Lifshitz tails for some random Schrödinger operators or an aspect of random walk in random traps

F. Klopp

Université Paris 13
and
Institut Universitaire de France

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The setting and the questions

On \(\mathbb{R}^d\), consider a stationary ergodic random field \(x \mapsto V_\omega(x)\).

**Spectral theory** On \(L^2(\mathbb{R}^d)\), consider the random Schrödinger operator

\[
H_\omega = -\frac{1}{2} \Delta + V_\omega
\]

and the associated evolution equation

\[
\begin{cases}
  i\partial_t \psi_t = H_\omega \psi_t, \\
  \psi_{t|t=0} = \psi_0
\end{cases}
\]

where

- \(\Delta\) is the Laplace operator on \(\mathbb{R}^d\).

**Questions:**
- the spectral data of \(H_\omega\),
- the large time behavior of the semi-group.

**Probability theory** Consider the brownian motion in this random field i.e. the path measures

\[
Q_t = \frac{1}{S_{t,\omega}} \exp \left(- \int_0^t V_\omega(Z_s) ds \right) P_0
\]

\[
Q_{t,\omega} = \frac{1}{S_t} \exp \left(- \int_0^t V_\omega(Z_s) ds \right) P_0 \otimes \mathbb{P}
\]

where

- \(\mathbb{P}\) is the law of the random field \(V_\omega\),
- \(Z_s\) is the standard Brownian motion,
- \(P_0\) is the Wiener measure,
- \(S_t\) and \(S_{t,\omega}\) are normalizing constants.

**Questions:**
- the large \(t\) behavior of the path measures.

Various random potentials

1. **The Poisson model:** \(V_\omega(x) = \sum_{n \in \mathbb{N}} V(x - \xi_n)\) where
   - \(V : \mathbb{R}^d \to \mathbb{R}\) is continuous, non identically vanishing, real valued and compactly supported;
   - \((\xi_n)_n\) is a Poisson point process.

2. **The alloy type model:** \(V_\omega(x) = \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma V(x - \gamma)\) where
   - \(V : \mathbb{R}^d \to \mathbb{R}\) is continuous, non identically vanishing, real valued and compactly supported;
   - \((\omega_\gamma)_\gamma\) are real valued i.i.d random variables.

3. **The displacement model:** \(V_\omega(x) = \sum_{\gamma \in \mathbb{Z}^d} V(x - \gamma - \xi_\gamma)\)
   where
   - \(V : \mathbb{R}^d \to \mathbb{R}\) is continuous, non identically vanishing, real valued and compactly supported;
   - \((\xi_\gamma)_\gamma\) are \(\mathbb{R}^d\)-valued i.i.d random variables.
The integrated density of states and the annealed random walk

Define the integrated density of states (IDS) of $H_\omega$ as

$$N(E) = \lim_{L \to +\infty} \frac{1}{(2L)^d} \# \{ \text{eigenvalues of } H_\omega \in [-L,L]^d \text{ less that } E \}. $$

Almost surely, the limit exists, is independent of $\omega$ and non decreasing. The Pastur-Shubin formula:

$$N(E) = \begin{cases} \mathbb{E} \left[ 1_{(-\infty,E]}(H_\omega)(0,0) \right] & \text{when } V_\omega \text{ is } \mathbb{R}^d\text{-ergodic,} \\ \mathbb{E} \left[ \text{tr} \left( 1_{[0,1]^d} 1_{(-\infty,E]}(H_\omega) \right) \right] & \text{when } V_\omega \text{ is } \mathbb{Z}^d\text{-ergodic.} \end{cases}$$

Related to the heat kernel of $H_\omega$ by Laplace transform:

$$L(t) = \int_{\mathbb{R}} e^{-tE} dN(E) = \begin{cases} \mathbb{E} \left[ e^{-tH_\omega}(0,0) \right] & \text{when } V_\omega \text{ is } \mathbb{R}^d\text{-ergodic,} \\ \mathbb{E} \left[ \text{tr} \left( 1_{[0,1]^d} e^{-tH_\omega} \right) \right] & \text{when } V_\omega \text{ is } \mathbb{Z}^d\text{-ergodic} \end{cases}$$

$$= (2\pi t)^{-d/2} \left\{ E_{0,0}^t \left( \mathbb{E} \left[ \exp \left( -\int_0^t V_\omega(Z_s) ds \right) \right] \right) \text{ when } V_\omega \text{ is } \mathbb{R}^d\text{-ergodic,} \\ \int_{[0,1]^d} E_{x,x}^t \left( \mathbb{E} \left[ \exp \left( -\int_0^t V_\omega(Z_s) ds \right) \right] \right) dx \text{ when } V_\omega \text{ is } \mathbb{Z}^d\text{-ergodic} \right\}$$

Random operators

Under our assumptions, $H_\omega = -\Delta + V_\omega$ is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^d)$. It is a metrically transitive family of operators i.e. there exists

- $(U_\alpha)_\alpha$ a family of unitary transform of $L^2(\mathbb{R}^d)$
- $(\tau_\alpha)_\alpha$, an ergodic family of transformation such that

$$H_{\tau_\alpha \omega} = U_\alpha H_\omega U_\alpha^*. $$

The family $(H_\omega)_\omega$ admits an almost sure spectrum, say $\Sigma$ such that $\Sigma = \text{supp } dN$. Typically $\Sigma$ is a union of bands

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One wants to study the behavior of $N(E)$ near spectral edges, in particular near $E_- = \inf(\Sigma)$. It is known that the behavior of this function is instrumental in the study of the nature spectrum of $H_\omega$ (Lifshitz ‘63).
The monotonous alloy type model

On $\mathbb{R}^d$, consider the alloy type (or Anderson) model

$$H_\omega = -\Delta + V_\omega$$

where

$$V_\omega(x) = \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma V(x - \gamma)$$

where

- $V : \mathbb{R}^d \to \mathbb{R}$ is continuous, non identically vanishing, real valued and compactly supported; assume, moreover, $V \geq 0$;
- $(\omega_\gamma)_\gamma$ are i.i.d random variables distributed in $[0, a]$, $a > 0$.

To fix ideas let us assume that $\log |\log P(\{\omega_0 \leq \varepsilon\})| = o(\log \varepsilon)$ when $\varepsilon \to 0^+$. Then, $\Sigma = [0, +\infty)$, i.e. $E_+ = 0$.

**Lifshitz tails:**

**Theorem (Lifshitz, Pastur, Kirsch, Simon, ...)**

One has

$$\lim_{E \to 0^+} \frac{\ln |\ln (N(E))|}{\ln (E)} = -\frac{d}{2}.$$  

Recall for $H_0 = -\Delta$: $N(E) = C_d \max(E, 0)^{d/2}$

An idea of the proof:

By Dirichlet-Neumann bracketing,

$$\mathbb{E} \left( \frac{1}{(2L)^d} \# \{n; \lambda_{n}(H_\omega^D_{[-L,L]^d}) \leq E \} \right) \leq N(E) \leq \mathbb{E} \left( \frac{1}{(2L)^d} \# \{n; \lambda_{n}(H_\omega^N_{[-L,L]^d}) \leq E \} \right).$$

One reduces the problem to estimating

$$\mathbb{P} \left( \{H_\omega^N_{[-L,L]^d} \text{ has an eigenvalue less than } \varepsilon \} \right)$$

for $L \sim \varepsilon^{-\alpha}$.

i.e. the probability that there exists $\psi \in H^1([-L, L]^d)$ such that

$$\langle -\Delta \psi, \psi \rangle + \langle V_\omega \psi, \psi \rangle \leq \varepsilon \|\psi\|^2.$$  

As $V_\omega \geq 0$ and $-\Delta \geq 0$, this implies

$$\langle -\Delta \psi, \psi \rangle \leq \varepsilon \|\psi\|^2 \quad \text{and} \quad \langle V_\omega \psi, \psi \rangle \leq \varepsilon \|\psi\|^2.$$  

So roughly, one has to estimate

$$\varepsilon^{d/2} \sum_{|\gamma| \leq \varepsilon^{-1/2}} \omega_\gamma \leq C \varepsilon,$$

and one concludes by large deviations.
The Poisson potential

On $\mathbb{R}^d$, consider the alloy type (or Anderson) model

$$H_\omega = -\Delta + V_\omega$$

where

$$V_\omega(x) = \sum_{n \in \mathbb{N}} V(x - x_n)$$

where

- $V : \mathbb{R}^d \to \mathbb{R}$ is continuous, non negative, non identically vanishing, real valued and compactly supported;
- $(x_n)_{n \in \mathbb{N}}$ are the support of a Poissonian cloud of positive density.

Then, $\Sigma = [0, +\infty)$ and $E_- = 0$.

Theorem (Pastur,Sznitman,...)

One has

$$\lim_{E \to 0^+} \frac{\ln(n(E))E^{d/2}}{E} = -C < 0.$$ 

The result is obtained by probabilistic methods.

Much more precise than the previous result obtained using spectral methods.

But the spectral methods are more flexible.

Internal Lifshitz tails:

Let $V_\omega$ be of alloy type.

Lifshitz tails also hold at $\inf(\Sigma)$ when $H_0 = -\Delta$ becomes $H_0 = -\Delta + V_0$ where $V_0$ is $\mathbb{Z}^d$-periodic.

Let $n(E)$ be the IDS of $H_0$. Assume that $\Sigma_p = \sigma(H_0)$, the spectrum of $H_0$ has a gap below energy 0.

Assume that, for $t \in [0, 1]$, $\sigma(H_0 + tV_\omega)$ has a gap below 0.

Theorem (K.,K.-Wolff)

Then

$$\lim_{E \to 0^+} \frac{\log |\log(n(E) - n(0))|}{\log E} = -\frac{d}{2} \iff \lim_{E \to 0^+} \frac{\log(n(E) - n(0))}{\log E} = \frac{d}{2},$$

When $d = 2$, then

$$\limsup_{E \to 0^+} \frac{\log |\log(n(E) - n(0))|}{\log E} < 0.$$
The non monotonous alloy type model:

On $\mathbb{R}^d$, consider the standard continuous alloy type (or Anderson) model

$$H_\omega = -\Delta + V_\omega$$

where

$$V_\omega(x) = \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma V(x - \gamma)$$

where

- $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is continuous, non identically vanishing, real valued and compactly supported;
- $(\omega_\gamma)_\gamma$ are i.i.d random variables distributed in $[a,b]$, $a$ and $b$ in the support.

One wants to study the spectrum or spectral quantities for $H_\omega$ near $E_- = \inf(\Sigma)$.

When $V$ has a fixed sign, it is clear that

- $E_- = \inf(\sigma(\Delta + V_\pi))$ if $V \leq 0$;
- $E_- = \inf(\sigma(\Delta + V_\pi))$ if $V \geq 0$.

We want to address the case when $V$ changes sign i.e. we assume

(H1) there exists $x_+ \neq x_-$ such that $V(x_-) \cdot V(x_+) < 0$.

We require one more assumption:

(H2) $V$ is supported in $(-1/2, 1/2)^d$ and reflection symmetric i.e. for any $\sigma = (\sigma_1, \ldots, \sigma_d) \in \{0,1\}^d$ and any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, one has

$$V(x_1, \ldots, x_d) = V((-1)^{\sigma_1}x_1, \ldots, (-1)^{\sigma_d}x_d).$$

Determining the bottom of the spectrum:

Consider the operator $H^N_{\lambda} = -\Delta + \lambda V$ with Neummann b. c. on $[-1/2, 1/2]^d$.

Its spectrum is discrete and let $E_-(\lambda)$ be its ground state energy.

It is a simple eigenvalue and $\lambda \mapsto E_-(\lambda)$ is a real analytic concave function.

**Proposition (K.-Nakamura)**

One has $E_- = \inf(\inf \sigma(H_{\pi}), \inf \sigma(H_{\bar{\pi}})) = \inf(E_-(a), E_-(b))$.

If $a$ and $b$ sufficiently small, Najar proved proposition assuming

$$\int_{\mathbb{R}^d} V(x) dx = E'_-(0) \neq 0$$

without (H2).
Lifshitz tails: when $E_-(a) \neq E_-(b)$

Denote by $N(E)$ the integrated density of states of $H_\omega$.

**Theorem (K.-Nakamura)**

Assume $E_-(a) \neq E_-(b)$. Then

$$-\frac{d}{2} - \alpha_- \leq \liminf_{E \to E_-^+} \frac{\log|\log N(E)|}{\log(E - E_-)} \leq \limsup_{E \to E_-^+} \frac{\log|\log N(E)|}{\log(E - E_-)} \leq -\frac{d}{2} - \alpha_+$$

where $c = a$ if $E_-(a) < E_-(b)$ and $c = b$ if $E_-(a) > E_-(b)$ and

$$\alpha_- = -\liminf_{\varepsilon \to 0} \frac{\log|\log \mathbb{P}(\{|c - \omega_0| \leq \varepsilon\})|}{\log \varepsilon} \geq 0,$$

$$\alpha_+ = -\limsup_{\varepsilon \to 0} \frac{\log|\log \mathbb{P}(\{|c - \omega_0| \leq \varepsilon\})|}{\log \varepsilon} \geq 0.$$

This result is similar to the one obtained in the monotonous case.

Lifshitz tails: when $E_-(a) = E_-(b)$

**Theorem (K.-Nakamura)**

Assume (H1) and (H2) and $E_- := E_-(a) = E_-(b)$. Then,

- If the random variables $(\omega_\gamma)_{\gamma}$ are not Bernoulli distributed i.e. if $\mathbb{P}(\omega_0 = a) + \mathbb{P}(\omega_0 = b) < 1$, then

  $$-\frac{d}{2} - \alpha_- \leq \liminf_{E \to E_-^+} \frac{\log|\log N(E)|}{\log(E - E_-)} \leq \limsup_{E \to E_-^+} \frac{\log|\log N(E)|}{\log(E - E_-)} \leq -\frac{d}{2} - \alpha_+. \quad (2.1)$$

- If $\mathbb{P}(\omega_0 = a) + \mathbb{P}(\omega_0 = b) = 1$, there exists potentials $V$ satisfying assumption (H1) and (H2) such that $E_-(a) = E_-(b)$ and, there exists $C > 0$ such that, for $E \geq E_-$,

  $$\frac{1}{C}(E - E_-)^{d/2} \leq N(E) \leq C(E - E_-)^{d/2}.$$
A random displacement model

Consider

\[ H_\omega = -\Delta + V_\omega \text{ where } V_\omega(x) = \sum_{\gamma \in \mathbb{Z}^d} V(x - \gamma - \xi_\gamma). \]

where

- \( V : \mathbb{R}^d \to \mathbb{R} \) is continuous, non identically vanishing and supported in \((-r, r)^d\), \(0 < r < 1/2\) and satisfies (H2);
- \((\xi_\gamma)_{\gamma}\) are independent identically distributed (i.i.d.) random variables distributed in \([-1/2 + r, 1/2 - r]^d\) such that all these points have a positive probability.

By work of Baker, Loss and Stolz, minimizing configurations given by a symmetric "clusterization".

For \( \xi \in \{-1/2 + r, 1/2 - r\}^d \), we define

\[ H_\xi = -\Delta + V(x - \xi) \text{ on } [-1/2, 1/2]^d \text{ with Neumann BC.} \]

All the \((H_\xi)_{\xi}\) have the same ground state energy, say \(E_-\).

\( H_{\xi_1} \) and \( H_{\xi_2} \) match in the direction \(e_j\) if \(E_-\) is also the ground state energy of

\[ -\Delta + V(\cdot - \xi_1) + V(\cdot - e_j - \xi_2) \text{ on } [-1/2, 1/2]^d \cup (e_j + [-1/2, 1/2]^d) \text{ (Neumann BC).} \]

**Theorem (K.-Nakamura)**

Let \( N(E) \) denote the IDS of \( H_\omega \). Then,

- if, at least, two of the \((H_\xi)_{\xi}\) do not match in, at least, one direction, one has
  \[ \limsup_{E \to E_-} \frac{\log |\log N(E)|}{\log(E - E_-)} \leq -\frac{d}{2}; \]

- if all the \((H_\xi)_{\xi}\) match in all directions, then \(N(E) \geq c(E - E_-)^{d/2}.\)
Finding the minimum: decoupling due to symmetry

Recall that $E_-(\lambda)$ is the ground state energy of the operator $H_N^{\lambda}$ i.e. $-\Delta + \lambda V$ on $[-1/2, 1/2]^d$ with Neumann boundary conditions.

To fix ideas, assume $E_-(a) \leq E_-(b)$.

Partitioning $\mathbb{R}^d$ into cubes $\gamma \times [-1/2, 1/2]^d$ for $\gamma \in \mathbb{Z}^d$, we get that

$$H_\omega \geq \bigoplus_{\gamma \in \mathbb{Z}^d} H_{\omega\gamma}^N$$

Hence, $H_\omega \geq E_-(a)$.

Consider $H_{\omega,L}^P$, the operator $H_\omega$ restricted to the cube $[-L - 1/2, L + 1/2]^d$ with periodic boundary conditions.

One proves

**Lemma**

$$\Sigma = \bigcup_{L \geq 1} \bigcup_{\omega \text{ admissible}} \sigma(H_{\omega,L}^P).$$

The characterization of the infimum of the almost sure spectrum follows from

$$\inf_{\omega \in [\alpha, \beta]} \inf_{C_L^d} \sigma(H_{\omega,L}^P) \leq E_-(\alpha) \quad \text{where} \quad C_L^d = \mathbb{Z}^d \cap [-L - 1/2, L + 1/2]^d.$$  

The normalized positive ground state of $H_{\alpha}^N$, say $\psi$, is simple and unique.

The reflection symmetry of the potential $V$ guarantees that $\psi$ is reflection symmetric.

For $\gamma \in \mathbb{Z}^d$ such that $|\gamma|_1 = 1$, we continue $\psi$ to the $\gamma + [-1/2, 1/2]^d$ by reflection symmetry with respect to the common boundary of $[-1/2, 1/2]^d$ and $\gamma + [-1/2, 1/2]^d$.

As $\psi$ is reflection symmetric, we obtain a continuation of $\psi$ that is $\mathbb{Z}^d$-periodic, positive and reflection symmetric with respect to any plane that is common boundary to two cubes of the form $\gamma + [-1/2, 1/2]^d$.

Moreover $\psi$ satisfies, for any $L \geq 0$, $H_{\alpha,L}^P \psi = H_{\alpha,0}^P \psi = H_{\alpha,0}^N \psi = E_-(a) \psi$. This proves that $E_-(a) \geq \inf \sigma(H_{\alpha,L}^P)$.

When the single site potential of fixed sign, $H_{\omega,L}^P$ is increasing/decreasing in any $\omega_\gamma$ $\implies$ one can optimize each random variables separately.

With symmetry assumption, also decoupling the dependence on the random variables.
The upper bound in the Lifshitz tails when \( E_-(a) \neq E_-(b) \)

**Theorem (K.-Nakamura)**

Suppose assumptions (H1) and (H2) are satisfied, and, that \( E_-(a) < E_-(b) \). Then, there exists \( C > 0 \) such that, for \( E \) close to \( E_-(a) \), one has \( N(E) \leq N_m(C(E - E_-(a))) \) where \( N_m \) is the integrated density of states of the random operator

\[
H^m_{\omega} = H_{\bar{a}} - E_-(a) + \sum_{\gamma \in \mathbb{Z}^d} (\omega_\gamma - a) \mathbf{1}_{[-1/2, 1/2]^d}(x - \gamma)
\]

and \( H_{\bar{a}} \) is defined above.

This is a consequence of Neumann BC and the simple

**Lemma**

Let \( H_0 \) be self-adjoint on \( \mathcal{H} \) a separable Hilbert space such that \( 0 = \inf \sigma(H_0) \). Let \( V_1 \) be a non trivial closed symmetric operator relatively bounded with respect to \( H_0 \) with bound \( 0 \). Set \( H_1 = H_0 + V_1 \) and \( E_1 = \inf \sigma(H_1) \). Assume \( E_1 > 0 \). Then, there exists \( C > 0 \) such that, for \( t \in [0, 1] \), one has

\[
C(H_0 + tV_1) \geq H_0 + t
\]

When \( E_-(a) = E_-(b) \): absence of Lifshitz tails.

Let \( \varphi \in \mathcal{C}^\infty((-1/2, 1/2)^d) \) be positive, reflection symmetric, constant near the boundary of \([-1/2, 1/2]^d\) and normalized in the cube.

Let \( V = \Delta \varphi / \varphi \). Then, \( \varphi \) is the positive normalized ground state of \(-\Delta + V\) on \([-1/2, 1/2]^d\) with Neumann boundary conditions.

Let \((\omega_\gamma)_{\gamma \in \mathbb{Z}^d}\) be Bernoulli r.v. with support \(\{0, 1\}\).

Let \( \varphi_L \) be ground state of \( H^{N}_{\omega, L} \): it is equal to

- in \( \gamma + [-1/2, 1/2]^d \), \( \varphi_L(\cdot) = \varphi(\cdot - \gamma) \) if \( \omega_\gamma = 1 \);
- in \( \gamma + [-1/2, 1/2]^d \), \( \varphi_L(\cdot) = \text{cst} \) if \( \omega_\gamma = 0 \).

As the ground state is uniformly bounded (in \( \omega \) and \( L \)), a result of [KiSi89] and a calculation imply that, there exists \( C_D \geq c_N > 0 \), for all \( \omega \),

- the second eigenvalue of the Neumann problem is larger than \( c_N L^{-2} \);
- the ground state of the Dirichlet problem is smaller than \( C_D L^{-2} \).

As

\[
\frac{1}{L^d} \mathbb{E}(\#\{\text{eigenvalues of } H^{D}_{\omega, L} \leq E\}) \leq N(E) \leq \frac{1}{L^d} \mathbb{E}(\#\{\text{eigenvalues of } H^{N}_{\omega, L} \leq E\})
\]

for \( L = cE^{-1/2} \), we get \( C^{-1} E^{d/2} \leq N(E) \leq CE^{d/2} \).
The upper bound in the Lifshitz tails when $E_-(a) = E_-(b)$

Assume that $(\omega_\gamma)_\gamma$ are not Bernoulli distributed i.e. $P(\omega_0 = a) + P(\omega_0 = b) < 1$. Pick $\varepsilon > 0$ such that

$$P(\omega_0 \leq a + \varepsilon) + P(\omega_0 \geq b - \varepsilon) < 1.$$ 

Let $H_{\omega, L}^N$ be the operator $H_\omega$ restricted to the cube $[-L - 1/2, L + 1/2]^d$ with Neumann boundary conditions.

Define

$$N_{L}^N(E) = (2L + 1)^{-d} \mathbb{E}(\#\{\text{eigenvalues of } H_{\omega, L}^N \leq E\}).$$

Well known : the sequence $N_{L}^N(E)$ is decreasing and converges to $N(E)$ (except possibly at countably many $E$).

Define $E_{-, L}(\omega) = \inf \sigma(H_{\omega, L}^N)$. One has $N_{L}^N(E) \leq C \mathbb{P}(\{E_{-, L}(\omega) \leq E\})$

Sufficient to prove a suitable upper bound for $\mathbb{P}(\{E_{-, L}(\omega) \leq E\})$ for a well chosen value of $L$.

Basic property:

**Lemma**

The function $\omega \mapsto E_{-, L}(\omega)$ is real analytic and strictly concave.

The function $\omega \mapsto E_{-, L}(\omega)$ is defined on $\mathbb{R}^{C_L}$.

The upper epigraphs of $\omega \mapsto E_{-, L}(\omega)$ i.e. the sets $\Omega_L(E) := \{\omega \in \Omega_L; E_{-, L}(\omega) > E\}$ are convex.

On $\Omega_L$, $E_{-, L}(\omega)$ reaches its minimum only at one or more vertices of $\Omega_L$.

One studies what happens at the vertices of $\Omega_L$ i.e. at the points of $\{a, b\}^{C_L}$. 
A local estimate on the ground state energy

Assume $E_\omega = E_\omega (a) = E_\omega (b)$.

**Lemma (K.-Nakamura)**

*Partition the discrete cube $C_L^d$ into strips*

$$C_L^d = \bigcup_{\gamma' \in C_{L-1}^d} S_{L,\gamma'}$$

where $S_{L,\gamma'} = \{(\gamma_1, \gamma') \mid -L \leq \gamma_1 \leq L\}$.

There exists $C > 0$ such, for all $L \geq 0$, if $\omega \in \{a, b, a+\varepsilon, b-\varepsilon\}^C_L$ is such that

(Prop) for all $\gamma' \in C_{L-1}^d$, there exists $\gamma \in S_{L,\gamma'}$ such that $\omega_\gamma \in \{a + \varepsilon, b - \varepsilon\}$

then

$$E_{-L}(\omega) \geq E_- + \frac{1}{CL^2}.$$

The proof of this result relies on Neumann decoupling and on the analysis of the ground state energy of a strip where all but one single site potential are the same.

**Corollary**

There exists $C > 0$, independent of $L \geq 0$ and $\omega \in \Omega_L$, such that if

(Prop (ter)) for all $\gamma' \in C_{L-1}^d$, there exists $\gamma \in S_{L,\gamma'}$ s.t. $\omega_\gamma \in [a + \varepsilon, b - \varepsilon]$

then

$$E_{-L}(\omega) \geq E_- + \frac{1}{CL^2}.$$

with the same constant as in the lemma.

Pick $E > E_-(a) = E_-(b)$, $L = c(E - E_-(a))^{-1/2}$ and $c > 0$ s.t. $Cc^2 < 1$. Corollary ensures that, if $\omega$ satisfies (Prop (ter)), then $E_-(\omega) > E$.

So, the set $\Omega_L(E) := \{\omega \in \Omega_L; E_-(\omega) > E\}$ satisfies

$$\Omega_L \setminus \Omega_L(E) \subset \{\omega \in \Omega_L; \exists \gamma' \in C_{L-1}^d, \forall \gamma \in S_{L,\gamma'}, \omega_\gamma \in [a, a+\varepsilon) \cup (b-\varepsilon, b]\}.$$

Hence,

$$\mathbb{P}(\Omega_L \setminus \Omega_L(E)) \leq \sum_{\gamma' \in C_{L-1}^d} \mathbb{P}([\forall \gamma \in S_{L,\gamma'}, \omega_\gamma \in [a, a+\varepsilon) \cup (b-\varepsilon, b])]$$

$$= (2L + 1)^{d-1}[\mathbb{P}(\omega_0 \in [a, a+\varepsilon]) + \mathbb{P}(\omega_0 \in (b-\varepsilon, b])]^{2L+1}$$

This yields exponential decay; to get the correct exponent is more involved.
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