gamma))$ of elements of $\mathrm{PSL}(n, \mathbb{R})$ (see Proposition 8.2.2). However, contrary to hyperbolic cross ratios, for $n \geq 3$ a simple computation of the dimensions shows that the pant gap function is no longer determined by just the eigenvalues of the monodromies of the three boundary components of the pants: it also depends on extra *internal* parameters. In the third part of this article, we give a description of the gap functions for Hitchin representations which nevertheless parallels our description of gap functions in hyperbolic geometry. We first recall the construction by Fock and Goncharov in [7] of *Fock–Goncharov coordinates* on the *Fock–Goncharov moduli space* which are far reaching generalisations of *Thurston's shear coordinates* on the *enhanced Teichmüller space* for rational laminations, again discussed by Bonahon in [3]. We use these coordinates to obtain coordinates on the space of Hitchin representations which we show are *positive* in the sense of Fock–Goncharov – see Definition 7.1.8 and Theorem 9.0.3. Actually, since the Fock–Goncharov moduli space is a "covering" of the space of Hitchin representations, we obtain $(n!)^3$ different sets of coordinates for the moduli of a pair of pants. In Theorem 10.3.1 we finally show that, for a suitable choice of coordinates, the pant gap function has a nice expression. On the other hand, using computer algebra software and the explicit description of the holonomies given by V. Fock and A. Goncharov in [8], in Section 11 we show that, even when $n = 3$, the pant gap function has a very complicated expression for other choices of coordinates – see for instance Formula (55).

In conclusion, recall that using her identities, M. Mirzakhani gives in [19] a recursive formula for the volumes of the moduli space of hyperbolic structures identified with the quotient of Teichmüller space by the mapping class group. From the work of the first author in [15], it follows that the mapping class group acts properly on the moduli space of Hitchin representations. It is quite possible that the formula obtained in Theorem 10.3.1 combined with the use of Fock–Goncharov coordinates can help to compute geometric quantities associated to the corresponding quotient. However the volume is not the right quantity to compute since for n at least 3, one can show it is infinite.

We also hope that some of our work could be generalised to $\mathrm{PSL}(n, \mathbb{C})$ as McShane's identities were generalised to $\mathrm{PSL}(2, \mathbb{C})$ by B Bowditch [6], H. Akiyoshi,

H. Miyachi, M. Sakuma [1] and Ser Peow Tan, Yan Loi Wong, Ying Zhang [23].

We conclude by saying that it is a striking fact that so many of the familiar ideas from the world of hyperbolic geometry translate naturally to the world of Hitchin representations. So much so that one is tempted to call the latter *a higher (rank) Teichmüller-Thurston theory*.

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Contents

2 Cross ratios and the boundary at infinity

We first recall the definition of cross ratio on the boundary at infinity of a closed surface according to [15] and then extend this notion to open surfaces. This extension requires defining a boundary at infinity for surfaces with cusps and boundary.

2.1 Boundary at infinity for open surfaces of finite type

Let Σ be a connected orientable open surface homeomorphic to

$$\Sigma_0 \setminus \{b_1, \dots, b_n, c_1, \dots c_p\},$$

where Σ_0 is closed and b_i, c_i are distinct points in Σ_0. We have deliberately labelled the points in two different ways. The points b_i will be referred as *boundary components* and the points c_i as *cusps*. We denote this data by $\Sigma(g, n, p)$ meaning that Σ_0 has genus g, and that there are n boundary components and p cusps. We assume that $2g - 2 + p + n > 0$.

Definition 2.1.1 [ADMISSIBLE METRIC] *A finite volume hyperbolic metric on Σ is* admissible *if the completion of Σ is a surface with n totally geodesic boundary components (corresponding to the neighbourhoods of the points b_i) and p cusps (corresponding to neighbourhoods of the the points c_i).*

Admissible metrics exist since $2g - 2 + p + n > 0$. The deck transformations associated to an admissible metric σ on an oriented surface yield an embedding ρ_σ from $\pi_1(\Sigma)$ into $\mathrm{PSL}(2, \mathbb{R})$ well defined up to conjugacy in $\mathrm{PSL}(2, \mathbb{R})$.

We recall the following fact

Proposition 2.1.2 [LIMIT SETS] *If σ_1 and σ_2 are two admissible metrics and let ρ_1 and ρ_2 be the embeddings from $\pi_1(\Sigma)$ associated as above. Let Λ_1 and Λ_2 be the the limit sets of $\rho_1(\pi_1(\Sigma))$ and $\rho_2(\pi_1(\Sigma))$ respectively. Then there exists a unique $\pi_1(\Sigma)$-equivariant Hölder homeomorphism ϕ from Λ_1 to Λ_2.*

This allows us to give the following

Definition 2.1.3 [BOUNDARY AT INFINITY FOR OPEN SURFACES] *The* boundary at infinity $\partial_\infty \pi_1(\Sigma)_p$ *of the open surface Σ with p cusps is the boundary at infinity of the universal cover $\tilde{\Sigma}$ of Σ equipped with some admissible metric g.*

REMARKS:

- Let ρ_g be the monodromy representation of an admissible hyperbolic metric on Σ. The universal cover $\tilde{\Sigma}$ is then isometric to the interior of the convex hull of the limit set of the group $\rho_g(\pi_1(\Sigma))$ in the hyperbolic plane. It follows that the limit set of the group $\rho_g(\pi_1(\Sigma))$ is identified to the boundary at infinity of $\tilde{\Sigma}$, hence to $\partial_\infty\pi_1(\Sigma)_p$ by definition.

- By Proposition 2.1.2 above, this definition is independent of the choice of an admissible metric. Moreover, by the same proposition, $\partial_\infty\pi_1(\Sigma)_p$ is equipped with a Hölder structure (coming from the various choices of embedding it as a limit set) so that it makes sense to speak of Hölder functions on $\partial_\infty\pi_1(\Sigma)_p$ as well as of sets of zero Hausdorff dimension. Finally, $\pi_1(\Sigma)$ acts on $\partial_\infty\pi_1(\Sigma)_p$ by Hölder homeomorphisms.

- If there are no cusps and Σ is open, $\partial_\infty\pi_1(\Sigma)_0$ is precisely the boundary at infinity of the free group $\pi_1(\Sigma)$: the Cantor set of ends of any Cayley graph of $\pi_1(\Sigma)$.

- The set $\partial_\infty\pi_1(\Sigma)_p$ has a *cyclic ordering on points* depending on the orientation of the surface Σ. By construction of $\partial_\infty\pi_1(\Sigma)_p$, an admissible metric gives rise to an inclusion i of $\partial_\infty\pi_1(\Sigma)_p$ into the boundary at infinity of the hyperbolic plane which is homeomorphic to the circle $\mathbb{T}$, and so it inherits a cyclic ordering from the circle. The orientation of Σ being fixed, this inclusion i is not canonical but only defined up to a orientation preserving homeomorphism of the circle, therefore the cyclic order is a *topological* invariant. If we choose the reverse orientation on Σ, the cyclic ordering is reversed. This cyclic ordering (see Section 3.1.1) describes which conjugacy classes of $\pi_1(\Sigma)$ represent simple curves on the surface Σ.

- Even though we shall not use this construction in the sequel, in order to complete this presentation, we sketch a way to obtain the topological space $\partial_\infty\pi_1(\Sigma)_p$ from $\partial_\infty\pi_1(\Sigma)_0$. Choose an admissible metric on Σ with $n+p$ boundary components and no cusps. Each geodesic boundary component lifts to a union of disjoint geodesics in $\mathbb{H}$. We consider the equivalence relation $\mathcal{R}$ on $\partial_\infty\pi_1(\Sigma)_0$ which identifies the two end points of each of the geodesic that comes from points we wish to declare as cusps. Then the quotient space $\partial_\infty\pi_1(\Sigma)/\mathcal{R}$ is homeomorphic to $\partial_\infty\pi_1(\Sigma)_p$. At this stage however, the Hölder structure is missing.

Nontrivial elements of $\pi_1(\Sigma)$ are of two types according to the number of their fixed points on $\partial_\infty\pi_1(\Sigma)_p$.

Definition 2.1.4 [Parabolic and hyperbolic elements] *The nontrivial element γ of $\pi_1(\Sigma)$ is* hyperbolic *if a loop representing γ is not freely homotopic to a curve in a neighbourhood of a cusp. Such a γ has precisely two fixed points on $\partial_\infty\pi_1(\Sigma)_p$: one attractive γ^+ and one repulsive γ^-.*

The element γ is parabolic, *if it is nontrivial and if the loop representing γ is freely homotopic to a curve in a neighbourhood of a cusp. Then γ has precisely one fixed point on $\partial_\infty\pi_1(\Sigma)_p$ called the* cusp *of γ.*

In what follows, we adopt the convention that both γ^+ and γ^- denotes the cusp of a parabolic element of γ. Every non trivial element of $\pi_1(\Sigma)$ is either parabolic or hyperbolic.

2.2 Cross ratio for metric spaces

2.2.1 Cross ratios

Let S be a metric space equipped with the action of a group Γ by Hölder homeomorphisms. Let

$$S^{4*} = \{(x, y, z, t) \in S^4 \mid x \neq t, \text{ and } y \neq z\}.$$

We equip S^{4*} with the diagonal action of Γ.

Our main example will be the boundary at infinity $\partial_\infty \pi_1(\Sigma)_p$ of a surface with p cusps equipped with the natural action of $\pi_1(\Sigma)$ by Hölder homeomorphisms.

Definition 2.2.1 [Cross ratio] *A* cross ratio *on S is a Γ-invariant Hölder function $\mathbf{B}$ on S^{4*} with real values which satisfies the following rules*

$$\begin{aligned}
\mathbf{B}(x,y,z,t) &= 0 \quad \Leftrightarrow x = y \quad \text{or } z = t, &(3)\\
\mathbf{B}(x,y,z,t) &= 1 \ \ \Leftrightarrow x = z \quad \text{or } y = t, &(4)\\
\mathbf{B}(x,y,z,t) &= \mathbf{B}(x,y,w,t)\mathbf{B}(w,y,z,t), &(5)\\
\mathbf{B}(x,y,z,t) &= \mathbf{B}(x,y,z,w)\mathbf{B}(x,w,z,t). &(6)
\end{aligned}$$

The first cocycle identity (5) *is a multiplicative cocycle identity on the first and third arguments, and the* second cocycle identity (6) *is a multiplicative cocycle identity on the second and fourth arguments.*

These rules imply the following symmetries

$$\begin{aligned}
\mathbf{B}(x,y,z,t) &= \mathbf{B}(z,t,x,y), &(7)\\
\mathbf{B}(x,y,z,t) &= \mathbf{B}(z,y,x,t)^{-1}, &(8)\\
\mathbf{B}(x,y,z,t) &= \mathbf{B}(x,t,y,z)^{-1}. &(9)
\end{aligned}$$

The classical cross ratio $\mathbf{b}$ on $\mathbb{RP}^1$ defined in an affine chart by

$$\mathbf{b}(x,y,z,t) = \frac{(x-y)(z-t)}{(x-t)(z-y)},$$

is an example of a cross ratio with respect to the action of $\mathrm{PSL}(2,\mathbb{R})$.

We will recall in Section 8.2 results of [15] which associate cross ratios to representations in $\mathrm{PSL}(n,\mathbb{R})$ as well as other related constructions. In Section 2.2.4, we explain how a hyperbolic metric on Σ gives rise to a cross ratio.

Definition 2.2.2 [PERIOD] *Let* $\mathbf{B}$ *be a cross ratio on* $\partial_\infty\pi_1(\Sigma)_p$ *and* γ *be a hyperbolic element in* $\pi_1(\Sigma)$*. The* period *of* γ *with respect to* $\mathbf{B}$ *is*

$$\ell_{\mathbf{B}}(\gamma) := \log|\mathbf{B}(\gamma^-, \gamma y, \gamma^+, y)|, \tag{10}$$

where γ^+ *and* γ^- *are respectively the attracting and repelling fixed points of* γ *on* $\partial_\infty\pi_1(\Sigma)$ *and* y *is any element of* $\partial_\infty\pi_1(\Sigma)$ *different from* γ^+ *and* γ^-*.*

Relation (6) and the invariance under the action of γ imply that $\ell_{\mathbf{B}}(\gamma)$ does not depend on y. Moreover, by Equation (7), $\ell_{\mathbf{B}}(\gamma) = \ell_{\mathbf{B}}(\gamma^{-1})$.

REMARK:

The definition given above **does not coincide** with the definition given for instance in [12, 16, 22] (even after taking an exponential): indeed, we do not require $\mathbf{B}(x, y, z, t) = \mathbf{B}(y, x, t, z)$. However we may observe that if $\mathbf{B}(x, y, z, t)$ is a cross ratio with our definition, so is $\mathbf{B}^*(x, y, z, t) = \mathbf{B}(y, x, t, z)$, and finally $\log|\mathbf{B}\mathbf{B}^*|$ is a cross ratio according to the definitions quoted above.

2.2.2 Cross ratios and cyclic orderings

We observed in the previous paragraph that $\partial_\infty\pi_1(\Sigma)_p$ has a natural cyclic ordering coming from the orientation of the surface. We impose an extra condition on the cross ratio defined on cyclically ordered sets.

Definition 2.2.3 [CROSS RATIO FOR CYCLICALLY ORDERED SETS] *Let* S *be a metric space equipped with a cyclic ordering and an action of a group* Γ *preserving this ordering. An* ordered cross ratio *for* S *is a cross ratio for the action of* Γ *on* S *such that if the cyclic ordering on the quadruple* (x, y, z, t) *coincides with the induced ordering, then both following inequalities are satisfied*

$$\mathbf{B}(x, z, t, y) > 1, \tag{11}$$

$$\mathbf{B}(x, y, z, t) < 0. \tag{12}$$

REMARKS:

- This extra condition is a mild requirement satisfied by all the cross ratios that are constructed in this article. It does not depend on the choice of the orientation on the surface.
- If Σ is a closed surface and $\mathbf{B}$ is a cross ratio on $\partial_\infty\pi_1(\Sigma)$, then, by continuity, either $\mathbf{B}$ or $\mathbf{B}^{-1}$ is ordered.
- In the sequel, when we speak of a cross ratio on $\partial_\infty\pi_1(\Sigma)_p$, we shall always assume that the cross ratio is ordered and hence implicitely assume Inequalities (12) and (11).
- Finally, if γ is a hyperbolic element of $\pi_1(\Sigma)$ and $\mathbf{B}$ be a (ordered) cross ratio, then $\mathbf{B}(\gamma^-, \gamma(x), \gamma^+, x) > 1$. Hence, the absolute values in the definition of the period could be removed and the periods are positive.

2.2.3 Cross ratio and subsurfaces

Any embedding of $\Sigma(g,n,p)$ in $\check{\Sigma}(\check{g},\check{n},\check{p})$ injective at the level of fundamental groups, sending cusps to cusps and no boundary component to cusp, can be realised by an isometric embedding for some choice of admissible metrics. Hence, such an embedding induces an Hölder embedding of the corresponding boundary at infinity. In particular, a cross ratio on $\partial_\infty\pi_1(\check{\Sigma})_{\check{p}}$ induces a cross ratio on $\partial_\infty\pi_1(\Sigma)_p$. This will be mostly used in the sequel when $\check{n}=\check{p}=0$.

2.2.4 Hyperbolic cross ratios

Assume that Σ is equipped with an admissible hyperbolic metric. In particular, deck transformations generate a representation ρ from $\pi_1(\Sigma)$ to $\mathrm{PSL}(2,\mathbb{R})$. Then, $\partial_\infty\pi_1(\Sigma)_p$ is identified equivariantly with the limit set of $\rho(\pi_1(\Sigma))$ which is a subset of the boundary at infinity of $\mathbb{H}^2$. In other words, an admissible metric gives rise to a ρ-equivariant map ι from $\partial_\infty\pi_1(\Sigma)_p$ to $\partial_\infty\mathbb{H}^2\simeq\mathbb{RP}^1$. Let $\mathbf{b}$ be the classical cross ratio. Then,

$$\mathbf{B}_\rho(x,y,z,t)=\mathbf{b}(\iota(x),\iota(y),\iota(z),\iota(t)),$$

is a cross ratio on $\partial_\infty\pi_1(\Sigma)_p$.

Definition 2.2.4 [HYPERBOLIC CROSS RATIO] *A cross ratio obtained through the previous construction is a* hyperbolic cross ratio*.*

For hyperbolic cross ratios, periods coincide with lengths of closed geodesics.

3 Gaps and gap functions

In this section, we discuss the geometry of the space of pairs of distinct points of the boundary at infinity. We first introduce a terminology to describe intersection of geodesics in terms of their end points. We then present the Birman–Series set and state their remarkable theorem [2]. Finally, we describe *gaps* and recall that they are related to embedded pairs of pants.

3.1 The boundary at infinity and embedding of geodesics

If Σ is equipped with an admissible metric, a geodesic in its universal cover gives rise to a pair of points in the boundary at infinity. We introduce a terminology based purely on properties of configurations of points in $\partial_\infty\pi_1(\Sigma)_p$ relative to its cyclic ordering, then translate this terminology into properties of the associated geodesics (compare [18, 19])).

Definition 3.1.1 [SUBSETS OF $\partial_\infty\pi_1(\Sigma)_p$]

- *Let a and b be distinct points in $\partial_\infty\pi_1(\Sigma)_p$. An* open interval $]a,b[$ *in $\partial_\infty\pi_1(\Sigma)_p$ is the set of those elements t in $\partial_\infty\pi_1(\Sigma)_p$ different from a and b and such that (a,t,b) is positively oriented. The* closed interval $[a,b]$ *is $]a,b[\cup\{a,b\}$.*

- *Let X, Y be subsets of $\partial_\infty \pi_1(\Sigma)_p$. We say that X* does not separate *Y if there exists disjoint closed intervals I and J such that I contains X and J contains Y.*

- *The subset X* injects *if for all elements γ in $\pi_1(\Sigma)$, X does not separate $\gamma(X)$.*

- *The pair $X = \{x, y\}$ is* simple *if it injects and is fixed by a nontrivial element of $\pi_1(\Sigma)$.*

Observe that intervals may not be connected.

If x_1, y_1, x_2 and y_2 are pairwise distinct points of $\partial_\infty \pi_1(\Sigma)_p$ and $\mathbf{B}$ is a cross ratio – which is now by convention ordered – on $\partial_\infty \pi_1(\Sigma)_p$, observe that $\{x_1, x_2\}$ does not separate $\{y_1, y_2\}$ if and only if

$$\mathbf{B}(x_1, y_1, x_2, y_2) > 0.$$

Indeed, by Properties (8) and (9) this statement is independent under renumbering of $\{x_1, x_2\}$ and $\{y_1, y_2\}$. But, after renumbering, either (x_1, x_2, y_1, y_2) is oriented in which case $\{x_1, x_2\}$ does not separate $\{y_1, y_2\}$ and $\mathbf{B}(x_1, y_1, x_2, y_2) > 0$, or (x_1, y_1, x_2, y_2) is oriented in which case $\{x_1, x_2\}$ separates $\{y_1, y_2\}$ and $\mathbf{B}(x_1, y_1, x_2, y_2) < 0$.

If X is a simple pair, there exists an element α in $\pi_1(\Sigma)$, well defined up to multiplicity in $\mathbb{Z}^*$, so that $X = \{\alpha^+, \alpha^-\}$.

Definition 3.1.2 [Peripheral elements] *A non trivial element α in $\pi_1(\Sigma)$ is* peripheral *if it is not parabolic and if the pair $\{\alpha^+, \alpha^-\}$ does not separate $\partial_\infty \pi_1(\Sigma)_p$. By extension, we also say that α^+ and α^- are* peripheral *points.*

Note that if α is peripheral then $\{\alpha^+, \alpha^-\}$ is simple.

The following dictionary relates the previous definitions to intersection properties of geodesics. Let $X = \{x_1, x_2\}$ be a pair. If we equip Σ with an admissible metric and denote by $\widetilde{\gamma}_X$ the geodesic joining x_1 to x_2 in the universal cover of Σ and by γ_X its projection on Σ, then we have the dictionary

- Let Y be a pair. Then, "*X does not separate Y*" is equivalent to "*$\widetilde{\gamma}_X$ and $\widetilde{\gamma}_Y$ do not intersect*".

- "*X is simple*" is equivalent to "*γ_X is a simple closed geodesic*".

- "*X injects*" is equivalent to "*γ_X has no transverse self intersection*", in particular the closure of γ_X is a geodesic lamination.

3.1.1 Pairs of pants

Let P be a sphere minus three disjoint open discs, together with a base point and a choice of orientation. We choose once and for all a *marking* of P, that is three elements α_0,β_0, and γ_0 of $\pi_1(P)$ represented by three loops each of which is freely homotopic to one of the boundary loops (with the orientation induced from the sphere) and such that $\alpha_0\gamma_0\beta_0 = 1$.

We now define what we mean by an embedded pairs of pants or more properly an isotopy class of embedding of a pair of pants.

Definition 3.1.3 [Embedded pairs of pants] *An* isotopy class of pairs of pants *in an orientable surface Σ, possibly with boundary, is an isotopy class of orientation preserving incompressible embeddings of P in Σ, i.e. the induced morphism on the fundamental groups is an injection. We write α, β, γ respectively for the images in $\pi_1(\Sigma)$ of $\alpha_0, \beta_0, \gamma_0$; the triple (α, β, γ) is well defined up to conjugacy by $\pi_1(\Sigma)$ and satisfies $\alpha\gamma\beta = 1$. For any such a triple we shall call α the* first *boundary component of the pair of pants and, for brevity, we shall say that the triple (α, β, γ)* represents *the pair of pants.*

Remark: Note that, with our definition above, if (α, β, γ) represents an embedding of P then $(\alpha, \alpha\beta\alpha^{-1}, \alpha\gamma\alpha^{-1})$ still represents the same isotopy class of embedding of P. Though this is a departure from the usual definition for surfaces with boundary (which we give below and refer to as pointed pairs of pants) it will make the statement of our identities as well as their proofs slightly more "compact". Informally, an embedded pair of pants is the orbit under the Dehn twist round the boundary loop α of an embedded pointed pair of pants. This distinction is important particularly in the proof of Theorem 4.1.2.

Definition 3.1.4 [Embedded pointed pair of pants] *Let Σ be a surface with boundary. Fix a basepoint p for the pair of pants P on the boundary loop representing α_0 and a fixed point p' for Σ on a distinguished boundary component α. We define an isotopy class of pointed pairs of pants in Σ to be an isotopy class of orientation preserving embeddings as above, but with the additional hypothesis that, for each such embedding and each such isotopy, the image of p is p'. With this definition if (α, β, γ) represents an embedding of P then $(\alpha, \alpha\beta\alpha^{-1}, \alpha\gamma\alpha^{-1})$ represents a distinct (from (α, β, γ)) isotopy class of embedding of P.*

Observe then that the sextuplet $(\alpha^-, \alpha^+, \beta^-, \beta^+, \gamma^-, \gamma^+)$ is positively oriented as in Figure 1.

3.2 Birman–Series sets

3.2.1 The Birman–Series Theorem

Let
$$\partial_\infty \pi_1(\Sigma)^{2*}_p = \{(x, y) \in \partial_\infty \pi_1(\Sigma)^2_p \mid x \neq y\}.$$

Definition 3.2.1 *The* Birman–Series set *is the set of couples in $\partial_\infty \pi_1(\Sigma)^{2*}_p$ whose associated pairs inject.*

The Birman–Series set is a $\pi_1(\Sigma)$-invariant set and by the previous paragraph a couple $X = (x, y)$ belongs to the Birman–Series set if and only if the projection

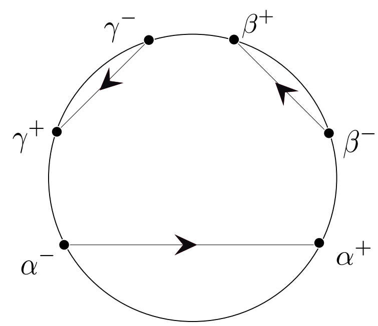

Figure 1: Associated sextuplet to a pair of pants

to the surface Σ of the geodesic joining x to y for an admissible metric is a complete simple geodesic. Joan Birman and Caroline Series studied in [2] the set of all complete simple geodesic on a hyperbolic surface and showed that the points of this set are somewhat sparse.

Theorem 3.2.2 [J.Birman–C.Series] *The union of the simple complete geodesics of a compact hyperbolic surface is closed and has Hausdorff dimension* 1*. In particular the Birman–Series set has zero Hausdorff dimension.*

3.2.2 Relative Birman–Series sets

Let α be a peripheral or parabolic element of $\pi_1(\Sigma)$.

Definition 3.2.3 [Relative Birman–Series set] *The* first relative Birman–Series set with respect to α *is the subset K_α of $\partial_\infty \pi_1(\Sigma)_p \setminus \{\alpha^+, \alpha^-\}$ of those elements t such that $\{\alpha^+, t\}$ injects.*

The second relative Birman–Series set with respect to α *is the subset K^*_α of K_α of those elements which do not belong to the $\pi_1(\Sigma)$ orbit of α^+.*

In our dictionary, K_α corresponds to the set of (lifts) of geodesics that are simple and spiral to the boundary component defined by α in a given direction if α is hyperbolic, or go to the cusp defined by α if α is parabolic. These relative sets are subsets of the Birman–Series set, and hence have zero Hausdorff dimension.

Proposition 3.2.4 *The sets K_α and K^*_α are closed (for the relative topology) subsets of $]\alpha^+, \alpha^-[$. If x is a fixed point of a peripheral or parabolic element of $\pi_1(\Sigma)$ in K_α then x is isolated in K_α.*

Proof: Note first, that the Birman-Series set is closed so $K_\alpha \cup \{\alpha^+, \alpha^-\}$ is immediately seen to be closed too.

The fact that K^*_α is a closed subset will follow from the second statement, since every element in the orbit of α^+ is peripheral or parabolic and hence isolated. One shows this second statement as follows. Let $x = \beta^+$ be the attractive fixed point of a peripheral hyperbolic element β. We assume that $(\alpha^+, \beta^+, \beta^-, \alpha^-)$ is positively oriented, the other case being treated similarly. If y belongs to $]\beta(\alpha^+), \beta^+[$, then $\{\alpha^+, y\}$ does not inject since

$$(\alpha^-, \alpha^+, \beta(\alpha^+), y, \beta(y), \beta^+),$$

is positively oriented and, hence $\{\alpha^+, y\}$ separates $\{\beta(\alpha^+), \beta(y)\}$. Hence the intersection of K_α with the interval $]\beta(\alpha^+), \beta-[$ contains only β^+. When β is a parabolic, we restrict ouselves to the case $(\alpha^+, \beta^+, \alpha^-)$ is oriented and we show using the same argument that $]\beta(\alpha^+), \beta^{-1}(\alpha^-)[$ only contains β^+. Q.E.D.

3.3 Gaps and pair of pants

Let α be a peripheral or parabolic element.

Definition 3.3.1 [Gap] *A* gap *with respect to α is a couple (x, y) of elements of K^*_α so that the interval $]x, y[$ does not intersect K^*_α.*

Note that $\partial_\infty \pi_1(\Sigma)_p \setminus K^*_\alpha$ is a disjoint union of intervals associated to gaps. In the sequel, we shall abusively refer to the interval $]\alpha, \beta[$ when (α, β) is a gap also as a gap.

There are exactly two types of gap, those arising from pair of pants, those arising from peripheral elements. The following result is a rephrasing of the corresponding statements in [18] and [19].

Proposition 3.3.2 [Gaps and pants] *If (α, β, γ) represents a pair of pants then:*

1. *The pair (β^+, γ^-) is a gap.*
2. *If furthermore β is peripheral, then (β^-, β^+) is a gap.*

Conversely, let (x, y) be a gap. Then, there exist primitive elements β and γ in $\pi_1(\Sigma)$ such that $(x, y) = (\beta^+, \gamma^-)$. Moreover:

1. *If $\beta \neq \gamma$ then, after possibly taking inverses whenever γ or β are parabolic, (α, β, γ) represents a possibly degenerate pair of pants. Such group elements β and γ are then unique.*
2. *If $\beta = \gamma$, then β is peripheral and there exists a unique embedded (pointed) pair of pants P so that α and β are boundary loops of P.*

Figure 2 represents a gap arising form a pair of pants represented by (α, β, γ): the left figure represents the gap (β^+, γ^-) in the universal cover, with the point $\beta(\alpha^+)$ which is the unique point of K^*_α that belongs to the gap and the lift of the geodesic laminations in dotted lines, the right figure represents the pant and the geodesic laminations which are the projections in the compact surface of the same objects in the previous picture.

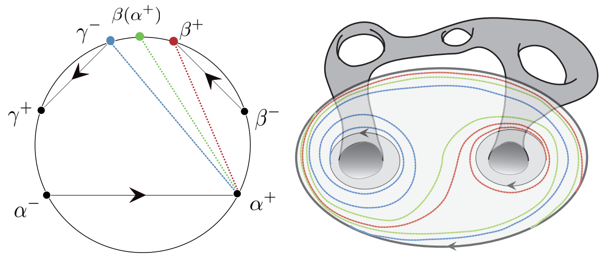

Figure 2: Gap associated to a pair of pants

PROOF: We choose an admissible metric on Σ. Let π be the orthonormal projection from $\partial_\infty\pi_1(\Sigma)_p$ to the boundary component represented by α if α is peripheral, or to an horosphere centred at the corresponding cusp if α is parabolic. Then one checks that x belongs to K_α if and only if the orthogonal geodesic – to either the boundary component or the horosphere –starting from $\pi(x)$ has no self intersection.

Hence, the result follows from Theorem 4.6 and the discussion afterwards in [19] when α is peripheral and the corresponding discussion in [18] when α is parabolic. Q.E.D.

4 Identities

We prove in this section our main results Theorems 4.1.2 and 4.2.4 which state an identity at a boundary component and at a cusp respectively. To treat the cusp case, we shall have to impose an extra regularity assumption on the cross ratio. These identities involve *gap functions* which we describe in preliminary paragraphs.

4.1 The identity at a boundary component

4.1.1 Gap functions

Let $\mathbf{B}$ be an ordered cross ratio on $\partial_\infty\pi_1(\Sigma)_p$. Let $\mathcal{P}_\alpha$ denote the set of (embeddings of) pairs of pants (possibly degenerate) up to isotopy which have α

as first boundary component. Let $\mathcal{S}_\alpha$ denote the set of pairs of pants up to isotopy which have α as first boundary component as well as another boundary component of Σ.

Observe that since we allow isotopies that do not fix α pointwise, the two triples $(\alpha, \beta_0, \gamma_0)$ and $(\alpha, \beta_1, \gamma_1)$ of elements of $\partial_\infty \pi_1(\Sigma)^3$ represent the same class in $\mathcal{P}_\alpha$ if and only if there exists an integer n such that

$$\begin{aligned} \beta_0 &= \alpha^n \beta_1 \alpha^{-n}, \\ \gamma_0 &= \alpha^n \gamma_1 \alpha^{-n}. \end{aligned}$$

Definition 4.1.1 [Gap functions]

- *Let P be a pair of pants in $\mathcal{P}_\alpha$ represented by the triple (α, β, γ). The value of the* pant gap function $G_{\mathbf{B}}$ *at P is*

$$G_{\mathbf{B}}(P) = \log(\mathbf{B}(\alpha^+, \gamma^-, \alpha^-, \beta^+)).$$

- *Let P be a pair of pants in $\mathcal{S}_\alpha$ represented by the triple $(\alpha, \gamma_0, \gamma_1)$ so that $\beta = \gamma_i$ is another peripheral element for $i = 0$ or $i = 1$. The value of the* boundary gap function $G^r_{\mathbf{B}}$ *at P is*

$$G^r_{\mathbf{B}}(P) = \log(\mathbf{B}(\alpha^+, \beta^+, \alpha^-, \beta^-)).$$

Note in the second case that if β is parabolic then $\beta^+ = \beta^-$ and $G^r_{\mathbf{B}}(P) = 0$. Finally, the gap functions are well defined and have positive values: indeed, according to Figure 1, $(\alpha^+, \beta^-, \beta^+, \alpha^-)$ and $(\alpha^+, \beta^+, \gamma^-, \alpha^-)$ are positively oriented hence by Inequality (11), $\mathbf{B}(\alpha^+, \gamma^-, \alpha^-, \beta^+) > 1$ and $\mathbf{B}(\alpha^+, \beta^+, \alpha^-, \beta^-) > 1$.

4.1.2 The main result for a boundary component

We can now state the identity. We use the terminology of the beginning of Section 2.

Theorem 4.1.2 *Let Σ be a connected oriented surface of finite type with boundary components and p cusps. Let α be a primitive peripheral element in $\pi_1(\Sigma)$ which represents an oriented boundary component. Let $\mathbf{B}$ be an ordered cross ratio on $\partial_\infty \pi_1(\Sigma)_p$. Then*

$$\ell_{\mathbf{B}}(\alpha) = \sum_{P \in \mathcal{P}_\alpha} G_{\mathbf{B}}(P) + \sum_{P \in \mathcal{S}_\alpha} G^r_{\mathbf{B}}(P).$$

Remarks:

- Our identities in the Fuchsian case are equivalent to those of Mirzakhani although the way we count contributions from pants is slightly different. Mirzakhani separates the set of embeddings of pants $\mathcal{P}_\alpha$ into those containing two boundary components of Σ ($\mathcal{S}_\alpha$ in our notation) and those that have a single boundary component ($\mathcal{P}_\alpha \setminus \mathcal{S}_\alpha$ in our notation). On the other hand, our first series counts a contribution from every embedding whether it has one or two boundary components and the second is a "correction term" which corresponds to the contribution due to geodesics that "escape the surface" via the second boundary component of an embedding in $\mathcal{S}_\alpha$. This is only a matter of convention and taste.

- The reader may wonder why the pair of pants whose boundary components are cusps do not appear in $\mathcal{S}_\alpha$. The answer is that adding them to $\mathcal{S}_\alpha$ has no effect: these pair of pants have zero relative gaps functions.

The strategy of the proof, as in [18], is to compute the length of a circle (the quotient of $]\alpha^+, \alpha^-[$ by $\langle\alpha\rangle$) in terms of the lengths of the complementary regions, which are gaps, of the set K^*_α.

The identity holds for more general functions than cross ratios. Indeed, we shall not use the first cocycle identity on the first and third arguments, and only use the second cocycle identity on the second and fourth arguments.

Proof: Let $\mathbf{B}$ be a cross ratio and α be a primitive peripheral element of $\pi_1(\Sigma)$ as in the statement. Observe first that because α is peripheral, $\partial_\infty\pi_1(\Sigma)_p \setminus \{\alpha^-, \alpha^+\}$ is an interval $]\alpha^+, \alpha^-[$. Let ζ be a reference point in $]\alpha^+, \alpha^-[$. Observe that for any point y in $]\alpha^+, \alpha^-[$, $\mathbf{B}(\alpha^+, y, \alpha^-, \zeta)$ is positive by Inequality (11) and Equality (9).

Let B be the map from $]\alpha^+, \alpha^-[$ to $\mathbb{R}$ defined by $B(y) = \log(\mathbf{B}(\alpha^+, y, \alpha^-, \zeta))$; Note that, since $\mathbf{B}$ is Hölder, B is Hölder. Note further that $B(K^*_\alpha)$ has zero Hausdorff dimension, since by Theorem 3.2.2, K^*_α has zero Hausdorff dimension. Moreover, B is injective: if $\mathbf{B}(\alpha^+, y, \alpha^-, \zeta) = \mathbf{B}(\alpha^+, y', \alpha^-, \zeta))$, then $\mathbf{B}(\alpha^+, y, \alpha^-, y') = 1$ by (5) and so $y = y'$ by (4). Observe moreover that B preserves the ordering by Inequality (11) – hence is a homeomorphism – so we identify $]\alpha^+, \alpha^-[$ with its image under B.

Let μ be the Lebesgue measure on $\mathbb{R}$. Recall that a set of Hausdorff zero dimension has zero Lebesgue measure. Hence $\mu(K^*_\alpha) = 0$.

By the second cocycle identity

$$B(\alpha(z)) = B(z) - \ell_{\mathbf{B}}(\alpha).$$

Let $\mathbb{T} = \mathbb{R}/\ell_{\mathbf{B}}(\alpha)\mathbb{Z}$, and π be the projection from $\mathbb{R}$ to $\mathbb{T}$. The set K^*_α is also invariant by α and we continue to denote by K^*_α its projection on $\mathbb{T}$. Let $\widehat{I}_P = [\beta^+, \gamma^-] \subset \mathbb{R}$ for each pair of pants P represented by (α, β, γ) and $\widehat{J}_P = [\beta^-, \beta^+]$ for each pair of pants P represented by (α, β, γ) where β is peripheral (see Proposition 3.3.2 and Figure 1).

Observe that by Condition (4) and since $\alpha\widehat{J}_P$ and $\widehat{J}_P$ are disjoint, π is injective from $\widehat{J}_P$ to $J_P = \pi\widehat{J}_P$, similarly from $\widehat{I}_P$ to $I_P = \pi\widehat{I}_P$. Moreover, the set of intervals I_P is in bijection with $\mathcal{P}_\alpha$ and, since we prefer β over γ, there is a two to one map from $\mathcal{S}_\alpha$ to the set of intervals J_P. By Proposition 3.3.2

$$\mathbb{T} \setminus K^*_\alpha = (\sqcup_{P\in\mathcal{P}_\alpha} I_P) \sqcup (\sqcup_{P\in\mathcal{S}_\alpha} J_P).$$

By construction and the second cocycle identity (6.)

- $\mu(I_P) = G_b(P)$, if P belongs to $\mathcal{P}_\alpha$,
- $\mu(J_P) = G_b^r(P)$, if P belongs to S_α.

The identity then follows from

$$\begin{aligned}\ell_{\mathbf{B}}(\alpha) &= \mu(\mathbb{T}) = \mu(\mathbb{T} \setminus K_\alpha^*) \\ &= \sum_{P \in \mathcal{P}_\alpha} \mu(I_P) + \sum_{P \in \mathcal{S}_\alpha} \mu(J_P) = \sum_{P \in \mathcal{P}_\alpha} G_{\mathbf{B}}(P) + \sum_{P \in \mathcal{S}_\alpha} G_b^r(P).\end{aligned}$$

Q.E.D.

4.2 The identity at a cusp

In order to state and prove the identity at a cusp, we make in this section the following further assumption on the cross ratio $\mathbf{B}$.

Definition 4.2.1 [Regularity Hypothesis]
There is an orientation preserving Hölder embedding of $\partial_\infty \pi_1(\Sigma)_p$ in the circle $\mathbb{T}$, such that the action of $\pi_1(\Sigma)$ turns out to be C^1 and that moreover for every pair of distinct points s, t in $\partial_\infty \pi_1(\Sigma)_p$ the function

$$(x, y) \mapsto \mathbf{B}(x, s, y, t),$$

is the restriction of a C^1 function with non zero Hölder derivatives along the diagonal in $\mathbb{T}^2$.

4.2.1 Auxiliary functions at a cusp

Let α be a correctly oriented primitive parabolic element of $\pi_1(\Sigma)$ with fixed point α^+. By *correctly oriented* we mean that the orientation of a loop in the surface representing α coincides with the orientation around the boundary component. We define for any points s, t, s_0 in $\partial_\infty \pi_1(\Sigma)_p \setminus \{\alpha^+\}$

$$W_\alpha(s, t) = \left.\frac{\partial_y \log \mathbf{B}(\alpha^+, s, y, t)}{\partial_y \log \mathbf{B}(\alpha^+, s_0, y, \alpha(s_0))}\right|_{y=\alpha^+}.$$

By the hypothesis above, W_α is well defined and Hölder.

Proposition 4.2.2 *The function $W_\alpha(s, t)$ does not depend on the choice of s_0. Moreover,*

$$\begin{aligned} W_\alpha(\alpha(s), \alpha(t)) &= W(s, t), & (13)\\ W_\alpha(s, \alpha(s)) &= 1, & (14)\\ W_\alpha(s, u) &= W_\alpha(s, t) + W_\alpha(t, u). & (15)\end{aligned}$$

Finally, if (α^+, s, t) is positively oriented, then $W_\alpha(s, t)$ is negative.

Proof: We first remark that since α is parabolic the differential of α at α^+ is the identity. Indeed, maybe after exchanging α and α^{-1}, the triple $(y, \alpha(y), \alpha^+)$ is always oriented, hence the right derivative of α at α^+ is no more than 1, and the left derivative is no less than 1. Therefore

$$\partial_y \log \mathbf{B}(\alpha^+, s, y, t)\big|_{y=\alpha^+} = \partial_y \log \mathbf{B}(\alpha^+, s, \alpha(y), t)\big|_{y=\alpha^+}.$$

This, together with the invariance of $\mathbf{B}$, shows Equality (13).

We define

$$R_x(s, t, s_0, t_0) = \left.\frac{\partial_y \log \mathbf{B}(x, s, y, t)}{\partial_y \log \mathbf{B}(x, s_0, y, t_0)}\right|_{y=x}.$$

Observe that $R_x(s_0, t_0, s_0, t_0) = 1$ and that by the second cocycle identity,

$$R_x(s, u, s_0, t_0) \quad = \quad R_x(s, t, s_0, t_0) + R_x(t, u, s_0, t_0).$$

We claim that $R_{\alpha^+}(s, t, s_0, \alpha(s_0))$ does not depend on the choice of s_0. Indeed, by the second cocyle identity and the invariance under α

$$\begin{aligned}
& \frac{1}{R_{\alpha^+}(s, t, s_0, \alpha(s_0))} - \frac{1}{R_{\alpha^+}(s, t, t_0, \alpha(t_0))} \\
= \quad & \frac{1}{R_{\alpha^+}(s, t, s_0, t_0)} - \frac{1}{R_{\alpha^+}(s, t, \alpha(s_0), \alpha(t_0))} \\
= \quad & \left.\frac{\partial_y \log \mathbf{B}(\alpha^+, s_0, y, t_0)}{\partial_y \log \mathbf{B}(\alpha^+, s, y, t)}\right|_{y=\alpha^+} - \left.\frac{\partial_y \log \mathbf{B}(\alpha^+, \alpha(s_0), y, \alpha(t_0))}{\partial_y \log \mathbf{B}(\alpha^+, s, y, t)}\right|_{y=\alpha^+} \\
= \quad & 0.
\end{aligned}$$

Since $W_\alpha(s, t) = R_{\alpha^+}(s, t, s_0, \alpha(s_0))$, this concludes the proof of the first part of the proposition. The last statement follows from the previous observation. Q.e.d.

4.2.2 Cusp gap function

Let again $\mathcal{P}_\alpha$ denotes the set of pair of pants on Σ which have the cusp of α as a boundary component and $\mathcal{S}_\alpha$ the set of pair of pants which have the cusp of α as a boundary component as well as some other boundary β. As before, we consider pairs of pants up to isotopies that leave α invariant.

Definition 4.2.3 *The* cusp gap function *of the pair of pants P represented by (α, β, γ) in $\pi_1(\Sigma)^3$ is*

$$W_{\mathbf{B}}(P) = W_\alpha(\gamma^-, \beta^+).$$

For P in $\mathcal{S}_\alpha$, the boundary cusp gap function *is*

$$W^r_{\mathbf{B}}(P) = W_\alpha(\beta^+, \beta^-).$$

Since W_α is invariant under the action of α, it follows that $W_{\mathbf{B}}(P)$ and $W^r_{\mathbf{B}}(P)$ only depend on the isotopy class of P.

4.2.3 The identity

Our identity for cusps is

Theorem 4.2.4

$$1 = \sum_{P\in\mathcal{P}_\alpha} W_{\mathbf{B}}(P) + \sum_{P\in\mathcal{S}_\alpha} W^r_{\mathbf{B}}(P) \tag{16}$$

Observe again that pair of pants with another cusp as boundary component have zero relative gap function.

PROOF: The proof is almost exactly the same as that of Theorem 4.1.2. Let α be a correctly oriented primitive parabolic element of $\pi_1(\Sigma)$. Since α is parabolic $\alpha^+ = \alpha^-$. One then defines a map of $\partial_\infty\pi_1(\Sigma)_p \setminus \{\alpha^+\}$ in $\mathbb{R}$ by

$$B(y) = W_\alpha(s, y).$$

where $s \in \partial_\infty\pi_1(\Sigma)_p\setminus\{\alpha^+\}$ is an arbitrary point. Moreover, by the last assertion in Proposition 4.2.2, if (α^+, t, y) is oriented, then

$$B(y) - B(t) = W_\alpha(t, y) < 0.$$

Thus B is monotone and injective.

Noting that $B(\alpha(y)) - B(y) = W_\alpha(y, \alpha(y)) = 1$, we replace $\ell_{\mathbf{B}}(\alpha)$ by 1 in the proof of Theorem 4.1.2 to obtain the equation. Q.E.D.

5 Gaps and coordinates in hyperbolic geometry

In this section, we restrict ourselves to hyperbolic cross ratios and express the gap functions of a pair of pants as a function of the lengths of the boundary components, and more generally of the *shear coordinates* of the pair of pants. We recover the previous results by G. McShane and M. Mirzakhani [19, 18] and do not claim any originality about these results. Nevertheless, the method is new and involves only the formal properties of hyperbolic cross ratios instead of hyperbolic trigonometry as in the original proofs. Moreover, these computations will be helpful later. This approach also emphasises the importance of the notion of cross ratio.

5.1 Gap function for a pair of pants

The main result of this section is the following

Theorem 5.1.1 *Let* $\mathbf{b}$ *be a hyperbolic cross ratio. Let* P *be a pair of pants represented by* (α, β, γ)*. For the boundary gap functions, we assume that* β *is also a peripheral element. Let* $\ell(\alpha)$*,* $\ell(\beta)$ *and* $\ell(\gamma)$ *be the lengths of the corresponding boundary components. If* α *is a correctly oriented primitive peripheral element* *then the gap and boundary gap functions are given by*

$$G_{\mathbf{b}}(P) = \log\left(\frac{e^{\frac{\ell(\beta)+\ell(\gamma)}{2}} + e^{\frac{\ell(\alpha)}{2}}}{e^{\frac{\ell(\beta)+\ell(\gamma)}{2}} + e^{-\frac{\ell(\alpha)}{2}}}\right), \tag{17}$$

$$G^r_{\mathbf{b}}(P) = \log\left(\frac{\cosh\left(\frac{\ell(\gamma)}{2}\right) + \cosh\left(\frac{\ell(\beta)-\ell(\alpha)}{2}\right)}{\cosh\left(\frac{\ell(\gamma)}{2}\right) + \cosh\left(\frac{\ell(\beta)+\ell(\alpha)}{2}\right)}\right). \tag{18}$$

Moreover, assume α is a correctly oriented primitive peripheral element in P. Then, the cusp gap functions are given by

$$W_{\mathbf{b}}(P) = \frac{1}{1+e^{-\frac{\ell(\beta)+\ell(\gamma)}{2}}},$$

$$W^r_{\mathbf{b}}(P) = \frac{\sinh\left(\frac{\ell(\beta)}{2}\right)}{\cosh\left(\frac{\ell(\gamma)}{2}\right) + \cosh\left(\frac{\ell(\beta)}{2}\right)}.$$

We split the theorem in Propositions 5.3.1 and 5.4.1. We prove them using Thurston's shear coordinates [3, 4, 24] that we describe in the next section. We recover this way Formulae (1) and (2) given in the introduction.

The proofs will make a heavy use of the extra identity satisfied by a hyperbolic cross ratio $\mathbf{b}$, namely

$$1 - \mathbf{b}(f, v, e, u) = \mathbf{b}(u, v, e, f), \tag{19}$$

or equivalently

$$\mathbf{b}(x, y, z, t) = 1 - \frac{1}{\mathbf{b}(y, z, x, t)} = \frac{1}{1 - \mathbf{b}(z, x, y, t)}. \tag{20}$$

These rules implies the following symmetry

$$\mathbf{b}(x, y, z, t) = \mathbf{b}(y, x, t, z). \tag{21}$$

5.2 Length functions and shear coordinates

Let P be the sphere minus three points and let (α, β, γ) be a marking given by three elements of $\pi_1(P)$ such that $\alpha\gamma\beta = 1$. We assume that P is equipped with a finite area hyperbolic metric whose completion has p cusps and $3-p$ totally geodesic boundary components. Let $\alpha_0, \beta_0, \gamma_0$ be fixed points of α, β, γ on $\partial_\infty\pi_1(P)_p$ which we consider as a subset of $\mathbb{RP}^1$. We shall denote α_1 the other fixed point of α if α is hyperbolic. If α is parabolic, we set as usual $\alpha_0 = \alpha_1$. Observe that $\alpha^{-1}(\beta_0) = \gamma(\beta_0)$. Let finally $\mathbf{b}$ be the hyperbolic cross ratio.

Definition 5.2.1 [Shear coordinates] *The* shear coordinates *of the hyperbolic pair of pants P are the positive numbers A, B and C defined by*

$$\begin{aligned}
B &= -\mathbf{b}(\alpha_0, \beta_0, \gamma_0, \alpha^{-1}(\beta_0)) = -\mathbf{b}(\alpha_0, \beta_0, \gamma_0, \gamma(\beta_0)),\\
C &= -\mathbf{b}(\beta_0, \gamma_0, \alpha_0, \beta^{-1}(\gamma_0)) = -\mathbf{b}(\beta_0, \gamma_0, \alpha_0, \alpha(\gamma_0)),\\
A &= -\mathbf{b}(\gamma_0, \alpha_0, \beta_0, \gamma^{-1}(\alpha_0)) = -\mathbf{b}(\gamma_0, \alpha_0, \beta_0, \beta(\alpha_0)).
\end{aligned}$$

We observe that these shear coordinates depend on the choice of the fixed points, hence there are 8 choices of shear coordinates. We prove now

Proposition 5.2.2 [LENGTHS AND SHEARS] *Let* $\ell_0(\alpha) = \log \mathbf{b}(\alpha_0, \alpha(z), \alpha_1, z)$. *Then*

$$\begin{aligned} &e^{\ell_0(\alpha)} = BC, \quad e^{\ell_0(\beta)} = AC, \quad e^{\ell_0(\gamma)} = AB, \\ &A = e^{\frac{-\ell_0(\alpha)+\ell_0(\beta)+\ell_0(\gamma)}{2}}, \quad B = e^{\frac{-\ell_0(\beta)+\ell_0(\alpha)+\ell_0(\gamma)}{2}}, \quad C = e^{\frac{-\ell_0(\gamma)+\ell_0(\beta)+\ell_0(\alpha)}{2}}. \end{aligned}$$

More generally if α_0 *and* α_1 *are fixed by* α *and distinct, then for any points* z, t *and* s *in* $\partial_\infty \pi_1(P)_p$ *not fixed by* α, *with* $s \neq t$, *we have*

$$\mathbf{b}(\alpha_0, \alpha(z), \alpha_1, z) = \mathbf{b}(\alpha_0, s, t, \alpha^{-1}(s))\mathbf{b}(s, t, \alpha_0, \alpha(t)). \tag{22}$$

PROOF: It suffices to prove the first equality, the others follow from cyclic permutation. We introduce the following functions of a point x in $\partial_\infty \pi_1(P)_p$

$$\begin{aligned} B(x) &= -\mathbf{b}\left(x, \beta_0, \gamma_0, \alpha^{-1}(\beta_0)\right), \\ C(x) &= -\mathbf{b}\left(\beta_0, \gamma_0, x, \alpha(\gamma_0)\right). \end{aligned}$$

By Relation (20) and (19), we have

$$\left(1 + \frac{1}{B(x)}\right)(1 + C(x)) = \mathbf{b}\left(\alpha(\gamma_0), \gamma_0, x, \beta_0\right) \mathbf{b}\left(\gamma_0, x, \beta_0, \alpha^{-1}(\beta_0)\right).$$

Using the symmetry (21) as well as the second cocycle identity, we get

$$\left(1 + \frac{1}{B(x)}\right)(1 + C(x)) = \mathbf{b}\left(\gamma_0, \alpha(\gamma_0), \beta_0, \alpha^{-1}(\beta_0)\right).$$

Thus the following quantity

$$\left(1 + \frac{1}{B(x)}\right)(1 + C(x))$$

does not depend on x. We observe that, again using the second identity,

$$\begin{aligned} B(\alpha_1) &= e^{-\ell_0(\alpha)} B(\alpha_0) = e^{-\ell_0(\alpha)} B, \\ C(\alpha_1) &= e^{-\ell_0(\alpha)} C(\alpha_0) = e^{-\ell_0(\alpha)} C. \end{aligned}$$

Hence, we have

$$\left(1 + \frac{1}{B}\right)(1 + C) = \left(1 + \frac{e^{\ell_0(\alpha)}}{B}\right)\left(1 + \frac{C}{e^{\ell_0(\alpha)}}\right).$$

Finally, we remark that the equation

$$\left(1 + \frac{1}{B}\right)(1 + C) = \left(1 + \frac{u}{B}\right)\left(1 + \frac{C}{u}\right),$$

is quadratic in u and its two obvious solutions are $u = 1$ and $u = BC$. Equation (22) is just another way to restate this last equality. Q.E.D.

5.3 Gap functions in terms of shear coordinates

We now compute the gap function in terms of the shear coordinates and lengths.

Proposition 5.3.1 *We have the following expression of the pant gap functions*

$$\begin{aligned} G_{\mathbf{b}}(P) &= \log\left(\frac{1+B^{-1}}{1+C}\right) = \log\left(\frac{e^{\frac{\ell(\beta)+\ell(\gamma)}{2}}+e^{\frac{\ell(\alpha)}{2}}}{e^{\frac{\ell(\beta)+\ell(\gamma)}{2}}+e^{-\frac{\ell(\alpha)}{2}}}\right), \\ G^{r}_{\mathbf{b}}(P) &= \log\left(\frac{\cosh\left(\frac{\ell(\gamma)}{2}\right)+\cosh\left(\frac{\ell(\beta)-\ell(\alpha)}{2}\right)}{\cosh\left(\frac{\ell(\gamma)}{2}\right)+\cosh\left(\frac{\ell(\beta)+\ell(\alpha)}{2}\right)}\right). \end{aligned}$$

Proof: It follows from the first cocycle identity and Relation (20) that

$$\mathbf{b}(v,f,u,e) = \frac{\mathbf{b}(w,f,u,e)}{\mathbf{b}(w,f,v,e)} = \frac{1-\mathbf{b}(e,f,u,w)}{1-\mathbf{b}(e,f,v,w)}.$$

Using the first cocycle identity (5), we get

$$\mathbf{b}(v,f,u,e) = \frac{1-\mathbf{b}(e,f,v,w)\,\mathbf{b}(v,f,u,w)}{1-\mathbf{b}(e,f,v,w)}.$$

Hence, we have

$$\begin{aligned} \mathbf{b}(\alpha_0,\gamma_0,\alpha_1,\beta_0) &= \frac{1-\mathbf{b}(\beta_0,\gamma_0,\alpha_0,\alpha(\gamma_0))\,\mathbf{b}(\alpha_0,\gamma_0,\alpha_1,\alpha(\gamma_0))}{1-\mathbf{b}(\beta_0,\gamma_0,\alpha_0,\alpha(\gamma_0))} \\ &= \frac{1+Ce^{-\ell_0(\alpha)}}{1+C} = \frac{1+B^{-1}}{1+C} \\ &= \frac{1+e^{\frac{-\ell_0(\gamma)+\ell_0(\beta)-\ell_0(\alpha)}{2}}}{1+e^{\frac{-\ell_0(\gamma)+\ell_0(\beta)+\ell_0(\alpha)}{2}}} \\ &= \frac{e^{\frac{-\ell_0(\beta)+\ell_0(\gamma)}{2}}+e^{-\frac{\ell_0(\alpha)}{2}}}{e^{\frac{-\ell_0(\beta)+\ell_0(\gamma)}{2}}+e^{\frac{\ell_0(\alpha)}{2}}}. \end{aligned} \tag{23}$$

We obtain our first result by taking $\alpha_0 = \alpha^+$, $\beta_0 = \beta^+$ and $\gamma_0 = \gamma^-$. Indeed, in this case we have

$$\ell_0(\alpha) = -\ell(\alpha), \quad \ell_0(\beta) = -\ell(\beta), \quad \ell_0(\gamma) = \ell(\gamma).$$

When changing β_0 to β_1, we get

$$\mathbf{b}(\alpha_0,\gamma_0,\alpha_1,\beta_1) = \frac{e^{\frac{\ell_0(\beta)+\ell_0(\gamma)}{2}}+e^{\frac{\ell_0(\alpha)}{2}}}{e^{\frac{\ell_0(\beta)+\ell_0(\gamma)}{2}}+e^{-\frac{\ell_0(\alpha)}{2}}}. \tag{24}$$

Hence combining Equations (23) and (24), and using the second cocycle identity, we get

$$\mathbf{b}(\alpha_0,\beta_1,\alpha_1,\beta_0) = \frac{\left(e^{-\frac{\ell_0(\beta)+\ell_0(\gamma)}{2}}+e^{-\frac{\ell_0(\alpha)}{2}}\right)}{\left(e^{\frac{-\ell_0(\beta)+\ell_0(\gamma)}{2}}+e^{\frac{\ell_0(\alpha)}{2}}\right)}\frac{\left(e^{\frac{\ell_0(\beta)+\ell_0(\gamma)}{2}}+e^{\frac{\ell_0(\alpha)}{2}}\right)}{\left(e^{\frac{\ell_0(\beta)+\ell_0(\gamma)}{2}}+e^{-\frac{\ell_0(\alpha)}{2}}\right)}$$

$$
\begin{aligned}
&= \frac{e^{\ell_0(\gamma)}+1+e^{\frac{-\ell_0(\beta)+\ell_0(\gamma)+\ell_0(\alpha)}{2}}+e^{\frac{+\ell_0(\beta)+\ell_0(\gamma)-\ell_0(\alpha)}{2}}}{e^{\ell_0(\gamma)}+1+e^{\frac{-\ell_0(\beta)+\ell_0(\gamma)-\ell_0(\alpha)}{2}}+e^{\frac{\ell_0(\beta)+\ell_0(\gamma)-\ell_0(\alpha)}{2}}} \\
&= \frac{e^{\frac{-\ell_0(\gamma)}{2}}+e^{\frac{-\ell_0(\beta)+\ell_0(\alpha)}{2}}+e^{\frac{\ell_0(\beta)-\ell_0(\alpha)}{2}}+e^{\frac{\ell_0(\gamma)}{2}}}{e^{\frac{\ell_0(\gamma)}{2}}+e^{-\frac{\ell_0(\beta)-\ell_0(\alpha)}{2}}+e^{\frac{\ell_0(\beta)+\ell_0(\alpha)}{2}}+e^{-\frac{\ell_0(\gamma)}{2}}} \\
&= \frac{\cosh\left(\frac{\ell(\gamma)}{2}\right)+\cosh\left(\frac{\ell(\beta)-\ell(\alpha)}{2}\right)}{\cosh\left(\frac{\ell(\gamma)}{2}\right)+\cosh\left(\frac{\ell(\beta)+\ell(\alpha)}{2}\right)}.
\end{aligned}
$$

The proposition follows. Q.E.D.

5.4 Cusp gap functions

We use in this paragraph the notations of Paragraph 4.2.2.

Proposition 5.4.1 *If the hyperbolic pair of pants P has a cusp at α, we have*

$$
W_{\mathbf{b}}(P)=\frac{1}{1+e^{\frac{\ell(\beta)+\ell(\gamma)}{2}}}, \qquad W^r_{\mathbf{b}}(P)=\frac{\sinh\left(\frac{\ell(\beta)}{2}\right)}{\cosh\left(\frac{\ell(\gamma)}{2}\right)+\cosh\left(\frac{\ell(\beta)}{2}\right)}.
$$

Proof: We introduce as before the *shear coordinates* of P associated to some choice of fixed points:

$$
\begin{aligned}
B &= -\mathbf{b}(\alpha^+,\beta^+,\gamma^-,\alpha^{-1}(\beta^+)) = -\mathbf{b}(\alpha^+,\beta^+,\gamma^-,\gamma(\beta^+)),\\
C &= -\mathbf{b}(\beta^+,\gamma^-,\alpha^+,\beta^{-1}(\gamma^-)) = -\mathbf{b}(\beta^+,\gamma^-,\alpha^+,\alpha(\gamma^-)),\\
A &= -\mathbf{b}(\gamma^-,\alpha^+,\beta^+,\gamma^{-1}(\alpha^+)) = -\mathbf{b}(\gamma^-,\alpha^+,\beta^+,\beta(\alpha^+)).
\end{aligned}
$$

We now have

$$
\begin{aligned}
&1=BC,\quad e^{-\ell(\beta)}=AC,\quad e^{\ell(\gamma)}=AB,\\
&A=e^{\frac{-\ell(\beta)+\ell(\gamma)}{2}},\quad B=e^{\frac{\ell(\beta)+\ell(\gamma)}{2}},\quad C=e^{\frac{-\ell(\gamma)-\ell(\beta)}{2}}.
\end{aligned}
$$

By the definition in 4.2.1,

$$
W_{\mathbf{b}}(P)=W_\alpha(\gamma^-,\beta^+).
$$

Recall that

$$
R_x(s,t,s_0,t_0)=\left.\frac{\partial_y\log\mathbf{b}(x,s,y,t)}{\partial_y\log\mathbf{b}(x,s_0,y,t_0)}\right|_{y=x}.
$$

In projective coordinates, we obtain by a direct computation of derivatives that

$$
R_x(s,t,s_0,t_0)=\frac{\frac{1}{x-t}-\frac{1}{x-s}}{\frac{1}{x-t_0}-\frac{1}{x-s_0}}.
$$

Hence

$$
R_x(s,t,s_0,t_0)=\frac{(x-t_0)(x-s_0)(t-s)}{(x-t)(x-s)(t_0-s_0)}=\mathbf{b}(t_0,x,t,s)\mathbf{b}(s_0,x,s,t_0).
$$

Therefore, taking $s_0 = \gamma^-$, yields by definition of W_α that

$$W_\alpha(\gamma^-, \beta^+) = R_{\alpha^+}(\gamma^-, \beta^+, \gamma^-, \alpha(\gamma^-)) = \mathbf{b}(\alpha(\gamma^-), \alpha^+, \beta^+, \gamma^-). \tag{25}$$

Hence

$$\begin{aligned}
W_{\mathbf{b}}(P) &= \frac{1}{\mathbf{b}(\alpha(\gamma^-), \gamma^-, \beta^+, \alpha^+)} \quad \text{by (5)} \\
&= \frac{1}{1 - \mathbf{b}(\alpha^+, \gamma^-, \beta^+, \alpha(\gamma^-))} \text{ by (20)} \\
&= \frac{1}{1 + C^{-1}} \quad \text{by (5)} \\
&= \frac{1}{1 + e^{\frac{\ell(\beta)+\ell(\gamma)}{2}}}.
\end{aligned} \tag{26}$$

This proves the first equality in the proposition.

Similarly, taking β^- instead of β^+, yields

$$W_\alpha(\gamma^-, \beta^-) = \frac{1}{1 + e^{\frac{-\ell(\beta)+\ell(\gamma)}{2}}}. \tag{27}$$

Recall that

$$W^r_{\mathbf{b}}(P) = W_\alpha(\beta^+, \beta^-).$$

The additive cocycle identity (15) yields

$$W^r_{\mathbf{b}}(P) = W_\alpha(\gamma^-, \beta^-) - W_\alpha(\gamma^-, \beta^+).$$

Using Equations (26) and (27), we obtain

$$\begin{aligned}
W^r_{\mathbf{b}}(P) &= \frac{1}{1 + e^{\frac{-\ell(\beta)+\ell(\gamma)}{2}}} - \frac{1}{1 + e^{\frac{\ell(\beta)+\ell(\gamma)}{2}}} \\
&= \frac{\sinh\left(\frac{\ell(\beta)}{2}\right)}{\cosh\left(\frac{\ell(\gamma)}{2}\right) + \cosh\left(\frac{\ell(\beta)}{2}\right)}.
\end{aligned}$$

This last equality concludes the proof. Q.E.D.

6 Fuchsian and Hitchin representations

Let Σ be a compact oriented connected surface with or without boundary. If Σ is closed we assume it has genus at least two; if Σ has a boundary, we assume that its double along its boundary has genus at least two.

Definition 6.0.2 [Fuchsian and Hitchin homomorphisms] *An* n-Fuchsian *homomorphism from* $\pi_1(\Sigma)$ *to* $\mathrm{PSL}(n,\mathbb{R})$ *is a homomorphism* ρ *which factorises as* $\rho = \iota \circ \rho_0$*, where* ρ_0 *is a convex cocompact discrete faithful homomorphism with values in* $\mathrm{PSL}(2,\mathbb{R})$ *and* ι *is an irreducible homomorphism from* $\mathrm{PSL}(2,\mathbb{R})$ *to* $\mathrm{PSL}(n,\mathbb{R})$.

*An n-*Hitchin homomorphism *from $\pi_1(\Sigma)$ to* $\mathrm{PSL}(n,\mathbb{R})$ *is a homomorphism which may be deformed into an n-Fuchsian homomorphism in such a way that the image of each boundary component stays purely loxodromic at each stage of the deformation.*

In the case of a closed surface, these definitions agree with the definitions in [14]. Recall that for a closed surface the image of every non trivial element is purely loxodromic, hence the restriction of a Hitchin representation for a closed surface to a subsurface is again Hitchin.

Let Σ be a compact connected oriented surface with k boundary components $C_1,\ldots,C_k$. Let $A_1,\ldots,A_k$ be conjugacy classes of purely loxodromic elements in $\mathrm{PSL}(n,\mathbb{R})$. We denote by

$$\mathrm{Hom}_H(\pi_1(\Sigma),\mathrm{PSL}(n,\mathbb{R});A_1,\ldots,A_k)$$

the space of Hitchin homomorphisms from $\pi_1(\Sigma)$ to $\mathrm{PSL}(n,\mathbb{R})$ whose holonomy along the boundary component C_j is conjugated to A_j. Observe that $\mathrm{PSL}(n,\mathbb{R})$ acts on this space by conjugation. Let

$$\begin{aligned}&\mathrm{Rep}_H(\pi_1(\Sigma),\mathrm{PSL}(n,\mathbb{R});A_1,\ldots,A_k)\\ =\ &\mathrm{Hom}_H(\pi_1(\Sigma),\mathrm{PSL}(n,\mathbb{R});A_1,\ldots,A_k)/\mathrm{PSL}(n,\mathbb{R}),\end{aligned}$$

be *the moduli space of Hitchin representations* whose holonomies along the boundary component C_j is conjugated to A_j.

7 Positivity

In this section, we recall definitions used by Volodia Fock and Sasha Goncharov in Section 9 of [7].

7.1 Positivity in flag manifolds

Definition 7.1.1 [FLAGS AND BASIS OF FLAGS] *A* flag *F in $\mathbb{R}^n$ is a family $(F^1,\ldots,F^{n-1})$ such that F^k is a k-dimensional vector space and $F^k\subset F^{k+1}$. A* basis for the flag *F is a basis $\{f_i\}$ of $\mathbb{R}^n$ such that F^k is generated by $\{f_1,\ldots,f_k\}$. We denote by $\mathcal{F}(\mathbb{R}^n)$ the space of all flags.*

7.1.1 Triple ratios and positive triple of flags

Let (F,G,H) be a triple of flags in *general position* that is so that for every triple of positive integers (m,l,p) with $m+l+p=n$ the sum $F^m+G^l+H^p$ is direct. Let $\{f_i\}$, $\{g_i\}$, and $\{h_i\}$ be bases respectively for F, G and H. We define if $1\leq p\leq n$

$$\widehat{f}^p=f_1\wedge\ldots\wedge f_p,$$

and by convention $\widehat{f}^0=1$.

Definition 7.1.2 [POSITIVITY AND (m,l,p)-TRIPLE RATIO] *The* (m,l,p)-triple ratio *of* (F,G,H) *is*

$$T^{m,l,p}(G,F,H)=\frac{\Omega(\widehat{f}^{m+1}\wedge\widehat{g}^{l}\wedge\widehat{h}^{p-1})\Omega(\widehat{f}^{m-1}\wedge\widehat{g}^{l+1}\wedge\widehat{h}^{p})\Omega(\widehat{f}^{m}\wedge\widehat{g}^{l-1}\wedge\widehat{h}^{p+1})}{\Omega(\widehat{f}^{m+1}\wedge\widehat{g}^{l-1}\wedge\widehat{h}^{p})\Omega(\widehat{f}^{m}\wedge\widehat{g}^{l+1}\wedge\widehat{h}^{p-1})\Omega(\widehat{f}^{m-1}\wedge\widehat{g}^{l}\wedge\widehat{h}^{p+1})},$$

where Ω *is any volume form on* $\mathbb{R}^n$*.*

The triple (F,G,H) *is* positive *if the* (m,l,p)*-triple ratios are positive for all* (m,l,p)*.*

Finally, the following result holds by the remark after Lemma 9.1 in [7] in dimension 3 and an induction in the dimension for the general case

Proposition 7.1.3 *The collection of functions* $T=(T^{m,l,p})_{m+l+p=n}$ *defines a homeomorphism from the space of positive triple of flags – up to the action of* $\mathrm{PSL}(n,\mathbb{R})$*– to* $(\mathbb{R}^+)^{\frac{(n-1)(n-2)}{2}}$*.*

The reader is at least encouraged to check that the dimensions are the same. This is a special case and, as we just said, a step in the proof of Theorem 9.1 of [7]. For $n=3$, there is just one triple ratio and we have

Proposition 7.1.4 *The* triple ratio *of the three flags* $F_1=(L_1,P_1)$*,* $F_2=(L_2,P_2)$ *and* $F_3=(L_3,P_3)$ *is*

$$T(F_1,F_2,F_3)=\frac{\langle\widehat{L}_1|\widehat{P}_2\rangle\langle\widehat{L}_2|\widehat{P}_3\rangle\langle\widehat{L}_3|\widehat{P}_1\rangle}{\langle\widehat{L}_1|\widehat{P}_3\rangle\langle\widehat{L}_3|\widehat{P}_2\rangle\langle\widehat{L}_2|\widehat{P}_1\rangle}.$$

where $\widehat{L}_i$ *and* $\widehat{P}_i$ *are nonzero vectors in* L_i *and* $P_i^{\perp}$ *respectively.*

REMARK: The reader can check from this proposition or the previous definition that the positivity of a triple is invariant under *all* permutations. Therefore positivity has nothing to do with orientation.

7.1.2 Cross ratios and positive quadruple of flags

We now consider a quadruple of flags $Q=(X,Y,Z,T)$ in $\mathbb{R}^n$. Throughout this paragraph, we suppose that both $Q_1=(X,Y,Z)$ and $Q_2=(T,X,Z)$ are positive triples of flags. We associate $n-1$ numbers to Q which, together with the triple ratios of Q_1 and Q_2, completely determine the configuration Q up to the action of $\mathrm{PSL}(n,\mathbb{R})$.

As before we choose a basis $\{x_1,\ldots,x_n\}$ adapted to the flag X and likewise for Y, Z and T. We say a quadruple (X,Y,Z,T) is in *general position* if the triples (X,Y,Z) and (T,X,Z) are in general position. Following [7],

Definition 7.1.5 [EDGE FUNCTIONS] *The value of the* edge functions δ_i *for* $i=1,\ldots,n-1$ *at a quadruple in general position* (X,Y,Z,T) *is*

$$\delta_i(X,Y,Z,T)=\frac{\Omega(\widehat{x}^{i}\wedge\widehat{z}^{n-i-1}\wedge t_1)\Omega(\widehat{x}^{i-1}\wedge y_1\wedge\widehat{z}^{n-i})}{\Omega(\widehat{x}^{i}\wedge y_1\wedge\widehat{z}^{n-i-1})\Omega(\widehat{x}^{i-1}\wedge\widehat{z}^{n-i}\wedge t_1)},$$

where Ω is any volume form on $\mathbb{R}^n$.

For $n = 2$, it is worth noticing that

$$\delta_1(X, Y, Z, T) = \frac{\Omega(x_1 \wedge t_1)\Omega(y_1 \wedge z_1)}{\Omega(x_1 \wedge y_1)\Omega(z_1 \wedge t_1)} = -\mathbf{b}(X, T, Z, Y). \tag{28}$$

We also observe that

$$\delta_1(U, V, W, R) = \frac{1}{\delta_{n-1}(W, V, U, R)}. \tag{29}$$

We write Δ for the $(n-1)$-tuple of functions $(\delta_1, \ldots, \delta_{n-1})$. Let π be the projection from $\mathbb{R}^n$ to the two-dimensional vector space $P = \mathbb{R}^n/(X^{i-1} \oplus Z^{n-i-1})$. Let $\mathbf{b}_P$ be the cross ratio in this plane. One deduces from Equation (28) that

$$\delta_i(X, Y, Z, T) = -\mathbf{b}_P(\pi(X^i), \pi(T^1), \pi(Z^{n-i})), \pi(Y^1)). \tag{30}$$

Definition 7.1.6 [Positive quadruple] *The quadruple of flags*

$$Q = (X, Y, Z, T),$$

is positive *if it is in general position and if all the edge functions δ_i are positive and if both $Q_1 = (X, Y, Z)$ and $Q_2 = (T, X, Z)$ are positive triples of flags.*

Positivity of quadruples is invariant under cyclic permutations.
An easy induction on dimension yields.

Proposition 7.1.7 *The mapping*

$$Q \longrightarrow (T(Q_1), T(Q_2), \Delta(Q)),$$

is a homeomorphism from the space of positive quadruples of flags (up to the action of the projective special linear group) into $(\mathbb{R}^+)^{(n-1)^2}$.

7.1.3 Positive maps and representations

Following [7], we have

Definition 7.1.8

- *A* positive map *from a cyclically ordered set to the space of flags is a map such that the image of every positively ordered quadruple is a positive quadruple.*
- *A representation from $\pi_1(\Sigma)$ to $\mathrm{PSL}(n, \mathbb{R})$ is* positive *if there exists a positive continuous ρ-equivariant map from $\partial_\infty \pi_1(\Sigma)_p$ to $\mathcal{F}(\mathbb{R}^n)$ for some p.*

One can easily check that the limit curve ξ of a Fuchsian representation is positive. We shall prove later that all Hitchin representations are positive.

8 Frenet curves

In this section, we define Frenet curves and recall results from [15, 14] that link representations of closed surface groups to cross ratios using Frenet curves. We give a simpler expression of the cusp gap function in the case of the cross ratio associated to a Frenet curve. In the last paragraph, we recall the relation between positivity and Frenet curves.

8.1 Frenet curves and cross ratios

Definition 8.1.1 [Frenet curve and osculating flag] *A curve ξ defined from $\mathbb{T}$ to $\mathbb{P}(\mathbb{R}^n)$ is a* Frenet curve *if there exists a curve $(\xi^1, \xi^2, \ldots, \xi^{n-1})$ defined on $\mathbb{T}$, called the* osculating flag curve*, with values in the flag variety such that for every x in $\mathbb{T}$, $\xi(x) = \xi^1(x)$, and moreover*

- *For every pairwise distinct points $(x_1, \ldots, x_l)$ in $\mathbb{T}$ and positive integers $(n_1, \ldots, n_l)$ such that $\sum_{i=1}^{i=l} n_i \leq n$, then the following sum is direct*

$$\xi^{n_i}(x_i) + \ldots + \xi^{n_l}(x_l). \tag{31}$$

- *For every x in $\mathbb{T}$ and positive integers $(n_1, \ldots, n_l)$ such that $p = \sum_{i=1}^{i=l} n_i \leq n$, then*

$$\lim_{\substack{(y_1,\ldots,y_l)\to x,\\ y_i \, all \ distinct}} \left(\bigoplus_{i=1}^{i=l} \xi^{n_i}(y_i)\right) = \xi^p(x). \tag{32}$$

We call ξ^{n-1} the osculating hyperplane.

By Condition (32), the image of a Frenet curve is a C^1-submanifold and the tangent line to $\xi^1(x)$ is $\xi^2(x)$. Moreover, a Frenet curve ξ is *hyperconvex* in the following sense: for any n-tuple of pairwise distinct points $(x_1, \ldots, x_n)$ we have

$$\xi(x_1) + \xi(x_2) + \ldots + \xi(x_n) = \mathbb{R}^n.$$

Definition 8.1.2 [Associated weak cross ratio] *Let ξ be a Frenet curve and ξ^* be its associated osculating hyperplane curve. The* weak cross ratio *associated to this pair of curves is the function on $\mathbb{T}^{4*}$ defined by*

$$\mathbf{B}_{\xi,\xi^*}(x, y, z, t) = \frac{\langle\widehat{\xi}(x)|\widehat{\xi}^*(y)\rangle\langle\widehat{\xi}(z)|\widehat{\xi}^*(t)\rangle}{\langle\widehat{\xi}(z)|\widehat{\xi}^*(y)\rangle\langle\widehat{\xi}(x)|\widehat{\xi}^*(t)\rangle},$$

where for every u, we choose an arbitrary nonzero vector $\widehat{\xi}(u)$ and $\widehat{\xi}^*(u)$ in respectively $\xi(u)$ and $\xi^*(u)$.

Observe that the weak cross ratio associated to a Frenet curve is not necessarily a cross ratio. However, we shall see that it is so for Frenet curves that we shall be interested in.

Remarks:

1. Let $V = \xi(x) \oplus \xi(z)$. Let $\eta(m) = \xi^*(m) \cap V$. Let $\mathbf{b}_V$ be the classical cross ratio on $\mathbb{P}(V)$, then

$$\mathbf{B}_{\xi,\xi^*}(x,y,z,t) = \mathbf{b}_V(\xi(x), \eta(y), \xi(z), \eta(t)). \tag{33}$$

2. Symmetrically, let $H = \xi^*(y) \cap \xi^*(t)$, let π be the projection on $P = \mathbb{R}^n/H$, let $\mathbf{b}_P$ be the classical cross ratio on $\mathbb{P}(P)$, then

$$\mathbf{B}_{\xi,\xi^*}(x,y,z,t) = \mathbf{b}_P(\pi(\xi(x)), \pi(\xi^*)(y)), \pi(\xi(z)), \pi(\xi^*(t))). \tag{34}$$

8.2 Frenet curves and representations

The following summarises some of the results of [14, 15].

Theorem 8.2.1 [F. Labourie] *Let ρ be an n-Hitchin representation of the fundamental group of a closed connected oriented surface of genus at least two. Then, there exists a ρ-equivariant Frenet curve from $\partial_\infty \pi_1(\Sigma)$ to $\mathbb{P}(\mathbb{R}^n)$. Moreover, the osculating flag curve is Hölder.*

Finally, the weak cross ratio $\mathbf{B}_\rho = \mathbf{B}_{\xi,\xi^*}$ is a cross ratio – called the cross ratio associated to the Hitchin representation ρ *– and the period of γ is given by*

$$\ell_{\mathbf{B}}(\gamma) = \log\left(\left|\frac{\lambda_{\max}(\rho(\gamma))}{\lambda_{\min}(\rho(\gamma))}\right|\right),$$

where $\lambda_{\max}(\rho(\gamma))$ and $\lambda_{\min}(\rho(\gamma))$ are the eigenvalues of respectively maximum and minimum absolute values of the element $\rho(\gamma)$.

The first main result in [15] characterises which cross ratios appear in this theorem. In Corollary 9.2.6, we will show that every Hitchin representation ρ for a surface Σ with boundary is the restriction of a Hitchin representation, called the Hitchin double, for the double surface. It follows that we can also associate to such a representation a cross ratio: the restriction of the cross ratio associated to the Hitchin double by Theorem 1.0.2. Finally, the following observation follows from the construction and from the density of attracting points of elements of the fundamental group.

Proposition 8.2.2 *Let S be a closed surface. Let Σ be an incompressible connected surface embedded in S. Let ρ_0 and ρ_1 be two Hitchin representations of $\pi_1(S)$ whose restrictions to $\pi_1(\Sigma)$ coincide.*

Let $\mathbf{B}_{\rho_0}$ and $\mathbf{B}_{\rho_1}$ be the associated cross ratios on $\partial_\infty \pi_1(S)$. Then $\mathbf{B}_{\rho_0}$ and $\mathbf{B}_{\rho_1}$ coincide on $\partial_\infty \pi_1(\Sigma)$.

In particular, the gap functions for a pair of pants defined by the triple (α, β, γ) of elements of $\pi_1(\Sigma)$ and a cross ratio associated to a Hitchin representation ρ only depend on the triple $(\rho(\alpha), \rho(\beta), \rho(\gamma))$ of elements of $\mathrm{PSL}(n, \mathbb{R})$.

8.3 Frenet curves and cusps

In this paragraph, we compute the cusp gap function whenever the cross ratio arises from a Frenet curve. For the sake of simplicity, we consider a surface Σ with only one cusp. Let ρ be a representation from $\pi_1(\Sigma)$ to $\mathrm{PSL}(\mathbb{R}^n)$.

We assume that there exists a ρ-equivariant Frenet curve ξ from $\partial_\infty\pi_1(\Sigma)_1$ to $\mathbb{P}(\mathbb{R}^n)$ with Hölder osculating flag curve. .

Note that we do not know if this is the case in general except for Fuchsian representation, since the results of the first author do not cover the cusp case.

Let $\mathbf{B}_\rho = \mathbf{B}_{\xi,\xi^*}$ be the associated cross ratio on $\partial_\infty\pi_1(\Sigma)_1$. Since ξ is Frenet, there exists a C^1 structure on $\partial_\infty\pi_1(\Sigma)_1$ such that the map $(x,y)\mapsto \mathbf{B}_\rho(x,s,y,t)$, is C^1 with Hölder derivatives along the diagonal. In other words, the associated cross ratio satisfies the regularity hypothesis 4.2.1.

We evaluate in this paragraph the quantity $W_\alpha(s,t)$ defined in Paragraph 4.2. Let $F=(L,P)$ be a two-flag in a vector space, that is a pair (L,P) where L is a line included in a two-plane P. Let S,T,S_0,T_0 be four hyperplanes. We denote by $\mathbf{B}$ the classical cross ratio in $\mathbb{P}(P)$. We also denote hyperplanes by uppercase letters and their intersection with P by the corresponding gothic lowercase so that $\mathfrak{h}=H\cap P$. Then we define for generic hyperplanes,

$$\widehat{W}_F(S,T,S_0,T_0)=\mathbf{B}(\mathfrak{t},\mathfrak{s},L,\mathfrak{s}_0)\mathbf{B}(\mathfrak{t}_0,L,\mathfrak{t},\mathfrak{s}_0).$$

We now prove

Theorem 8.3.1 *Let $(\xi^1,\xi^2,\ldots,\xi^{n-1})$ be the osculating flag curve of ξ. Let $F(\alpha)=(\xi^1(\alpha^+),\xi^2(\alpha^+))$. Then*

$$W_\alpha(s,t)=\widehat{W}_{F(\alpha)}\left(\xi^{n-1}(s),\xi^{n-1}(t),\xi^{n-1}(s_0),\xi^{n-1}(\alpha(s_0))\right).$$

Moreover, when $n=2$,

$$W_\alpha(s,t)=\mathbf{B}_\rho(\alpha(s),\alpha^+,t,s). \tag{35}$$

Let us make the following remarks when $n=2$

- One recover directly from Formula (35) that when $n=2$

  $$W_\alpha(s,t)=W_\alpha(s,u)+W_\alpha(u,t).$$

  Indeed, we choose projective coordinates so that $\alpha^+=+\infty$. In this case α is the translation by a constant τ and

  $$W_\alpha(s,t)=\frac{t-s}{\tau}.$$

- Formula (35) coincides with Formula (25) obtained through a direct computation.

8.3.1 A preliminary proposition

We introduce some notations and definitions. Let u_1, u_2 be lines in E and P_1, P_2 be hyperplanes, we define

$$\mathbf{b}(u_1, P_1, u_2, P_2) = \frac{\langle \widehat{u}_1 | \widehat{P}_1 \rangle \langle \widehat{u}_2 | \widehat{P}_2 \rangle}{\langle \widehat{u}_1 | \widehat{P}_2 \rangle \langle \widehat{u}_2 | \widehat{P}_1 \rangle},$$

where $\widehat{u}_i$ and $\widehat{P}_i$ are nonzero vectors in u_i and $P_i^{\perp}$ respectively.

Now let (L, P) be a two-flag, let $\underline{L}$ be a line so that P is generated by L and $\underline{L}$.

Proposition 8.3.2 *We have*

$$\widehat{W}_F(S, T, S_0, T_0) \quad = \quad \frac{\mathbf{b}(L, S_0, \underline{L}, T) - \mathbf{b}(L, S_0, \underline{L}, S)}{\mathbf{b}(L, S_0, \underline{L}, T_0) - 1}. \tag{36}$$

Proof: We denote by A the right hand term. Let $\mathbf{B}$ be the cross ratio in $\mathbb{P}(P)$. We again denote hyperplanes by uppercase letters and their intersection with P by the corresponding gothic lowercase so that $\mathfrak{h} = H \cap P$. Since L and $\underline{L}$ generate P, for all hyperplanes U and V, we have similarly to Equation (33)

$$\mathbf{b}(L, U, \underline{L}, V) = \mathbf{B}(L, \mathfrak{u}, \underline{L}, \mathfrak{v}).$$

By applying first Relation (19), then the first cocycle identity, we get

$$1 - \mathbf{b}(L, S_0, \underline{L}, T_0) = \mathbf{B}(\mathfrak{t}_0, \mathfrak{s}_0, \underline{L}, L) = \mathbf{B}(\mathfrak{t}_0, \mathfrak{s}_0, \mathfrak{t}, L)\mathbf{B}(\mathfrak{t}, \mathfrak{s}_0, \underline{L}, L).$$

Moreover, using first the second cocycle identity then Relation (19), we get

$$\begin{aligned} \mathbf{B}(L, \mathfrak{s}_0, \underline{L}, \mathfrak{s}) - \mathbf{B}(L, \mathfrak{s}_0, \underline{L}, \mathfrak{t}) &= \mathbf{B}(L, \mathfrak{s}_0, \underline{L}, \mathfrak{s}) \left(1 - \mathbf{B}(L, \mathfrak{s}, \underline{L}, \mathfrak{t})\right) \\ &= \mathbf{B}(L, \mathfrak{s}_0, \underline{L}, \mathfrak{s}) \mathbf{B}(\mathfrak{t}, \mathfrak{s}, \underline{L}, L). \end{aligned}$$

Hence,

$$A = \left(\frac{\mathbf{B}(L, \mathfrak{s}_0, \underline{L}, \mathfrak{s})}{\mathbf{B}(\mathfrak{t}_0, \mathfrak{s}_0, t, L)} \right) \left(\frac{\mathbf{B}(\mathfrak{t}, \mathfrak{s}, \underline{L}, L)}{\mathbf{B}(\mathfrak{t}, \mathfrak{s}_0, \underline{L}, L)} \right).$$

Applying the second cocycle identity to the second factor, we obtain

$$A = \frac{\mathbf{B}(L, \mathfrak{s}_0, \underline{L}, \mathfrak{s})}{\mathbf{B}(\mathfrak{t}, \mathfrak{s}_0, \underline{L}, \mathfrak{s}) \mathbf{B}(\mathfrak{t}_0, \mathfrak{s}_0, \mathfrak{t}, L)} = \mathbf{B}(\mathfrak{t}, \mathfrak{s}, L, \mathfrak{s}_0) \mathbf{B}(\mathfrak{t}_0, L, \mathfrak{t}, \mathfrak{s}_0).$$

The result follows Q.E.D.

8.3.2 Proof of the theorem

With $t_0 = \alpha(s_0)$, $x = \alpha^+$, let $S = \xi^{n-1}(s)$, $S_0 = \xi^{n-1}(s_0)$, $T = \xi^{n-1}(t)$, $T_0 = \xi^{n-1}(t_0)$, $L = \xi(x)$, $L + \underline{L} = \xi^2(x)$.

Let also $\widehat{y}$ be a nonzero vector of $\xi(y)$, z be a nonzero vector of $\underline{L}$, $(\widehat{s}_0, \widehat{t}_0, \widehat{s}, \widehat{t})$ be linear forms whose kernels are respectively (S_0, T_0, S, T). Then, by Equation (4),

$$R_x(s,t,s_0,t_0) = \left.\frac{\partial_y \log \mathbf{B}_\rho(x,s,y,t)}{\partial_y \log \mathbf{B}_\rho(x,s_0,y,t_0)}\right|_{y=x} = \lim_{y\to x} \frac{\log(\mathbf{B}_\rho(x,s,y,t))}{\log(\mathbf{B}_\rho(x,s_0,y,t_0))}.$$

Using that $\lim_{s\to 1} \frac{\log s}{1-s} = 1$, we get

$$\begin{aligned}
R_x(s,t,s_0,t_0) &= \lim_{y\to x} \frac{1-\mathbf{B}_\rho(x,s,y,t)}{1-\mathbf{B}_\rho(x,s_0,y,t_0)} \\
&= \lim_{y\to x} \frac{1 - \frac{\langle \widehat{x}|\widehat{s}\rangle}{\langle \widehat{x}|\widehat{t}\rangle}\frac{\langle \widehat{y}|\widehat{t}\rangle}{\langle \widehat{y}|\widehat{s}\rangle}}{1 - \frac{\langle \widehat{x}|\widehat{s}_0\rangle}{\langle \widehat{x}|\widehat{t}_0\rangle}\frac{\langle \widehat{y}|\widehat{t}_0\rangle}{\langle \widehat{y}|\widehat{s}_0\rangle}} \\
&= \lim_{y\to x} \frac{\langle \widehat{x}|\widehat{t}_0\rangle}{\langle \widehat{x}|\widehat{t}\rangle}\frac{\langle \widehat{y}|\widehat{s}_0\rangle}{\langle \widehat{y}|\widehat{s}\rangle}\left(\frac{\langle \widehat{x}|\widehat{t}\rangle\langle \widehat{y}|\widehat{s}\rangle - \langle \widehat{x}|\widehat{s}\rangle\langle \widehat{y}|\widehat{t}\rangle}{\langle \widehat{x}|\widehat{t}_0\rangle\langle \widehat{y}|\widehat{s}_0\rangle - \langle \widehat{x}|\widehat{s}_0\rangle\langle \widehat{y}|\widehat{t}_0\rangle}\right) \\
&= \frac{\langle \widehat{x}|\widehat{t}_0\rangle}{\langle \widehat{x}|\widehat{t}\rangle}\frac{\langle \widehat{x}|\widehat{s}_0\rangle}{\langle \widehat{x}|\widehat{s}\rangle}\left(\frac{\langle \widehat{x}|\widehat{t}\rangle\langle z|\widehat{s}\rangle - \langle \widehat{x}|\widehat{s}\rangle\langle z|\widehat{t}\rangle}{\langle \widehat{x}|\widehat{t}_0\rangle\langle z|\widehat{s}_0\rangle - \langle \widehat{x}|\widehat{s}_0\rangle\langle z|\widehat{t}_0\rangle}\right) \\
&= \frac{\langle \widehat{x}|\widehat{t}_0\rangle}{\langle \widehat{x}|\widehat{t}\rangle}\frac{\langle z|\widehat{t}\rangle}{\langle z|\widehat{t}_0\rangle}\left(\frac{\frac{\langle \widehat{x}|\widehat{t}\rangle}{\langle z|\widehat{t}\rangle}\frac{\langle z|\widehat{s}\rangle}{\langle \widehat{x}|\widehat{s}\rangle} - 1}{\frac{\langle \widehat{x}|\widehat{t}_0\rangle}{\langle \widehat{x}|\widehat{s}_0\rangle}\frac{\langle z|\widehat{s}_0\rangle}{\langle \widehat{x}|\widehat{s}_0\rangle} - 1}\right).
\end{aligned}$$

Using the definition of $\mathbf{b}$, we get

$$R_x(s,t,s_0,t_0) = \mathbf{b}(L,T_0,\underline{L},T)\frac{1-\mathbf{b}(L,T,\underline{L},S)}{1-\mathbf{b}(L,T_0,\underline{L},S_0)}.$$

Applying the second cocycle identity thrice yields

$$\begin{aligned}
R_x(s,t,s_0,t_0) &= \frac{\mathbf{b}(L,S_0,\underline{L},T)(1-\mathbf{b}(L,T,\underline{L},S))}{\mathbf{b}(L,S_0,\underline{L},T_0)(1-\mathbf{b}(L,T_0,\underline{L},S_0))} \\
&= \frac{\mathbf{b}(L,S_0,\underline{L},T) - \mathbf{b}(L,S_0,\underline{L},S)}{\mathbf{b}(L,S_0,\underline{L},T_0)-1}.
\end{aligned}$$

The first part of the result follows from the previous proposition. We obtain the result for $n = 2$, by taking $s = s_0$. Q.E.D.

8.4 Frenet curves and positivity

We state the elementary Lemma 8.4.2 which shows that the osculating flag curve to a Frenet curve is positive.

Definition 8.4.1 [Compatible flags] *Two flags F_1, F_2 are* transverse *if the sum $F_1^k + F_2^{n-k}$ is direct for all $1 \leq k \leq n$. If F_1 and F_2 are transverse flags, we* *say a flag F is* compatible *with F_1 with respect to F_2 – or in short with (F_1, F_2) – if there exist an integer p, such that*

$$\begin{aligned} k \leq p \quad &\implies \quad F^k = F_1^k, \\ k > p \quad &\implies \quad F^k = F_1^p \oplus F_2^{k-p}. \end{aligned}$$

The following lemma will follow easily from the definitions

Lemma 8.4.2 [Frenet curve and positivity] *Let $\xi\colon\ \mathbb{T} \to \mathbb{P}(\mathbb{R}^n)$ be a Frenet curve. Then its osculating flag curve $\widehat{\xi}$ is positive. More generally*

- *Let $K \subset \mathbb{T}$ be a closed set.*
- *Let $G = \bigcup_{n\in\mathbb{N}}\{x_i^+, x_i^-\}$ be a collection of points of K.*
- *Let $K_0 = K \setminus G$.*

Assume that, for every i, K_0 lies in one of the connected component of $\mathbb{T} \setminus \{x_i^+, x_i^-\}$. For each i, let F_i be a compatible flag with $(\widehat{\xi}(x_i^+), \widehat{\xi}(x_i^-))$. Then the following map is positive :

$$\left\{ \begin{array}{rcl} K_0 \cup_{n\in\mathbb{N}} \{x_i^+\} & \to & \mathcal{F}, \\ x_i^+ & \mapsto & F_i, \\ x \in K_0 & \mapsto & \widehat{\xi}(x). \end{array} \right.$$

Remark:

- This lemma is an immediate consequence of Proposition 13.2.2 which we prove in an Appendix.
- Conversely, we may ask whether a positive curve is always the osculating flag curve of a Frenet curve. This is indeed the case if the positive curve is smooth enough (see Theorem 1.8 of Fock and Goncharov in [7]). However, this does not imply directly that a positive representation preserves a Frenet curve since the osculating flag curve is only Hölder unless the representation is Fuchsian.
- The related discussion in Section 9 of [7] covers the case $K = \mathbb{T}$ and this general case could also be deduced from general discussion about positivity as in G. Lusztig work [17].

9 Hitchin representations for open surfaces

The aim of this section is to extend Theorem 8.2.1 to representations of the fundamental group of surfaces with boundary and to prove the following

Theorem 9.0.3 *Let Σ be a compact connected oriented surface with or without boundary components, but without cusps. Let ρ be an n-Hitchin representation of $\pi_1(\Sigma)$. Then there exists a positive ρ-equivariant Hölder map from $\partial_\infty\pi_1(\Sigma)$ to the flag manifold, which is furthermore the restriction of the osculating flag curve of a Frenet curve. Moreover, for all nontrivial elements γ in $\pi_1(\Sigma)$, $\rho(\gamma)$ is purely loxodromic.*

In particular, every Hitchin representation is a positive representation. Furthermore, let S be an incompressible surface embedded in Σ. Then ρ restricted to $\pi_1(S)$ is a positive representation.

Finally, there exists an ordered cross ratio $\mathbf{B}$ on $\partial_\infty\pi_1(\Sigma)$, such that the period of any non trivial element γ satisfies

$$\ell_{\mathbf{B}}(\gamma) = \log\left(\left|\frac{\lambda_{\max}(\rho(\gamma))}{\lambda_{\min}(\rho(\gamma))}\right|\right),$$

where $\lambda_{\max}(\rho(\gamma))$ and $\lambda_{\min}(\rho(\gamma))$ are the eigenvalues of respectively maximum and minimum absolute values of the element $\rho(\gamma)$.

Observe that, as a corollary, this result yields better information about positives curves appearing in [7] for positive representations: these positive curves are Hölder and are the restriction of the osculating flag curve of a Frenet curve.

Theorem 9.0.3 is a consequence of the "doubling" Corollary 9.2.6 of Theorem 8.2.1 which shows that every Hitchin representation is the restriction of a Hitchin representation of the fundamental group of a closed surface.

We prove Theorem 9.0.3 in Paragraph 9.2.3. The main part of this section is devoted to the doubling construction and the proof of Theorem 8.2.1.

9.1 Homomorphisms and boundary components

9.1.1 Good homomorphisms

Let Σ be a compact surface with boundary. Every boundary component is supposed to be oriented and we abusively identify each boundary component with a loop.

We introduce technical definitions whose purpose is to define "better" homomorphisms in a given conjugacy class by fixing some boundary data.

Definition 9.1.1 [Good homomorphism] *A* good homomorphism *based at a point v of a boundary component ∂_v is a homomorphism $\mathbf{R}$ from $\pi_1(\Sigma, v)$ into* $\mathrm{PSL}(n,\mathbb{R})$*, so that*

- *The matrix $\mathbf{R}(\partial_v)$ is diagonal with decreasing entries.*
- *The image of all boundary components are purely loxodromic.*

The purpose of this definition is to study Hitchin representations and we say

Definition 9.1.2 [GOOD REPRESENTATIVE] *A* good representative*, based at a point v of a boundary component ∂_v, of a Hitchin representation ρ is a good homomorphism* $\mathbf{R}$*, based at v, from $\pi_1(\Sigma, v)$ into* $\mathrm{PSL}(n,\mathbb{R})$*, which belongs to the conjugacy class defined by ρ.*

By the definition, the image of every boundary component by a Hitchin representation is purely loxodromic. It follows that every Hitchin representation admits a good representative.

9.1.2 Changing boundary components

Given ρ and v, a good representative is uniquely defined up to conjugation by a diagonal matrix.

The next proposition explains how good representatives based at different boundary components are related.

Let $\mathcal{D}$ be the group of diagonal matrices. We denote by $*$ the composition of paths, so that $a * b$ is the path a followed by b. Let $\mathcal{B}_v$ be the set of classes of paths joining v to a boundary component up to the following equivalence: the path a joining v to a boundary component ∂_w is equivalent to a path b joining v to ∂_w if there exists a path d along ∂_w so that $a * d$ is homotopic to b.

We now prove

Proposition 9.1.3 *Let* $\mathbf{R}$ *be a good homomorphism. Then there exists a unique map*

$$\underline{\mathbf{R}} : \mathcal{B}_v \times \mathcal{D} \to \mathrm{PSL}(n, \mathbb{R}),$$

such that the following properties hold

- *For every path c, there exists an element K_c of* $\mathrm{PSL}(n,\mathbb{R})$ *so that for any diagonal matrix Δ,*
$$\underline{\mathbf{R}}(c, \Delta) = \mathrm{K}_c \Delta \mathrm{K}_c^{-1}.$$
- *Let c be an arc joining v to a point w in another boundary component ∂_w of S. Let* $\mathbf{F}$ *be any good homomorphism based at w conjugated to* $\mathbf{R}$*, then*
$$\mathbf{R}\left(c * \partial_w * c^{-1}\right) = \underline{\mathbf{R}}\left(c, \mathbf{F}\left(\partial_w\right)\right). \tag{37}$$

Note that, fixing c, the map $\Delta \mapsto \underline{\mathbf{R}}(c, \Delta)$ is a conjugation between two maximal split tori of $\mathrm{PSL}(n, \mathbb{R})$.

PROOF: Let c be an element of $\mathcal{B}_v$ that is a path joining v to a point w in a boundary component ∂_w of S. We observe that

$$\mathbf{R}\left(c * \partial_w * c^{-1}\right) \quad = \quad \mathrm{K}\Delta_w \mathrm{K}^{-1},$$

where Δ_w is a diagonal matrix with decreasing entries and K is an element of $\mathrm{PSL}(n, \mathbb{R})$ well defined up to multiplication by a diagonal matrix on the right. It follows that for every diagonal matrix Δ

$$\underline{\mathbf{R}}(c, \Delta) := \mathrm{K}\Delta\mathrm{K}^{-1},$$

depends only on the equivalence class of c and Δ. We observe that this map $\underline{\mathbf{R}}$ satisfies the properties of the proposition and is characterised by them. Q.E.D.

The following proposition summarises the properties of the map that we have just constructed.

Proposition 9.1.4 *With the notation of the previous proposition, the map $\underline{\mathbf{R}}$ enjoys the following properties.*

1. *For all loop γ based at v, all path c and all diagonal matrix Δ*

$$\underline{\mathbf{R}}\left(\gamma * c, \Delta\right) = \mathbf{R}\left(\gamma\right) \underline{\mathbf{R}}\left(c, \Delta\right) \mathbf{R}\left(\gamma\right)^{-1}. \tag{38}$$

2. *Fixing a path c and a diagonal matrix Δ, the map $\mathbf{R} \mapsto \underline{\mathbf{R}}(c, \Delta)$ is a continuous map from the space of good homomorphims to* $\mathrm{PSL}(n, \mathbb{R})$.

3. *If Δ is a diagonal matrix*

$$\begin{aligned} \underline{\Delta \mathbf{R} \Delta^{-1}} &= \Delta \underline{\mathbf{R}} \Delta^{-1}, & (39)\\ \underline{\mathbf{R}}\left(c, \Delta\right)^{-1} &= \underline{\mathbf{R}}\left(c, \Delta^{-1}\right). & (40) \end{aligned}$$

PROOF: We fix a path c joining v to a point w in a boundary component ∂_w as well as a good homomorphism $\mathbf{F}$ based at w and conjugated to $\mathbf{R}$.

One then checks that Equation (38) holds for $\Delta = \mathbf{F}(\partial_w)$ hence for all diagonal matrices Δ by the construction of $\underline{\mathbf{R}}$. The continuity follows a similar argument. The last properties follow easily from the construction. Q.E.D.

9.2 The doubling construction

We present the main doubling construction. Let Σ be a compact connected orientable surface with non empty boundary. Let $\widehat{\Sigma}$ be its double. Let j_0 and j_1 be the two injections of Σ in $\widehat{\Sigma}$, and $j : x \mapsto \bar{x}$ be the involution of $\widehat{\Sigma}$ fixing all points in the boundary of Σ such that $j \circ j_0 = j_1$.

Let $v \in \partial\Sigma$ and $\widehat{v} = j(v)$. We denote $\pi_1(\Sigma) = \pi_1(\Sigma, v)$ and $\pi_1(\widehat{\Sigma}) = \pi_1(\widehat{\Sigma}, \widehat{v})$.

9.2.1 Extensions of good homomorphisms

We first define

Definition 9.2.1 *Let* J *be an involution of* $\mathrm{PGL}(n, \mathbb{R})$ *commuting with all diagonal matrices. Let $\mathbf{R}$ be a good homomorphism from $\pi_1(\Sigma)$ into* $\mathrm{PSL}(n, \mathbb{R})$. *A* J-extension *of $\mathbf{R}$ is a homomorphism $\widehat{\mathbf{R}}$ from $\pi_1(\widehat{\Sigma})$ into* $\mathrm{PSL}(n, \mathbb{R})$ *so that*

1. *We have*

$$\widehat{\mathbf{R}} \circ (j_0)_* = \mathbf{R}. \tag{41}$$

2. *For all element γ in $\pi_1(\widehat{\Sigma})$, we have*
$$\widehat{\mathbf{R}}\left(\bar{\gamma}\right) = \mathrm{J}\widehat{\mathbf{R}}\left(\gamma\right)\mathrm{J}. \tag{42}$$
3. *For all arc c in Σ joining v to a boundary component of Σ, we have*
$$\widehat{\mathbf{R}}\left(c * \bar{c}^{-1}\right) = \underline{\mathbf{R}}\left(c, \mathrm{J}\right)\mathrm{J}, \tag{43}$$
where $\underline{\mathbf{R}}$ is defined in Proposition 9.1.3.

The following proposition shows that we have a uniquely defined extension in the case of good homomorphims.

Proposition 9.2.2 *Let Σ be a compact connected orientable surface with non empty boundary. Let $\widehat{\Sigma}$ be its double. Let J be an involution of $\mathrm{PGL}(n,\mathbb{R})$ commuting with all diagonal matrices. Then any good homomorphism admits a unique J-extension.*

Furthermore the map which associates to a good homomorphism its J-extension is continuous.

Proof: The uniqueness follows from the fact that $\pi_1(\widehat{\Sigma})$ is generated by the groups $(j_0)_*\pi_1\left(\Sigma\right)$ and $(j_1)_*\pi_1\left(\Sigma\right)$ as well as the homotopy classes of loops of the form $c * \bar{c}^{-1}$ where c describes the set of arcs joining v to another boundary component.

We now prove the existence of the J-extension. We first recall the description of the fundamental group of the double in combinatorial terms. Let $\partial_0, \ldots, \partial_m$ be the boundary components of Σ such that v belongs to ∂_0. Let $c_1, \ldots, c_m$ be arcs joining v to the boundary components $\partial_1, \ldots, \partial_m$. Let ∂_i^c be the elements of $\pi_1\left(\Sigma\right)$ given by
$$\partial_i^c = c_i * \partial_i * c_i^{-1}.$$
Let F_m be the free group on m generators $x_1, \ldots, x_m$. Let
$$G = \pi_1\left(\Sigma\right) \star \pi_1\left(\Sigma\right) \star F_m$$
where $A \star B$ denotes the free product of A and B. Let i_0, and i_1 be the injections of $\pi_1\left(\Sigma\right)$ in G given by the first and second factor respectively. Let H be the group normally generated by $i_0\left(\partial_0\right)^{-1} \cdot i_1\left(\partial_0\right)$ and the elements
$$i_0\left(\partial_p^c\right)^{-1} \cdot x_p \cdot i_1\left(\partial_p^c\right) \cdot x_p^{-1},$$
for $1 \leq p \leq m$. Let
$$\Gamma = G/H.$$
We identify x_p with its image in Γ, and i_0 and i_1 with injections of $\pi_1(\Sigma)$ in Γ. Let i be the involution of G such that for all p in $\{1, \ldots, m\}$
$$\begin{aligned} i\left(x_p\right) &= x_p^{-1}, \\ i \circ i_0 &= i_1. \end{aligned}$$

Observe that i preserves H so that i gives rise to an involution also denoted i on Γ. We now recall that Γ is isomorphic with $\pi_1(\widehat{\Sigma})$ and that an isomorphism is given by the map φ defined by

$$\begin{aligned}\varphi\left(x_p\right) &= c_p * \left(\bar{c}_p\right)^{-1}, \\ \varphi \circ i_k &= (j_k)_*, \\ \varphi \circ i &= j_* \varphi,\end{aligned}$$

for all p in $\{1, \ldots, m\}$ and k in $\{0,1\}$. We identify once and for all $\pi_1(\widehat{\Sigma})$ with Γ using φ.

We now construct $\widehat{\mathbf{R}}$. Let $\mathbf{R}$ be a good homomorphism of ρ. Using the notation of Proposition 9.2.4, we define $\widetilde{\mathbf{R}}$ as the morphism from G to $\mathrm{PSL}(n, \mathbb{R})$ uniquely characterised by

$$\begin{aligned}\widetilde{\mathbf{R}} \circ i_0 &= \mathbf{R}, \\ \widetilde{\mathbf{R}} \circ i_1 &= \mathrm{J} \mathbf{R} \mathrm{J}, \\ \widetilde{\mathbf{R}}\left(x_p\right) &= \underline{\mathbf{R}}\left(c_p, \mathrm{J}\right) \mathrm{J},\end{aligned}$$

for all p in $\{1, \ldots, m\}$. We now prove that $\widetilde{\mathbf{R}}$ vanishes on elements of H. First, we have for all p in $\{1, \ldots, m\}$

$$\begin{aligned}\widetilde{\mathbf{R}}\left(i_0\left(\partial_p^c\right)^{-1} \cdot x_p \cdot i_1\left(\partial_p^c\right) \cdot x_p^{-1}\right) &= \widetilde{\mathbf{R}}\left(i_0\left(\partial_p^c\right)\right)^{-1} \cdot \widetilde{\mathbf{R}}\left(x_p\right) \cdot \widetilde{\mathbf{R}}\left(i_1\left(\partial_p^c\right)\right) \cdot \widetilde{\mathbf{R}}\left(x_p^{-1}\right) \\ &= \mathbf{R}\left(\partial_p^c\right)^{-1} \underline{\mathbf{R}}\left(c_p, \mathrm{J}\right) \mathbf{R}\left(\partial_p^c\right) \underline{\mathbf{R}}\left(c_p, \mathrm{J}\right)^{-1}.\end{aligned}$$

Recall now from the definition of $\underline{\mathbf{R}}$ that we have for every p, a diagonal matrix Δ and a matrix K such that

$$\mathbf{R}\left(\partial_p^c\right) = \mathrm{K} \Delta \mathrm{K}^{-1},\ \underline{\mathbf{R}}\left(c_p, \mathrm{J}\right) = \mathrm{K} \mathrm{J} \mathrm{K}^{-1}.$$

Hence

$$\begin{aligned}\widetilde{\mathbf{R}}\left(i_0\left(\partial_p^c\right)^{-1} \cdot x_p \cdot i_1\left(\partial_p^c\right) \cdot x_p^{-1}\right) &= \mathrm{K} \Delta^{-1} \mathrm{K}^{-1} \mathrm{K} \mathrm{J} \mathrm{K}^{-1} \mathrm{K} \Delta \mathrm{K}^{-1} \mathrm{K} \mathrm{J} \mathrm{K}^{-1} \\ &= \mathrm{K} \Delta^{-1} \mathrm{J} \Delta \mathrm{J} \mathrm{K}^{-1} = \mathrm{Id}.\end{aligned}$$

Moreover

$$\begin{aligned}\widetilde{\mathbf{R}}\left(i_0\left(\partial_0\right)^{-1} \cdot i_1\left(\partial_0\right)\right) &= \mathbf{R}(\partial_0)^{-1} \mathrm{J} \mathbf{R}(\partial_0) \mathrm{J} \\ &= \mathrm{Id}.\end{aligned}$$

Therefore $\widetilde{\mathbf{R}}$ vanishes on the elements generating H normally. Hence, $\widetilde{\mathbf{R}}$ gives rise to a homomorphism $\widehat{\mathbf{R}}$ from Γ to $\mathrm{PSL}(n, \mathbb{R})$.

The homomorphism $\widehat{\mathbf{R}}$ satisfies by construction the first condition to be a J-extension.

Observe that by Equation (40) and the construction of the generators of Γ, we have

$$\widehat{\mathbf{R}} \circ i = \mathrm{J} \mathbf{R} \mathrm{J}. \tag{44}$$

Hence $\widehat{\mathbf{R}}$ satisfies the second property (42) of the definition of J-extension.

Finally, we check Property (43). Let c be a curve in Σ joining v to a boundary component ∂_p. We observe that there exists a loop in Σ based γ at v such that as elements of $\mathcal{B}_v$ $c = \gamma * c_p$, thus

$$c * \bar{c}^{-1} = \gamma * c_p * \bar{c}_p^{-1} * \bar{\gamma}^{-1}.$$

Hence, using Equation (44) and the definition of $\widehat{\mathbf{R}}$

$$\widehat{\mathbf{R}}\left(c * \bar{c}^{-1}\right) = \widehat{\mathbf{R}}\left(\gamma\right) \underline{\mathbf{R}}\left(c_p, \mathrm{J}\right) \mathrm{J} \widehat{\mathbf{R}}\left(\bar{\gamma}^{-1}\right) = \mathbf{R}\left(\gamma\right) \underline{\mathbf{R}}\left(c_p, \mathrm{J}\right) \mathbf{R}\left(\gamma^{-1}\right) \mathrm{J}.$$

It follows from Equation (38) that

$$\widehat{\mathbf{R}}\left(c * \bar{c}^{-1}\right) \quad = \quad \underline{\mathbf{R}}\left(c, \mathrm{J}\right) \mathrm{J}. \tag{45}$$

Thus we have completed the proof of the existence and uniqueness of J-extension.

The continuity statement follows from the construction and the continuity statement in Proposition 9.1.4. Q.E.D.

9.2.2 Doubling Hitchin representations

Let again Σ be a compact connected orientable surface with non empty boundary. Let $\widehat{\Sigma}$ be its double. Let J be an involution of $\mathrm{PGL}(n, \mathbb{R})$ commuting with all diagonal matrices.

Definition 9.2.3 *Let ρ be a Hitchin representation from $\pi_1(\Sigma)$ to $\mathrm{PSL}(n, \mathbb{R})$. A* J*-extension of ρ is a representation $\widehat{\rho}$ from $\pi_1(\widehat{\Sigma})$ to $\mathrm{PSL}(n, \mathbb{R})$ such that the* J*-extension of any good representative of ρ belongs to the conjugacy class of $\widehat{\rho}$.*

We now prove the main result of this section.

Theorem 9.2.4 [Doubling] *Let ρ be a Hitchin representation from $\pi_1(\Sigma)$ to $\mathrm{PSL}(n, \mathbb{R})$. Then there exists a unique – up to conjugacy –* J*-extension $\widehat{\rho}$ of ρ.*

Furthermore, the map $\rho \mapsto \widehat{\rho}$ is continuous.

Definition 9.2.5 [Hitchin double] *Let*

$$\mathrm{J}_n = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & -1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

The J_n*-extension of a Hitchin representation ρ is the* Hitchin double *of ρ.*

The discussion extends to any real reductive split group. The following Corollary (proved in Paragraph 9.2.3) is important

Corollary 9.2.6 *Let Σ be a compact connected oriented surface with non empty boundary whose double $\widehat{\Sigma}$ has genus at least two. Let ρ be a Hitchin representation of $\pi_1(\Sigma)$. Then the Hitchin double of ρ is a Hitchin representation of a closed surface. In particular, there exists a Hitchin representation $\widehat{\rho}$ of $\pi_1(\widehat{\Sigma})$ whose restriction to $\pi_1(\Sigma)$ is ρ.*

Proof: We prove the first assertion of Theorem 9.2.4. The uniqueness up to conjugacy of $\widehat{\rho}$ is the consequence of the uniqueness of J-extensions.

To prove the existence, we show that the J-extension $\widehat{\mathbf{R}}$ of a good representative $\mathbf{R}$ based at v is independent – up to conjugation – of the choices made for $\mathbf{R}$. We have two degrees of freedom in the construction of $\mathbf{R}$

- the choice of a good representative based at v,
- the choice of v.

The choice of a different good representative based at v amounts in conjugating $\mathbf{R}$ by a diagonal matrix Δ. It follows from Equation (39) that $\Delta\widehat{\mathbf{R}}\Delta^{-1}$ is the J-extension of $\Delta\mathbf{R}\Delta^{-1}$. Thus the conjugacy class of $\widehat{\mathbf{R}}$ is invariant under the choice of a good representative when the based point is fixed.

Secondly, we can choose another base point w in a boundary component ∂_w of Σ. Let γ be an arc from v to w. Let K be a matrix such that

$$\begin{aligned}\mathbf{R}\left(\gamma*\partial_w*\gamma^{-1}\right) &= \mathrm{K}\Delta_w\mathrm{K}^{-1}, && (46)\\ \underline{\mathbf{R}}(\gamma,\Delta) &= \mathrm{K}\Delta\mathrm{K}^{-1} && (47)\end{aligned}$$

where Δ_w is a diagonal matrix with decreasing entries. It follows that the homomorphism $\mathbf{M}$ given by

$$\mathbf{M}(h) = \mathrm{K}^{-1}\mathbf{R}\left(\gamma*h*\gamma^{-1}\right)\mathrm{K},$$

is a good representative based at w. We are going to prove that the homomorphism $\mathbf{F}$ defined by

$$\mathbf{F}(c) = \mathrm{K}^{-1}\widehat{\mathbf{R}}\left(\gamma*c*\gamma^{-1}\right)\mathrm{K}, \qquad (48)$$

is the J-extension of $\mathbf{M}$. First, we note that if c is an arc from v to a component ∂_u, then

$$\mathbf{M}\left(\gamma^{-1}*c*\partial_u*c^{-1}*\gamma\right) = \mathrm{K}^{-1}\mathbf{R}\left(c*\partial_u*c^{-1}\right)\mathrm{K}.$$

It follows that

$$\underline{\mathbf{M}}\left(\gamma^{-1}*c,\Delta\right) = \mathrm{K}^{-1}\underline{\mathbf{R}}(c,\Delta)\,\mathrm{K}. \qquad (49)$$

We now prove that $\mathbf{F}$ is the J-extension of $\mathbf{M}$. The first property of J-extension is obviously satisfied. For the second one, observe that by definition.

$$\begin{aligned}\mathbf{F}(\bar{c}) &= \mathrm{K}^{-1}\widehat{\mathbf{R}}\left(\gamma*\bar{c}*\gamma^{-1}\right)\mathrm{K}\\ &= \mathrm{K}^{-1}\widehat{\mathbf{R}}\left(\gamma*\bar{\gamma}^{-1}\right)\widehat{\mathbf{R}}\left(\overline{\gamma*c*\gamma^{-1}}\right)\widehat{\mathbf{R}}\left(\overline{\gamma*\bar{\gamma}^{-1}}\right)\mathrm{K}.\end{aligned}$$

It follows by the second and third properties of J-extensions (42) and (43) that

$$\begin{aligned}\mathbf{F}(\bar{c}) &= \mathrm{K}^{-1}\widehat{\mathbf{R}}\left(\gamma*\bar{\gamma}^{-1}\right)\mathrm{J}\widehat{\mathbf{R}}\left(\gamma^{-1}*c*\gamma\right)\mathrm{JJ}\widehat{\mathbf{R}}\left(\gamma*\bar{\gamma}^{-1}\right)\mathrm{JK}\\ &= \mathrm{K}^{-1}\underline{\mathbf{R}}(\gamma,\mathrm{J})\,\widehat{\mathbf{R}}\left(\gamma^{-1}*c*\gamma\right)\underline{\mathbf{R}}(\gamma,\mathrm{J})\,\mathrm{K}.\end{aligned}$$

Finally using (47) for $\Delta=\mathrm{J}$,

$$\mathbf{F}(\bar{c})=\mathrm{JK}^{-1}\widehat{\mathbf{R}}\left(\gamma^{-1}*c*\gamma\right)\mathrm{KJ}=\mathrm{J}\mathbf{F}(c)\,\mathrm{J}.$$

Thus the second property (42) of the definition of a J-extension is satisfied by $\mathbf{F}$. For the last property (43) of the definition of J-extension, let c be a curve from w to another boundary component, then by definition

$$\begin{aligned}\mathbf{F}\left(c*\bar{c}^{-1}\right) &= \mathrm{K}^{-1}\widehat{\mathbf{R}}\left(\gamma*c*\bar{c}^{-1}*\gamma^{-1}\right)\mathrm{K}\\ &= \mathrm{K}^{-1}\widehat{\mathbf{R}}\left(\gamma*c*\overline{(\gamma*c)}^{-1}\right)\widehat{\mathbf{R}}\left(\bar{\gamma}*\gamma^{-1}\right)\mathrm{K}.\end{aligned}$$

Hence, using (42),

$$\begin{aligned}\mathbf{F}\left(c*\bar{c}^{-1}\right) &= \mathrm{K}^{-1}\underline{\mathbf{R}}(\gamma*c,\mathrm{J})\,\mathrm{J}\widehat{\mathbf{R}}\left(\bar{\gamma}*\gamma^{-1}\right)\mathrm{K}\\ &= \mathrm{K}^{-1}\underline{\mathbf{R}}(\gamma*c,\mathrm{J})\,\widehat{\mathbf{R}}\left(\gamma*\bar{\gamma}^{-1}\right)\mathrm{JK}\\ &= \mathrm{K}^{-1}\underline{\mathbf{R}}(\gamma*c,\mathrm{J})\,\underline{\mathbf{R}}(\gamma,\mathrm{J})\,\mathrm{K}\\ &= \mathrm{K}^{-1}\underline{\mathbf{R}}(\gamma*c,\mathrm{J})\,\mathrm{KJ}\end{aligned}$$

It follows that by (49)

$$\mathbf{F}\left(c*\bar{c}^{-1}\right) = \underline{\mathbf{M}}(c,\mathrm{J})\,\mathrm{J}.$$

It follows that $\mathbf{F}$ is the J-extension of $\mathbf{M}$. Hence $\widehat{\rho}$ is well defined as a conjugacy class. We have completed the proof of the existence and uniqueness of $\widehat{\rho}$.

The continuity statement follows from the continuity statement in Proposition 9.2.2. Q.E.D.

9.2.3 Proof of Theorem 9.0.3 and Corollary 9.2.6

A Fuchsian representation ρ from $\pi_1(\Sigma)$ to $\mathrm{PSL}(2,\mathbb{R})$ is a monodromy representation of hyperbolic metric on Σ with totally geodesic boundary. The monodromy of the closed hyperbolic surface obtained by isometric gluing of Σ along its boundary is the J_2-extension of ρ. Indeed J_2 is the symmetry with respect to a geodesic when we identify the group of isometry of the hyperbolic plane with $\mathrm{PGL}(2,\mathbb{R})$ through its action on the boundary at infinity identified with a real projective line. Hence, the Hitchin double $\widehat{\rho}$ is the monodromy of the closed hyperbolic surface homeomorphic to the topological double of Σ and, in particular, it is also Fuchsian.

We consider the irreducible representation of dimension n given by the action of $\mathrm{GL}(2,\mathbb{R})$ on homogeneous polynomials of two variables of degree $n-1$. Then the image of J_2 is J_n. It follows that the Hitchin double of an n-Fuchsian representation is an n-Fuchsian representation and so belongs to the Hitchin component. Now recall that any Hitchin representation ρ_1 of $\pi_1(\Sigma)$ is obtained by continuous deformation of some Fuchsian representation ρ_0; that is there is a continuous path $\rho_t, t \in [0,1]$ of representations connecting ρ_0 to ρ_1. The "doubling map" $\rho \mapsto \widehat{\rho}$ is continuous so that $\widehat{\rho}_t$ is a path of representations of the fundamental group of the topological double of Σ. This, in fact, proves Corollary 9.2.6 as $\widehat{\rho}_t$ starts in the Hitchin component at $\widehat{\rho}_0$ and so remains in the Hitchin component for all $t \in [0,1]$; in particular $\widehat{\rho}_1$ is a Hitchin representation.

Now, since the Hitchin double $\widehat{\rho}$ is a Hitchin representation, Theorem 8.2.1 applies and we see that $\widehat{\rho}$ preserves a Frenet curve, hence a positive curve by Lemma 8.4.2. Hence, the Hitchin double is a positive representation. Finally, it follows directly from the definition of positivity that the restriction of the Hitchin double $\widehat{\rho}$ to the fundamental group of any connected incompressible surface in the double surface is a positive representation.

The statement involving the cross ratio is a consequence of the same result for closed surfaces – see Theorem 1.0.2. This concludes the proof of Theorem 9.0.3. Q.E.D.

10 Gap functions and coordinates for Hitchin representations

Let Σ be a compact orientable surface with or without boundary different from the sphere, the disk, the cylinder or the torus.

Let ρ be a Hitchin representation of $\pi_1(\Sigma)$ and P be a pair of pants in Σ. By Theorem 9.0.3, $\partial_\infty\pi_1(\Sigma)$ inherits a cross ratio $\mathbf{B}_\rho$.

In order to complete our circle of ideas and fully generalise the results of Section 5, we wish to describe the gap functions for P. Recall that by Proposition 8.2.2, these gap functions only depend on $\rho(\pi_1(P))$. The aim of this section is to compute these functions using coordinates – generalising shear coordinates – on the space of Hitchin representations of $\pi_1(P)$ which come from coordinates on the Fock–Goncharov moduli space $\mathcal{M}_{FG}(P)$ – see below for definitions.

Therefore, the aim of this section – parallel to Section 5 – is threefold

- Following [7], we describe the *Fock–Goncharov moduli space* of the pair of pants and its *Fock-Goncharov coordinates* in the next paragraph. This Fock-Goncharov moduli space is mapped onto the space of positive representations by a *holonomy map.*

- In Paragraph 10.2.1, we describe the preimage in the Fock–Goncharov moduli space of a Hitchin representation under the holonomy map. This construction induces various *Fock–Goncharov coordinates for the moduli space of Hitchin representation* of the fundamental group of a pair of pants Paragraph 10.1.1.

- We finally express in Theorem 10.3.1 the pant gap function for a good choice of coordinates on the space of Hitchin representations.

10.1 Fock–Goncharov moduli space for the pair of pants

We start with the canonical ideal triangulation $\mathcal{T}$ of the 3-punctured sphere P obtained by gluing two triangles X and Z. This ideal triangulation has two faces X and Z, three ideal vertices α, β and γ, and three edges A, B and C where A is the edge opposite the vertex α *etc.*

Definition 10.1.1 [Fock–Goncharov Moduli space] *Following [7], an element of the* Fock–Goncharov moduli space $\mathcal{M}_{FG}(P)$ *is a configuration of six flags*

$$\mathcal{S} = (X_\alpha, Z_\gamma, X_\beta, Z_\alpha, X_\gamma, Z_\beta) \in \mathcal{F}(\mathbb{R}^n)^6/\mathrm{PSL}(n,\mathbb{R}),$$

such that

1. *The triple $T_X = (X_\alpha, X_\beta, X_\gamma)$ is positive.*
2. *The three triples $(Z_\alpha, X_\gamma, X_\beta)$, $(X_\alpha, X_\gamma, Z_\beta)$ and $(X_\alpha, Z_\gamma, X_\beta)$ are positive and are equivalent under the action of* $\mathrm{PSL}(n,\mathbb{R})$ *to some positive triple T_Z.*
3. *The three quadruples*

$$\begin{aligned} Q_A &= (X_\gamma, Z_\alpha, X_\beta, X_\alpha), \\ Q_B &= (X_\alpha, Z_\beta, X_\gamma, X_\beta), \\ Q_C &= (X_\beta, Z_\gamma, X_\alpha, X_\gamma), \end{aligned}$$

   are positive.

We associate to an element of the Fock–Goncharov moduli space a representation of the fundamental group of the pair of pants in $\mathrm{PSL}(n,\mathbb{R})$ as follows.

We choose a complete finite volume hyperbolic metric on P such that the edges or $\mathcal{T}$ are geodesics. Let $\widetilde{P}$ be the universal cover of P. Let $\widehat{\mathcal{T}}$ be the pull back of the ideal triangulation $\mathcal{T}$ on $\widetilde{P}$. Then let $\widehat{\mathcal{V}}$ denote the set of ideal vertices of $\widehat{\mathcal{T}}$. We consider $\widehat{\mathcal{V}}$ as a subset of $\partial_\infty\pi_1(P)_3$, and observe that this identification is independent of the choice of the hyperbolic metric.

Note that the fundamental group $\pi_1(P)$ acts on $\widehat{\mathcal{V}}$. We describe this action as follows. For every lift X^i of our original triangle X – respectively Z^i of Z – we denote by $(X^i_\alpha, X^i_\beta, X^i_\gamma)$ – respectively $(Z^i_\alpha, Z^i_\beta, Z^i_\gamma)$ – the lift of the ideal vertices (α,β,γ), and follow a similar convention for edges.

We now choose a particular lift X^0 of X. Let then Z^A be the lift of Z adjacent to X^0_A and similarly for the other edges B and C. We then denote – again abusively – by α the element of $\pi_1(P)$ that fixes the vertex X^0_α and sends X^0_γ to Z^C_γ, and symmetrically for β and γ as in Figure 3.

According to Theorem 7.1 of [7] – which is another incarnation of the general principle associating to a geometric structure a holonomy representation – an

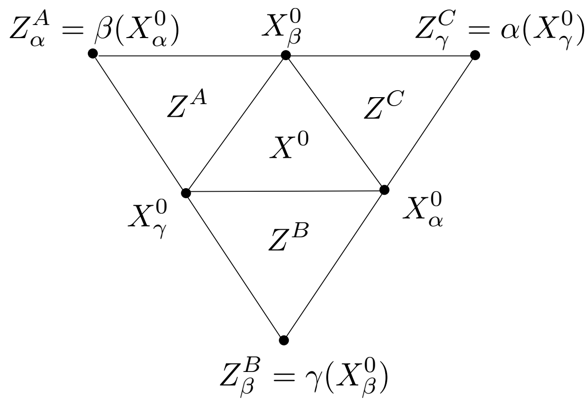

Figure 3: *The action of the fundamental group*

element of the Fock–Goncharov moduli space defines a pair (f, ρ) where ρ is a homomorphism from $\pi_1(P)$ to $\mathrm{PSL}(n, \mathbb{R})$ and f is a ρ-equivariant map from $\widehat{\mathcal{V}}$ to $\mathcal{F}(\mathbb{R}^n)$ – well defined up to congugacy by $\mathrm{PSL}(n, \mathbb{R})$. The pair (f, ρ) is characterised by $\mathcal{S}$ in the following way

$$\begin{aligned} X_\alpha &= f(X^0_\alpha), & Z_\gamma &= \rho(\beta)\left(f(X^0_\alpha)\right), \\ X_\beta &= f(X^0_\beta), & Z_\alpha &= \rho(\gamma)\left(f(X^0_\beta)\right), \\ X_\gamma &= f(X^0_\gamma), & Z_\beta &= \rho(\alpha)\left(f(X^0_\gamma)\right). \end{aligned}$$

The existence of $\rho(\alpha)$, $\rho(\beta)$ and $\rho(\gamma)$, as well as the fact that ρ is indeed a homomorphism, is guaranteed by Condition 2 of Definition 10.1.1.

Then, by construction, if (U, V, W) and (Y, U, W) are pairs of adjacent triangles in $\widehat{\mathcal{T}}$ with $W \neq Y$, then $(f(U), f(V), f(W), f(Y))$ is a positive quadruple.

Definition 10.1.2 *The representation ρ – denoted by* $\mathrm{Hol}(\mathcal{S})$ *– described above is the* holonomy *of the element $\mathcal{S}$ of the Fock–Goncharov moduli space.*

This representation is positive and, moreover, it follows from Proposition 5.8 and Theorem 7.1 in [7] that all positive representations are obtained this way.

10.1.1 Fock–Goncharov coordinates for a pair of pants

Following [7] and by Proposition 7.1.7, a parametrisation of $\mathcal{M}_{FG}(P)$ is given by the following collections of functions of $\mathcal{S}$

$$(X^{m,l,p}, Z^{m,l,p}, \Delta^A_k, \Delta^B_k, \Delta^C_k),$$

for $1 \leq k \leq n-1$ and $m+l+p = n$, which we define as follows using the notations of Section 7.1

- triple ratios describing T_X : $X^{m,l,p}(\mathcal{S}) = T^{m,l,p}(T_X)$ for $m+l+p = n$,

- triple ratios describing T_Z : $Z^{m,l,p}(\mathcal{S}) = T^{m,l,p}(T_Z)$ for $m+l+p=n$,
- edges functions

$$\begin{array}{rcl} \Delta_k^A(\mathcal{S}) & = & \delta_k(Q_A), \\ \Delta_k^B(\mathcal{S}) & = & \delta_k(Q_B), \\ \Delta_k^C(\mathcal{S}) & = & \delta_k(Q_C), \end{array}$$

for $1 \leq k \leq n-1$.

10.2 Hitchin representations for the pair of pants

10.2.1 Lifts of Hitchin representations

Let $\mathrm{Hom}_H(P)$ be the space of Hitchin homomorphism of $\pi_1(P)$ into $\mathrm{PSL}(n,\mathbb{R})$. The space $\mathrm{Hom}_H(P)$ have one or two connected components depending on the parity of n, and the group of outer automorphisms of $\mathrm{PSL}(n,\mathbb{R})$ acts transitively by conjugation on its set of connected components. Let

$$\mathrm{Rep}_H(P) = \mathrm{Hom}_H(P)/\,\mathrm{PGL}(n,\mathbb{R}),$$

be the *space of Hitchin representations* of $\pi_1(P)$ into $\mathrm{PSL}(n,\mathbb{R})$. Let $\mathcal{M}_{FG}(P)$ be the Fock–Goncharov moduli space. Let (u,v,w) be a triple of elements of the Weyl group $\mathcal{W}$ of $\mathrm{PSL}(n,\mathbb{R})$.

Observe first that given a purely loxodromic element M, we have a well defined action of the Weyl group – identified with the symmetric group of $\{1,\ldots,n\}$ – on the set of invariant flags by M characterised as follows: let $(L_1, L_2, \ldots, L_n)$ be the tuples of eigenlines ordered by decreasing eigenvalues; let u be an element of the Weyl group; then the image of the attracting flag M^+ is

$$\mathrm{M}^u = (L_{u(1)}, L_{u(1)} \oplus L_{u(2)}, \ldots).$$

Proposition 10.2.1 *Let (u,v,w) be a triple of elements of the Weyl group. There exists a map*

$$\mathcal{S}^{u,v,w} : \left\{ \begin{array}{rcl} \mathrm{Rep}_H(P) & \rightarrow & \mathcal{M}_{FG}(P), \\ \rho & \mapsto & \mathcal{S}_\rho^{u,v,w}, \end{array} \right.$$

such that $\mathrm{Hol} \circ \mathcal{S}^{u,v,w} = \mathrm{Id}$. *This map is onto a connected component of* $\mathrm{Hol}^{-1}(\mathrm{Rep}_H(P))$.

PROOF: Recall first that by Corollary 9.2.6, the double $\widehat{\rho}$ preserves a Frenet curve, which we denote by ξ.

Let now α be a boundary component of P. Recall that since $\rho(\alpha)$ is purely loxodromic, its dynamics on the flag manifolds has the following property: the action of $\rho(\alpha)$ on the flag manifold has exactly one attractive point that we call the *attractive flag* and denote by A^+. The attractive flag is $A^+ = (L_1, L_1 \oplus L_2, \ldots)$ where L_i is the eigenspace for the eigenvalue λ_i, where $\lambda_1 > \lambda_2 > \ldots$.

Finally it follows from [14] that the attractive flag A^+ is the osculating flag of the curve ξ at the point $\xi(\alpha^+)$.

Symmetrically the action of $\rho(\alpha)$ has exactly one repulsive point that we call the *repulsive flag* and denote by A^-.

Let $\mathcal{W}$ be the Weyl group of $\mathrm{PSL}(n,\mathbb{R})$. Then for every element u in $\mathcal{W}$ in $\mathrm{PSL}(n,\mathbb{R})$, we denote as in the introduction of this section

$$A^u = (L_{u(1)}, L_{u(1)} \oplus L_{u(2)}, \ldots),$$

the image of A^+ by u. Similarly we have flags B^v and C^w for the boundary components associated to β and γ and the elements v and w of the Weyl group.

It follows from Section 7 and 9 of [7], that the configuration given by the following sextuplet is positive

$$\mathcal{S}_\rho^{u,v,w} = (A^u, \rho(\beta)(A^u), B^v, \rho(\gamma)(B^v), C^w, \rho(\alpha)(C^w)).$$

Hence $\mathcal{S}_\rho^{u,v,w}$ determines an element of the Fock–Goncharov moduli space for the pair of pants whose image under Hol is ρ.

Conversely it follows from the description in the previous paragraph that for any $\mathcal{S}$ in $\mathcal{M}_{FG}(P)$ such that

$$\mathrm{Hol}(\mathcal{S}) = \rho,$$

there exists a unique triple (u, v, w) in the Weyl group such that

$$\mathcal{S} = \mathcal{S}_\rho^{u,v,w}.$$

Indeed any flag invariant by $\rho(\alpha)$ is of the form A^u for some u in the Weyl group. Thus, the map $(\rho, u, v, w) \mapsto \mathcal{S}_\rho^{u,v,w}$ from $\mathrm{Rep}_H(P) \times \mathcal{W}^3$ to $\mathrm{Hol}^{-1}(\mathrm{Rep}_H(P))$ is a homeomorphism. Since $\mathrm{Rep}_H(P)$ is connected it follows that $S^{u,v,w}$ is onto a connected component of $\mathrm{Hol}^{-1}(\mathrm{Rep}_H(P))$. Q.E.D.

We describe the configuration of flags $\mathcal{S}_\rho^{u,v,w}$ in Figure 4 analoguously to Figure 3, where we identify the fundamental group with its image under ρ, and use the following notation

$$\begin{aligned} Z_\alpha &= \beta(X_\alpha) = \gamma^{-1}(X_\alpha), \\ Z_\beta &= \gamma(X_\beta) = \alpha^{-1}(X_\beta), \\ Z_\gamma &= \alpha(X_\gamma) = \beta^{-1}(X_\gamma), \end{aligned}$$

where $X_\alpha = A^u$, $X_\beta = B^v$ and $X_\gamma = C^w$ are so that

$$\mathcal{S}_\rho^{u,v,w} = (X_\alpha, Z_\gamma, X_\beta, Z_\alpha, X_\gamma, Z_\beta),$$

is positive.

In the case $n = 2$, the Fock–Goncharov moduli space is nothing but the enhanced Teichmüller space of the pair of pants [3]. Recall that a point of the enhanced Teichmüller space of a surface with boundary consists of a point of the usual Teichmüller space plus a choice of orientation for each of the boundary components. The discussion of the previous paragraph is exactly a generalisation of this discussion.

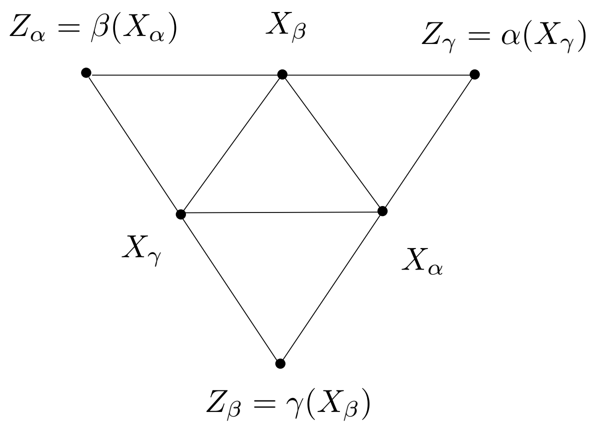

Figure 4: *The configuration of flags*

10.2.2 Coordinates on the moduli space of Hitchin representations

Therefore using the notations of Paragraph 10.1.1 and 10.2.1, we obtain a system of coordinates on Hitchin representations by choosing a triple (u, v, w) of elements of the Weyl group – one for each boundary component – and considering the functions

$$\begin{array}{rcl} X^{m,l,p}_{u,v,w} : \rho & \mapsto & X^{m,l,p}_{u,v,w}(\rho) = X^{m,l,p}(\mathcal{S}^{u,v,w}_\rho), \\ Y^{m,l,p}_{u,v,w} : \rho & \mapsto & Y^{m,l,p}_{u,v,w}(\rho) = Y^{m,l,p}(\mathcal{S}^{u,v,w}_\rho), \\ A^{u,v,w}_k : \rho & \mapsto & A^{u,v,w}_k(\rho) = \Delta^A_k(\mathcal{S}^{u,v,w}_\rho), \\ B^{u,v,w}_k : \rho & \mapsto & B^{u,v,w}_k(\rho) = \Delta^B_k(\mathcal{S}^{u,v,w}_\rho), \\ C^{u,v,w}_k : \rho & \mapsto & C^{u,v,w}_k(\rho) = \Delta^C_k(\mathcal{S}^{u,v,w}_\rho). \end{array}$$

We call this set of functions the *Fock–Goncharov coordinates on the moduli space of Hitchin representations* associated to the triple (u, v, w). This is an extension of Thurston shear coordinates described in Section 5.2 for Fuchsian representations when $n = 2$. Observe that this construction leads to $(n!)^3$ different coordinate systems on the space of Hitchin representations. To the knowledge of the authors, there is no nice formula and explicit for these changes of coordinates.

10.3 Gap functions, periods and coordinates

In general, it seems difficult to obtain a closed form for the gap functions for each of these coordinate systems. However, for particular choices of coordinates, we have a nice formula for the pant gap function.

Theorem 10.3.1 *Let u, v and w be elements of the Weyl group of* $\mathrm{PSL}(n, \mathbb{R})$*, identified with the group of permutations of* $\{1, \ldots, n\}$*, one for each boundary* *component. We assume that*

$$u(n) = n,\ u(n-1) = 1,\ v(1) = 1,\ w(1) = n.$$

Let ρ-be an n-Hitchin representation $\pi_1(P)$ into $\mathrm{PSL}(n, \mathbb{R})$. *Let ξ be Frenet curve with values in $\mathbb{P}(\mathbb{R}^n)$ associated to ρ and ξ^* its hyperplane osculating curve. Let $\mathbf{B} = \mathbf{B}_{\xi^*,\xi}$ be the cross ratio associated to the pair (ξ^*, ξ). Then the pant gap function of the cross ratio $\mathbf{B}$ has the following expression in the Fock–Goncharov coordinates associated to the triple* (u, v, w)

$$G_{\mathbf{B}}(P) \quad = \quad \log\Big(\frac{1 + C_1(\rho)e^{\ell_{\mathbf{B}}(\alpha)}}{1 + C_1(\rho)}\Big) = \log\Big(\frac{1 + B_{n-1}(\rho)}{B_{n-1}(\rho)(1 + C_1(\rho))}\Big).$$

(We have used the notation introduced in the previous paragraph, but ignored the superscripts u, v, w in order to obtain a readable formula.)

Remarks:

1. In other words, we choose invariant flags A, B and C so that
$$\begin{aligned} A^{n-2} &= \xi_{n-1}(\alpha^+) \cap \xi_{n-1}(\alpha^-), \\ A^{n-1} &= \xi_{n-1}(\alpha^+), \\ B^1 &= \xi_1(\beta^+), \\ C^1 &= \xi_1(\gamma^-). \end{aligned}$$
where ξ is the limit curve from $\partial_\infty \tilde{P}$ to $\mathbb{P}(\mathbb{R}^n)$, where $\tilde{P}$ is the universal cover of P equipped with an admissible hyperbolic metric.

2. We shall explain in Section 11 how to obtain, for $\mathrm{PSL}(3, \mathbb{R})$, and the choice of the identity for the elements of the Weyl group, the formula for the pant gap function using a computer assisted proof.

3. So far, it remains a challenge to obtain a closed formula for gap functions in all coordinates. The same remark holds for the boundary gap function, for which we do not have a similar nice formula. However, one could prove – but we shall not do it here – that the gap functions are logarithms of rational functions.

Proof: Let P be a pair of pants represented by a triple (α, β, γ) of elements of $\pi_1(\Sigma)$. Let us consider the three flags

$$X_\gamma = C, \quad X_\alpha = A, \quad X_\beta = B.$$

We complete this configuration by

$$Z_\gamma = \alpha(C), \quad Z_\beta = \gamma(C), \quad Z_\alpha = \beta(A).$$

Now let (u, v, w) be three elements of the Weyl group as described in the hypothesis of the theorem. Let

$$V = \mathbb{R}^n/\xi_{n-1}(\alpha^+) \cap \xi_{n-1}(\alpha^-) = \mathbb{R}^n/A^{n-2}.$$

To simplify the notation, we will write α for $\rho(\alpha)$. We observe that α acts on V. Let $\mathbf{b}_V$ be the cross ratio on the projective line $\mathbb{P}(V)$. Let π be the projection on V. By definition, the gap function is

$$G_{\mathbf{B}}(P) \quad = \quad \log(\mathbf{B}_{\xi^*,\xi}(\alpha^+, \gamma^-, \alpha^-, \beta^+)),$$

where

$$\mathbf{B}_{\xi^*,\xi}(\alpha^+, \gamma^-, \alpha^-, \beta^+) \quad = \quad \frac{\Omega(\xi_1(\gamma^-) \wedge \xi_{n-1}(\alpha^+))\Omega(\xi_1(\beta^+) \wedge \xi_{n-1}(\alpha^-))}{\Omega(\xi_1(\gamma^-) \wedge \xi_{n-1}(\alpha^-))\Omega(\xi_1(\beta^+) \wedge \xi_{n-1}(\alpha^+))}.$$

Let

$$\begin{aligned} c = \pi(\xi_1(\gamma^-)), &\qquad b = \pi(\xi_1(\beta^+)), \\ a^+ = \pi(\xi_{n-1}(\alpha^+)), &\qquad a^- = \pi(\xi_{n-1}(\alpha^-)). \end{aligned}$$

By Equation (34), we have

$$\mathbf{B}_{\xi^*,\xi}(\alpha^+, \gamma^-, \alpha^-, \beta^+) \quad = \quad \mathbf{b}_V(a^+, c, a^-, b)$$

Then, using the fact that $\mathbf{b}_V$ is a cross ratio on a projective line and following the proof of Equation (23) in the $\mathrm{PSL}(2, \mathbb{R})$ case, we obtain

$$\mathbf{B}_{\xi^*,\xi}(\alpha^+, \gamma^-, \alpha^-, \beta^+) \quad = \quad \frac{1 - \mathbf{b}_V(b, c, a^+, \alpha(c))\mathbf{b}_V(a^+, c, a^-, \alpha(c))}{1 - \mathbf{b}_V(b, c, a^+, \alpha(c))}. \tag{50}$$

By Equation (30) and since $\alpha(c) = \pi(\alpha(\xi_1(\gamma^-)))$

$$\begin{aligned} \mathbf{b}_V(b, c, a^+, \alpha(c)) &= -\delta_1(B, \alpha(C), A, C) \\ &= -\Delta_1^C(\mathcal{S}_\rho) \\ &= -C_1(\rho). \end{aligned}$$

Finally, we observe that the two eigenvalues of α on V are the largest and smallest eigenvalues of α on $\mathbb{R}^n$. Hence

$$\mathbf{b}_V(a^+, c, a^-, \alpha(c)) = e^{\ell_{\mathbf{B}}(\alpha)}.$$

This concludes the proof of the first equality. For the second equality, we observe first using Equation (22) that

$$\mathbf{b}_V(a^+, c, a^-, \alpha(c)).\mathbf{b}_V(a^+, b, c, \alpha^{-1}(b)).\mathbf{b}_V(b, c, a^+, \alpha(c)) = 1.$$

Multiplying the numerator and denominator of the left hand side of Equation (50) by $\mathbf{b}_V(a^+, c, a^-, \alpha(c))$, we get that

$$\mathbf{B}_{\xi^*,\xi}(\alpha^+, \gamma^-, \alpha^-, \beta^+) = \frac{\mathbf{b}_V(a^+, c, a^-, \alpha(c)) - 1}{\mathbf{b}_V(a^+, c, a^-, \alpha(c))\,(1 - \mathbf{b}_V(b, c, a^+, \alpha(c)))} \quad (51)$$

Finally, by Equation (30)

$$\begin{aligned} \mathbf{b}_V(a^+, b, c, \alpha^{-1}(b)) &= -\delta_{n-1}(A, \alpha^{-1}(B), C, B) \\ &= -B_{n-1}(\rho). \end{aligned}$$

The second formula follows. Q.E.D.

11 Appendix A: the three dimensional case

We use in this section the notations and results of Paragraph 5.2 of [8] which describes the monodromy associated to the Fock–Goncharov coordinates in the case of $\mathrm{PSL}(3,\mathbb{R})$ as that of a local system on a graph.

Our aim is to explain how one computes explicitly the pant gap function for various choices of elements in the Weyl group for the boundary components. Using the notations of Paragraph 10.2.2 and in order to simplify our notations in this particular case, we set

$$\begin{aligned} x &= X^{u,v,w}_{u,v,w}(\rho), & y &= Y^{u,v,w}_{u,v,w}(\rho), \\ a &= A^{u,v,w}_1(\rho), & A &= A^{u,v,w}_2(\rho), \\ b &= B^{u,v,w}_1(\rho), & B &= B^{u,v,w}_2(\rho), \\ c &= C^{u,v,w}_1(\rho), & C &= C^{u,v,w}_2(\rho). \end{aligned}$$

We denote by

$$M(x, y, a, A, b, B, c, C),$$

The element of the Fock–Goncharov moduli space $\mathcal{M}_{FG}(P)$ whose coordinates are x, y, a, A, b, B, c, C, and by

$$R(x, y, a, A, b, B, c, C),$$

its holonomy representation. By construction

$$\mathcal{S}^{u,v,w}(R(x, y, a, A, b, B, c, C)) = M(x, y, a, A, b, B, c, C).$$

We will actually explicit the gap functions for three different choices of coordinates which give very different formulae: Equations (55), (56) and (57). On the arXiv version of this article, we give the computer assisted proof of this result and explain the instructions to obtain the 18 different formulae.

We consider as usual a pair of pants whose boundary are α, β, γ, so that the corresponding elements in $\pi_1(P)$ satisfy

$$\alpha\gamma\beta = 1.$$

We consider positive representations as holonomies of discrete connections on a graph.

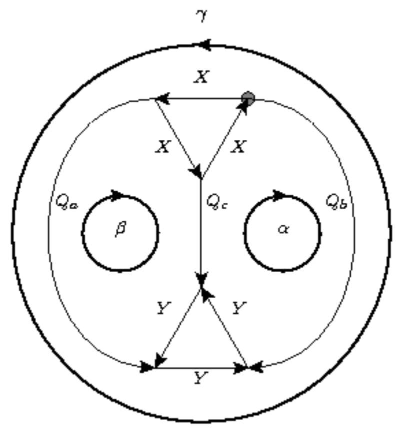

Figure 5: *Local system on a graph*

11.1 The construction

11.1.1 Some matrices

We consider the following matrices where we adopt the notation of Fock and Goncharov [8]

$$X = T(x), \quad Y = T(y), \qquad Q_a = E(A, a), \quad Q_b = E(B, b), \quad Q_c = E(C, c),$$

where

$$T(x) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ x & 1+x & 1 \end{pmatrix}, \quad E(w, z) = \begin{pmatrix} 0 & 0 & \frac{1}{z} \\ 0 & -1 & 0 \\ w & 0 & 0 \end{pmatrix}.$$

We observe that $T(x)^3 = x \cdot \mathrm{Id}$.

11.1.2 A discrete connection on a graph

Following [8], we label the edges of a graph by the above matrices following Figure 5. This way we obtain a discrete connection on this graph and we can compute the holonomies of the boundary components using (x, y, A, a, B, b, C, c). Then, the holonomies (in $PGL(3, \mathbb{R})$) corresponding to the boundary components α, β, γ starting at the point M are respectively

$$\begin{aligned} \rho(\alpha) = \mathrm{Hol}(A) &= X \cdot Q_c^{-1} \cdot Y \cdot Q_b, && (52) \\ \rho(\beta) = \mathrm{Hol}(B) &= X \cdot X \cdot Q_a^{-1} \cdot Y \cdot Q_b \cdot X \cdot X, && (53) \\ \rho(\gamma) = \mathrm{Hol}(C) &= Q_b^{-1} \cdot Y \cdot Q_a \cdot X. && (54) \end{aligned}$$

A computer assisted calculation shows that $\mathrm{Hol}(A)$ is a lower triangular matrix, and that $\mathrm{Hol}(C)$ is upper triangular.

11.1.3 Flags

We now make the link with the description of the coordinates as in Section 10.1.1. First we introduce the following notation: if (U,V) is a pair of independent vectors, we denote by $F(U,V)$ the flag $(\mathbb{R}U, \mathbb{R}U+\mathbb{R}V)$. We now observe that the flags

$$\begin{aligned} X_\alpha &= F((0,0,1),(0,1,0)), \\ X_\gamma &= F((1,0,0),(0,1,0)). \end{aligned}$$

are invariant by $\mathrm{Hol}(A)$ and $\mathrm{Hol}(C)$ respectively, which are respectively lower and upper triangular matrices. Furthermore

$$X_\beta = X.X_\alpha = F((1,-1,1),(0,-1,1+x)),$$

is invariant by $\mathrm{Hol}(C)$, which is the conjugate by X of a lower triangular matrix.

It is useful at this stage to calculate the dual invariant flags. We have

$$\begin{aligned} X^*_\alpha &= F((1,0,0),(0,1,0)), \\ X^*_\beta &= F((0,0,1),(0,1,0)), \\ X^*_\gamma &= F((x,1+x,1),(0,1+x,1)). \end{aligned}$$

The full configuration of flags as in Section 10.1.1 is now completed by

$$\begin{aligned} Z_\alpha &= \mathrm{Hol}(B)(X_\alpha) = \mathrm{Hol}(C)^{-1}(X_\alpha), \\ Z_\beta &= \mathrm{Hol}(C)(X_\beta) = \mathrm{Hol}(A)^{-1}(X_\beta), \\ Z_\gamma &= \mathrm{Hol}(A)(X_\gamma) = \mathrm{Hol}(B)^{-1}(X_\gamma). \end{aligned}$$

Then, by the result of Fock and Goncharov in [8], we have

Proposition 11.1.1 *The configuration of flags*

$$\mathcal{S} = (X_\alpha, Z_\gamma, X_\beta, Z_\alpha, X_\gamma, Z_\beta),$$

is $M(x,y,A,a,B,b,C,c)$.

11.1.4 Eigenvectors and flags

We use a computer program to compute the eigenvectors and eigenvalues for $(\mathrm{Hol}(A))^*$, $\mathrm{Hol}(B)$ and $\mathrm{Hol}(C)$.

- We denote by A_1, A_2, A_3 the eigenvectors of $\mathrm{Hol}(A)^*$ of eigenvalues $\frac{xy}{bC}, 1, Bc$ respectively.
- We denote by B_1, B_2, B_3 the eigenvectors of $\mathrm{Hol}(B)$ of eigenvalues $\frac{x^2y}{Ac}, x, xCa$ respectively.
- We denote by C_1, C_2, C_3 the eigenvectors of $\mathrm{Hol}(C)$ of eigenvalues $bC, 1, \frac{xy}{Bc}$ respectively.

The results are given in the next section. We observe that

$$\begin{aligned} X_\beta &= F(B_1, B_2), \\ X_\gamma &= F(C_3, C_2), \\ X^*_\alpha &= F(A_3, A_2). \end{aligned}$$

11.1.5 Coordinates for Hitchin representations and inequalities

We have explained in Paragraph 10.2.1 that various choices of elements (u, v, w) of the Weyl group gives rise to different lifts $\mathcal{S}^{u,v,w}$ from $\mathrm{Rep}_H(P)$ to $\mathcal{M}_{FG}(P)$.

We explain now how to describe the image of $S^{u,v,w}$ in the coordinates (x, y, a, A, b, B, c, C). The answer follows from considering the converse question.

Given (x, y, a, A, b, B, c, C) so that the holonomy ρ is Hitchin, what is the triple (u, v, w) of elements of the Weyl group so that

$$S^{u,v,w}_\rho = M(x, y, a, A, b, B, c, C) \quad ?$$

To answer this question, we have to identify the elements of the Weyl group which send the attractive flags of the corresponding boundary components to the invariant flags X_α, X_β and X_β. More precisely, the Weyl group acts on the set of invariant flags by the monodromy of a boundary component – see the beginning Paragraph 10.2.1– and we want to find the elements u, v and w so that the X_α is the image by u of the attractive flags of the corresponding monodromy etc ... This depends on the ordering of the eigenvalues of $\mathrm{Hol}(A)$, $\mathrm{Hol}(B)$ and $\mathrm{Hol}(A)$. In other words, if we denote by α^+ the eigenvector with the largest eigenvalue of $\mathrm{Hol}(A)$, α^0 the eigenvector with the middle eigenvalue of $\mathrm{Hol}(A)$ and α^- the eigenvector with the smallest eigenvalue of $\mathrm{Hol}(A)$, we need to identify who, amongst A_1, A_2, A_3, is α^+. We will do that using identities involving functions of x, y, a, A, b, B, c, C, in the next paragraph and write down the corresponding gap functions.

11.2 Pant gap functions in dimension 3

We have explained that the ordering of the eigenvalues of $\mathrm{Hol}(A)$, $\mathrm{Hol}(B)$ and $\mathrm{Hol}(C)$ – described by inequalities depending on x, y, a, A, b, B, c, C – corresponds to different choices of elements of the Weyl group on every boundary component, that is to the various parametrisations of Hitchin representations.

In this paragraph, we give the explicit expressions corresponding to various choices of parametrisation in the case of $\mathrm{PSL}(3, \mathbb{R})$. The formulae were obtained using a computing software and the instructions can be found online on the ArXiv version.

We just want to make two informal observations

- The formulae are highly sensitive on the choice of elements of the Weyl group and it seems difficult to hope for a simple expression of the gap function in all possible cases.

- From the cluster algebra point of view, the gap functions are positive functions. Therefore the identities are relations in a completion of the cluster algebra. Does these identities have a pure cluster meaning ?

THE IDENTITY ON THE BOUNDARY: The case $(u,,v,w) = (\mathrm{Id},\mathrm{Id},\mathrm{Id})$ is obtained whenever (x,y,a,A,b,B,c,C) satisfies the following inequalities :

$$Bc > 1 > \frac{xy}{bC}, \quad \frac{xy}{Ac} > 1 > Ca, \quad \frac{xy}{Ba} > 1 > bA.$$

Indeed, in this case $X_\alpha = F(\alpha^+,\alpha^0)$, $X^*_\alpha = F(\alpha^+,\alpha^0)$, $X_\gamma = F(\gamma^+,\gamma^0)$ and $X_\beta = F(\beta^+,\beta^0)$. Then

$$B_1 = \beta^+, \quad C_1 = \gamma^-, \ A_3 = \alpha^+, \ A_1 = \alpha^-.$$

The pant gap function is then the logarithm of

$$\mathbf{B}(B_1, A_1, C_1, C_3).$$

Using a computing software whose instructions can be found online on the ArXiv version, we obtained that the exponential of the pant gap function is equal to the following rational fraction

$$\begin{aligned} e^{G(P)} = \Big(& (ayA + yA + xyA + xy + A^2ab + abA + aybA + ybA) \\ & (xby + xy + xb + xbBC + xCb + bBx + b^2BC + bBC)xy\Big) / \Big(bx^3y^3 + bx^3y^2 \\ & +bx^3Cy^2 + x^2y^2ACb + b^2Bx^2y^2C + bBx^2y^2C + bBx^3Cy^2 + bBx^3y^2 \\ & +x^3y^3 + 2b^2Bx^2y^2AC + b^2Bx^2y^2AcC + b^3BxayAcC^2 + x^2y^2b^2ABaC \\ & +xb^3AB^2acC^2 + CbABax^2y^2 + x^3y^2ACb + xb^3A^2B^2acC^2 \\ & +b^4A^2B^2acC^2 + b^2AB^2acxyC + b^2ABaxyC + b^2Ax^2y^2C \\ & +x^2y^2AcBCb + b^3A^2B^2acC^2 + x^3y^2bA + x^3y^2bAB + 2xb^2Bay^2AcC \\ & +xb^3BA^2acCy + xy^2bBAC + xb^3By^2AcC + 2xb^3BaAcCy \\ & +xb^3BaAcy^2C + xb^3AB^2acyC + xy^2b^3ABaC + 2xb^2By^2AcC \\ & +2xb^2By^2AC + xb^3By^2AC + xb^3BA^2acC + xb^3A^2B^2acC \\ & +xb^3AB^2acC + xb^3BaAcC^2 + xb^3BaAcC + xy^2bBaAcC \\ & +2xyb^2BaAcC + xyb^2BaAcC^2 + xyb^3ABaC + xyb^3BAcC \\ & +xyb^2BA^2acC + xyb^2BAcC + x^3y^2bBAC + 2x^2y^2bBAC \\ & +x^2y^2b^2BA + x^2y^2bAB + xy^2CbABa + xy^2bBAcC + xb^3BA^2acC^2 \\ & +2xy^2b^2ABaC\Big). \end{aligned} \tag{55}$$

THE CASE OF THEOREM 10.3.1, we have

$$B_1 = \beta^+, \quad C_3 = \gamma^- \quad A_2 = \alpha^+, \quad A_3 = \alpha^-.$$

which gives the expected reasonable formula

$$G(P) = \log\left(\frac{1+B}{B(1+c)}\right). \tag{56}$$

An intermediate choice A somewhat intermediate choice is

$$B_1 = \beta+, \quad C_3 = \gamma^- \quad A_1 = \alpha^+, \quad A_3 = \alpha^-.$$

which gives

$$G(P) = \log\left(\frac{(xby + xy + xb + xbBC + xCb + bBx + b^2BC + bBC)y}{bB(bcC^2 + cCb + bcyC + yCb + cyC + yC + Cxy + xy)}\right). \quad (57)$$

12 Appendix B: computer instructions

```
> restart;
> with(linalg):
```

12.1 Matrices and holonomies

```
> Id:=linalg[matrix](3,3,[1,0,0,0,1,0,0,0,1]):
> Xf:=x->linalg[matrix](3,3,[0,0,1,0,-1,-1,x,1+x,1]):
> Af:=(a,b)->linalg[matrix](3,3,[0,0,1/b,0,-1,0,a,0,0]):
> X:=Xf(x):
> Y:=Xf(y):
> Qb:=Af(B,b):
> Qa:=Af(A,a):
> Qc:=Af(C,c):
> HolA:=evalm(X&*inverse(Qc)&*Y&*Qb);
> HolB:=evalm(X&*X&*inverse(Qa)&*Y&*Qc&*X&*X):
> HolC:=evalm(inverse(Qb)&*Y&*Qa&*X);
```

$$HolA := \begin{bmatrix} c\,B & 0 & 0 \\ (-c-1)\,B & 1 & 0 \\ (c+1+x+\frac{x}{C})\,B & -1-x-\frac{x\,(1+y)}{C} & \frac{x\,y}{C\,b} \end{bmatrix}$$

$$HolC := \begin{bmatrix} \frac{y\,x}{B\,a} & \frac{1+y}{B} + \frac{y\,(1+x)}{B\,a} & \frac{A}{B} + \frac{1+y}{B} + \frac{y}{B\,a} \\ 0 & 1 & A+1 \\ 0 & 0 & b\,A \end{bmatrix}$$

12.2 Eigenvectors and eigenvalues

```
> IA:=inverse(transpose(HolA)):
> egA:=eigenvalues(IA):
> A3:=kernel(IA-scalarmul(Id,1/(c*B)))[1];
> A2:=kernel(IA-scalarmul(Id,1))[1];
> A1:=kernel(IA-scalarmul(Id,(C*b)/(x*y)))[1];
```

$$A3 := [1,\, 0,\, 0]$$

$$A2 := \left[\frac{(c+1)\,B}{c\,B-1},\,1,\,0\right]$$

$$A1 := \left[\frac{b\,B\,x\,(b\,c\,C^2+c\,C\,b+b\,c\,y\,C+y\,C\,b+c\,y\,C+y\,C+C\,x\,y+x\,y)}{c\,B\,C^2\,b^2-c\,B\,x\,y\,C\,b-x\,y\,C\,b+x^2\,y^2},\right.$$
$$\left.-\frac{b\,(C+x\,C+x+x\,y)}{-C\,b+x\,y},\,1\right]$$

```
> IB:=HolB:
> B3:=kernel(IB-scalarmul(Id,x*C*a))[1];
> B2:=kernel(IB-scalarmul(Id,x))[1];
> B1:=kernel(IB-scalarmul(Id,(x*x*y)/(A*c)))[1];
> egB:=eigenvalues(IB):
```

$$B3 := \left[-\frac{c\,(a\,A\,x+x\,y\,a+x\,C\,a\,A+a\,x+x\,y+x\,C\,a+C\,a^2\,A+C\,a\,A)}{x\,(a\,c+y+a\,A\,c+C\,a\,A\,c+a\,c\,y+C\,a\,c+a\,y+c\,y)},\,1,\right.$$
$$\left.-\frac{C^2\,a\,c+C\,a\,c+C\,a\,c\,y+C\,a\,y+c\,y\,C+y\,C+C\,x\,y+x\,y}{(a\,c+y+a\,A\,c+C\,a\,A\,c+a\,c\,y+C\,a\,c+a\,y+c\,y)\,C}\right]$$

$$B2 := \left[-\frac{(A+x\,A+x+x\,y)\,c}{x\,(c+y+c\,y+A\,c)},\,1,\,-\frac{x\,y+y+c\,y+c}{c+y+c\,y+A\,c}\right]$$

$$B1 := [-1,\,1,\,-1]$$

```
> IC:=HolC:
> C3:=kernel(IC-scalarmul(Id,(x*y)/(B*a)))[1];
> C2:=kernel(IC-scalarmul(Id,1))[1];
> C1:=kernel(IC-scalarmul(Id,(b*A)))[1];
> egC:=eigenvalues(IC):
```

$$C3 := [1,\,0,\,0]$$

$$C2 := \left[\frac{a+a\,y+y+x\,y}{B\,a-x\,y},\,1,\,0\right]$$

$$C1 := \left[\frac{a\,y\,A+y\,A+x\,y\,A+x\,y+A^2\,a\,b+a\,b\,A+a\,y\,b\,A+y\,b\,A}{-x\,y\,A-x\,y+b\,A^2\,B\,a+b\,A\,B\,a},\,1,\,\frac{-1+b\,A}{A+1}\right]$$

12.3 Pant Gap Function

```
> S:=(u,v)->multiply(transpose(convert(u,vector)),convert(v,vector)):
> bir:=(u,v,w,z)->eval(S(u,v)*S(w,z)/(S(u,z)*S(w,v))):
> pantgap:=factor(simplify(evalm(bir(B1,A2,C3,A3))));
```

$$birapport := \frac{B+1}{(c+1)\,B}$$

```
> pantgap:=factor(simplify(evalm(bir(B1,A1,C1,A3))));
```

$$pantgap := (a\,y\,A + y\,A + x\,y\,A + x\,y + A^2\,a\,b + a\,b\,A + a\,y\,b\,A + y\,b\,A)$$
$$(x\,b\,y + x\,y + x\,b + x\,b\,B\,C + x\,C\,b + b\,B\,x + b^2\,B\,C + b\,B\,C)\,x\,y/(b\,x^3\,y^3 + b\,x^3\,y^2$$
$$+ b\,x^3\,C\,y^2 + x^2\,y^2\,A\,C\,b + b^2\,B\,x^2\,y^2\,C + b\,B\,x^2\,y^2\,C + b\,B\,x^3\,C\,y^2 + b\,B\,x^3\,y^2$$
$$+ x^3\,y^3 + 2\,b^2\,B\,x^2\,y^2\,A\,C + b^2\,B\,x^2\,y^2\,A\,c\,C + b^3\,B\,x\,a\,y\,A\,c\,C^2 + x^2\,y^2\,b^2\,A\,B\,a\,C$$
$$+ x\,b^3\,A\,B^2\,a\,c\,C^2 + C\,b\,A\,B\,a\,x^2\,y^2 + x^3\,y^2\,A\,C\,b + x\,b^3\,A^2\,B^2\,a\,c\,C^2$$
$$+ b^4\,A^2\,B^2\,a\,c\,C^2 + b^2\,A\,B^2\,a\,c\,x\,y\,C + b^2\,A\,B\,a\,x\,y\,C + b^2\,A\,x^2\,y^2\,C$$
$$+ x^2\,y^2\,A\,c\,B\,C\,b + b^3\,A^2\,B^2\,a\,c\,C^2 + x^3\,y^2\,b\,A + x^3\,y^2\,b\,A\,B + 2\,x\,b^2\,B\,a\,y^2\,A\,c\,C$$
$$+ x\,b^3\,B\,A^2\,a\,c\,C\,y + x\,y^2\,b\,B\,A\,C + x\,b^3\,B\,y^2\,A\,c\,C + 2\,x\,b^3\,B\,a\,A\,c\,C\,y$$
$$+ x\,b^3\,B\,a\,A\,c\,y^2\,C + x\,b^3\,A\,B^2\,a\,c\,y\,C + x\,y^2\,b^3\,A\,B\,a\,C + 2\,x\,b^2\,B\,y^2\,A\,c\,C$$
$$+ 2\,x\,b^2\,B\,y^2\,A\,C + x\,b^3\,B\,y^2\,A\,C + x\,b^3\,B\,A^2\,a\,c\,C + x\,b^3\,A^2\,B^2\,a\,c\,C$$
$$+ x\,b^3\,A\,B^2\,a\,c\,C + x\,b^3\,B\,a\,A\,c\,C^2 + x\,b^3\,B\,a\,A\,c\,C + x\,y^2\,b\,B\,a\,A\,c\,C$$
$$+ 2\,x\,y\,b^2\,B\,a\,A\,c\,C + x\,y\,b^2\,B\,a\,A\,c\,C^2 + x\,y\,b^3\,A\,B\,a\,C + x\,y\,b^3\,B\,A\,c\,C$$
$$+ x\,y\,b^2\,B\,A^2\,a\,c\,C + x\,y\,b^2\,B\,A\,c\,C + x^3\,y^2\,b\,B\,A\,C + 2\,x^2\,y^2\,b\,B\,A\,C$$
$$+ x^2\,y^2\,b^2\,B\,A + x^2\,y^2\,b\,A\,B + x\,y^2\,C\,b\,A\,B\,a + x\,y^2\,b\,B\,A\,c\,C + x\,b^3\,B\,A^2\,a\,c\,C^2$$
$$+ 2\,x\,y^2\,b^2\,A\,B\,a\,C)$$

```
>  birapport:=factor(simplify(evalm(bir(B1,A1,C3,A3))));
```

$$pantgap := \frac{(x\,b\,y + x\,y + x\,b + x\,b\,B\,C + x\,C\,b + b\,B\,x + b^2\,B\,C + b\,B\,C)\,y}{b\,B\,(b\,c\,C^2 + c\,C\,b + b\,c\,y\,C + y\,C\,b + c\,y\,C + y\,C + C\,x\,y + x\,y)}$$

13 Appendix B: positivity of Frenet curves

13.1 Positivity of triples

We first prove.

Proposition 13.1.1 *Let ξ^1 be a Frenet curve. Let I_1, I_2 and I_3 be three disjoint subintervals of $\mathbb{T}$. For $i = 1, 2, 3$, let $X_i = (x_1^i, \ldots, x_{n-1}^i)$ be an $(n-1)$-tuple of distinct points of I_i. Let F_i be the flag given by*

$$F_i^k = \sum_{j=1}^{k} \xi^1(x_j^i).$$

Then the triple (F_1, F_2, F_3) is positive.

Proof: Let t be a point in $\mathbb{T} \setminus (I_1 \cup I_2 \cup I_3)$. We choose a continuous map $\widehat{\xi}$

$$\begin{cases} \mathbb{T} \setminus \{t\} & \to & \mathbb{R}^n \setminus \{0\}, \\ u & \mapsto & \widehat{\xi}(u) \in \xi(u). \end{cases}$$

We choose an orientation Ω on $\mathbb{R}^n$ such that for any positively oriented n-tuple of points $\{y_1, \ldots, y_n\}$ in $\mathbb{T} \setminus \{t\}$ we have

$$\Omega(\widehat{\xi}(y_1) \wedge \ldots \wedge \widehat{\xi}(y_n)) > 0. \tag{58}$$

We write

$$X_i^p = \widehat{\xi}^1(x_1^i) \wedge \ldots \wedge \widehat{\xi}^1(x_p^i).$$

For each $i, 1 \leq i \leq 3$ and each $p, 1 \leq p \leq n$ let $\sigma_{i,p}$ be a permutation such that $(x^i_{\sigma_{i,p}(1)}, \ldots, x^i_{\sigma_{i,p}(p)})$, is positively oriented.

We also define $\alpha(I_1, I_2, I_3) = +1$, if (a_1, a_2, a_3) is positively oriented for any a_i in I_i and $\alpha(I_1, I_2, I_3) = -1$ in the opposite case. Thus the sign of

$$\Omega(X_1^m \wedge X_2^l \wedge X_3^p),$$

coincides with the sign of

$$\varepsilon(\sigma_{1,m}).\varepsilon(\sigma_{2,l}).\varepsilon(\sigma_{3,p}))\alpha(I_1, I_2, I_3)^{pl},$$

where $\varepsilon(\sigma)$ is the signature of the permutation σ.

When we report this information in the ratio given in Definition 7.1.2, we obtain that each of the signatures above appear twice, and thus the sign of $T^{m,l,p}(F_1, F_2, F_3)$ coincides with the sign of

$$\alpha(I_1, I_2, I_3)^{(p-1)l+p(l+1)+(p+1)(l+1)-p(l-1)-(p-1)(l+1)-(p+1)l} = \alpha(I_1, I_2, I_3)^{2p+2}.$$

Thus the triple (F_1, F_2, F_3) is positive. Q.E.D.

The following is essentially a corollary

Proposition 13.1.2 *Let ξ^1 be a Frenet curve from $\mathbb{T}$ to $\mathbb{P}(\mathbb{R}^n)$ with osculating flag curve $\xi = (\xi^1, \xi^2, \ldots, \xi^{n-1})$. Then $(\xi(y_1), \xi(y_2), \xi(y_3))$ is a positive triple of flags whenever (y_1, y_2, y_3) is a triple of distinct points of $\mathbb{T}$.*

More generally, if $(y_1^+, y_1^-, y_2^+, y_2^-, y_3^+, y_3^-)$ is a positively oriented sextuplet of points in S^1 and Y_i, $i = 1, 2, 3$ is a flag compatible with $(\xi(y_i^+), \xi(y_i^-))$ then (Y_1, Y_2, Y_3) is a positive flag.

Proof: For the first part, let I_1, I_2 and I_3 be three disjoint subintervals of $\mathbb{T}$ and $X_i = (x_1^i, \ldots, x_{n-1}^i)$ for $i = 1, 2, 3$ be three $(n-1)$-tuples of distinct points $X_i \subset I_i$. As before set

$$F_i^k = \sum_{j=1}^{k} \xi^1(x_j^i).$$

It follows that the triple of flags (F_1, F_2, F_3) satisfy the hypothesis of Proposition 13.1.1 above. Hence, this triple is positive.

Now we let x_j^i tend to y_j. Recall that ξ is a Frenet curve and so satisfies Conditions (31) and (32). Firstly, since ξ satisfies Condition (31) by Definition 7.1.2, all the triple ratios associated to $(\xi(y_1), \xi(y_2), \xi(y_3))$ are nonzero. Secondly, since ξ satisfies Condition (32), $\xi(y_i)$ is a limit of F_i, hence all the triple ratios of the $(\xi(y_1), \xi(y_2), \xi(y_3))$ are nonnegative. As a conclusion, $(\xi(y_1), \xi(y_2), \xi(y_3))$ is a positive triple.

The proof of the second part is a natural extension of this argument. One merely observes that a flag compatible with $(\xi(x_+), \xi(x_-))$ is a limit of flags constructed from direct sums of $\xi(x_+^i)$ and $\xi(x_-^j)$ for points x_+^i close to x_+ and x_-^j close to x_-. Q.E.D.

13.2 Positivity of quadruples

The same argument as in the previous paragraph yields

Proposition 13.2.1 *Let ξ^1 be a Frenet curve. Let I_1, I_2, I_3 and I_4 be four disjoint subintervals of $\mathbb{T}$ such that some (and so any) quadruple of points $t_k \in I_k$ is cyclically ordered. For $i = 1, 2, 3, 4$ let $X_i = (x_1^i, \ldots, x_{n-1}^i)$ be an $(n-1)$-tuple of distinct points of I_i. Let F_i be the flag given by*

$$F_i^k = \sum_{j=1}^{k} \xi^1(x_j^i).$$

Then the quadruple (F_1, F_2, F_3, F_4) is positive.

PROOF: By construction, the sign of $\delta_i(F_1, F_2, F_3, F_4)$ depends continuously on (X_1, X_2, X_3, X_4). Moreover, when x_1^1 converges to x_1^4, $\delta_i(F_1, F_2, F_3, F_4)$ converges to 1 by Equation (28). The proposition follows from this remark and Proposition 13.1.1. Q.E.D.

Proposition 13.2.2 *Let ξ^1 be a Frenet curve from S^1 to $\mathbb{P}(\mathbb{R}^n)$ with osculating flag curve $\xi = (\xi^1, \xi^2, \ldots, \xi^{n-1})$. Then for every positively oriented quadruple of distinct points (y_1, y_2, y_3, y_4) in S^1, $(\xi(y_1), \xi(y_2), \xi(y_3), \xi(y_4))$ is a positive quadruple of flags. More generally, let*

$$(y_1^+, y_1^-, y_2^+, y_2^-, y_3^+, y_3^-, y_4^+, y_4^-),$$

be a cyclically ordered octuplet of points in S^1. Let Y_i be a flag compatible with $(\xi(y_i^+), \xi(y_i^-))$, then (Y_1, Y_2, Y_3, Y_4) is a positive quadruple.

PROOF: We use a similar argument as in Proposition 13.1.2 in order to obtain the flags Y_i as limits of flags of the type described in Proposition 13.2.1. Then, the edge functions for the flags Y_i are nonnegative by a limiting argument, and nonzero by the Frenet property. The result follows. Q.E.D.

References

[1] Hirotaka Akiyoshi, Hideki Miyachi, and Makoto Sakuma, *Variations of McShane's identity for punctured surface groups*, Spaces of Kleinian groups, London Math. Soc. Lecture Note Ser., vol. 329, Cambridge Univ. Press, Cambridge, 2006, pp. 151–185.

[2] Joan S. Birman and Caroline Series, *Geodesics with bounded intersection number on surfaces are sparsely distributed*, Topology **24** (1985), no. 2, 217–225.

[3] Francis Bonahon, *Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form*, Ann. Fac. Sci. Toulouse Math. (6) **5** (1996), no. 2, 233–297.

[4] Francis Bonahon and Yaşar Sözen, *The Weil-Petersson and Thurston symplectic forms*, Duke Math. J. **108** (2001), no. 3, 581–597.

[5] Marc Bourdon, *Sur le birapport au bord des* CAT(-1)*-espaces*, Inst. Hautes Études Sci. Publ. Math. (1996), no. 83, 95–104.

[6] Brian H. Bowditch, *A proof of McShane's identity via Markoff triples*, Bull. London Math. Soc. **28** (1996), no. 1, 73–78.

[7] Vladimir V. Fock and Alexander B. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Inst. Hautes Études Sci. Publ. Math. (2006), no. 103, 1–211.

[8] ______, *Moduli spaces of convex projective structures and surfaces*, Adv. in Math. **208** (2007), 249–273.

[9] W. J. Floyd. *Group completions and limit sets of Kleinian groups*, Invent. Math. , **57** (1980), no. 3, 205–218.

[10] Mikhael Gromov. Hyperbolic manifolds, groups and actions. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 183–213. Princeton Univ. Press, Princeton, N.J., 1981.

[11] Mikhael Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[12] Ursula Hamenstädt, *Cocycles, Hausdorff measures and cross ratios*, Ergodic Theory Dynam. Systems **17** (1997), no. 5, 1061–1081.

[13] François Labourie, *Cross ratios, Anosov representations and the energy functional on Teichmüller space.*, Ann. Sci. École Norm. Sup. (4) (To appear).

[14] ______, *Anosov flows, surface groups and curves in projective space*, Invent. Math. **165** (2006), no. 1, 51–114.

[15] ______, *Cross ratios, surface groups,* $\mathrm{PSL}(n,\mathbf{R})$ *and diffeomorphisms of the circle*, Publ. Math. Inst. Hautes Études Sci. (2007), no. 106, 139–213.

[16] François Ledrappier, *Structure au bord des variétés à courbure négative*, Séminaire de Théorie Spectrale et Géométrie, No. 13, Année 1994–1995, Sémin. Théor. Spectr. Géom., vol. 13, Univ. Grenoble I, Saint, 1995, pp. 97–122.

[17] George Lusztig, *Total positivity in reductive groups*, Lie theory and geometry, Progr. Math., vol. 123, Birkhäuser Boston, Boston, MA, 1994, pp. 531–568.

[18] Gregory McShane, *Simple geodesics and a series constant over Teichmüller space*, Invent. Math. **132** (1998), no. 3, 607–632.

[19] Maryam Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, Invent. Math. **167** (2007), no. 1, 179–222.

[20] Jacob Nielsen. *Untersuchungen zur Topologie der geschlossenen zweiseitigen Flächen*, Acta Math., **50** (1927), no. 1 :189–358.

[21] Jean-Pierre Otal, *Le spectre marqué des longueurs des surfaces à courbure négative*, Ann. of Math. (2) **131** (1990), no. 1, 151–162.

[22] ______, *Sur la géometrie symplectique de l'espace des géodésiques d'une variété à courbure négative*, Rev. Mat. Iberoamericana **8** (1992), no. 3, 441–456.

[23] Ser Peow Tan, Yan Loi Wong, and Ying Zhang, *McShane's identities for classical Schottky groups*, ArXiv: math.GT/0411628, 2004.

[24] William P. Thurston, *Minimal stretch maps between hyperbolic surfaces*, ArXiv: math.GT/9801039, 1984.