CYCLIC SURFACES AND HITCHIN COMPONENTS IN RANK 2

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Abstract. We prove that given a Hitchin representation in a real split rank 2 group G_0, there exists a unique equivariant minimal surface in the corresponding symmetric space. As a corollary, we obtain a parametrization of the Hitchin components by a Hermitian bundle over Teichmüller space. The proof goes through introducing holomorphic curves in a suitable bundle over the symmetric space of G_0. Some partial extensions of the construction hold for cyclic bundles in higher rank.

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1. Introduction

We will study in this article minimal surfaces in rank 2 symmetric spaces. More precisely, let G_0 be a real split simple Lie group of rank 2 and $\mathsf{S}(\mathsf{G}_0)$ be the associated symmetric space. We will take G_0 to be the connected component of the isometry group of $\mathsf{S}(\mathsf{G}_0)$. The group G_0 is in particular locally isomorphic to $\mathsf{SL}(3,\mathbb{R})$, $\mathsf{Sp}(4,\mathbb{R})$ or G_2.

Let Σ be a connected oriented closed surface of genus greater than 2. We consider *Hitchin representations* from $\pi_1(S)$ with values in G_0. Recall that those are deformations *Fuchsian representations*, that is discrete faithful representations in the principal SL_2 in G_0 (See Paragraph 5.2.1 for details). The *Hitchin component* $\mathcal{H}(\Sigma,\mathsf{G}_0)$ is then set the space

of Hitchin representations up to conjugation by the automorphism group of G_0. By Hitchin [22], the Hitchin component is a smooth manifold consisting of irreducible representations. From [29] for $\mathsf{PSL}(n,\mathbb{R})$ (and the split groups contained in such) completed by Fock and Goncharov [13] for the remaining cases, a Hitchin representation is discrete faithful.

Hitchin representation have a geometric interpretation. For $\mathsf{PSL}(2,\mathbb{R})$, Hitchin representations are monodromies of hyperbolic structures, for $\mathsf{PSL}(3,\mathbb{R})$ they are monodromies of convex real projective structures by Choi and Goldman [17], in general Guichard and Wienhard have shown they are monodromies of geometric structures on higher dimensional manifolds [20]. The special case of $\mathsf{SP}(4,\mathbb{R})$ has been dealt by these latter authors as convex foliated projective structures in [19].

1.1. **Minimal surfaces.** One of our two main results is the following

Theorem 1.1.1. *Given a Hitchin representation δ, there exist a unique δ-equivariant minimal mapping from Σ to $\mathsf{S}(\mathsf{G}_0)$.*

The existence was proved by the author in [31] without any assumption on the rank.

The case of $\mathsf{SL}(3,\mathbb{R})$ was proved by the author in [30]. The new cases are thus $\mathsf{Sp}(4,\mathbb{R})$ and G_2, however the proof is general. Interestingly enough, the theorem is also valid when G_0 is semisimple, that is $\mathsf{G}_0 = \mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R})$. This was done by R. Schoen in [36] and see also [6] for generalizations.

1.2. **Parametrisation of Hitchin components.** Using *Hitchin parametrisation* of the space of minimal surfaces [22, 31] one obtains equivalently the following Theorem

Theorem 1.2.1. *There exists an analytic diffeomorphism, equivariant under the mapping class group action, from the Hitchin component $\mathcal{H}(\Sigma,\mathsf{G}_0)$ for G_0, when G_0 is of real rank 2, to the space of pairs (J,Q) where J is a complex structure on Σ and Q a holomorphic differential with respect to J of degree $\frac{\dim(\mathsf{G}_0)-2}{2}$.*

For $\mathsf{SL}(3,\mathbb{R})$, this corollary was obtained by Loftin in [33] and the author in [30] (announced in [28]) Theorem 1.2.1 holds for compact surfaces. In the non compact case, the natural question is to extend the remarkable results that have been obtained in the case of $\mathsf{SL}(3,\mathbb{R})$, on one hand for polynomial cubic differentials (Dumas and Wolf – announced) and on the other hand for the unit disk (Benoist and Hulin [3] .

1.3. **Kähler structures.** Using the theory of positive bundles and the work of Bo Berndtsson [4], we extend a result obtained by Inkang Kim and Genkai Zhang [24] for cubic holomorphic differential to get

Proposition 1.3.1. *Let* $\mathrm{m} = (m_1, \dots, m_p)$ *be a p-tuples integers greater than 1. Let* $\mathcal{E}(\mathrm{m})$ *be the holomorphic line bundle over Teichmüller space whose fiber at a Riemann surface* Σ *is*

$$\mathcal{E}(\mathrm{m})_\Sigma := \bigoplus_{n=1,\dots,p} H^0\left(\Sigma, \mathcal{K}^{m_i}\right). \tag{1}$$

Then $\overline{\mathcal{E}(\mathrm{m})}$ *carries a* $p-1$*-dimensional family of mapping class group invariant Kähler metrics, linear along the fibers and whose restriction to the zero section is the Weil–Petersson metric.*

The metric and its properties are given explicitly in Section 9.

Corollary 1.3.2. *The Hitchin component* $\mathcal{H}(\Sigma, \mathsf{G}_0)$*, when* G_0 *is of real rank 2, carries a* 1*-dimensional family of mapping class group invariant Kähler metrics for which the Fuchsian locus is totally geodesic and whose restriction to the Fuchsian locus is the Weil–Petersson metric.*

Still the relation of this metric and complex structure with other objects such as the Atiyah–Bott–Goldman symplectic form [1, 16], the pressure metric of [9] or the metric exhibited by Qiongling Li [32] for convex projective structures is rather mysterious.

1.4. **Area rigidity.** Let $\mathcal{T}(S)$ be the Teichmüller space of S. For a Hitchin representation δ, let us define as in [31]

$$\mathrm{MinArea}(\delta) := \inf\{e_\delta(J) \mid J \in \mathcal{T}(S)\},$$

where $e_\delta(J)$ is the energy of the unique δ-equivariant harmonic map from Σ equipped with J to $\mathsf{S}(\mathsf{G}_0)$, equipped with the symmetric metric normalised so that the principal hyperbolic plane has curvature -1.

Motivated by a question of Anna Wienhard, we obtain

Theorem 1.4.1. [Area Rigidity] *The following inequality holds*

$$\mathrm{MinArea}(\delta) \geqslant -2\pi \cdot \chi(\Sigma). \tag{2}$$

Moreover, the equality holds if and only if δ *is a fuchsian representation.*

By Katok Theorem [23], since the intrinsic metric of any minimal surface is non positively curved, we have the inequality

$$\mathrm{MinArea}(\delta) h(S) \geqslant -2\pi \cdot \chi(\Sigma). \tag{3}$$

Where $h(S)$ is the entropy of the induced metric on S. On the other hand $h(S) \geqslant h(\delta)$, where $h(\delta)$ is the *entropy* of $\mathsf{S}(\mathsf{G}_0)/\delta(\pi_1(S)$ seen as

the asymptotic growth of the length of closed geodesics. Thus one immediately gets

$$\mathrm{MinArea}(\delta) \geqslant -\frac{2\pi \cdot \chi(\Sigma)}{h(\delta)}. \tag{4}$$

Thus the previous result would also a consequence of the *entropy rigidity conjecture* for Hitchin representations: $h(\delta) \leqslant 1$ with equality of and only if δ is Fuchsian.

1.5. **Cyclic Higgs bundles.** For higher rank, we only have a very partial result. Let m_i is the highest exponent of G_0. Following Baraglia [2], let us call $\mathcal{E}_{m_i}$ the space of *cyclic bundles*. The Hitchin section gives an analytic map Ψ from the space of cyclic bundles to the Hitchin component $\mathcal{H}(\Sigma, \mathsf{G}_0)$. We then have,

Theorem 1.5.1. *The map* $\Psi : \mathcal{E}_{m_i} \to \mathcal{H}(\Sigma, \mathsf{G}_0)$ *is an immersion.*

It would be nice to understand in a geometric way the image of Ψ.

1.6. **Cyclic surfaces and the idea of the proof.** The main idea of the proof is to work with *cyclic surfaces* which are holomorphic curves (in a certain sense) in a bundle X over the symmetric space. First we show in Section 6 that the minimal surfaces constructed by Hitchin actually lift to X as cyclic surfaces, and conversely every projection of a cyclic surface is minimal. This is strongly related to a work of Baraglia [2] and cousin to a construction by Bolton, Pedit and Woodward in [5]. Then, complexifying the situation and treating cyclic surfaces as solutions to a Pfaffian system, the core of the proof is to prove an infinitesimal rigidity result for cyclic surfaces in Section 7. To conclude, we use in Section 8 the main result of [31].

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2. Lie theory preliminaries

In this section, we review for the convenience of the reader, without proof, root systems with applications to the construction of the maximal compact subgroup, and that of the split form of a complex simple Lie group.

We explain that the choice of a Cartan sub algebra, a positive root system and a Chevalley system define naturally two commuting anti linear automorphims of the Lie algebra, whose fixed points are respectively the maximal compact subgroup, and the maximal split form. We study the basic properties of these objects.

We also introduce the cyclic roots set and prove Proposition 2.2.2 that will play a central rôle in the proof.

The material comes form Baraglia [2], Kostant [26], Hitchin [22], Bourbaki [7]. The only non standard material (which is probably common lore) is in Section 2.2.1.

We will use the following typographic convention: for any Lie group H, we shall denote by $\mathfrak{h}$ its Lie algebra.

2.1. **Roots.** We recall the notations for the root systems. Let G be a complex simple Lie group. Let H be its maximal abelian semisimple subgroup and $\mathfrak{h}$ its Lie algebra called the *Cartan sub algebra*. The dimension of $\mathfrak{h}$, denoted $\mathrm{rank}(\mathsf{G})$, is the *rank of* G. We denote by $\langle .|. \rangle$ the Killing form of $\mathfrak{g}$, and by the same symbol the restriction of the Killing form to $\mathfrak{h}$ and its dual extension to $\mathfrak{h}^* := \hom(\mathfrak{h}, \mathbb{C})$.

2.1.1. *Roots, positive roots.* For any α in $\mathfrak{h}^*$, we denote by $\mathfrak{g}_\alpha$, the subset of $\mathfrak{g}$ defined by

$$\mathfrak{g}_\alpha := \{u \in \mathfrak{g} | , \forall x \in \mathfrak{h}, \ \ [x, u] = \alpha(x).u\}.$$

A *root* in an element of $\mathfrak{h}^*$ such that $\dim(\mathfrak{g}_\alpha) > 0$. Let Δ be the set of roots. Then we have

Proposition 2.1.1. *The set Δ spans $\mathfrak{h}^*$. We have the direct sum decomposition, called* root space decomposition

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha. \tag{5}$$

Moreover for any root

$$\dim(\mathfrak{g}_\alpha) = 1.$$

2.1.2. *Positive roots.* We can also define (up to a choice) a subset Δ^+ of *positive roots* and a subset Π of *simple* roots, satisfying the following conditions

(1) $\Pi \subset \Delta^+ \subset \Delta$.
(2) There exists $x \in \mathfrak{h}$ so that $\Delta^+ = \{\alpha \in \Delta | \alpha(x) > 0\}$,
(3) Every positive root can be written uniquely as a weighted sum with positive coefficients of simple roots.
(4) if α and β are simple, then $\alpha - \beta$ is not a root.

In particular, the cardinal of Π is the rank of G. For any root α, the *coroot* h_α in $\mathfrak{h}$ is so that

$$\langle \mathrm{h}_\alpha | u \rangle = \frac{2}{\langle \alpha | \alpha \rangle} \alpha(u).$$

Given a positive root $\beta = \sum_{\alpha \in \Pi} n_\alpha \cdot \alpha$, the *degree* of β is

$$\deg(\beta) := \sum_{\alpha \in \Pi} n_\alpha. \tag{6}$$

The *longest root* is the uniquely defined positive root η so that for any positive root β, $\beta + \eta$ is not a root.

We recall that if $N(\mathfrak{h})$ is the normalizer of $\mathfrak{h}$ in $\mathrm{Aut}\,\mathsf{G}$), and if $w \in N(\mathfrak{h})$ satisfies $w \cdot \mathrm{h}_\eta = \mathrm{h}_\eta$, then w fixes $\mathfrak{h}$ pointwise.

2.1.3. *Chevalley systems.* A *Chevalley base* is a collection of non zero vectors $\{\mathrm{x}_\alpha\}_{\alpha \in \Delta}$ in $\mathfrak{g}$ so that

(1) there exists integers $N_{\alpha,\beta}$ such that,

$$\begin{aligned} \mathrm{x}_\alpha &\in \mathfrak{g}_\alpha \\ [\mathrm{x}_\alpha, \mathrm{x}_{-\alpha}] &= \mathrm{h}_\alpha, \\ [\mathrm{x}_\alpha, \mathrm{x}_\beta] &= N_{\alpha,\beta} \mathrm{x}_{\alpha+\beta}, \end{aligned}$$

(2) the anti linear endomorphism preserving $\mathfrak{h}$ and sending x_α to $\mathrm{x}_{-\alpha}$ is an anti linear automorphism of $\mathfrak{g}$.

By ([7] Ch. 7, proposition 7), Chevalley basis exist. Moreover $N_{\alpha,\beta}$ only depends on the root system.

2.2. **Cyclic roots and projections.** Let G, Δ, Δ^+ and Π be as above and η the longest positive root. Recall that we have

$$\forall \alpha \in \Delta^+ : \quad \deg(\alpha) \leqslant \deg(\eta),$$

with equality only if $\eta = \alpha$.

Definition 2.2.1. [Cyclic root sets]

(1) The conjugate cyclic root set *is*

$$Z^\dagger := \Pi \cup \{-\eta\}. \tag{7}$$

(2) The cyclic root set *is*

$$Z := \{\alpha \in \Delta \mid -\alpha \in Z^\dagger\}. \tag{8}$$

Observe that $\Delta \setminus \left(Z \sqcup Z^\dagger\right) = \{\alpha, |\deg(\alpha)| \neq 1, \deg(\eta)\}$.

2.2.1. *Projections.* We consider the following projections (whose pairwise disjoint product are zero) from $\mathfrak{g}$ to itself that comes from the decomposition (5)

$$\pi_0 : \mathfrak{g} \to \mathfrak{h}, \tag{9}$$

$$\pi : \mathfrak{g} \to \mathfrak{g}_Z := \bigoplus_{\alpha \in Z} \mathfrak{g}_\alpha, \tag{10}$$

$$\pi^\dagger : \mathfrak{g} \to \mathfrak{g}_{Z^\dagger} := \bigoplus_{\alpha \in Z^\dagger} \mathfrak{g}_\alpha, \tag{11}$$

$$\pi_1 : \mathfrak{g} \to \mathfrak{g}_1 := \bigoplus_{\alpha \notin Z \cup Z^\dagger} \mathfrak{g}_\alpha. \tag{12}$$

$$\tag{13}$$

Obviously

$$\pi + \pi^\dagger + \pi_0 + \pi_1 = \mathrm{Id}.$$

2.2.2. *Brackets of cyclic roots.* The main observation about this decomposition is the following trivial but crucial observation

Proposition 2.2.2. [Brackets] *We have*

$$[\mathfrak{h}, \mathfrak{g}_{Z^\dagger}] \subset \mathfrak{g}_{Z^\dagger}, \tag{14}$$

$$[\mathfrak{h}, \mathfrak{g}_Z] \subset \mathfrak{g}_Z, \tag{15}$$

$$[\mathfrak{h}, \mathfrak{g}_1] \subset \mathfrak{g}_1, \tag{16}$$

$$[\mathfrak{g}_Z, \mathfrak{g}_{Z^\dagger}] \subset \mathfrak{h}, \tag{17}$$

$$[\mathfrak{g}_Z, \mathfrak{g}_1] \subset \mathfrak{g}_1 \oplus \mathfrak{g}_{Z^\dagger}, \tag{18}$$

$$[\mathfrak{g}_{Z^\dagger}, \mathfrak{g}_1] \subset \mathfrak{g}_1 \oplus \mathfrak{g}_Z, \tag{19}$$

$$[\mathfrak{g}_Z, \mathfrak{g}_Z] \subset \mathfrak{g}_1 \oplus \mathfrak{g}_{Z^\dagger} \tag{20}$$

$$[\mathfrak{g}_{Z^\dagger}, \mathfrak{g}_{Z^\dagger}] \subset \mathfrak{g}_1 \oplus \mathfrak{g}_Z. \tag{21}$$

Proof. In this proof, we write for any $w \in \mathfrak{g}$

$$w = w_0 + \sum_{\alpha \in \Delta} w_\alpha,$$

where $w_0 \in \mathfrak{h}$ and $w_\alpha \in \mathfrak{g}_\alpha$. Observe that Assertions (14),(15) and (16) follows from the fact that

$$[\mathfrak{h}, \mathfrak{g}_\alpha] \subset \mathfrak{g}_\alpha.$$

We will now use the following two facts in the proof

(1) If α and β are distinct simple roots, then

$$\alpha - \beta = \gamma,$$

is not a root.

(2) If α is a positive root and η the longest root then $\alpha+\eta$ is not a root.

Combining the two, we get that if $\alpha \in Z$ and $\beta \in Z^{\dagger}$ then $\alpha+\beta$ is not a root unless $\alpha = -\beta$. Thus if $v \in \mathfrak{g}_Z$ and $u \in \mathfrak{g}_{Z^{\dagger}}$ then

$$[u, v] = [u_\eta, v_{-\eta}] + \sum_{\beta\in\Pi}[u_\beta, v_{-\beta}] \in \mathfrak{h}$$

This proves (17).

Next we observe, let α be a simple root and $\gamma \notin Z \cup Z^{\dagger}$ a root of length a, then the length of $\alpha+\gamma$ (if it is a root) is $a+1$. Next we observe

- Since $a \neq -1$, then $a+1 \neq 0$, thus $[\mathfrak{g}_\alpha, \mathfrak{g}_\gamma] \cap \mathfrak{h} = \{0\}$,
- Since $a \neq 0$, then $a+1 \neq 1$, moreover $a+1 \neq -\ell(\eta)$, thus $[\mathfrak{g}_\alpha, \mathfrak{g}_Z] \cap = \{0\}$.

Similarly

- Since $a \neq \ell(\eta)$, then $a-\ell(\eta) \neq 0$, thus $[\mathfrak{g}_{-\eta}, \mathfrak{g}_\gamma] \cap \mathfrak{h} = \{0\}$,
- Since $a \neq 0$, then $a-\ell(\eta) \neq -\ell(\eta)$, moreover $a-\ell(\eta) \neq -1$, thus $[\mathfrak{g}_{-\eta}, \mathfrak{g}_Z] \cap = \{0\}$.

This finishes proving Assertion (18). Assertion (19) follows by symmetry.

Finally, Assertion (20) follows from the fact that if α and β belong to Z, then $\alpha+\beta \notin Z$: if α and β are both simple, then $\alpha+\beta$ is not simple and positive, if α is simple and $\beta = -\eta$, then $\alpha+\beta$ is negative and not the longest. Assertion (20) follows by symmetry. □

2.3. **The principal 3-dimensional subalgebras.** We begin by recalling the existence and properties of the principal $\mathfrak{sl}(2,\mathbb{C})$. Let a in $\mathfrak{h}$ and r_α in $\mathbb{R}$ be defined by

$$a := \frac{1}{2}\sum_{\alpha\in\Delta^+} \mathrm{h}_\alpha =: \sum_{\alpha\in\Pi} r_\alpha \cdot \mathrm{h}_\alpha.$$

Let now

$$X := \sum_{\alpha\in\Pi} \sqrt{r_\alpha}\mathrm{x}_\alpha,\ Y := \sum_{\alpha\in\Pi} \sqrt{r_\alpha}\mathrm{x}_{-\alpha}.$$

Then, we have

Proposition 2.3.1. [Kostant] *The span of (a, X, Y) is a subalgebra $\mathfrak{s}$ isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ so that*

$$[a, X] = X,\ \ [a, Y] = -Y,\ \ [X, Y] = a.$$

Moreover for any root α, we have

$$\deg(\alpha) = \alpha(a). \tag{22}$$

We remark that we follow here the convention by Kostant and not the one by Bourbaki on the canonical basis for the Lie algebra of $\mathfrak{sl}_2$: they differ by a factor 2.

Definition 2.3.2. [Principal subalgebras]

- *A 3-dimensional subalgebra $\mathfrak{s}$ isomorphic to $\mathfrak{sl}_2$ is a* principal subalgebra *if it contains an element conjugate to a.*
- *A principal subalgebra is an $\mathfrak{h}$-*principal subalgebra*, if it intersects non trivially a Cartan subalgebra $\mathfrak{h}$.*
- *A* principal SL_2 *in a complex simple group is a group whose Lie algebra is a principal subalgebra.*
- *A* principal SL_2 *in a split real group is a group whose complexification is a principal* SL_2.

As an example, the Lie algebra $\mathfrak{s}$ generated by (a, X, Y) is an $\mathfrak{h}$-principal sub algebra of $\mathfrak{g}$. We then have from Theorem 4.2 in [26],

Proposition 2.3.3. *Any two principal subalgebra are conjugate, and moreover any two principal subalgebra intersecting $\mathfrak{h}$ are conjugated by an element of $\mathfrak{h}$.*

2.3.1. *Exponents and decomposition under the principal subalgebra.* We use the notation of the previous paragraph. Let $\ell := \operatorname{rank}(\mathsf{G})$. Then have

Proposition 2.3.4. [Kostant] *There exists a increasing sequence of integers $\{m_1, \dots, m_\ell\}$ called the* exponents *of G such that the Lie algebra $\mathfrak{g}$ decomposes as the sum of ℓ irreducible representations of $\mathfrak{s}$*

$$\mathfrak{g} = \sum_{i=1}^{\ell} \mathfrak{v}_i,$$

with $\dim(\mathfrak{v}_i) = 2m_i + 1$. *Moreover* $\mathfrak{v}_1 = \mathfrak{s}$.

Observe that $m_1 = 1$. For a rank 2 group, we furthermore have

$$m_2 = \frac{\dim(\mathsf{G}) - 4}{2}$$

Thus for G_2, $m_2 = 5$, for $\mathsf{Sp}(4, \mathbb{R})$, $m_2 = 3$, for $\mathsf{SL}(3, \mathbb{R})$, $m_2 = 2$.

Fixing now a Chevalley base and the associated generators (a, X, Y) of $\mathfrak{s}$, we recall that a *highest weight vector* in $\mathfrak{v}_i$ is an eigenvector e_i of a satisfying $[e_i, X] = 0$. The highest weight vectors in $\mathfrak{v}_i$ generates a line. Observe that we have

Proposition 2.3.5. *Let η be the longest root. Then*

$$m_\ell = \deg(\eta), \tag{23}$$

and x_η is a highest weight vector in $\mathfrak{v}_\ell$.

Proof. Observe that a highest weight vector is an eigenvector of $\mathrm{ad}(a)$ with eigenvalue m_i being the largest eigenvalue of a in $\mathfrak{v}_i$. It follows that m_ℓ is the the largest eigenvalue of a on $\mathfrak{g}$, and thus $m_\ell = \deg(\eta)$. This proves the first part.

Next, we observe that

$$[\mathrm{x}_\eta, X] = \sum_{\alpha\in\Pi} \sqrt{r_\alpha}[\mathrm{x}_\eta, \mathrm{x}_\alpha] = 0,$$

since for all positive root $\eta + \alpha$ is not a root. Since moreover

$$[\mathrm{x}_\eta, a] = \deg(\eta) \cdot \mathrm{x}_\eta = m_\ell \cdot \mathrm{x}_\eta.$$

It thus follows that x_η is a highest weight vector in $\mathfrak{v}_\ell$. □

2.4. **The maximal compact subgroup and the first conjugation.** Let $\mathfrak{h}$ be a Cartan subalgebra.

Definition 2.4.1. *An anti-linear involution ρ which*

- *is a Lie algebra automorphism,*
- *globally preserves $\mathfrak{h}$,*
- *is so that $(u, v) \mapsto -\langle u|\rho(v)\rangle$ is definite positive,*

is called an $\mathfrak{h}$-Cartan involution.

We then have

Proposition 2.4.2. *The set of fixed points of a Cartan involution is the Lie algebra $\mathfrak{k}$ of a maximal compact subgroup K,*

2.4.1. *Existence of a Cartan involution.* We have the following proposition from [7]

Proposition 2.4.3. *Given a Chevalley base $\{\mathrm{x}_\alpha\}_{\alpha\in\Delta}$, there exists a unique $\mathfrak{h}$-Cartan involution ρ so that*

$$\rho(\mathrm{x}_\alpha) = \mathrm{x}_{-\alpha}. \tag{24}$$

Moreover this Cartan involution preserves the principal subalgebra associated to the Chevalley basis as in Proposition 2.3.1

We then have the orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where by definition $\mathfrak{p} = \mathfrak{k}^\perp$. Recall that

$$\begin{aligned} [\mathfrak{p}, \mathfrak{p}] &\subset \mathfrak{k} \\ [\mathfrak{k}, \mathfrak{p}] &\subset \mathfrak{p}. \end{aligned} \tag{25}$$

2.4.2. *Basic properties.* The following proposition summarizes some useful properties.

Proposition 2.4.4. *Any two $\mathfrak{h}$-Cartan involutions are conjugated by a an element of $\mathfrak{h}$. Any two $\mathfrak{h}$-Cartan involutions have the same restriction to $\mathfrak{h}$. An $\mathfrak{h}$-Cartan involution send roots to opposite roots and $\mathfrak{g}_1$ (cf. Section2.2.1) to itself.*

2.4.3. *Symmetric space.* Let $\mathsf{S}(\mathsf{G})$ be the *symmetric space* of Cartan involutions.

Proposition 2.4.5. *The group G acts transitively on $\mathsf{S}(\mathsf{G})$, and the stabilizer at a Cartan involution ρ is conjugated to a maximal compact subgroup K so that $\mathsf{S}(\mathsf{G})$ is isomorphic as a transitive G-space with G/K.*

The symmetric space is equipped with an interesting geometry. Let $\mathcal{G}$ be the trivial $\mathfrak{g}$ bundle over $\mathsf{S}(\mathsf{G})$ with its trivial connection D. We define the *Maurer–Cartan* form $\omega \in \Omega^1(\mathsf{S}(\mathsf{G}), \mathcal{G})$ as the identification of $\mathsf{T}_\rho(\mathsf{S}(\mathsf{G}))$ with $\mathfrak{p} = \{u \mid \rho(u) = -u\}$. Observe also that by construction we have a section ρ of $\mathrm{Aut}(\mathcal{G})$ such that $\rho(x) = x$ where in the right term x is considered as an involution of $\mathfrak{g}$. We thus have a Riemannian metric g on $\mathcal{G}$, defined by $g(u, v) = -\langle u|\rho(v)\rangle$.

Then we have

Proposition 2.4.6. *The connection $\nabla := \mathsf{D} - \mathrm{Ad}(\omega)$ preserves the metric g.*

2.5. **The real split form and the second conjugation.** In this subsection we review the construction the maximal real split form following Kostant [26]. We will prove.

Proposition 2.5.1. *Let $\mathfrak{h}$ be a Cartan subalgebra equipped with a positive root systems. Let ρ be an $\mathfrak{h}$-Cartan involution preserning and $\mathfrak{h}$-principal subalgebra. Then there exists a linear involution σ of $\mathfrak{g}$ with the following properties.*

1. *σ is an automorphism of $\mathfrak{g}$ and preserves globally $\mathfrak{h}$ and an $\mathfrak{h}$-principal subalgebra.*
2. *The involution σ commutes with ρ*
3. *The set of fixed point of $\sigma \circ \rho$ is the Lie algebra of the real split form G_0 of G.*
4. *if η is the longest root then $\sigma(\mathrm{h}_\eta) = \mathrm{h}_\eta$.*
5. *the automorphism σ preserves globally $\mathfrak{g}_Z$ and $\mathfrak{g}_{Z^\dagger}$.*

2.5.1. *Existence of σ.* We sketch the construction of σ due to Kostant and prove the existence part of Proposition 2.5.1 . Let $\mathfrak{h}$ be a Cartan subalgebra with its set of roots Δ, set of positive roots Δ^+ and set of positive roots Π. We also fix a Chevalley base $\{\mathrm{x}_\alpha\}_{\alpha\in\Delta}$.

Let $\mathfrak{z}$ be the vector space generated by the highest weight vectors for all $\mathfrak{v}_i$. Observe that $\mathfrak{z}$ and $\mathrm{ad}(Y)$ generates $\mathfrak{g}$. This data thus defines an involution σ on $\mathfrak{g}$ characterized by

$$\sigma|_{\mathfrak{z}} = -1, \quad \sigma(Y) = -Y. \tag{26}$$

Let now ρ be the unique Cartan involution associated to $\mathfrak{h}$ according to Proposition 2.4.3 and the choice of the Chevalley base. From [22], we gather that

Proposition 2.5.2. [Hitchin involution]

(1) *The involution σ and the anti-antilinear involution ρ commutes.*
(2) *The set of fixed points $\mathfrak{g}_0$ of the anti-linear involution $\sigma \circ \rho$ is the Lie algebra of a real split form G_0.*
(3) *The set of fixed point of σ is the complexification of the Lie algebra of maximal compact subgroup of G_0 and contains an $\mathfrak{h}$-principal subalgebra*
(4) *Finally, the elements $a_i = \mathrm{ad}(Y)^{m_i} e_i$, where m_i are the exponents and e_i highest weight vectors of $\mathfrak{v}_i$, generates $\mathfrak{h}$.*

A corollary of the last statement is

Proposition 2.5.3. *The union of $\mathfrak{h}$ and $\mathfrak{s}$ generates $\mathfrak{g}$.*

We will usually write for any $u \in \mathfrak{g}$,

$$\lambda(u) := \sigma \circ \rho(u).$$

We finally observe the following fact that concludes the proof of the existence part in Proposition 2.5.1 .

Proposition 2.5.4. [Involution and the longest root] *Let η be the longest root. Then*

$$\begin{aligned} \sigma(\mathrm{x}_\eta) &= -\mathrm{x}_\eta \\ \sigma(\mathrm{h}_\eta) &= \mathrm{h}_\eta. \end{aligned} \tag{27}$$

Moreover σ globally preserves $\mathfrak{g}_Z$ and $\mathfrak{g}_{Z^\dagger}$. Similarly

$$\begin{aligned} \lambda(Y) &= -X, \\ \lambda(X) &= -Y, \\ \lambda(a) &= -a. \end{aligned} \tag{28}$$

Proof. Since for all positive root α, $\alpha+\eta$ is not a root and thus $[\mathrm{x}_\alpha, \mathrm{x}_\beta] = 0$, we get that

$$\begin{aligned} [\mathrm{x}_\eta, X] &= \sum_{\alpha \in \Pi} \sqrt{r_\alpha}[\mathrm{x}_\alpha, \mathrm{x}_\eta] = 0, \\ [\mathrm{x}_\eta, a] &= \deg(\eta).\mathrm{x}_\eta. \end{aligned} \tag{29}$$

In particular, x_η generates an irreducible representation of $\mathfrak{s}$ for which x_η is a highest weight vector. In particular, $\sigma(x_\eta) = -x_\eta$.

Then since σ commutes with ρ we have

$$\sigma(x_{-\eta}) = \sigma(\rho(x_\eta)) = \rho(\sigma(x_\eta)) = -\rho(x_\eta) = -x_{-\eta}.$$

Finally, σ being an automorphism of the Lie algebra, we have that

$$\sigma(h_\eta) = \sigma([x_\eta, x_{-\eta}]) = [\sigma(x_\eta), \sigma(x_{-\eta}] = (-1)^2[x_\eta, x_{-\eta}] = h_\eta.$$

Finally, σ is an automorphism of $\mathfrak{g}$ preserving $\mathfrak{h}$, thus sending roots to roots. Since σ preserves the longest coroot, it preserves the degree and thus send simple roots to simple roots. The last statement follows. □

The next proposition follows immediately the definition of σ,

Proposition 2.5.5. [Involution and the principal subgroup] *We have*

$$\begin{aligned} \lambda(Y) &= -X, \\ \lambda(X) &= -Y, \\ \lambda(a) &= -a. \end{aligned} \tag{30}$$

2.5.2. *Uniqueness of σ.* We now prove the uniqueness part of Proposition 2.5.1

Proposition 2.5.6. *Let $\mathfrak{h}$ equipped with a positive root system and $\mathfrak{s}$ be an $\mathfrak{h}$-principal subalgebra. Then there exists a unique linear involution σ such that*

(1) σ is an automorphism of $\mathfrak{g}$,
(2) σ preserves globally $\mathfrak{s}$ and $\mathfrak{h}$,
(3) If η is the longest root then $\sigma(h_\eta) = h_\eta$.

Proof. If σ_1 and σ_2 are two such involutions. Let $I = \sigma_1 \circ \sigma_2$. Then I normalizes $\mathfrak{h}$ and fixes h_η. Thus by Section 2.1.2, I fixes $\mathfrak{h}$ pointwise. Since I fixes $\mathfrak{s}$ pointwise, it follows that I fixes the Lie algebra generated by $\mathfrak{h}$ and $\mathfrak{s}$ that is $\mathfrak{g}$ by Proposition 2.5.3. □

3. Hitchin triple

In this section, we introduce the basic algebraic concept used by Hitchin in the construction of the *Hitchin section* that we explain in the next section.

3.1. **Definitions.** Let G be a complex simple group.

Definition 3.1.1. *A* Hitchin triple *is a triple* $(\mathfrak{h}, \rho, \lambda)$ *where*

- $\mathfrak{h}$ *is a Cartan subalgebra equipped with a positive root system,*
- ρ *is an anti-linear involution globally fixing* $\mathfrak{h}$ *whose set of fixed points is the Lie algebra of a maximal compact subgroup,*
- λ *is an anti-linear involution commuting with* ρ*, globally fixing* $\mathfrak{h}$ *whose set of fixed points is the Lie algebra of a maximal real split form,*
- ρ *and* λ *both fixes globally an* $\mathfrak{h}$*-principal subalgebra*

We will also use the notation $\sigma := \lambda \circ \rho$.

By Proposition 2.5.1, Hitchin triples exist. Moreover:

Proposition 3.1.2. *Any two Hitchin triples are conjugate.*

Proof. Let $(\mathfrak{h}_1, \rho_1, \lambda_1)$ and $(\mathfrak{h}_2, \rho_2, \lambda_2)$. Let $\mathfrak{s}_i$ be the $\mathfrak{h}_i$-principal subalgebra fixed globally by ρ_i and λ_i. Let $\sigma_i = \lambda_i \circ \rho_i$. By Proposition 2.3.3, we can as well assume after conjugation that

$$(\mathfrak{h}_1, \mathfrak{s}_1) = (\mathfrak{h}_2, \mathfrak{s}_2) =: (\mathfrak{h}, \mathfrak{s}).$$

Thus by Proposition 2.5.6, $\sigma_1 = \sigma_2$. Applying Proposition 2.4.4 to $\mathfrak{s}$, we can further use a conjugation by an element of $\mathfrak{h} \cap \mathfrak{s}$ so that the restriction of ρ_1 and ρ_2 coincide on $\mathfrak{s}$. Thus $\rho_1 \circ \rho_2$ is the identity on $\mathfrak{s}$ and is also the identity on $\mathfrak{h}$ by Proposition 2.4.4. Since $\mathfrak{s}$ and $\mathfrak{h}$ generates $\mathfrak{g}$ by Proposition 2.5.3, it follows that $\rho_1 = \rho_2$, thus $\lambda_1 = \lambda_2$ and the result follows. □

3.2. **The stabiliser of a Hitchin triple.** Let $(\mathfrak{h}, \rho, \lambda)$ be a Hitchin triple. Let K the maximal compact subgroup of G fixed by ρ and similarly G_0 the split real form, fixed by λ. From Section 6 in Hitchin [22], we have

Proposition 3.2.1. [Hitchin] *The group* $\mathsf{K}_0 := \mathsf{K} \cap \mathsf{G}_0$ *is the maximal compact subgroup of* G*. The algebra* $\mathfrak{t} = \mathfrak{g}_0 \cap \mathfrak{h} \cap \mathfrak{k}$ *is Lie algebra of the maximal torus* T *of* K_0*.*

Observe that G acts by conjugation on the space X of Hitchin triples.

Proposition 3.2.2. *The stabiliser in* G *of the Hitchin triple* $(\mathfrak{h}, \rho, \lambda)$ *is* T*.*

Proof. The normaliser of $\mathfrak{h}$ equipped with a positive root system is H. The normaliser of λ is included in G_0 since it normalizes the set of fixed points of λ. The normaliser of ρ is similarly included in K. The result follows. □

Proposition 3.2.3. *We have the following*

(1) We have $\mathrm{h}_\eta \in \mathfrak{t}_{\mathbb{C}}$,
(2) For $\mathsf{G} = \mathsf{SL}(3,\mathbb{R})$, *for all simple root* α, $\mathrm{h}_\alpha \notin \mathfrak{t}_{\mathbb{C}}$.
(3) If G *is* G_2 *or* $\mathsf{Sp}(4,\mathbb{C})$ *Then* $\sigma|_{\mathfrak{h}} = 1$, *or in other words* $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$.

Proof. We first prove Statement (1). Since $\sigma(\mathrm{h}_\eta) = \mathrm{h}_\eta$ by Proposition 3.1.2, and $\mathfrak{t}_{\mathbb{C}}$ is the set of fixed points of σ in $\mathfrak{h}$ statement follows.

We now prove Statement (2). When $\mathsf{G} = \mathsf{SL}(3,\mathbb{C})$, $\mathfrak{t}_{\mathbb{C}}$ is of rank 1; it follows from Property (1) that $\mathfrak{t}_{\mathbb{C}} = \mathrm{h}_\eta.\mathbb{C}$. Since for all simple root α, h_α is not collinear to h_η, the result follows.

We finally prove Statement (3). In that case, since G is of rank 2, we have only 2 representations of $\mathfrak{s}$ appearing in $\mathfrak{g}$. Let x_1 and x_2 be the corresponding highest weight vectors. In that case the exponent m_1 and m_2 are both odd. Thus let $a_i = \mathrm{ad}(Y)^{m_i}\mathrm{x}_i$, then $\sigma(a_i) = a_i$. But the a_i generate $\mathfrak{h}$ by the last assertion of Proposition 2.5.2. The result now follows. □

4. The space of Hitchin triples

Our goal now is to describe the geometry of the transitive G-space of Hitchin triples that we shall denote by X and which is isomorphic to G/T, where G is a complex simple group and T is the maximal torus of a maximal compact of the real split form G_0. In particular we which to describe a Lie algebra bundle over X which come equipped with a connexion, a metric and other differential geometric devices.

4.1. **Preliminary: forms with values in a Lie algebra bundle.** We shall in the sequel study form on a manifold M with values in a Lie algebra bundle $\mathcal{G}$. We store in this paragraph the formulas that we shall use later.

We denote by $\Omega^*(M,\mathcal{G})$ the graded vector space of forms on M with values in $\mathcal{G}$. We say a form α in $\Omega^*(M,\mathcal{G})$ is *decomposable* if $\alpha = \hat{\alpha} \otimes A$ where A is a section of $\mathcal{G}$ and $\widehat{\alpha}$ a form on M.

We recall the existence of a unique linear binary operation $\wedge$

$$\Omega^p(M,\mathcal{G}) \otimes \Omega^q(M,\mathcal{G}) \to \Omega^{p+q}(M,\mathcal{G}),$$

so that if $\alpha = \widehat{\alpha} \otimes A$, $\beta = \widehat{\beta} \otimes B$ are decomposable forms then

$$\alpha \wedge \beta = \left(\widehat{\alpha} \wedge \widehat{\beta}\right) \otimes [A,B]. \tag{31}$$

Similarly, if α and β are $\mathcal{G}$ valued forms of degree, $\langle \alpha|\beta\rangle$ is the form defined for decomposable forms $\alpha = \widehat{\alpha} \otimes A$ and $\beta = \widehat{\beta} \otimes B$ by

$$\langle \alpha|\beta\rangle := (\widehat{\alpha} \wedge \widehat{\beta}) \otimes \langle A|B\rangle$$

We will use the following proposition freely

Proposition 4.1.1. *If α and β are respectively of degree p and q, and if $\xi \in \chi^\infty(M)$*

$$\alpha \wedge \beta = (-1)^{pq+1}\beta \wedge \alpha \tag{32}$$

$$i_\xi(\alpha \wedge \beta) = i_\xi\alpha \wedge \beta + (-1)^p \alpha \wedge i_\xi\beta \tag{33}$$

$$\langle \gamma | \beta \wedge \alpha \rangle = (-1)^{pq+1}\langle \gamma \wedge \alpha | \beta \rangle. \tag{34}$$

If α and β are 1-form and γ a 0-form then

$$\alpha \wedge (\gamma \wedge \beta) + \beta \wedge (\gamma \wedge \alpha) = \gamma \wedge (\alpha \wedge \beta). \tag{35}$$

if furthermore $\mathcal{G}$ is equipped with a connection and if d *is the corresponding exterior derivative on $\Omega^*(M, \mathcal{G})$, then*

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + \alpha \wedge \mathrm{d}\beta. \tag{36}$$

The proposition follows by considering these formula first in the context of decomposable forms, then extending these formulas by linearity.

Finally recall a convention of notation: if α belongs to $\Omega(\mathsf{X}, \mathcal{G})$, and f is a map from Σ to X, then $f^*(\alpha)$ is a form with values in $f * (\mathcal{G})$.

4.1.1. *An important sign.* Let S be a Riemann surface, $\mathcal{G}$ a Lie algebra bundle over S equipped with a section of Cartan involution ρ. Then we have

Proposition 4.1.2. *Let α be a $(1,0)$ form with value in $\mathcal{G}$, then*

$$i \cdot \int_S \langle \alpha | \rho(\alpha) \rangle \leqslant 0. \tag{37}$$

Conversely if α is of type $(0,1)$ then

$$i \cdot \int_S \langle \alpha | \rho(\alpha) \rangle \geqslant 0. \tag{38}$$

Proof. It is enough to consider a form $\alpha = A.a$, where a is a section of $\mathcal{G}$ and A a $(1,0)$-form on S. Then $\rho(\alpha) = \overline{A}.\rho(a)$. Thus

$$i \cdot \int_S \langle \alpha | \rho(\alpha) \rangle = i \cdot \int_S A \wedge \overline{A} \cdot \langle a | \rho(a) \rangle \leqslant 0. \tag{39}$$

□

4.2. **Geometry of the space of Hitchin triples.**

4.2.1. *Vector subbundles.* Let X be the space of Hitchin triples in G as in Section 3.

The group G act by conjugation on X. Moreover once we fix a Hitchin triple $(\mathfrak{h}, \rho, \lambda)$ then X is identifed with G/T by Proposition 3.2.2, where $\mathsf{T} \subset \mathsf{H}$ be the torus fixed by the involutions λ and ρ. Recall then that T is compact and is the maximal torus of the maximal compact of G_0 by Proposition 3.2.1.

Let $\mathcal{G}$ be the trivial bundle $\mathcal{G} := \mathfrak{g} \times \mathsf{X}$ equipped with the trivial connection D.

The following definition introduces some of the geometry of X.

Definition 4.2.1. *We denote by $\mathcal{H}, \mathcal{T}, \mathcal{H}_0$, the ∇-parallel subdundles of $\mathcal{G}$ whose fiber at $(\mathfrak{h}, \rho, \sigma)$ are respectively*

$$\mathfrak{h}, \tag{40}$$

$$\mathfrak{t} := \{u \in \mathfrak{h} \mid \sigma(u) = u, \rho(u) = u\}, \tag{41}$$

$$\mathfrak{h}_0 := \{u \in \mathfrak{h} \mid \forall v \in \mathfrak{t}, \langle u|v\rangle = 0\}. \tag{42}$$

We also have a decomposition using the root system that we write

$$\mathcal{G} = \mathcal{H} \bigoplus_{\alpha \in \Delta} \mathcal{G}_\alpha, \tag{43}$$

such that at a point $x = (\mathfrak{h}, \rho, \lambda)$, we have $\mathcal{H}_x = \mathfrak{h}$, and $(\mathcal{G}_\alpha)_x = \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ come from the root space decomposition (5) associated to $\mathfrak{h}$.

4.2.2. *The Maurer–Cartan form.*

Definition 4.2.2. [Maurer–Cartan Form] *The* Maurer–Cartan *form on X is the form $\omega \in \Omega^1(X, \mathcal{G})$ defined as follows: at a point $x = (\mathfrak{h}, \rho, \lambda)$ of X, $\mathsf{T}_x\mathsf{X}$ is identified with $\mathfrak{g}/\mathfrak{t}$. Let $\mathcal{T}^\perp$ be the orthogonal of $\mathcal{T}$ in $\mathcal{G}$ with respect to the Killing form. Let $\omega \in \Omega^1(X, \mathcal{T}^\perp)$ be the inverse of the projection from the $\mathfrak{t}^\perp$ to $\mathfrak{g}/\mathfrak{t} = \mathsf{T}_x\mathsf{X}$.*

Observe that the Maurer–Cartan form satisfies

$$\forall u \in \mathfrak{t}, \ \ \langle u|\omega\rangle = 0, \tag{44}$$

In the sequel we will sloppily identify ω with $\mathrm{ad}(\omega)$.

4.2.3. *Vector bundle and connexions.* We begin with a preliminary Lemma. Let $b : \mathsf{G} \to \mathrm{End}(V)$, be a linear representation. Let $\mathcal{V} = V \times \S(G)$ be the associated trivial bundle equipped with the trivial connection D. We say a section σ of $\mathcal{V}$ is G-equivraiant if

$$\sigma(gx) = b(g)\sigma(x).$$

Then

Lemma 4.2.3. *A* G*-equivariant section is parallel under the connection*

$$\nabla = \mathsf{D} - \mathsf{D}b(\omega(u)).$$

Proof. Since $\sigma(g.x) = b(g).\sigma(x)$, then for every $u \in \mathfrak{g}$,

$$\mathsf{D}_u\sigma = \mathsf{D}b(u).\sigma.$$

Observe that if $u = \rho(u)$, then $u = 0$ as a vector field along $S(\mathsf{G})$. Thus the result follows. □

Proposition 4.2.4. *The subbundles* $\mathcal{H}$*,* $\mathcal{T}$ *and* $\mathcal{H}_0$ *are parallel for the connection* $\nabla := \mathsf{D} - \omega$*, where* ω *is the Maurer–Cartan form. Finally the connection* R^∇ *belongs to* $\Omega^2(\mathsf{X}, \mathcal{T})$*.*

In particular, the curvature of ∇ is given by the equation

$$\mathrm{d}^\nabla\omega + \omega \wedge \omega + R^\nabla = 0. \tag{45}$$

Proof. We denote momentarily by $\mathcal{H}, \mathcal{T}$ and $\mathcal{H}_0$ the orthogonal projections on $\mathcal{H}, \mathcal{T}$ and $\mathcal{H}_0$. Now, these projections are G-equivariant sections of $\mathrm{End}(\mathfrak{g})$. By the previous Lemma they are parrallel under the connection ∇^0 such that

$$\nabla^0_u S = \mathsf{D}_u S - [\mathrm{ad}(\omega(u), S].$$

Now observe that

$$\begin{aligned}(\nabla_u S)(v) &= \nabla_u(S(v)) - S\nabla_u(v)\\ &= (D_u S)(v)) - \mathrm{ad}(\omega(u)).Sv + S.\,\mathrm{ad}(\omega(u))(v)\\ &= \nabla_u S(v).\end{aligned}$$

The first part of the proposition follows.

Since ρ and λ are parrallel, it follows that the Lie algebra of k (associated to to the maximal compact of G) and g_0 (associated to to the real split form of G) are both parralel, since these two algebras are both self normalizing, it follows that $R^\nabla \in \Omega^2(\mathsf{X}, \mathfrak{g}_0 \cap \mathfrak{k})$. Similarly since $\mathfrak{h}$ is parrallel and self normalizingh, we further have that

$$R^\nabla \in \Omega^2(\mathsf{X}, \mathfrak{g}_0 \cap \mathfrak{k} \cap \mathfrak{h}).$$

The last statement of the proposition follows from the fact that

$$\mathfrak{g}_0 \cap \mathfrak{k} \cap \mathfrak{h} = \mathfrak{t}.$$

□

Conversely, we have

Proposition 4.2.5. *Let* M *be a simply connected manifold together with a Lie algebra bundle* $\widehat{\mathcal{G}}$ *equipped with*

(1) *a smoothly varying Hitchin triple* $m \mapsto (\widehat{\mathfrak{h}}_m, \widehat{\rho}_m, \widehat{\lambda}_m)$ *in every fiber.*
(2) *a connexion* $\widehat{\nabla}$ *for which the Hitchin triple is parallel.*
(3) *an element* $\widehat{\omega} \in \Omega^1(M, \mathcal{G})$*, such that* $\widehat{D} := \widehat{\nabla} + \mathrm{ad}(\widehat{\omega})$ *is flat and moreover*

$$\forall u \in \widehat{\mathfrak{t}}, \ \langle u | \widehat{\omega} \rangle = 0, \tag{46}$$

where $\widehat{\mathfrak{t}} := \{u \in \widehat{h} \mid \widehat{\rho}(u) = u = \widehat{\sigma}(u)\}$.

Then there exists a map f *from* M *in* X*, unique up to postcomposition by an element of* G*, such that* $\widehat{\mathcal{G}}$, $\widehat{\mathfrak{h}}$, $\widehat{\rho}$, $\widehat{\lambda}$, $\widehat{\omega}$,$\widehat{\nabla}$ *and* $\widehat{\sigma}$ *are the pulled back of the corresponding objects in* $\mathcal{G}$*.*

As an immediate corollary, we get

Corollary 4.2.6. *Let* M *be a manifold together with a Lie algebra bundle* $\widehat{\mathcal{G}}$ *equipped with the same structure as in Proposition 4.2.5, then there exists*

(1) *a representation* ρ *of* $\pi_1(M)$ *in* G *unique up to conjugation,*
(2) *a* ρ*-equivariant map* f *from the universal cover* $\widetilde{M}$ *of* M*, in* X *satisfying the properties in the conclusion of Proposition 4.2.5.*

Proof. Since $\widehat{\mathsf{D}}$ is flat and M simply connected, we may as well assume that $\widehat{\mathcal{G}}$ is the trivial bundle $\mathcal{G} = \mathfrak{g} \times M$. Thus the map $f : m \mapsto (\widehat{\mathfrak{h}}_m, \widehat{\rho}_m, \widehat{\lambda}_m)$ is now a map from M to X.

By construction $\widehat{\mathcal{G}}$, $\widehat{\mathfrak{h}}$, $\widehat{\rho}$, $\widehat{\mathsf{D}}$ are the pullback by f of $\mathcal{G}$, $\mathfrak{h}$, ρ, λ and D. Thus $(\widehat{h}, \widehat{\rho}, \widehat{\lambda})$ is parrallel both for $\widehat{\nabla}$ and $f^*\nabla$. Thus since the stabilzer in $\mathfrak{g}$ of $(\mathfrak{h}, \rho, \lambda)$ is $\mathfrak{t}$,

$$f^*(\omega) - \widehat{\omega} = \widehat{\nabla} - f^*\nabla \in \Omega^1(M, \mathfrak{t}).$$

However by Hypothesis (46) and Equation (44),

$$f^*(\omega) - \widehat{\omega} \in \Omega^1(M, \mathfrak{g}/\mathfrak{t}).$$

Thus $f^*(\omega) = \widehat{\omega}$. Then $f^*\nabla = \widehat{\nabla}$ and the proof of the proposition is completed. □

4.2.4. *The real structure on the space of Hitchin triples.* Since T is a subgroup of G_0, it follows that each leaf of the foliation by right G_0 orbits of G is invariant by the action of T, thus giving rise to a foliation $\mathbf{F}$ of G/T whose leaves are all isomorphic to G_0/T. Since this foliation is left invariant by the action of G, it gives a foliation, that we also denote $\mathbf{F}$ on X.

Since λ preserves $\mathfrak{t}$ and thus $\mathfrak{t}^\perp$, we obtain a *real structure* $v \to \overline{v}$ on X by setting

$$\omega(v) = \lambda(\omega(v)).$$

One then immediately have

Proposition 4.2.7. *The tangent distribution* $\mathsf{T}\mathbf{F}$ *of the foliation* $\mathbf{F}$ *is given by*

$$\mathsf{T}\mathbf{F} := \{u \in \mathsf{T}\mathsf{X} \mid \overline{u} = u\}.$$

Proof. Indeed, $\mathfrak{g}_0$ is the set of fixed points of λ in $\mathfrak{g}$. □

4.2.5. *The space of Hitchin triples and the symmetric space.* Let $\mathsf{S}(\mathsf{G})$ be as in Proposition 2.4.5 the symmetric space of Cartan involution. The map $p : (\mathfrak{h}, \rho, \lambda) \mapsto \rho$ defines a natural G equivariant projection p from X to $\mathsf{S}(\mathsf{G})$. The fibers of this projection are described as follows. Since T is a subgroup of the maximal compact K, each leaf of the foliation by the right K orbits on G is invariant by T, giving rise to a foliation on G/T. This foliation is invariant under the left G action and thus gives a foliation $\mathbf{K}$ on X. The leaves of $\mathbf{K}$ are precisely the preimages of the projections p from X to $\mathsf{S}(\mathsf{G})$.

We should remark that the existence of the Maurer–Cartan form is not a specific feature of the space of Hitchin triples, but the same construction holds for many G, space, in particular the symmetric space $\mathsf{S}(\mathsf{G})$. In particular one immediately gets

Proposition 4.2.8. *The canonical* $\mathfrak{g}$*-bundle over* $\mathsf{S}(\mathsf{G})$ *identifies with* $\mathsf{T}_{\mathbb{C}}\mathsf{S}(\mathsf{G})$. *Moreover if* α *is the identity map of* $\mathsf{T}\mathsf{S}(\mathsf{G})$ *that we see as an element of* $\Omega^1(\mathsf{S}(\mathsf{G}), \mathsf{T}\mathsf{S}(\mathsf{G}))$, *then*

- $\alpha_{\mathbb{C}}$ *is the Maurer–Cartan form of* $\mathsf{S}(\mathsf{G})$,
- *Moreover* $p^*(\alpha_{\mathbb{C}}) = \frac{1}{2}(\omega + \rho(\omega))$.

4.3. **The cyclic decomposition of the Maurer-Cartan form.** Using Proposition 4.2.4, we obtain a decomposition of $\mathcal{G}$ as

Let ω be the Maurer–Cartan on X as in Definition 4.2.2 with value in the bundle $\mathcal{G}$. We use the decomposition (43) to write

$$\omega = \omega_0 + \sum_{\alpha \in \Delta} \omega_\alpha. \tag{47}$$

Actually one has by Equation (44) that

$$\omega_0 \in \Omega(\mathsf{X}, \mathcal{H}_0). \tag{48}$$

From Equation (45), we obtain that for all $\alpha \neq 0$,

$$-\mathrm{d}^\nabla \omega_\alpha = \omega_0 \wedge \omega_\alpha + \sum_{\substack{\beta,\gamma \in \Delta \\ \beta+\gamma=\alpha}} \omega_\beta \wedge \omega_\gamma. \tag{49}$$

We consider the following projections (whose pairwise product are zero) coming from the projection on the Lie algebra defined in equations (19) and that we denote by the same symbol by a slight abuse of notations.

$$\begin{aligned}
\pi_0 &: \mathcal{G} \to \mathcal{H}, \\
\pi &: \mathcal{G} \to \mathcal{G}_Z := \bigoplus_{\alpha \in Z} \mathcal{G}_\alpha, \\
\pi^\dagger &: \mathcal{G} \to \mathcal{G}_{Z^\dagger} := \bigoplus_{\alpha \in Z^\dagger} \mathcal{G}_\alpha, \\
\pi_1 &: \mathcal{G} \to \mathcal{G}_1 := \bigoplus_{\alpha \notin Z \cup Z^\dagger} \mathcal{G}_\alpha.
\end{aligned} \tag{50}$$

Observe that

$$\pi + \pi^\dagger + \pi_0 + \pi_1 = 1.$$

Obviously, Proposition 2.2.2 extends word for word for the various brackets of the vector subundles described in the equations (50).

Definition 4.3.1. *The* cyclic decomposition *of the Maurer–Cartan form ω is*

$$\omega = \omega_0 + \omega_1 + \phi + \phi^\dagger, \tag{51}$$

where $\omega_0 = \pi_0(\omega)$, $\omega_1 = \pi_1(\omega)$, $\phi = \pi(\omega)$ and $\phi^\dagger = \pi_\dagger(\omega)$. Remark that by Equation (44),

$$\omega_0 \in \Omega^1(X, \mathcal{H}_0) \subset \Omega^1(X, \mathcal{H}) \tag{52}$$

We will use the following

Proposition 4.3.2. *Let ω be the Maurer-Cartan form then*

$$\pi_0(\omega \wedge \omega) = 2 \cdot \phi \wedge \phi^\dagger + \pi_0(\omega_1 \wedge \omega_1), \tag{53}$$

$$\begin{aligned}
\pi_1(\omega \wedge \omega) = 2 \cdot \pi_1(\omega_0 \wedge \omega_1 + \omega_1 \wedge \phi + \omega_1 \wedge \phi^\dagger) \\
+ \pi_1(\omega_1 \wedge \omega_1 + \phi \wedge \phi + \phi^\dagger \wedge \phi^\dagger).
\end{aligned} \tag{54}$$

$$\begin{aligned}
\pi(\omega \wedge \omega) = 2 \cdot \omega_0 \wedge \phi + 2 \cdot \pi(\omega_1 \wedge \phi^\dagger) \\
+ \pi(\phi^\dagger \wedge \phi^\dagger) + \pi(\omega_1 \wedge \omega_1).
\end{aligned} \tag{55}$$

Proof. Let us consider the cyclic decomposition

$$\omega = \omega_0 + \omega_1 + \phi + \phi^\dagger. \tag{56}$$

Then

$$\begin{aligned}\omega = \omega_0 \wedge \omega_0 + \omega_1 \wedge \omega_1 + \phi \wedge \phi + \phi^\dagger \wedge \phi^\dagger \\ + 2\omega_0 \wedge \omega_1 + 2\omega_0 \wedge \phi + 2\omega_0 \wedge \phi^\dagger \\ + 2\omega_1 \wedge \phi + \omega_1 \wedge \phi^\dagger \\ + 2\phi \wedge \phi^\dagger. \end{aligned} \tag{57}$$

According to Proposition 2.2.2, we have that

$$\begin{aligned}\pi_0(\phi \wedge \phi) = \pi_0(\phi^\dagger \wedge \phi^\dagger) = 0 \\ \pi_0(\phi \wedge \zeta_1) = \pi_0(\phi^\dagger \wedge \zeta_1) = 0 \\ \pi_0(\phi \wedge \phi^\dagger) = \phi \wedge \phi^\dagger.\end{aligned}$$

Thus, using the fact that $\mathfrak{h}$ is commutative, and normalizes $\mathcal{G}_1$, $\mathcal{G}_Z$ and $\mathcal{G}_{Z^\dagger}$ we get Equation (53).

We use again Proposition 2.2.2 to get that

$$\begin{aligned}\pi_1(\omega_0 \wedge \omega_0) = 0, \quad \pi_1(\omega_0 \wedge \phi) = 0, \\ \pi_1(\phi \wedge \phi^\dagger) = 0, \quad \pi_1(\omega_0 \wedge \phi^\dagger) = 0.\end{aligned} \tag{58}$$

Thus

$$\begin{aligned}\pi_1(\omega \wedge \omega) = 2 \cdot \pi_1(\omega_0 \wedge \omega_1 + \omega_1 \wedge \phi + \omega_1 \wedge \phi^\dagger) \\ + \pi_1(\omega_1 \wedge \omega_1 + \phi \wedge \phi + \phi^\dagger \wedge \phi^\dagger).\end{aligned} \tag{59}$$

Then finally, by Proposition 2.2.2 we have that

$$\begin{aligned}\pi(\omega_0 \wedge \omega_0) = 0, \ \ \pi(\omega_0 \wedge \omega_1) = 0, \\ \pi(\omega_0 \wedge \phi^\dagger) = 0, \ \ \pi(\phi \wedge \phi) = 0, \\ \pi(\omega_1 \wedge \phi) = 0, \ \pi(\phi \wedge \phi^\dagger) = 0.\end{aligned} \tag{60}$$

Thus

$$\pi(\omega \wedge \omega) = 2\pi(\omega_0 \wedge \phi) + 2\pi(\omega_1 \wedge \phi^\dagger) + \pi(\omega_1 \wedge \omega_1) + \pi(\phi^\dagger \wedge \phi^\dagger) \tag{61}$$

The proof of the proposition is completed. □

5. Higgs bundles and Hitchin theory

In this section, we will recall the definition of a Higgs bundle and sketch some of Hitchin theory. Higgs bundles have been studied extensively by many authors. The special case of $\mathsf{Sp}(4, \mathbb{R})$ has been in particular studied by Bradlow, García-Prada, Gothen and Mundet Riera in [15], [8] and [14].

5.1. **Higgs bundles and the self duality equations.** We recall some definition and results from Hitchin [21]. We recall that a *Higgs bundle* over a Riemann surface Σ, is a pair $E = (\widehat{\mathcal{G}}, \Phi)$ where

(1) $\widehat{\mathcal{G}}$ is a holomorphic Lie algebra bundle over Σ,
(2) Φ is a holomophic section –called the *Higgs field*– of $\widehat{\mathcal{G}} \otimes K$, where K is the canonical bundle of Σ.

The *slope* of $(\widehat{\mathcal{G}}, \Phi)$ is $\mu(E) := \deg(E)/\operatorname{rank}(E)$.

A Higgs bundle is *stable*, if for all Lie algebra (strict) subbundle $\widehat{\mathcal{H}}$ such that $[\Phi, \widehat{\mathcal{H}}] \subset \widehat{\mathcal{H}} \otimes K$, then

$$\mu\left(\widehat{\mathcal{H}}, \Phi|_{\widehat{\mathcal{H}}}\right) < \mu(E). \tag{62}$$

We finally say that E is *polystable* if it is the sum of Higgs bundle of the same slope.

Let ∇ be a connection on $\widehat{\mathcal{G}}$ compatible with the holomorphic structure and $\widehat{\rho}$ a section of of the bundle of automorphisms of $\widehat{\mathcal{G}}$ such that the restriction to every fiber is a Cartan involution with respect to a maximal compact. Let $\Phi^* = -\widehat{\rho}(\Phi)$,

and R^∇ the curvature of ∇. We say that $(\nabla, \widehat{\rho})$ is a *solution the self duality equations* if

$$\begin{aligned} \nabla\widehat{\rho} &= 0, \\ d^\nabla\Phi &= 0, \\ d^\nabla\Phi^* &= 0 \\ R^\nabla &= 2 \cdot \Phi \wedge \Phi^*. \end{aligned} \tag{63}$$

The last three equations are equivalent to the fact that $\nabla + \Phi + \Phi^*$ is flat and the curvature of ∇ is of type $(1,1)$. Observe also that $\widehat{\rho}$ totally determines ∇: ∇ is the Chern connection of the Hermitian bundle $(\widehat{G}, \widehat{\rho})$.

Hitchin and Simpson proved in [21], [38].

Theorem 5.1.1. [Hitchin, Simpson] *Given a polystable Higgs bundle over a closed Riemann surface Σ, there exists a unique solution of the self duality equations.*

Given a polystable Higgs bundle $E = (\widehat{\mathcal{G}}, \Phi)$ over a closed Riemann surface Σ, let (∇, ρ) be the solution of the self duality equations. We then denote by $\rho(E)$ the monodromy of the flat connection $\nabla + \Phi + \Phi^*$ and call it the *representation associated to the Higgs bundle E*.

The *Hopf differential* of the Higgs bundle is the quadratic holomorphic differential $\langle \Phi | \Phi \rangle$ where $\langle | \rangle$ denotes the Killing form. From [12],

solutions of the self duality equation are interpreted as equivariant harmonic mappings, and those whose Hopf differential vanishes as conformal harmonic mapping, that is a minimal surface.

5.2. **The Hitchin section.** Let us now recall the construction by Hitchin [22] of the Higgs bundle from holomorphic differentials, using the notation of our preliminary paragraph. Let Σ be a closed surface. Given a complex Lie group G. We choose a Cartan subalgebra $\mathfrak{h}$ and an $\mathfrak{h}$-principal Lie algebra generated by (X, a, Y) as in Section 2.3. We decompose the Lie algebra under the irreducible action of $\mathfrak{s}$,

$$\mathfrak{g} = \bigoplus_{i=-1}^{i=\ell} S^{2m_i}(V). \tag{64}$$

Here $m_1, \dots, m_\ell$ are the exponents of G. Let e_i be an element of $S^{2m_i}(V)$ of highest weight with respect to the action of the principal Lie algebra generated by (X, a, Y).

Let us also write the decomposition under the grading by the element a as

$$\mathfrak{g} := \bigoplus_{i=-m_\ell}^{i=m_\ell} \mathfrak{g}_m, \tag{65}$$

where $\mathfrak{g}_m := \{u \in \mathfrak{g} \mid [a, u] = m \cdot u\}$. Observe that $e_i \in \mathfrak{g}_{m_i}$ and

$$Y \in \mathfrak{g}_{-1} = \bigoplus_{\alpha \in \Pi} \mathfrak{g}_{-\alpha}. \tag{66}$$

Moreover $\mathfrak{g}_0 = \mathfrak{h}$ is the centralizer of a. Let us now consider the Lie algebra bundle

$$\widehat{\mathcal{G}} := \bigoplus_{i=-m_\ell}^{i=m_\ell} \widehat{\mathcal{G}}_m, \tag{67}$$

where $\widehat{\mathcal{G}}_m := \mathfrak{g}_m \otimes K^m$ and K is the canonical bundle of Σ. We write $\widehat{\mathcal{G}_0} =: \widehat{\mathfrak{h}}$. The fiber of $\widehat{\mathfrak{h}}$ at a Cartan sub algebra equipped with a choice of positive roots (given by the element a). We then denote by

$$\widehat{\mathcal{G}} := \bigoplus_{\alpha \in \Delta} \widehat{\mathcal{G}_\alpha},$$

the corresponding root space decomposition where $\widehat{\mathcal{G}_\alpha}$ is the eigenspace associated to the root α.

The *Hitchin section* then associate to a family of holomorphic differentials $\mathrm{q} := (q_1, \dots, q_\ell)$ where q_i is of degree $m_i + 1$ the Higgs bundle $H(\mathrm{q}) := (\mathcal{G}, \Phi_q)$

$$\Phi_{\mathrm{q}} := Y + \sum_{i=1}^{\ell} e_i \otimes q_i \in H^0(\Sigma, \widehat{\mathcal{G}} \otimes K). \tag{68}$$

By Section 5 of [22] based on Theorem 7 of [27], we have

Proposition 5.2.1. *There exists homogeneous invariant polynomials p_i on $\mathfrak{g}$ of degree $m_i + 1$ such that $p_i(\Phi_{\mathrm{q}}) = q_i$.*

Observe also that

$$\sigma(\Phi_q) = -\Phi_q \tag{69}$$

where σ is the unique involution associated to $\mathfrak{s}$ by Proposition 2.5.6. Hitchin then proved in [22]

Theorem 5.2.2. [Hitchin] *The Higgs bundle $\mathcal{H}(\mathrm{q})$ is stable. Moreover if $(\nabla, \widehat{\rho})$ is the solution of the self duality equations, then $\nabla\sigma = 0$. In particular the monodromy is with values in G_0. Finally if $\mathrm{q} = 0$, then the monodromy is with values in the principal SL_2.*

The second assertion follows at once form the uniqueness of the solutions of the self duality equations.

5.2.1. *Hitchin component.* Finally, Hitchin as proved

Theorem 5.2.3. [Hitchin] *Given a Riemann surface Σ, the map which associates to q the monodromy associated to the Higgs bundle $H(\mathrm{q})$ is a parametrisation of a connected component of the character variety of representations from $\pi_1(S)$ to G_0.*

The case of $\mathsf{G}_0 = \mathsf{SL}(2, \mathbb{R})$ had been done independently by M. Wolf in [39]. The connected component described by the previous theorem is now called the *Hitchin component* and we will denote it $\mathcal{H}(\Sigma, \mathsf{G}_0)$. A representation in $\mathcal{H}(\Sigma, \mathsf{G}_0)$ will be called *Hitchin representations*. The *Fuchsian locus*, which is the set of *Fuchsian representations*, is the set of those representations that are discrete faithful and with values in a principal SL_2. By [29] and [13] Hitchin representations are discrete faithful.

5.3. **Cyclic Higgs bundles.** A *cyclic Higgs bundle* is by definition the image $H(\mathrm{q})$ by the Hitchin section of a family of holomorphic differential $\mathrm{q} := (q_1, \dots, q_\ell)$ where $q_i = 0$ when $i \neq \ell$. The corresponding Higgs Field Φ_{q} is called *cyclic*. Cyclic Higgs bundles were studied by Baraglia in [2] in relation with the affine Toda lattice.

It follows immediately from the construction that

Proposition 5.3.1. *For a cyclic Higgs field Φ_{q}, we have*

$$\Phi_{\mathrm{q}} \in \Omega^1\left(\Sigma, \bigoplus_{\alpha\in Z} \widehat{\mathcal{G}}_\alpha\right) \tag{70}$$

Proof. By Proposition 2.3.5, $e_\ell \in \mathcal{G}_\eta$. By Equation (66)

$$Y \in \mathcal{G}_{-1} \otimes K = \bigoplus_{\alpha\in\Pi} \widehat{\mathcal{G}}_{-\alpha}.$$

The proposition follows. □

The following is implicit in Baraglia's paper [2] but not stated as such. We write the proof using arguments borrowed from this article. Similar results are found in [10].

Proposition 5.3.2. [Baraglia] *Let $(\nabla, \widehat{\rho})$ be the solution of the self duality equation associated to a cyclic Higgs bundle. Then $\widehat{\mathfrak{h}}$ is parallel under ∇ and globally invariant by $\widehat{\rho}$.*

Proof. Let ω be a complex number such that $\omega^{m_\ell+1} = 1$. Let ψ be the automorphism of $\mathfrak{g}$ whose restriction $\mathfrak{g}_m$ is the multiplication by ω^m. Observe now that

$$\psi(\Phi_{\mathrm{q}}) = \omega^{-1} \cdot \Phi_{\mathrm{q}}.$$

It follows that $(\nabla, \widehat{\rho})$, being also a solution of Hitchin equation for $(E, \omega^{-1} \cdot \Phi_{\mathrm{q}})$ is then a solution for $(E, \psi(\Phi_{\mathrm{q}}))$. It follows that

$$\psi^*\nabla = \nabla,$$

and thus ψ is parallel for ∇ and in particular so is $\widehat{\mathfrak{h}}$ which is an eigenspace of ψ. Similalry, we obtain that ψ commutes with $\widehat{\rho}$ and thus the eigenspaces of ψ – in particular $\widehat{\mathfrak{h}}$ – are globally preserved by $\widehat{\rho}$. □

5.4. **Harmonic mappings and Higgs bundles.** We recall the following facts relating equivariant harmonic mappings and Higgs bundles (see [12] in the case of $\mathsf{SL}(2,\mathbb{C})$. Let as usual G be a complex Lie group, $\mathcal{G}$, $\omega \in \Omega^1(\mathsf{S}(\mathsf{G}))$ and $\rho \in \Gamma(\mathrm{Aut}(\mathcal{G}))$ as defined in paragraph 2.4.3.

Theorem 5.4.1. *Let f be an harmonic mapping from a Riemann surface S in $\mathsf{S}(\mathsf{G})$, then $(f^*(\mathcal{G}), (f^*\omega)^{(1,0)})$ is a Higgs bundle. Moreover $f^*(\rho)$ satisfies the self duality equations. Conversely, let S be a simply connected surface, (E, Φ) a Higgs bundle and ρ_E a solution of the self-duality equations. Then there exists an harmonic mapping f from S to $\mathsf{S}(\mathsf{G})$ unique up to the action of G so that*

$$(E, \Phi, \rho_E) = (f^*\mathcal{G}, (f^*\omega)^{(1,0)}, f^*\rho).$$

Moreover using the notation of paragraph 4.1, we have

Proposition 5.4.2. *Let f be an harmonic mapping defined from a Riemannian surface S, equipped with the area form $\mathrm{d}\mu$, to $\mathsf{S}(\mathsf{G})$. Let $e(f)$ be the energy density on S, then*

$$\mathrm{Energy}(f) := \frac{1}{2}\int_S e(f)\mathrm{d}\mu = i\cdot\int_S \langle (f^*\omega)^{(1,0)}|(f^*\omega)^{(0,1)}\rangle.$$

Proof. We have that

$$e(f)\mathrm{d}\mu = -\langle f^*\omega | f^*\omega\circ J\rangle = 2i\cdot\langle (f^*\omega)^{(1,0)}|(f^*\omega)^{(0,1)}\rangle.$$

□

6. Cyclic surfaces

Let $\rho : u \mapsto \overline{u}^k$ be the real structure on $\mathfrak{g}$ coming from the complexification of $\mathfrak{k}$. Let also σ be the involution constructed in Section 2.5. Let finally $\lambda = \sigma\circ\rho$ be the real structure on $\mathfrak{g}$ coming from the complexification of $\mathfrak{g}_0$. We will use in this section the decomposition (51).

Definition 6.0.3. [cyclic maps] *A map f from a surface Σ to X is* cyclic *if*

(1) $f^*(\omega_1) = 0$,
(2) $f^*(\omega_0) = 0$,
(3) $f^*(\phi\wedge\phi) = 0$
(4) $f^*\left(\rho(\phi)\right) = -f^*(\phi^\dagger)$,
(5) if β is a simple root, $f^*(\omega_\beta)$ never vanishes.
(6) $f^*(\lambda(\omega)) = f^*(\omega)$.

Observe that Assertion (3) is equivalent to : for all β and α in Z, we have

$$f^*(\omega_\alpha\wedge\omega_\beta) = 0. \tag{71}$$

The notion of cyclic surfaces is cousin to that of τ-maps studied in [5]. However, the latter notion is in the context of compact Lie groups.

6.0.1. *The reality condition.* Let G_0 be the real split form associated to the Cartan sub algebra and its positive root

Proposition 6.0.4. *Assume that $f : \Sigma\to X$ is a cyclic surface. Then the image of Σ lies in a G_0-orbit in X.*

Proof. By Assertion (6) $f^*(\lambda(\omega)) = f^*(\omega)$. In other words $\overline{\mathsf{T}f(u)} = \mathsf{T}f(u)$ for all u in S. Thus by Proposition 4.2.7, $f(\Sigma)$ is tangent to the foliation $\mathbf{F}$ defined by the "right" G_0 orbits. □

6.0.2. *First example: the Fuchsian case.* Let $x = (\mathfrak{h}, \rho, \lambda)$ be a point in X. Let S be the principal SL_2 in G_0 associated to $\mathfrak{h}$. By definition, the *Fuchsian surface* though x is the orbit of S.

Then we have our first examples of cyclic surfaces.

Proposition 6.0.5. *If S is a Fuchsian surface in X, then S is a cyclic surface such that $\omega_\eta\big|_S = 0$.*

Proof. By construction, the Lie algebra of the complexification of S is generated by (a, C, Y), where

$$a = \frac{1}{2}\sum_{\alpha\in\Delta^+} \mathrm{h}_\alpha = \sum_{\alpha\in\Pi} r_\alpha \cdot \mathrm{h}_\alpha,$$
$$X = \sum_{\alpha\in\Pi} \sqrt{r_\alpha}\mathrm{x}_\alpha,\ Y = \sum_{\alpha\in\Pi} \sqrt{r_\alpha}\mathrm{x}_{-\alpha}.$$

Thus, by Proposition 2.5.5, the Lie algebra of S is generated by $X - Y, iX - iY, ia$. Since ia belongs to $\mathfrak{h}$ and generates a compact subgroup, $ia \in \mathfrak{t}$.

Therefore, the orbit of S in X is with values in the complex 2-dimensional distribution $\mathcal{W}$ such that $\mathcal{V} := \omega(\mathcal{W})$ is generated by $X - Y, iX + iY$. Moreover, since S is real, the orbit of S in X is tangent to the real 2-dimensional distribution (W_0) such that

$$\omega(\mathcal{W}_0) := \{u \in \mathcal{V} \mid \lambda(u) = u\} =: \mathcal{V}_0.$$

In particular, we now observe that

$$\mathcal{V}_0 \subset \mathcal{V} \subset \mathcal{Q} := \sum_{\alpha\in\Pi} \mathcal{G}_\alpha \oplus \sum_{\alpha\in\Pi} \mathcal{G}_{-\alpha} \subset \mathcal{G}_Z \oplus \mathcal{G}_{Z^\dagger}.$$

Observe that $\mathcal{V}$ is stable by $-\rho$, thus if ϕ and $\phi^\dagger$ is the projection from $\mathcal{V}$ to $\mathcal{G}_Z$ and $\mathcal{G}_{Z^\dagger}$ respectively, $-\rho(\phi) = \phi^\dagger$. Thus a Fuchsian surface is a cyclic surface on which ω_η vanishes. □

6.1. **From cyclic surfaces to Higgs bundles.** We emphasize that in the next paragraph, the result is local: the surface Σ is not assumed to be closed. For a cyclic map f from Σ to X, we write $\Phi = f^*\phi, \Phi^\dagger = f^*\phi^\dagger$. We denote by $\widehat{\rho}$ the pullback of ρ on $f^*\mathcal{G}$. By the definition of cyclic maps $\Phi^\dagger = -\widehat{\rho}(\Phi) = \Phi^*$. By a slight abuse of language we also denote by ∇ the induced connexion $f^*\nabla$ on $f^*\mathcal{G}$

Proposition 6.1.1. *Let f be a cyclic map from Σ to X. Then*

1. *there exists a unique complex structure on the surface so that Φ is of type $(1,0)$ and $\Phi^\dagger$ is of type $(0,1)$.*
2. *the data $\Psi = (f^*\mathcal{G}, \Phi)$ defines a Higgs bundle whose Hopf differential is zero.*

(3) *The pair* $(\nabla, \widehat{\rho})$ *on* $f^*\mathcal{G}$ *is the solution of the self duality equations:*

$$\nabla\widehat{\rho} = 0, \tag{72}$$

$$R^{\nabla} = 2 \cdot \Phi \wedge \Phi^*, \tag{73}$$

$$\mathrm{d}^{\nabla}\Phi = 0, \tag{74}$$

$$\mathrm{d}^{\nabla}\Phi^* = 0. \tag{75}$$

(4) *Finally, if* H *is the Hitchin map from the space of Higgs bundle to holomorphic differentials,* $H(\Psi)$ *is a holomorphic differential of highest possible degree.*

As a corollary, we immediately get

Corollary 6.1.2. *Let* f *be a cyclic map. Let* p *be the projection from* X *to the symmetric space* $\mathsf{S}(\mathsf{G})$*, then* $p \circ f$ *is a minimal surface. Moreover*

$$\mathrm{Area}(p \circ f) = i \cdot \int_S \langle \Phi | \Phi^\dagger \rangle \tag{76}$$

Proof. The proof relies on the following observations coming from [12] and Section 5 of [31]. For a smooth map $g : M \to N$ between manifold, we consider $\mathsf{T}g$ as an element of $\Omega^1(M, g^*(\mathsf{T}N))$.

We saw in Proposition 4.2.8, that $\mathsf{T}_{\mathbb{C}}\mathsf{S}(\mathsf{G})$ is identified with the canonical $\mathfrak{g}$ bundle over $\mathsf{S}(\mathsf{G})$. As a consequence of the second assertion of Proposition 4.2.8, we have

$$\frac{1}{2}(\Phi + \Phi^*) = \mathsf{T}_{\mathbb{C}}(p \circ f),$$

and thus

$$\mathsf{T}_{\mathbb{C}}^{(1,0)}(p \circ f) = \frac{1}{2}\Phi,$$

$$\mathsf{T}_{\mathbb{C}}^{(0,1))}(p \circ f) = \frac{1}{2}\Phi^*.$$

Now the equation of $\overline{\partial}\Phi = 0$, exactly says that $p \circ f$ is harmonic (See [12] and Proposition 8.1.2. of [31]). Moreover by the last assertion of the Proposition 6.1.1, the Hopf differential of f is zero and thus $p \circ f$ is a minimal mapping (See Proposition 8.1.4 of [31]). Equation (76) follows at once from Proposition 5.4.2 and the fact that the energy of a minimal mapping is the area. □

We now proceed to the proof of the proposition.

Proof. Since for any simple root $f^*\omega_\alpha$ is non zero, it follows that there exists at most exactly one complex J_α structure so that $f^*\omega_\alpha$ is of type $(1,0)$. Since furthermore $f^*\omega_\alpha \wedge f^*\omega_\beta = 0$, it follows that $J_\alpha = J_\beta$. This

proves the uniqueness and shows that there exists a unique complex structure such that for all simple root α, $f^*\omega_\alpha$ is of type $(1,0)$. It remains to understand the type of $f^*\omega_{-\eta}$.

Since $f^*\phi \wedge f^*\phi = 0$, decomposing along roots we obtain that for all simple root α,

$$f^*\omega_\alpha \wedge f^*\omega_{-\eta} = 0 \tag{77}$$

Since there exist a simple root α so that $\eta - \alpha$ is a root and in particular

$$[\mathcal{G}_\alpha, \mathcal{G}_{-\eta}] \neq 0.$$

The equation (77) implies that $f^*(\omega_{-\eta})$ is of type $(1,0)$. We thus obtain that ϕ is of type $(1,0)$ and by the reality condition that

$$\phi^\dagger = \phi^*,$$

is of type $(0,1)$. This finishes the proof of statement (1). Statement(2) is just an immediate consequence of the previous statement.

Let us now prove statement (3). Let f be a cyclic map. In the sequel of this proof to avoid cumbersome notations, we omit the symbol f^* and all equalities are supposed to be taken restricted to the surface.

Recall that for a cyclic surface

$$f^*(\omega \wedge \omega) = 2.\Phi \wedge \Phi^* \in \Omega^2(\Sigma, f^*(\mathcal{H})). \tag{78}$$

Thus the curvature equation (45)

$$R^\nabla + \mathrm{d}^\nabla\Phi + \mathrm{d}^\nabla\Phi^\dagger + 2 \cdot \Phi \wedge \Phi^\dagger = 0,$$

yields the self duality field equation by taking the projections, namely π_0 for the first equation and π and $\pi^\dagger$ for the two last equations:

$$\begin{aligned} R^\nabla + 2 \cdot \Phi \wedge \Phi^* &= 0, \\ \mathrm{d}^\nabla\Phi &= 0, \\ \mathrm{d}^\nabla\Phi^* &= 0. \end{aligned}$$

Finally, statement (4) follows from Hitchin's construction in Section 5 of [22] (see also Baraglia [2]) and Proposition 5.2.1.

. □

6.2. **From cyclic Higgs bundles to cyclic surfaces.** In this section, contrarily to the previous section, where the construction was purely local, the surface is now assumed to be closed. The main result of this section is

Theorem 6.2.1. *Let $(\widehat{\mathcal{G}}, \Phi_q)$ be a cyclic Higgs bundle over a closed surface Σ. Then there exists a unique cyclic map f from Σ to X such that*

$$\begin{aligned} \widehat{\mathcal{G}} &= f^*(\mathcal{G}) \\ \Phi_q &= f^*(\phi). \end{aligned} \tag{79}$$

Proof. Let $(\mathcal{G}, \Phi_q)$ be a cyclic Higgs bundle as defined in Paragraph 5.3. Let $(\nabla, \widehat{\rho})$ be the solution of the self duality equations. Let then $\mathfrak{h}$ be as in Proposition 5.3.2. In particular $\mathfrak{h}$ is parallel. Thus, following Hitchin Theorem 5.2.2, the associated involution $\widehat{\sigma}$ is parallel. Thus the Hitchin triple $(\widehat{\mathfrak{h}}, \widehat{\rho}, \widehat{\sigma})$ is parallel.

Observe now that by Proposition 5.3.1,

$$\Omega := \Phi_q + \Phi_q^* \in \Omega^1\left(\Sigma, \widehat{\mathcal{G}}_Z \oplus \widehat{\mathcal{G}}_{Z^\dagger}\right).$$

In particular

$$\forall u \in \widehat{\mathfrak{h}}, \ \ \widehat{\rho}(u) = \widehat{\sigma}(u) = u \implies \langle u | \Omega \rangle = 0.$$

Recall finally that from the self duality equations $\nabla + \Omega$ is flat.

Thus we can apply Proposition 4.2.5, to obtain a map f from Σ to X so that

$$\begin{aligned} \widehat{\mathcal{G}} &= f^*(\mathcal{G}) \\ \Omega &= f^*(\omega). \end{aligned} \tag{80}$$

It remains to prove that f is cyclic. This follows at once from the following two facts

(1) By construction, we have that $\Phi_q = f^*(\phi)$, $\Phi_q^* = f^*(\phi^\dagger)$, $f^*\omega_0 = 0$, $f^*\omega_1 = 0$, and $f^*\omega_\alpha \neq 0$ for all simple roots.
(2) By Equation (69), $\sigma(\Phi_q) = -\Phi_q$ and since ρ commutes with σ, $\sigma(\Phi_q^*) = -\Phi_q^*$.
(3) Moreover $\Phi_q \wedge \Phi_q$ is of type $(2,0)$ hence vanishes.

We thus have verified Assertions (1), (2), (4), (6) and (3) of the definition of cyclic surfaces. Assertion (5) follows from the construction of Φ_q. □

6.3. **Cyclic surfaces as holomorphic curves.** The purpose of this section to give another interpretation of cyclic surfaces as holomorphic cuves in some homogeneous quotient of G_0.

We use the notation fo the previous section. Let us revisit the definition of cyclic surfaces. Let us first consider the complex distributions in X given by $\mathcal{V}$ with $\omega(\mathcal{V}) = \mathcal{G}_Z$ with the complex structure given by

the multiplication by i, as well as $\mathcal{V}^\dagger$ with $\omega(\mathcal{V}^\dagger) = \mathcal{G}_{\mathbb{Z}^\dagger}$ given by the multiplication by $-i$. Let $\mathcal{W}$ be the complex distribution given by

$$\mathcal{W} = \mathcal{V} \oplus \mathcal{V}^\dagger.$$

Then the complex conjugation ρ becomes a complex involution of $\mathcal{W}$, and the Hitchin involution, preserving both $\mathcal{V}$ and $\mathcal{V}^\dagger$ is also a complex conjugation. We now consider the subdistribution

$$\mathcal{S} = \{u \in \mathcal{W} \mid \sigma(u) = u, \ \ \rho(u) = u\}.$$

Observe that $\mathcal{S}$ is a sub distribution of TF (see Proposition 4.2.7). Then we have

Proposition 6.3.1. *A cyclic surface is a surface everywhere tangent to $\mathcal{S}$ and whose tangent space is complex. Conversely, a surface everywhere tangent to $\mathcal{S}$ and whose tangent space is complex is a cyclic surface.*

Proof. The proof is just linear algebra. If $\Sigma \hookrightarrow \mathsf{X}$ is a cyclic surface, then for all u tangent to Σ

$$\begin{aligned} \omega(u) &= \Phi(u) + \Phi^\dagger(u)), \\ \lambda(\omega(u)) &= \omega(u) \\ \rho(\Phi(u))) &= \Phi^\dagger. \end{aligned} \tag{81}$$

It follows that a cyclic surface is tangent to $\mathcal{W}$. Let now now J be the complex structure on Σ so that Φ is of type $(1,0)$ we obtain that

$$\begin{aligned} \omega(Ju) &= \phi(Ju) + \rho(\phi(Ju)), \\ &= i.\Phi(u) + \rho(i.\Phi(u)), \\ &= i.\Phi(u) - i\rho(\Phi(u)), \\ &= i.\Phi(u) - i\Phi^\dagger(u)), \\ &. \end{aligned} \tag{82}$$

Let p and $p^\dagger$ be the projection of $\mathcal{W}$ to $\mathcal{V}$ and $\mathcal{V}^\dagger$ respectively. Recall that

$$\begin{aligned} \omega \circ p(u)) &= \Phi(u) \\ \omega \circ p^\dagger(u) &= \Phi^\dagger(u) = \rho(\Phi(u). \end{aligned} \tag{83}$$

Thus from Equation (82), we obtain that

$$\begin{aligned} p(Ju) &= ip(\mathsf{T}f(Ju)), \\ p^\dagger(Ju)) &= ip^\dagger(Ju). \end{aligned} \tag{84}$$

In other words, Σ is a complex subspace of $\mathcal{W}$.

Conversely, assume $\mathsf{T}\Sigma$ is a complex subspace of $\mathcal{W}$. Let us equip Σ with the induced complex structure. Then by construction, for all $u \in \mathsf{T}\sigma$

$$\begin{aligned} \omega_1(u) = \omega_0(u) = 0, \\ \lambda(u) = u. \end{aligned} \tag{85}$$

Since $\omega(u) = \Phi(u) + \Phi^\dagger(u)$ is fixed by ρ which exchanges $\mathcal{G}$ and $\mathcal{G}^\dagger$, it follows that $\rho(\Phi(u)) = \Phi^(\dagger)$. Finally, since $\Phi(Ju) = i \cdot \Phi(u)$, it follows that $\Phi \wedge \Phi = 0$ on Σ. In particular, Σ is a cyclic surface. □

7. Infinitesimal rigidity

In this section, we prove the infinitesimal rigidity for closed cyclic surfaces. The important corollary for us is the Theorem 1.5.1 that we restate here.

Theorem 7.0.2. *The map* $\Psi : \mathcal{E}_{m_i} \to \mathcal{H}(\Sigma, \mathsf{G}_0)$ *is an immersion.*

.

We exploit the fact that cyclic surfaces are solutions of a Pfaffian system, which means that a certain family of forms vanishes on them, as well as a reality condition.

After a preliminary section on Pfaffian systems, we define infinitesimal variation and state our main result, Proposition 7.3.2, in this language.

We prove Theorem 7.0.2 as the corollary of Proposition 7.3.2 (whose proof occupy most of the section) in Paragraph 7.3.1.

The proof of Proposition 7.3.2 – which occupy most of the section– then proceeds through obtaining formulas for the derivatives of the infinitesimal variation and a Böchner type formula.

7.1. Preliminary: variation of Pfaffian systems.

In this section, totally independent on the rest, we explain a useful proposition that we shall use in the sequel of the proof.

We shall consider the following setting. Let $\mathcal{E}$ be a vector bundle over a manifold M equipped with a connection ∇. Let $\omega := (\omega_1, \dots, \omega_n)$ be a family of forms with value in $\mathcal{E}$.

Definition 7.1.1. *A submanifold N of M is* a solution of the Pfaffian system *defined by ω if all i, ω_i vanishes on N.*

If $\mathcal{E}$ is a trivial line bundle, so that ω_i are ordinary forms, then we say the Pfaffian system is *elementary*. We can always reduce any system to an elementary one, by choosing a local trivialization of

$\mathcal{E}$ given by local sections (ξ_α), then the *associated elementary Pfaffian system* in the trivialisation is (ω_i^α) where

$$\omega_i = \sum_\alpha \omega_i^\alpha \cdot \xi_\alpha. \tag{86}$$

7.1.1. *Deformation of Pfaffian systems.* Let $F = (f_t)$ be a 1-parameter smooth family of deformations of maps from N to M so that f_0 is the identity. Let

$$\xi = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f_t. \tag{87}$$

Thus ξ is a vector field along N, called the *tangent vector to the family* (f_t). .

Definition 7.1.2. *The family (f_t) is a* first order deformation *of the Pfaffian solution N if, for all i*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f_t^*\omega_i = 0, \tag{88}$$

where using ∇, we have identified $f_t^*(\mathcal{E})$ with $f_0^*(\mathcal{E})$ for all t.

We observe that the definition does not depend on the choice of ∇ : indeed, equivalently, (f_t) is a first order deformation, if an only if it is a first order deformation for all elementary associated Pfaffian system in local trivialization.

Let us introduce the following definition

Definition 7.1.3. *A vector field ξ along a solution of a Pfaffian system $\omega = (\omega_1, \dots, \omega_n)$ is* an infinitesimal variation fo the Pfaffian system *if for all i*

$$i_\xi \mathrm{d}^\nabla \omega_i\big|_N = -\,\mathrm{d}^\nabla\left(i_\xi \omega_i\right)\big|_N. \tag{89}$$

The following relates the two definitions and will be an important technical tool

Proposition 7.1.4. *Assume that ξ is a tangent vector to a family of first order deformation of the Pfaffian system. Then ξ is an infinitesimal variation fo the Pfaffian system: for all i,*

$$i_\xi \mathrm{d}^\nabla \omega_i\big|_N = -\,\mathrm{d}^\nabla\left(i_\xi \omega_i\right)\big|_N. \tag{90}$$

Proof. It is enough to assume that $n = 1$, that is $\omega = (\omega)$. Assume first that ∇ is the trivial connection. We consider (f_t) as a map F from $P := N \times [0,1]$. Let ∂_t be the canonical vector on P, associated to the

flow $\phi_t : (n,s) \mapsto (n, s+t)$. Let also J be the injection $n \mapsto (n,0)$ from N into P. Let finally $\Omega = F^*\omega$. Observe first that for any form α,

$$J^*(i_{\partial_t} F^*\alpha) = J^*(i_\xi \alpha). \tag{91}$$

Since ξ is is a tangent vector to a family of first order deformation of the Pfaffian system, we have

$$J^*\Omega = 0, \qquad J^* L_{\partial_t}\Omega = 0. \tag{92}$$

By the Lie–Cartan formula,

$$L_{\partial_t}\Omega = \mathrm{d} i_{\partial_t}\Omega + i_{\partial_t}\mathrm{d}\Omega. \tag{93}$$

Using Equation (91), we get

$$\begin{aligned} 0 &= J^*\mathrm{d} i_{\partial_t}\Omega + J^* i_{\partial_t}\mathrm{d}\Omega. \\ &= \mathrm{d} J^* i_{\partial_t}\Omega + J^* i_{\partial_t}\mathrm{d}\Omega. \\ &= \mathrm{d} J^* i_\xi \omega + J^* i_\xi \mathrm{d}\omega \\ &= J^*\left(\mathrm{d} i_\xi \omega + i_\xi \mathrm{d}\omega\right). \end{aligned} \tag{94}$$

Since $\mathsf{T}J$ is injective, the last equation yields

$$L_{\partial_t}\Omega = \mathrm{d} i_{\partial_t}\Omega + i_{\partial_t}\mathrm{d}\Omega. \tag{95}$$

Using Equation (91), we get

$$0 = \mathrm{d} i_\xi \omega + i_\xi \mathrm{d}\omega. \tag{96}$$

Thus the conlusion of the proposition holds when ∇ is trivial. Assume now that ∇ is non trivial. The result is local and let $x_0 \in N$. We can find locally a base (ξ_α) of $\mathcal{E}$ giving a local trivialisation, such that $\nabla\xi_\alpha = 0$ at x_0. Let us write

$$\omega = \sum_\alpha \omega_\alpha \cdot \xi_\alpha. \tag{97}$$

Observe that N is also a solution of the Pfaffian system defined by (ω_α) and that ξ is an infinitesimal deformation of that Pfaffian system. Thus, at x_0,

$$\begin{aligned} i_\xi \mathrm{d}^\nabla \omega &= \sum_\alpha i_\xi \mathrm{d}\omega_\alpha \cdot \xi_\alpha \\ &= -\sum_\alpha \mathrm{d}(i_\xi \omega_\alpha) \cdot \xi_\alpha \\ &= -\mathrm{d}^\nabla(i_\xi \omega), \end{aligned} \tag{98}$$

where in the first equality we used that $\nabla\xi_\alpha = 0$ at x_0, in the second we used Equation (89) for the Pfaffian system (ω_α) and finally in the last equality we used $\nabla\xi_\alpha = 0$ at x_0 again. □

The careful reader could check that Equation (89) is independent of the choice of ∇ if ω vanishes along N.

7.2. **Cyclic surfaces as solutions of a Pfaffian system.** By definition, for a cyclic surface Σ, some forms with value in $\mathcal{G}$ vanishes namely $\phi \wedge \phi|_\Sigma = 0$, $\omega_0|_\Sigma = 0$ and $\omega_1|_\Sigma = 0$.

Another important form vanishes. Let $\mathcal{H}_0$ be the orthogonal in $\mathcal{H}$ with respect to $\widehat{\rho}$ of the subdistribution $\mathcal{T}$ corresponding to the lie algebra of T. We have the orthogonal decomposition

$$\mathcal{H} = \mathcal{T} \oplus \mathcal{H}_0.$$

We can thus write

$$\pi_0 = \pi_t + \widehat{\pi}_0.$$

where $\widehat{\pi}_0$ and π_t are the orthogonal projections respectively on $\mathcal{H}_0$ and $\mathcal{T}$. Since $\widehat{\pi}_0\left(R^\nabla\right) = 0$, it follows from the self duality equations (73) that $\widehat{\pi}_0(\phi \wedge \phi^\dagger)|_\Sigma = 0$.

Thus we may define

Definition 7.2.1. *The* cyclic Pfaffian system *is the family of forms* $\lambda := (\omega_0, \omega_1, \phi \wedge \phi, \phi^\dagger \wedge \phi^\dagger, \phi + \rho(\phi^\dagger)$.

7.3. **Infinitesimal deformation of cyclic surfaces.** Let ξ be a vector field along of a cyclic surface f.

Definition 7.3.1. *We say ξ is an* infinitesimal deformation of cyclic surfaces, *if ξ is an infinitesimal deformation of the cyclic Pfaffian system and if ξ is* real, *that is $\overline{\xi} = \xi$.*

In the rest of this section, ξ will be a fixed infinitesimal variation of cyclic surfaces. We will also consider for the sake of simplicity, the surface Σ as a sub manifold of X.

The following is our main result.

Proposition 7.3.2. *Let ξ be an infinitesimal deformation of a closed surface. Assume that there exists a simple root such that $\omega_\alpha(\xi) = 0$, then $\xi = 0$.*

In this proposition, ω_α is defined in decomposition (47).

7.3.1. *Proof of the transversality of the Hitchin map.* .

Proof. In this paragraph, we prove Theorem 7.0.2, assuming Proposition 7.3.2. The proof is standard. As a standard notation if $(x_t)_{t\in]-1,1[}$ is a C^1-curve in amanifold M, we write

$$\dot{x}_0 := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} x_t \in \mathsf{T}_{x_0}M. \tag{99}$$

Let $(J_t, \mathrm{q}_t)_{t\in]-1,1[}$ be a family of elements of $\mathcal{E}_n$. By Theorem 6.2.1, we associate to (J_t, q_t) a homomorphism δ_t of $\pi_1(\Sigma)$ in G_0 and a δ_t-equivariant cyclic map f_t from Σ from X. Then by definition $\Psi(J_t, \mathrm{q}_t) = [\delta_t]$ is the equivalence class (by conjugation) of δ_t.

Observe first that $\xi(s) := \dot{f}_0(s)$ is an infinitesimal deformation of cyclic surfaces in X.

We want to prove the injectivity of Ψ. Let us thus assume that

$$[\dot{\delta}_0] = 0.$$

Since the smooth manifold $\mathcal{H}(\Sigma, \mathsf{G}_0)$ only consists of irreducible representations,

$$\mathsf{T}_{\delta_0}\mathcal{H}(\Sigma, \mathsf{G}_0) = H^1_{\delta_0}(\Sigma, \mathfrak{g}).$$

Thus after possibly conjugating the family (δ_t) by a family (g_t) of elements of G_0 and a similar transformation for (f_t) we obtain that

$$\forall \gamma \in \pi_1(\Sigma), \quad \dot{f}_0(\gamma(s)) = \delta_0(\gamma) \cdot \dot{f}_0(s). \tag{100}$$

In particular, $\xi(s) := \dot{f}_0(s)$ is an infinitesimal deformation of closed cyclic surfaces in $\mathsf{X}/\delta_0(\pi_1(\Sigma))$.

Let us fix a simple root α. Recall that by definition $f_0^*\omega_\alpha$ is a bijection from $\mathsf{T}\Sigma$ to $f_0^*(\mathcal{G}_\alpha)$. Thus, let ν be the vector field along Σ so that $\zeta_\alpha = f^*\omega_\alpha(\nu)$. Since every vector field tangent to the surface is an infinitesimal deformation of cyclic surfaces, $\mathsf{T}f_0(\nu)$ is an infinitesimal deformation of cyclic surfaces.

Let us now consider $\widehat{\xi} = \xi - \mathsf{T}f_0(\nu)$. Then by construction $\widehat{\xi}$ is an infinitesimal deformation of cyclic surfaces whose component along $\mathcal{G}_\alpha$ is zero. Applying Proposition 7.3.2, we obtain that $\widehat{\xi} = 0$. It then follows that $\dot{J}_0 = 0$ and $\dot{\mathrm{q}}_0 = 0$, in particular $\mathsf{T}\Psi$ is injective. This completes the proof. □

7.3.2. *The cyclic decomposition of an infinitesimal deformation.* The cyclic decomposition of ξ is given by

$$\omega(\xi) = \zeta_0 + \zeta_1 + \zeta + \zeta^\dagger \tag{101}$$

where

$$\zeta_0 := i_\xi\omega_0 \in \mathcal{H} \quad \zeta_1 := i_\xi\omega_1 \in \mathcal{G}_1,$$
$$\zeta := i_\xi\phi \in \mathcal{G}_Z, \quad \zeta^\dagger := i_\xi\phi^\dagger \in \mathcal{G}_{Z^\dagger}.$$

Equation (48) implies that actually

$$\zeta_0 \in \mathcal{H}_0. \tag{102}$$

7.3.3. *Reality condition.* We assume that ξ is a real vector, meaning that $\overline{\xi} = \xi$. It then follows

Proposition 7.3.3. *We have*

$$\lambda(\zeta) = \zeta^{\dagger}, \;\; \lambda(\zeta_0) = \zeta_0, \;\; \lambda(\zeta_1) = \zeta_1. \tag{103}$$

Moreover

$$\rho(\zeta_0) = -\zeta_0. \tag{104}$$

Proof. The first equality comes from the fact that σ preserves $\mathcal{G}_Z$ and $\mathcal{G}_{Z^\dagger}$ respectively (last statement of Proposition 2.5.1) and ρ exchanges them.

For the second equality, remark that ρ is an involution that globally preserves $\mathcal{T}$, hence $\mathcal{H}_0$, as well as

$$\mathcal{H}_\lambda := \{u \in \mathcal{H}_0 \mid \lambda(u) = u\}.$$

Any fixed vector by ρ in $\mathcal{H}_\lambda$, belongs to $\mathcal{T}$ hence is null. It follows that ρ acts as -1 on $\mathcal{H}_\lambda$. □

7.3.4. *The root space decomposition.* We also write the following decomposition of ξ as

$$\xi = \zeta_0 + \sum_{\alpha \in \Delta} \zeta_\alpha, \quad \text{where} \quad \zeta_\alpha \in \mathcal{G}_\alpha.$$

7.4. **Computations of first derivatives.** From now on, we assume that ξ is an infinitesimal deformation of cyclic surfaces. We will first obtain expressions for the derivatives of ζ_0 and ζ_1 exploiting the fact that ω_0 and ω_1 vanish on cyclic surfaces. We will denote in the sequel

$$\partial = \left(\mathrm{d}^\nabla\right)^{(1,0)}, \;\; \overline{\partial} = \left(\mathrm{d}^\nabla\right)^{(0,1)}.$$

7.4.1. *Vanishing of ω_0 and the derivatives of ζ_0.* Here, we exploit the fact that $\omega_0|_\Sigma = 0$.

Proposition 7.4.1. *We have the following equality in* $\Omega^*(\Sigma, \mathcal{G})$

$$\partial\zeta_0 = 2 \cdot \widehat{\pi}_0(\zeta^\dagger \wedge \phi) = 2 \cdot \widehat{\pi}_0(\rho(\zeta) \wedge \phi), \tag{105}$$

$$\overline{\partial}\zeta_0 = 2 \cdot \widehat{\pi}_0(\zeta \wedge \phi^\dagger) = 2 \cdot \widehat{\pi}_0(\rho(\zeta^\dagger) \wedge \phi^\dagger). \tag{106}$$

Proof. We have $\zeta_0 = i_\xi \omega_0$. By the definition of infinitesimal variation, we have the following equality in $\Omega^*(\Sigma, \mathcal{G})$

$$\nabla\zeta_0 = \mathrm{d}^\nabla i_\xi \omega_0 = -i_\xi \mathrm{d}^\nabla \omega_0.$$

Hence using the fact that $\widehat{\pi}_0$ commutes with d^∇ and the interior product,

$$\begin{aligned}\nabla\zeta_0 &= \nabla\widehat{\pi}_0\,(\xi_0)\\ &= \widehat{\pi}_0\,(\nabla\xi_0)\\ &= -\widehat{\pi}_0\left(i_\xi \mathrm{d}^\nabla\omega_0\right)\\ &= -i_\xi \mathrm{d}^\nabla\widehat{\pi}_0\,(\omega_0)\\ &= -i_\xi \mathrm{d}^\nabla\widehat{\pi}_0\,(\omega)\\ &= -i_\xi\widehat{\pi}_0(\mathrm{d}^\nabla\omega).\end{aligned}$$

Thus, the curvature equation (45) yields

$$\nabla\zeta_0 = i_\xi\widehat{\pi}_0\left(R^\nabla + \omega\wedge\omega\right).$$

Then by Proposition 4.2.4, we have that $\widehat{\pi}_0(R^\nabla) = 0$. Thus we get

$$\nabla\zeta_0 = i_\xi\widehat{\pi}_0(\omega\wedge\omega). \tag{107}$$

Finally, since $\omega_1|_\Sigma = 0$, it follows that $(i_\xi(\omega_1\wedge\omega_1))|_\Sigma = 0$. Thus, combining Equations (53) and (107), we get the following equality in $\Omega^*(\Sigma,\mathcal{G})$

$$\nabla\zeta_0 = 2\cdot i_\xi\widehat{\pi}_0(\phi\wedge\phi^\dagger) = 2\cdot\widehat{\pi}_0(\zeta\wedge\phi^\dagger + \zeta^\dagger\wedge\phi).$$

Using the fact that ϕ and $\phi^\dagger$ are respectively of type $(1,0)$ and $(0,1)$, we get the first part of both equations in the proposition. To get the second part, we use that $\zeta_0 = -\rho(\zeta_0)$ and $\phi = -\rho(\phi^\dagger)$. □

7.4.2. *Vanishing of ω_1 and the derivatives of ζ_1.* We exploit the fact that $\omega_1|_\Sigma = 0$.

Proposition 7.4.2. *We have*

$$\begin{aligned}\partial\zeta_1 &= 2\cdot\pi_1((\zeta_1+\zeta)\wedge\phi),\\ \overline{\partial}\zeta_1 &= 2\cdot\pi_1((\zeta_1+\zeta^\dagger)\wedge\phi^\dagger).\end{aligned} \tag{108}$$

Proof. By the definition of infinitesimal variation, we have

$$\nabla\zeta_1 = \mathrm{d}^\nabla i_\xi\omega_1 = -i_\xi\mathrm{d}^\nabla\omega_1.$$

Thus, since $\pi_1(R^\nabla) = 0$ and π_1 is parallel, the curvature equation (45) yields

$$\begin{aligned}\nabla\zeta_1 &= -i_\xi\mathrm{d}^\nabla\pi_1(\omega) = -i_\xi\pi_1(\mathrm{d}^\nabla\omega)\\ &= i_\xi\pi_1(\omega\wedge\omega).\end{aligned} \tag{109}$$

Observe also that we have the following equality in $\Omega^*(\Sigma, \mathcal{G})$

$$i_\xi(\omega_1 \wedge \omega_0) = 0, \quad i_\xi(\omega_1 \wedge \omega_1) = 0. \tag{110}$$

Combining equations (54), (109), and (110) we get the following equality in $\Omega^*(\Sigma, \mathcal{G})$

$$\begin{aligned}\nabla\zeta_1 &= i_\xi \pi_1 \left(2.\omega_1 \wedge \phi + 2.\omega_1 \wedge \phi^\dagger + \phi \wedge \phi + \phi^\dagger \wedge \phi^\dagger\right) \\ &= 2 \cdot \pi_1 \left(\zeta_1 \wedge \phi + \zeta_1 \wedge \phi^\dagger + \zeta \wedge \phi + \zeta^\dagger \wedge \phi^\dagger)\right).\end{aligned}$$

Now we can decompose the last equation into types, using the fact that $\phi|_\Sigma$ is of type $(1,0)$ and $\phi^\dagger|_\Sigma$ is of type $(0,1)$ to get

$$\begin{aligned}\partial\zeta_1 &= 2 \cdot \pi_1(\zeta_1 \wedge \phi + \zeta \wedge \phi), \\ \overline{\partial}\zeta_1 &= 2 \cdot \pi_1(\zeta_1 \wedge \phi^\dagger + \zeta^\dagger \wedge \phi^\dagger).\end{aligned}$$

The proposition now follows. □

7.5. **Again, computation of first derivatives.** So far we have obtained direct information about the first derivatives of ζ_0 and ζ_1 using vanishing of the 1-forms ω_0 and ω_1. In this section, we obtain constraints on the derivatives of ζ and $\zeta^\dagger$ using the vanishing of 2-forms.

7.5.1. *A preliminary computation.* The next proposition does not use the fact that ξ is an infinitesimal deformation of cyclic surfaces.

Proposition 7.5.1. *We have the following equality in* $\Omega^*(\Sigma, \mathcal{G})$

$$(i_\xi \mathrm{d}^\nabla \phi)^{(0,1)} = -2 \cdot \pi((\zeta_1 + \zeta^\dagger) \wedge \phi^\dagger), \tag{111}$$

$$(i_\xi \mathrm{d}^\nabla \phi)^{(1,0)} = -2 \cdot \zeta_0 \wedge \phi. \tag{112}$$

Symmetrically

$$(i_\xi \mathrm{d}^\nabla \phi^\dagger)^{(0,1)} = -2 \cdot \zeta_0 \wedge \phi^\dagger, \tag{113}$$

$$(i_\xi \mathrm{d}^\nabla \phi^\dagger)^{(1,0)} = -2 \cdot \pi^\dagger((\zeta_1 + \zeta) \wedge \phi). \tag{114}$$

Proof. First observe that using Assertion (55) of Proposition 4.3.2

$$\begin{aligned}\mathrm{d}^\nabla \phi &= \mathrm{d}^\nabla \pi(\omega) = \pi(\mathrm{d}^\nabla \omega) \\ &= -\pi(\omega \wedge \omega + R^\nabla) \\ &= -\pi(\omega \wedge \omega) \\ &= -2 \cdot \omega_0 \wedge \phi - 2 \cdot \pi(\omega_1 \wedge \phi^\dagger) \\ &- \pi(\phi^\dagger \wedge \phi^\dagger) - \pi(\omega_1 \wedge \omega_1).\end{aligned} \tag{115}$$

Recall again that for a cyclic surface $\omega_i|_\Sigma = 0$ for $i = 0, 1$, and thus for $i, j = 0, 1$

$$i_\xi(\omega_i \wedge \omega_j)|_\Sigma = 0.$$

Thus Equation (115) yields the following equality in $\Omega^*(\Sigma, \mathcal{G})$

$$\begin{aligned} i_\xi \mathrm{d}^\nabla \phi &= -i_\xi \left(2 \cdot \omega_0 \wedge \phi + \pi\left(2 \cdot \omega_1 \wedge \phi^\dagger + \phi^\dagger \wedge \phi^\dagger\right)\right) \\ &= -2 \cdot \zeta_0 \wedge \phi - 2 \cdot \pi(\zeta_1 \wedge \phi^\dagger + \zeta^\dagger \wedge \phi^\dagger). \end{aligned}$$

Since ϕ is of type $(1, 0)$ and $\phi^\dagger$ is of type $(0, 1)$ the previous equation yields the first part of the proposition, where in the second equality, we use that $\pi(\zeta_0 \wedge \phi) = \zeta_0 \wedge \phi$. The second part follows by symmetry. □

7.5.2. *Vanishing of $\phi + \rho(\phi^\dagger)$.* Let $\mu = \zeta + \rho(\zeta^\dagger)$, then

Proposition 7.5.2. *We have the following equality in $\Omega^*(\Sigma, \mathcal{G})$.*

$$\partial \mu = 4 \cdot \zeta_0 \wedge \phi. \tag{116}$$

Proof. Let $\beta = \phi + \rho(\phi^\dagger)$. Using Proposition 7.5.1, we get

$$\begin{aligned} (i_\xi \mathrm{d}^\nabla \beta)^{1,0} &= -2 \cdot \zeta_0 \wedge \phi - 2 \cdot \rho(\zeta_0 \wedge \phi^\dagger) \\ &= -2 \cdot \zeta_0 \wedge \phi + 2 \cdot \rho(\zeta_0) \wedge \phi \\ &= -4 \cdot \zeta_0 \wedge \phi, \end{aligned} \tag{117}$$

where we used Equation (104) in the last equality. Then, by the vanishing of β along cyclic surfaces, we obtain

$$\begin{aligned} (i_\xi \mathrm{d}^\nabla \beta)^{1,0} &= -(\mathrm{d}^\nabla i_\xi \beta)^{1,0} \\ &= -\partial \mu. \end{aligned} \tag{118}$$

This proves the result □

7.5.3. *Vanishing of $\phi \wedge \phi$.*

Proposition 7.5.3. *We have the following equality in $\Omega^*(\Sigma, \mathcal{G})$*

$$\nabla \zeta \wedge \phi = 2 \cdot \phi \wedge \pi((\zeta_1 + \zeta^\dagger) \wedge \phi^\dagger), \tag{119}$$

$$\nabla \zeta^\dagger \wedge \phi^\dagger = 2 \cdot \phi^\dagger \wedge \pi^\dagger((\zeta_1 + \zeta) \wedge \phi). \tag{120}$$

Proof. Let $\Psi = \phi \wedge \phi$. By the definition of infinitesimal variation, the following equality in $\Omega^*(\Sigma, \mathcal{G})$ holds

$$\mathrm{d}^\nabla i_\xi \Psi = -i_\xi \mathrm{d}^\nabla \Psi = -2 \cdot i_\xi(\mathrm{d}^\nabla \phi \wedge \phi). \tag{121}$$

Recall that by Equation (74) for a cyclic surface $d^\nabla\phi|_\Sigma = 0$. Thus the last equation yields (after a type decomposition)

$$d^\nabla i_\xi \Psi = -2 \cdot \phi \wedge (i_\xi d^\nabla \phi)^{(0,1)}. \tag{122}$$

Then Equation (111) from Proposition 7.5.1 yields

$$d^\nabla i_\xi \Psi = 4 \cdot \phi \wedge \pi((\zeta_1 + \zeta^\dagger) \wedge \phi^\dagger).$$

Now, $i_\xi \Psi = 2 \cdot \zeta \wedge \phi$. Thus the previous equation combined with the fact that $d^\nabla\phi|_\Sigma = 0$ yields

$$\nabla\zeta \wedge \phi = 2 \cdot \pi((\zeta_1 + \zeta^\dagger) \wedge \phi^\dagger) \wedge \phi.$$

A symmetric argument (using $\phi^\dagger \wedge \phi^\dagger = 0$ along Σ) yields the last statement. □

7.6. **Computation of second order derivatives.** We now combine the two previous sections to obtain formulas for the second derivatives of ζ_1 and ζ_0.

Proposition 7.6.1. *We have*

$$d^\nabla \partial\zeta_1 = 4.\pi_1\left((\zeta_1 \wedge \phi^\dagger) \wedge \phi\right), \tag{123}$$

$$d^\nabla \bar{\partial}\zeta_1 = 4.\pi_1\left((\zeta_1 \wedge \phi) \wedge \phi^\dagger\right), \tag{124}$$

Proof. By Proposition 7.4.2

$$\partial\zeta_1 = 2 \cdot \pi_1((\zeta_1 + \zeta) \wedge \phi).$$

Thus,

$$d^\nabla \partial\zeta_1 = 2 \cdot d^\nabla \pi_1(\zeta_1 \wedge \phi) + 2 \cdot d^\nabla \pi_1(\zeta \wedge \phi). \tag{125}$$

Let us first consider the derivatives of $\pi_1(\zeta_1 \wedge \phi)$. Since $d^\nabla\phi = 0$ by Equation (74), we have, using Proposition 7.4.2 for the third equality,

$$\begin{aligned} d^\nabla(\pi_1(\zeta_1 \wedge \phi)) &= \pi_1(\nabla\zeta_1 \wedge \phi) \\ &= \pi_1(\bar{\partial}\zeta_1 \wedge \phi) \\ &= 2 \cdot \pi_1\left(\phi \wedge \pi_1\left((\zeta_1 + \zeta^\dagger) \wedge \phi^\dagger\right)\right). \end{aligned} \tag{126}$$

Similarly, now using Proposition 7.5.3

$$\begin{aligned} d^\nabla(\pi_1(\zeta \wedge \phi)) &= \pi_1(\nabla\zeta \wedge \phi) \\ &= 2 \cdot \pi_1\left(\phi \wedge \pi\left((\zeta_1 + \zeta^\dagger) \wedge \phi^\dagger\right)\right). \end{aligned} \tag{127}$$

Combining Equations (126) and (127), we get

$$d^\nabla \partial\zeta_1 = 4.\pi_1\left(\phi \wedge (\pi_1 + \pi)\left((\zeta_1 + \zeta^\dagger) \wedge \phi^\dagger\right)\right). \tag{128}$$

Since by Proposition 2.2.2,

$$[\mathcal{G}_1, \mathcal{G}_{Z^\dagger}] \subset \mathcal{G}_1 + \mathcal{G}_Z,$$
$$[\mathcal{G}_{Z^\dagger}, \mathcal{G}_{Z^\dagger}] \subset \mathcal{G}_1 + \mathcal{G}_Z, \tag{129}$$

we get

$$d^\nabla \partial \zeta_1 = 4.\pi_1\left(\phi \wedge \left((\zeta_1 + \zeta^\dagger) \wedge \phi^\dagger\right)\right). \tag{130}$$

The Jacobi identity yields

$$(\zeta^\dagger \wedge \phi^\dagger) \wedge \phi = \zeta^\dagger \wedge (\phi^\dagger \wedge \phi) + \phi^\dagger \wedge (\zeta^\dagger \wedge \phi) \in \Omega^2(\Sigma, \mathcal{G}_{Z^\dagger}).$$

Thus

$$\pi_1\left((\zeta^\dagger \wedge \phi^\dagger) \wedge \phi\right) = 0.$$

Thus in the end, Equation (130) yields

$$d^\nabla \partial \zeta_1 = 4.\pi_1\left(\phi \wedge (\zeta_1 \wedge \phi^\dagger)\right). \tag{131}$$

The proof of the second equation of the proposition follows by inverting the role of ϕ and $\phi^\dagger$. □

Proposition 7.6.2. *We have*

$$d^\nabla \overline{\partial} \zeta_0 = 4 \cdot \widehat{\pi}_0((\zeta_0 \wedge \phi) \wedge \phi^\dagger). \tag{132}$$

Proof. By Proposition 109, we have

$$\overline{\partial} \zeta_0 = \widehat{\pi}_0(\mu \wedge \phi^\dagger). \tag{133}$$

By Proposition 7.5.2, and using the fact that $d^\nabla \phi^\dagger = 0$, it follows that

$$d^\nabla(\overline{\partial} \zeta_0) = \widehat{\pi}_0(\partial \mu \wedge \phi^\dagger) \tag{134}$$
$$= 4 \cdot \widehat{\pi}_0((\zeta_0 \wedge \phi) \wedge \phi^\dagger). \tag{135}$$

□

7.7. **Proof of the infinitesimal rigidity.** Our goal is now to prove Proposition 7.3.2

7.7.1. *First step.*

Proposition 7.7.1. *Let ξ be an infinitesimal deformation on a closed surface. Then*

$$\zeta_1 \wedge \phi = \zeta_1 \wedge \phi^\dagger = 0, \tag{136}$$
$$\nabla \zeta_1 = 0, \tag{137}$$
$$\pi_1(\zeta \wedge \phi) = 0. \tag{138}$$

Proof. Let Σ be a closed surface. By Proposition 7.6.1,

$$d^{\nabla}\partial\zeta_1 = 4.\pi_1\left((\zeta_1 \wedge \phi^{\dagger}) \wedge \phi\right).$$

Denoting by $\langle\cdot|\cdot\rangle$ the Killing form, an integration yields

$$\int_{\Sigma}\langle\overline{\zeta_1}^k|d^{\nabla}\partial\zeta_1\rangle = 4\cdot\int_{\Sigma}\langle\overline{\zeta_1}^k|\pi_1\left((\zeta_1 \wedge \phi^{\dagger}) \wedge \phi\right)\rangle.$$

Observe now that since ρ preserves $\mathfrak{g}_1$, $\pi_1\left(\overline{\zeta_1}^k\right) = \overline{\zeta_1}^k$. Thus for all κ,

$$\int_{\Sigma}\langle\overline{\zeta_1}^k|\pi_1(\kappa)\rangle = \int_{\Sigma}\langle\overline{\zeta_1}^k|\kappa\rangle.$$

Thus

$$\int_{\Sigma}\langle\overline{\zeta_1}^k|d^{\nabla}\partial\zeta_1\rangle = 4\int_{\Sigma}\langle\overline{\zeta_1}^k|(\zeta_1 \wedge \phi^{\dagger}) \wedge \phi\rangle.$$

Using Equation (34) and the fact that $\phi^{\dagger} = -\overline{\phi}^k$, we get

$$\int_{\Sigma}\langle\overline{\zeta_1}^k|d^{\nabla}\partial\zeta_1\rangle = -4\int_{\Sigma}\langle\overline{\zeta_1}^k \wedge \phi|\zeta_1 \wedge \overline{\phi}^k\rangle.$$

An integration by part and Proposition 4.1.2 finally yields

$$0 \leqslant i\cdot\int_{\Sigma}\langle\overline{\partial\zeta_1}^k|\partial\zeta_1\rangle = 4i\cdot\int_{\Sigma}\langle\overline{\zeta_1}^k \wedge \phi|\zeta_1 \wedge \overline{\phi}^k\rangle \leqslant 0.$$

It then follows that

$$\overline{\partial}\zeta_1 = 0 \tag{139}$$

$$\zeta_1 \wedge \phi = 0. \tag{140}$$

Symmetrically

$$\partial\zeta_1 = 0 \tag{141}$$

$$\zeta_1 \wedge \phi^{\dagger} = 0 \tag{142}$$

Assertion (136) is just Equations (140) and (142).

Assertion (137) now follows from Equations (139) and (141).

Assertion (138) then follows from Proposition 7.4.2 and Equation (140). □

7.7.2. *Second step.*

Proposition 7.7.2. *Let Σ be a closed surface. Let ξ be an infinitesimal variation. Then*

$$\nabla\zeta_0 = 0 \tag{143}$$

$$\zeta_0 \wedge \phi = 0 \tag{144}$$

$$\widehat{\pi}_0(\zeta \wedge \phi^\dagger) = 0. \tag{145}$$

Proof. We will use freely in the sequel the fact that $\int_\Sigma \langle u|\widehat{\pi}_0(v)\rangle = \int_\Sigma \langle \widehat{\pi}_0(u)|v\rangle$.

Thus, using Proposition 7.6.2

$$\int_S \langle \mathrm{d}^\nabla \overline{\partial}\zeta_0 | \overline{\zeta_0}^k\rangle = 4 \cdot \int_S \langle \widehat{\pi}_0(\zeta_0 \wedge \phi) \wedge \phi^\dagger) | \overline{\zeta_0}^k\rangle \tag{146}$$

$$= 4 \cdot \int_S \langle (\zeta_0 \wedge \phi) \wedge \phi^\dagger | \overline{\zeta_0}^k\rangle \tag{147}$$

$$= 4 \cdot \int_S \langle \zeta_0 \wedge \phi | \overline{\zeta_0}^k \wedge \phi^\dagger\rangle, \tag{148}$$

where we used Equation (34) in the last equality. Thus after an integration by part we obtain

$$\begin{aligned}
\frac{1}{4}\int_S \langle \overline{\partial}\zeta_0 | \overline{\overline{\partial}\zeta_0}^k\rangle &= \frac{1}{4}\int_S \langle \overline{\partial}\zeta_0 | \partial\overline{\zeta_0}^k\rangle \\
&= -\int_S \langle \widehat{\pi}_0(\zeta_0 \wedge \phi) \wedge \phi^\dagger) | \overline{\zeta_0}^k\rangle \\
&= -\int_S \langle (\zeta_0 \wedge \phi) \wedge \phi^\dagger | \overline{\zeta_0}^k\rangle \\
&= -\int_S \langle \zeta_0 \wedge \phi | \overline{\zeta_0}^k \wedge \phi^\dagger\rangle, \\
&= \int_S \langle \zeta_0 \wedge \phi | \overline{\zeta_0 \wedge \phi}^k\rangle.
\end{aligned} \tag{149}$$

But, by Proposition 4.1.2,

$$0 \leqslant \frac{i}{4} \cdot \int_S \langle \overline{\partial}\zeta_0 | \overline{\overline{\partial}\zeta_0}^k\rangle = i \cdot \int_S \langle \zeta_0 \wedge \phi | \overline{\zeta_0 \wedge \phi}^k\rangle \leqslant 0. \tag{150}$$

Thus, $\zeta_0 \wedge \phi = 0$ and $\overline{\partial}\zeta_0 = 0$. Since $\overline{\zeta_0}^k = -\zeta_0$. It follows that

$$0 = \overline{\overline{\partial}\zeta_0}^k = \partial\overline{\zeta_0}^k = -\partial\zeta_0.$$

Thus $\partial\zeta_0 = 0$. It follows that $d^\nabla\zeta_0 = 0$.

□

7.7.3. *Proof of Proposition 7.3.2.* Recall that we have the decomposition

$$\omega(\xi) = \zeta_0 + \zeta_1 + \zeta + \zeta^\dagger.$$

We assume in this section that there exists a simple root α, so that the component $\zeta_\alpha = \omega_\alpha(\xi)$ of ξ along $\mathcal{G}_\alpha$ vanishes.

Proposition 7.7.3. *We have $\zeta_0 = 0$ and $\zeta_1 = 0$.*

Proof. From Equation (137) of Proposition 7.7.1, we have that $\nabla\zeta_1 = 0$. Observe that

$$\zeta_1 = \sum_{\gamma\in\Delta\setminus Z\cup Z^\dagger} \zeta_\gamma, \text{ where } \zeta_\gamma \in \mathcal{G}_\gamma.$$

Since the line bundles $\mathcal{G}_\gamma$ are parallel, it follows that $\nabla\zeta_\gamma = 0$ for all γ not in Z nor in $Z^\dagger$. Recall that $\mathcal{G}_\alpha$ is identified for all simple roots as a complex line bundle with $\mathsf{T}\Sigma$ thanks to ϕ_α. It then follows that $\mathcal{G}_\beta$ is identified non zero power of the complex line bundle $\mathsf{T}\Sigma$ for any root β. Then since Σ is not a torus, ζ_γ vanishes at some point, hence everywhere since it is parallel.

By Equation (7.7.2), $\zeta_0 \wedge \phi = 0$, we get that for all simple root α, $\zeta_0 \wedge \omega_\alpha = 0$. By Definition 6.0.3 Property (5), ω_α never vanishes. Thus $\alpha(\zeta_0) = 0$. Since the simple roots form a basis of $\mathfrak{h}^*$, $\zeta_0 = 0$. □

It follows from the previous proposition that

$$\xi = \zeta + \zeta^\dagger. \tag{151}$$

It remains to prove that $\zeta = 0$ as well as $\zeta^\dagger = 0$. We now split the proof in two cases.

7.7.4. *First case:* $\mathsf{G} = \mathsf{SL}(3,\mathbb{R})$. Remark that in this case we have three positive roots, two simple that we name α, β and one long $\eta = \alpha + \beta$. Also, $\widehat{\pi}_0 \neq 0$ and $\pi_1 = 0$.

Proposition 7.7.4. *Assume that $\mathsf{G} = \mathsf{SL}(3,\mathbb{R})$, then $\zeta = 0$ and $\zeta^\dagger = 0$.*

Proof. Let α and β be the two simple roots and $\eta = \alpha + \beta$ be the longest root. Let us choose locally on Σ a Chevalley frame $\{x_\alpha\}_{\alpha\in\Delta}$ such that $\rho(x_\alpha) = x_{-\alpha}$, then we write

$$\begin{aligned}
\phi^\dagger &= \psi_\eta \cdot x_{-\eta} + \psi_\alpha \cdot x_\alpha + \psi_\beta \cdot x_\beta,\\
\phi &= \overline{\psi_\eta} \cdot x_\eta + \overline{\psi_\alpha} \cdot x_{-\alpha} + \overline{\psi_\beta} \cdot x_{-\beta},\\
\zeta &= \mu_\eta \cdot x_\eta + \mu_\alpha \cdot x_{-\alpha} + \mu_\beta \cdot x_{-\beta}.
\end{aligned} \tag{152}$$

Our hypothesis in that Section is that $\mu_\alpha = 0$. Observe now that

$$\zeta \wedge \phi^\dagger = \mu_\eta.\psi_\eta \mathrm{h}_\eta - \mu_\beta.\psi_\beta \mathrm{h}_\beta - \mu_\alpha.\psi_\alpha \mathrm{h}_\alpha. \tag{153}$$

By Proposition 3.2.3 Property (1), $\mathrm{t}_{\mathbb{C}}$ is generated by h_η. Let us write

$$h_\beta = u + \lambda \cdot h_\eta,$$

with $\langle u | h_\eta \rangle = 0$, so that $\widehat{\pi}_0(u) = u$. Observe also that $\langle h_\beta | u \rangle = \langle u | u \rangle \neq 0$, since h_β is not proportional to h_η. Thus the equality $\widehat{\pi}_0(\zeta \wedge \phi^\dagger) = 0$ and $\mu_\alpha = 0$ implies that

$$0 = \langle \zeta \wedge \phi^\dagger | u \rangle = -\mu_\beta \psi_\beta \cdot \langle \mathrm{h}_\beta | u \rangle .$$

Since ψ_β never vanishes, it follows that $\mu_\beta = 0$. Thus $\zeta \in \mathcal{G}_\eta$.

Since by the reality condition (103), $\zeta^\dagger = \lambda(\zeta)$, it follows that $\zeta^\dagger = \mu_\eta \cdot \mathrm{x}_{-\eta} \in \mathcal{G}_{-\eta}$ and

$$\zeta^\dagger \wedge \phi^\dagger = \mu_\eta \psi_\alpha [\mathrm{x}_{-\eta}, \mathrm{x}_\alpha] + \mu_\eta \psi_\beta [\mathrm{x}_{-\eta}, \mathrm{x}_\beta] = \pi(\zeta^\dagger \wedge \phi^\dagger). \tag{154}$$

Then the component of $\phi \wedge \pi(\zeta^\dagger \wedge \phi^\dagger)$ along $\mathcal{G}_{-\eta}$ is

$$\mu_\eta \cdot \left(\overline{\psi_\alpha} \wedge \psi_\alpha \cdot [x_{-\alpha}, [x_{-\eta}, x_\alpha] + \overline{\psi_\beta} \wedge \psi_\beta \cdot [x_{-\beta}, [x_{-\alpha}, x_\beta] \right). \tag{155}$$

By Proposition 7.5.3, we have

$$\nabla \zeta \wedge \phi = 2 \cdot \phi \wedge \pi(\zeta^\dagger \wedge \phi^\dagger) \tag{156}$$

Since $\zeta \in \mathcal{G}_\eta$, the component on $\mathcal{G}_{-\eta}$ of $\nabla \zeta \wedge \phi$ is zero. Thus we obtain

$$0 = \mu_\eta \cdot \left(\overline{\psi_\alpha} \wedge \psi_\alpha \cdot [x_{-\alpha}, [x_{-\eta}, x_\alpha] + \overline{\psi_\beta} \wedge \psi_\beta \cdot [x_{-\beta}, [x_{-\alpha}, x_\beta] \right).$$

We can use an element $\mathfrak{w}$ of the Weyl group that fixes η and exchanges α and β, we then get that

$$\mathfrak{w}([x_{-\alpha}, [x_{-\eta}, x_\alpha]]) = [x_{-\beta}, [x_{-\eta}, x_\beta]$$

On the other hand, $[x_{-\alpha}, [x_{-\eta}, x_\alpha]]$ is along $\mathrm{x}_{-\eta}$ and thus fixed by $\mathfrak{w}$. It follows that

$$[x_{-\alpha}, [x_{-\eta}, x_\alpha] = [x_{-\beta}, [x_{-\eta}, x_\beta] \neq 0.$$

Thus Equation (155) yields

$$0 = \mu_\eta \cdot \left(\overline{\psi_\alpha} \wedge \psi_\alpha + \overline{\psi_\beta} \wedge \psi_\beta \right).$$

Since $\overline{\psi_\alpha} \wedge \psi_\alpha$ and $\overline{\psi_\beta} \wedge \psi_\beta$ are both of type $(1,1)$ and positive, we get that $\mu_\eta = 0$. We have finished proving that $\zeta^\dagger = \zeta = 0$. □

7.7.5. *The general case.* Now we assume that $\mathsf{G} \neq \mathsf{SL}(3,\mathbb{R})$. Then we prove

Proposition 7.7.5. *Assume that* $\mathsf{G} \neq \mathsf{SL}(3,\mathbb{R})$*, then* $\zeta = 0$ *and* $\zeta^{\dagger} = 0$.

Proof. From Equation (138)

$$\pi_1(\zeta \wedge \phi) = 0. \tag{157}$$

Let α_0 and β_0 be a pair of simple roots such that $\gamma = \alpha_0 + \beta_0$ is a (positive) root. Recall that $\mathsf{G} \neq \mathsf{SL}(3,\mathbb{R})$ and thus $\gamma \neq \eta$. Then by the previous equation, the component of $v = \zeta \wedge \phi$ along $\mathcal{G}_\gamma$ is zero. But

$$v = \sum_{\alpha,\beta \in Z | \alpha+\beta=\gamma} (\zeta_\alpha \wedge \phi_\beta + \zeta_\beta \wedge \phi_\alpha)$$

However since for a simple root $\alpha - \eta$ is not a positive root, and since every positive root can be written uniquely as a sum of simple roots. It follows that

$$0 = v = \zeta_{\alpha_0} \wedge \phi_{\beta_0} + \zeta_{\beta_0} \wedge \phi_{\alpha_0}.$$

Thus for every pair of simple roots α and β so $\alpha + \beta$ is a root,

$$\zeta_\alpha \wedge \phi_\beta = -\zeta_\beta \wedge \phi_\alpha.$$

Since the Dynkin diagram is connected and the ϕ_α are not vanishing, since $\zeta_\alpha = 0$ for some simple root, then $\zeta_\beta = 0$ for all simple roots β.

Finally, all this implies that $\zeta = \zeta_\eta \in \mathcal{G}_\eta$. Using Equation (138) again, we obtain that $\pi_1(\zeta_\eta \wedge \sum_{\alpha\in\Pi} \phi_{-\alpha}) = 0$. Since $\mathsf{G} \neq \mathsf{SL}(3,\mathbb{R})$, there exist a simple root α so that $\gamma := \eta - \alpha$ is a positive root not in $Z \cup Z^{\dagger}$. Thus taking the component along $\mathcal{G}_\gamma$ one gets that $\zeta \wedge \phi_{-\alpha} = 0$. Thus by the injectivity of ϕ_α, we get that $\zeta = 0$. A symmetric argument shows that $\zeta^{\dagger} = 0$. □

8. Properness and the final argument

Assume that G_0 is a split real simple group of rank 2 and M be the degree of its longest root. Let $\mathcal{T}^M$ be total space of the vector bundle over Teichmüller space whose fiber at a complex structure J is $H^0(\Sigma, K^{M+1})$.

Let now Ψ be the map which associates to (J, q) in $\mathcal{T}^M$, the representation associated to the Higgs bundle $(\widehat{\mathcal{G}}, \Phi_q)$ as in Paragraph 5.2. Our main result is now

Theorem 8.0.6. *The map* Ψ *is diffeomorphism.*

This result has Theorem 1.1.1 and Theorem 1.2.1 as immediate consequences.

8.1. **A theorem in differential calculus.** Let $\pi : P \to M$ be a fiber bundle. Assume that the fiber $P_m := \pi_1^{-1}(m)$ at any point m of M is connected. Let F be a smooth positive function from P to $\mathbb{R}$ and let $F_m = F|_m$. Let finally $N \subset P$ be the set of critical points of F_m for all m:

$$N = \{x \in P \mid \mathrm{d}_x F_{\pi(x)} = 0\}.$$

Assume that

(1) For all m, F_m is proper,
(2) N is a connected submanifold of P everywhere transverse to the fibers.

Then we have

Theorem 8.1.1. *Assume the above hypothesis, then we have*

- *π is a diffeomorphism of N into M*
- *F_m has a unique critical point which is an absolute minimum.*

8.1.1. *Some preliminary lemmas.* Let f be a smooth function defined on a manifold Q diffeomorphic to a closed ball. Let m be a critical point of f on $\mathbb{Q}$. Assume that f has no critical point in $U := Q \setminus \{m\}$.

The first Lemma is obvious,

Lemma 8.1.2. *Assume that m is a local minimum of f. There exists an neighborhood V of f in the C^1 topology, such that if $g \in V$, and g has a unique critical point in Q, then this critical point is a local minimum.*

The second Lemma is the following

Lemma 8.1.3. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions converging to f in the C^1 topology. Assume that for every n, f_n has a unique critical point m_n in Q. Assume furthermore that m_n is a local minimum for f_n and assume that $\{m_n\}_{n\in\mathbb{N}}$ converges to m. Then m is a local minimum of f.*

8.1.2. *Proof of Lemma 8.1.3.* For simplicity we may as well assume that for all n, $f_n(m_n) = 0$. Choose an auxiliary Riemannian metric. Let $(\phi_t^n)_{t\in\mathbb{R}}$ be the gradient flow of f_n.

Let Z_n be the set of those points z in ∂Q such that an f_n-gradient line issued from m_n hits z (maybe not the first time). Let B_n be a small closed neighbourhood of m_n, such that the inverse gradient line of any point in B_n converges to m_n.

Proposition 8.1.4. *The set Z_n is not empty*

Proof. let $x \neq m_n$ be a point in B_n. Let $t \to \gamma_n(t)$ be the gradient flow of x_n. After a finite time t_0, $\gamma_n(t)$ leaves B_n and never come back afterwards. Since f_n has no critical point outside B_n, the norm of $\dot{\gamma}_n(t)$

is uniformly bounded from below for $t > t_0$. Since f_n is bounded, it follows that $\gamma_n(t)$ hits the boundary (in Z) after a finite time. Thus Z_n is not empty. □

Proposition 8.1.5. *The set Z_n is closed.*

Proof. Since the gradient of f_n is bounded from below on $Q \setminus B_n$ and f is bounded, there exist some T so that for all $z \in Z_n$, $\phi^n_{-T}(z) \in B_n$. Thus if a sequence $(z_p)_{p\in\mathbb{N}}$ of points of Z converges to z, it follows by continuity that $\Phi^n_{-T}(z) \in B_n$. In particular $z \in Z_n$. Thus Z_n is closed. □

Proposition 8.1.6. *There exists ε_0 so that*

$$\forall n, \forall z \in Z_n, f_n(z) > \varepsilon_0 > 0. \tag{158}$$

Proof. Let us work by contradiction. and assume that we can find a sequence $\{p_n\}_{n\in\mathbb{N}}$ with $p_n \in Z_n$ so that $f_n(p_n) \to 0$. Let γ_n be the gradient of f_n issuing from p_n. We choose a subsequence so $\{p_n\}_{n\in\mathbb{N}}$ converges p respectively. Let γ be a Hausdorff limit of a subsequence of the sequence of closed connected sets $\{\overline{\gamma_n[-\infty, 0]}\}_{n\in\mathbb{N}}$. Then the hypothesis of the paragraph implies that f is constant on the connected set γ. Let ϕ_t be the gradient flow of f, then for all positive t, we also have

$$\phi_{-t}(\gamma) \subset \gamma. \tag{159}$$

It follows that that all points in the connected set γ are critical, which is a contradiction with our hypothesis on f. □

Let γ_n be an orbit of the gradient of f_n that connects m_n to a point z_n in ∂Q. Let $\alpha < \varepsilon_0$ and $L^n_\alpha = f_n^{-1}(\alpha)$. Observe that since $f_n(z_n) > \alpha$, L^n_α intersects γ_n in an unique point q_n. Let S^n_α be the connected component of L^n_α containing q_n.

Proposition 8.1.7. *For all α less than ε_0, S^n_α is a closed submanifold of $Q \setminus \partial Q$ bounding an open set B^n_α containing m_n.*

Proof. Since m_n is a local minimum for f_n there exists β (depending on n) with $0 < \beta < \varepsilon_0$ such that all gradient lines passing though S^n_β ends up at m_n. Then $S^n_\beta \cap \partial Q \subset Z_n$ but since $\beta < \varepsilon_n = \inf\{f_n(z) \mid z \in Z_n\}$ it follows that $S^n_\beta \cap \partial Q = \emptyset$. Thus S^n_β is a closed submanifold of $Q \setminus \partial Q$ and bounds an open set B^n_β containing m_n.

We may choose an auxiliary Riemannian metric (depending on n) so that the norm of the gradient of f_n is 1 outside B^n_β. It follows that if

$$t < \varepsilon_0 - \beta,$$

then $\phi_t(S^n_\eta)$ is closed submanifold of $Q \setminus \partial Q$, intersecting γ_n and on which f_n is equal to $t + \eta$. Thus

$$\phi_t(S^n_\eta) = S^n_{t+\eta}. \tag{160}$$

It follows that for all α less than ε_0, S^n_α bounds an open set B^n_α containing m_n. □

Proposition 8.1.8. *There exists an open set O, with $\overline{O} \subset Q \setminus \partial Q$, such that f is constant on ∂O.*

Proof. Since the gradient of f_n is uniformly bounded from above there exists some positive β such that for n large enough

$$d\left(S^n_{\varepsilon_0/2}, S^n_{\varepsilon_0/4}\right) > \beta. \tag{161}$$

In particular

$$d\left(\partial Q, S^n_{\varepsilon_0/4}\right) > \beta. \tag{162}$$

It follows that $(S^n_{\varepsilon_0/4})$ converge to a connected component $S_{\varepsilon_0/4}$ of a level set of f, which is a closed sub manifold in $Q \setminus \partial Q$, which thus bounds an open set O. □

This last proposition implies Lemma 8.1.3: f has a minimun on O which has to be m since f has a unique critical point in Q. Thus m is a local minimum for f.

8.1.3. *Proof of Theorem 8.1.1.* By assumption N is a closed connected submanifold transverse to the fibers.

For any point m in N the transversality hypothesis implies that we can find neighbourhoods U of $\pi(m)$, W of m so that we can identify W with $U \times Q$, where Q is diffeomorphic to an open ball and $\pi : U \times Q \to U$ is the projection on the first factor.

Let X be the subset of those x in N such that x is a minimal point of $F_{\pi(x)}$. Then X is non empty since F_m is proper.

Then using the neighborhood U and $W = U \times Q$ as above, we obtain that X is open by Lemma 8.1.2 and closed by Lemma 8.1.3. By connectedness, $X = N$.

Thus every critical point of f_m is a local minimum. Since f_m is proper positive and P_m is connected, f_m has a unique critical point which is an absolute minimum. In particular, the local diffeomeorphism π from N to M is injective and surjective, thus a global diffeomorphism.

8.2. **Proof of the main Theorem.** Let $P = \mathcal{T} \times \mathcal{H}(\Sigma, \mathsf{G}_0)$ where $\mathcal{T}$ is Teichmüller space and $\mathcal{H}(\Sigma, \mathsf{G}_0)$ is the Hitchin component of G_0. Let π be the projection on the second factor. Let F be the function which associates to every (J, δ) in P the energy of the unique δ-equivariant harmonic map in the symmetric space $\mathsf{S}(\mathsf{G}_0)$ of G_0. By [31], F is smooth positive and F_δ is a proper map. By [34], [35] and [37], the critical points of F_δ are minimal mappings.

Finally by Hitchin fundamental result in [22], P is diffeomorphic to the bundle over $\mathcal{T}$ whose fiber at every point is $\mathcal{E}_2 \oplus \mathcal{E}_{M+1}$. From Theorem 7.0.2 the map $\widehat{\Psi} : (J, \mathrm{q}) \mapsto (J, \Psi(J, \mathrm{q}))$ is transverse to the fiber. Moreover Ψ is an embedding and its image N is precisely the pairs (J, δ) such that δ-equivariant harmonic mapping from Σ equipped with J has a vanishing Hopf differential, that is minimal. Thus N satisfy the conditions of the Theorem 8.1.1.

The main Theorem then follows.

9. The Kähler structures

We give a more precise result using the notation of the introduction, For any integer greater than 1, let $\mathcal{E}_n$ be the holomorphic line bundle over Teichmüller space whose fiber at a Riemann surface Σ is

$$(\mathcal{E}_n)_\Sigma := H^0(\Sigma, \mathcal{K}^n).$$

Let $\mathrm{m} = (m_1, \ldots, m_n)$, with all $m_i > 1$. We denote by $\mathcal{E}(\mathrm{m})$ be the holomorphic line bundle over Teichmüller space whose fiber at a Riemann surface Σ is

$$\mathcal{E}(\mathrm{m})_\Sigma := \bigoplus_{n=1,\ldots,p} \mathcal{E}_{m_i}$$

Proposition 9.0.1. *The bundle* $\mathcal{E}(\mathrm{m})$ *carries a family of dimension* $\mathrm{rank}(\mathsf{G}_0)$ *of Kähler structures with the following properties*

1. *The complex structure is compatible with that of the holomorphic bundle* $\overline{\mathcal{E}(\mathrm{m})}$,
2. *The Kähler structure is the* L^2*-metric in every fiber,*
3. *The Kähler structure is invariant by the mapping class group action,*
4. *The zero section is totally geodesic,*
5. *The metric induced on the zero section is Weil–Petersson metric.*

The L^2-metric on $H^0(\Sigma, \mathcal{K}^n)$ is taken with respect of the hyperbolic metric on Σ.

We explain the remark of Kim and Zhang in [24] given in the cubic case which extends with only slight modifications to the general case. We reproduce the proof here, with some small simplifications, in order

to get more specific details on the property of the Kähler metrics that we construct.

9.1. **Positive hermitian bundles.** Let $E \to X$ be a holomorphic bundle over a complex manifold X equipped with a hermitian metric h. Let ∇ be the Chern connexion and R^∇ be the Chern curvature that we see as an element of $\Omega^2(\mathsf{T}_{\mathbb{C}}X, E)$. We then define a (real) 2-tensor θ on $\mathsf{T}_{\mathbb{C}}X \otimes E$ by

$$\theta(Y \otimes u, Z \otimes v) := i \cdot h\Big(R^\nabla(Y, Z) \cdot u, v\Big). \tag{163}$$

Using the symmetry of the curvature tensor, one gets that Θ is Hermitian quadratic. We now say that (See [18], Definition 3.9 in [11])

- the hermitian bundle E is *Nakano positive* if θ is hermitian positive,
- the hermitian bundle E is *Griffiths positive* if for all non zero decomposable vectors $X \otimes u$, $\theta(X \otimes u, X \otimes u) > 0$,
- the hermitian bundle E is *Griffiths negative* if for all non zero decomposable vectors $X \otimes u$, $\theta(X \otimes u, X \otimes u) < 0$,

From the definitions one immediately gets the following facts.

(1) a Nakano positive bundle is Griffiths positive,
(2) the sum of Nakano positive is Nakano positive,
(3) the dual of a Griffiths positive bundle is Griffiths negative,

As an easy consequence, the following seems well known to complex geometers.

Proposition 9.1.1. [Kähler Metric on the total space]
Let $\pi : \mathcal{E} \to X$ be a holomorphic bundle over a complex manifold equipped with a Griffiths negative hermitian metric h that we consider as a fibrewise quadratic function. Assume that $\mathcal{E}$ is equipped with an holomorphic action of some group Γ preserving the hermitian metric h and a Kähler metric g on X. Then for any $\varepsilon > 0$,

$$H := \varepsilon \cdot \partial\overline{\partial} h + \pi^* g,$$

is a Γ-invariant Kähler metric on $\mathcal{E}$. Furthermore

(1) H is linear along the fiber,
(2) the zero section is a totally geodesic isometric immersion.

Compare with the content of the proof of Theorem 5.4 in [24].

Proof. Let σ be a holomorphic section of $\mathcal{E}$. A classical computation (Proposition 3.1.5 of [25]) says that $\partial\overline{\partial}(h(\sigma))$ is positive. Thus h is a plurisubharmonic function on $\mathcal{E}$. Since $\varepsilon \cdot \partial\overline{\partial} h$ is positive along the

fibers, we get that H is a Kähler metric. Since $u \mapsto -u$ is an isometry of H whose fixed point is the zero section, the zero section is totally geodesic. The other statements are obvious by construction. □

9.2. **Pushforward bundles.** Theorem 1.2. in Bo Berndtsson [4] is a powerful way to assert the positivity of bundles.

Theorem 9.2.1. [Berndtsson] *Let $\pi : X \to Y$ be a holomorphic fibration with non singular and compact fibres. Assume X is Kähler. We denote by X_y the fiber over $y \in Y$. Let $\mathcal{L}$ be a positive line bundle on Y. Then the vector bundle over Y whose fiber at y is*

$$H^0(X_y, \mathcal{L} \otimes \mathcal{K}_{X/Y}),$$

equipped with the L^2-metric, is Nakano positive.

As a consequence, we obtain as in [24]

Proposition 9.2.2. [Inkang Kim–Genkhai Zhang] *The holomorphic bundle $\mathcal{E}_n$ equipped with the L^2 metric is Nakano positive.*

Proof. We apply Theorem 9.2.1 to the following situation: Y is Teichmüller space, X is the Teichmuller curve, $\mathcal{L} = \mathcal{K}_{X/Y}^{n-1}$ is the canonical bundle of the fibre to the power $n-1$. Then at a Riemann surface Σ,

$$H^0(X_y, \mathcal{L} \otimes \mathcal{K}_{X/Y}) = H^0(\Sigma, \mathcal{K}_\Sigma^n) = (\mathcal{E}_n)_\Sigma.$$

It remains to check the hypothesis. Indeed

- $\mathcal{L}$ is positive by Lemma 5.8. of [40],
- X is Kähler, as a consequence.

The result follows. □

Since the sum of Nakano positive is Nakano positive bundles is positive, we immediately get that the bundle $\mathcal{E}(\mathrm{m})$ is Nakano positive, where $\mathrm{m} = (m_1, \dots, m_n)$. Thus using our preliminary remarks, we have

Proposition 9.2.3. *The bundle $\overline{\mathcal{E}(\mathrm{m})}$ is Griffiths negative.*

9.3. **The Kähler property.** In order to get 9.0.1, we apply Proposition 9.1.1 to $\mathcal{F} = \overline{\mathcal{E}(\mathrm{m})}$ using Proposition 9.2.3.

Actually we have a family of Kähler metric since we have a natural holomorphic $\mathbb{C}^\ell$ action on $\overline{\mathcal{E}(\mathrm{m})}$, where $\ell = \mathrm{rank}(\mathsf{G}_0)$.

By construction the metric is invariant under the mapping class group.

Since furthermore the metric is invariant by rotation in the fibres, the zero section is totally geodesic. Furthermore, by construction the zero section is an isometry from X into $\mathcal{F}$.

10. Area rigidity

For any split real rank 2 group G_0, let $c(\mathsf{G}_0)$ be the curvature of the totally geodesic hyperbolic planes associated to the principal SL_2 in G_0. Our goal is to prove in this section the following

Theorem 10.0.1. *Let δ be a Hitchin representation of $\pi_1(S)$ in G_0 where G_0 has rank 2. Then*

$$\mathrm{MinArea}(\delta) \geqslant c(\mathsf{G}_0) \cdot \chi(S),$$

with equality only if δ is Fuchsian.

10.1. **Forms.** Using the notation given in the decomposition (47), let us consider for any $a \in \mathfrak{h}^*$ the 2-form

$$\Omega_a := \sum_{\alpha \in \Delta} \langle a \mid \omega_\alpha \wedge \omega_{-\alpha} \rangle \in \Omega^2(\mathsf{X}). \tag{164}$$

Then we have the following result

Proposition 10.1.1. *For any $a \in \mathfrak{h}^*$, the form Ω_a is closed. Moreover, for any δ in the Hitchin component, let Σ_δ be the unique minimal surface in $\mathsf{X}/\delta(\pi_1(\Sigma))$ then $\int_{\Sigma_\delta} \Omega_a$ does not depend on δ.*

Proof. We first observe that

$$\Omega_a = \langle a \mid \omega \wedge \omega \rangle .$$

By equation (45),

$$\omega \wedge \omega = -R^\nabla - \mathrm{d}^\nabla \omega.$$

Thus

$$\begin{aligned} -\Omega_a &= \langle a \mid \mathrm{d}^\nabla \omega \rangle + \langle a \mid R^\nabla \rangle \\ &= \mathrm{d} \langle a \mid \omega \rangle + \langle a \mid R^\nabla \rangle . \end{aligned} \tag{165}$$

Thus Ω_a is in the same cohomology class as $-\langle a \mid R^\nabla \rangle$. Since R^∇ is the curvature of the T-bundle $\mathsf{G} \to \mathsf{G}/\mathsf{T}$, it follows that exists a constant $c(G_0)$ and an S^1 bundle P over X so that $\frac{1}{c(\mathsf{G}_0)}\Omega$ is the curvature of P. In particular

$$f(\delta) := \frac{1}{c(\mathsf{G}_0)} \int_{\Sigma_\delta} \Omega \in \mathbb{Z}.$$

Since $f(\delta)$ depends continuously on δ, $f(\delta)$ is constant. Then the evaluation of $f(\delta)$ for Fuchsian representations give the result. □

Let us now consider the following 2-forms on X (using the convention of paragraph 4.1)

$$\begin{aligned}\Omega_0 &:= i \cdot \sum_{\alpha\in\Pi} \langle \omega_{-\alpha} | \omega_\alpha \rangle \\ \Omega_1 &:= i \cdot \langle \omega_\eta | \omega_{-\eta} \rangle. \end{aligned} \tag{166}$$

Then we have

Proposition 10.1.2. *The form Ω_0 and Ω_1 are positive on any cyclic surface. Moreover if Σ be a cyclic surface in X_0 and p is the projection of X_0 to $\mathsf{S}(G_0)$, then*

$$\int_\Sigma \Omega_0 + \Omega_1 = \mathrm{Area}(p(\Sigma)).$$

Proof. Recall that for a cyclic surface ω_η and $\omega_{-\alpha}$ are of type $(1,0)$ for $\alpha \in \Pi$ and $\omega_\beta = -\rho(\omega_\beta)$. Thus the positivity of Ω_0 and Ω_1 on cyclic surfaces follows from Proposition 4.1.2.

From Equation (76), it follows

$$\mathrm{Area}(p(\Sigma)) = i. \int_\Sigma \langle \Phi | \Phi^\dagger \rangle.$$

Recall that

$$\begin{aligned}\Phi &= \omega_\eta + \sum_{\alpha\in\Pi} \omega_{-\alpha}, \\ \Phi^\dagger &= \omega_{-\eta} + \sum_{\alpha\in\Pi} \omega_\alpha. \end{aligned} \tag{167}$$

Thus since $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$ are othogonal with respect to the Killing form unless $\alpha + \beta = 0$. It follows that

$$\begin{aligned} i \cdot \langle \Phi | \Phi^\dagger \rangle &= i \cdot \langle \omega_\eta | \omega_{-\eta} \rangle + i \cdot \sum_{\alpha\in\Pi} \langle \omega_{-\alpha} | \omega_\alpha \rangle \\ &= \Omega_0 + \Omega_1. \end{aligned}$$

□

We finally will need

Proposition 10.1.3. *Let Π be the set of simple roots in G_0. There there exists a unique element u_0 in $\mathfrak{h}$, such that for any simple root α and $X \in \mathfrak{g}_{-\alpha}$, $Y \in \mathfrak{g}_\alpha$, we have*

$$\langle X | Y \rangle = \langle u_0 \mid [X, Y] \rangle .$$

Moreover, there exists a positive constant k_0, so that $X \in \mathfrak{g}_{-\eta}$, $Y \in \mathfrak{g}_\eta$, we have

$$k_0 \cdot \langle X | Y \rangle = \langle u \mid [X, Y] \rangle .$$

Proof. Let us choose a Chevalley basis $\{\mathrm{x}_\alpha\}_{\alpha\in\Delta}$. Let us write $X = x\cdot \mathrm{x}_{-\alpha}$, $Y = y\cdot \mathrm{x}_\alpha$, then

$$\langle X|Y\rangle = x\cdot y\cdot \langle \mathrm{x}_\alpha|\mathrm{x}_{-\alpha}\rangle.$$

On the other hand

$$\langle u \mid [X,Y]\rangle = -x\cdot y\cdot \langle u|h_\alpha\rangle.$$

Thus u_0 is uniquely determined by

$$\langle u_0|h_\alpha\rangle = -\langle \mathrm{x}_\alpha|\mathrm{x}_{-\alpha}\rangle.$$

We may choose a Cartan involution so that $x_{-\alpha} = \rho(x_\alpha)$. Thus $\langle \mathrm{x}_\alpha|\mathrm{x}_{-\alpha}\rangle < 0$. Since

$$h_\eta = \sum_{\alpha\in\Pi} r_\alpha\cdot h_\alpha,$$

with $r_\alpha > 0$. It follows that

$$\langle u_0|h_\eta\rangle > 0.$$

In particular, if $X\in\mathfrak{g}_\eta$ and $Y\in\mathfrak{g}_{-\eta}$ we have

$$k_0\cdot\langle X|Y\rangle = \langle u \mid [X,Y]\rangle\,,$$

where

$$k_0 = -\frac{\langle u|h_\eta\rangle}{\langle \mathrm{x}_\eta|\mathrm{x}_{-\eta}\rangle} > 0.$$

□

As a corollary of this proposition one obtains immediately

Corollary 10.1.4. *Let u_0 be defined as in Proposition 10.1.3 and $v_0 = i\cdot u_0$, then*

$$\Omega_{v_0} = \Omega_0 - k_0\cdot\Omega_1.$$

10.2. **Proof of the Area Rigidity Theorem.** We can now prove Theorem 10.0.1. From Corollary 10.1.4 and Proposition 10.1.2, on gets that

$$\mathrm{Area}(\pi(\Sigma_\delta)) = \int_{\Sigma_\delta}\Omega_{v_0} + (K_0+1)\int_{\Sigma_\delta}\Omega_1 \tag{168}$$

If δ_0 is a Fuchsian representation, then the corresponding cyclic surface is Fuchsian (see Proposition 6.0.5) and thus ω_η and Ω_1 vanish. Thus we get that if δ_0 is Fuchsian

$$\int_{\Sigma_{\delta_0}}\Omega_{v_0}. = \mathrm{Area}(\Sigma_{\delta_0}) = c(\mathsf{G}_0)\chi(S). \tag{169}$$

By Proposition 10.1.1 $\int_{\Sigma_\delta} \Omega_{v_0}$ does not depend on δ. It thus follows that

$$\begin{aligned}\mathrm{MinArea}(\delta) &= \mathrm{Area}(\pi(\Sigma_\delta)) \\ &= c(\mathsf{G}_0) \cdot \chi(S) + (k_0 + 1) \cdot \int_{\Sigma_\delta} \Omega_1. \end{aligned} \tag{170}$$

The result now follows from the fact that Ω_1 is positive on cyclic surfaces by Proposition 10.1.2. Morover Ω_1 vanishes if and only if ω_η vanishes, but ω_η vanishes if and only if the Higgs field takes in $\sum_{\alpha\in\Pi} \mathcal{G}_\alpha$ that is δ is Fuchsian (see Theorem 5.2.2).

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