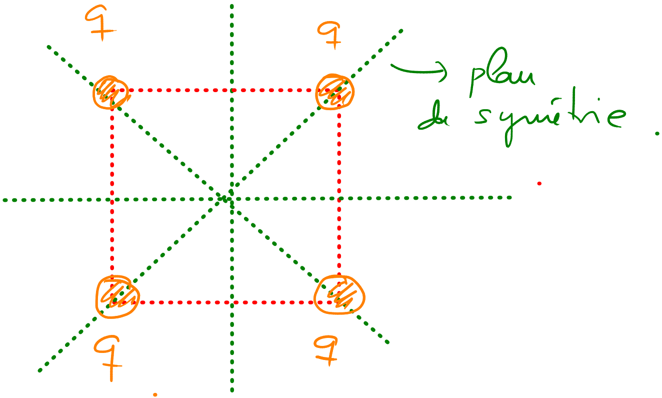
Symétries et invariances

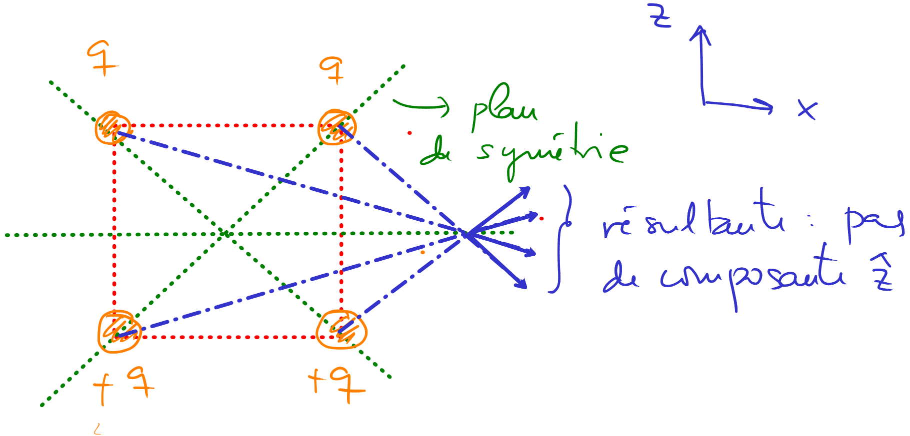
Symétries

Exemple



Proposition : le champ est inclus dans le plan de symétrie

Proposition : le champ est <u>tangent</u> au plan de symétrie



Démonstration mathématique

$$\frac{1}{4\pi\varepsilon_0} = K$$

(distribution continue)

$$\rho(x,y,z) = \rho(x,y,-z)$$

On doit démontrer que $E_z(x,y,0) = 0$

$$E_z(x,y,0) \overset{k}{=} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \, \rho(x',y',z') \frac{-z'}{\left[(x-x')^2+(y-y')^2+z'^2\right]^{3/2}} =$$

$$= k \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \left[\int_0^{\infty} dz' \, \rho(x',y',z') \frac{\overset{0}{\overset{\shortparallel}{z}}-z'}{[\quad]^{3/2}} + \int_{-\infty}^{0} dz' \, \rho(x',y',z') \frac{\overset{0}{\overset{\shortparallel}{z}}-z'}{[\quad]^{3/2}} \right]$$

Deuxième intégrale : $s' = -z'$, $ds' = -dz'$

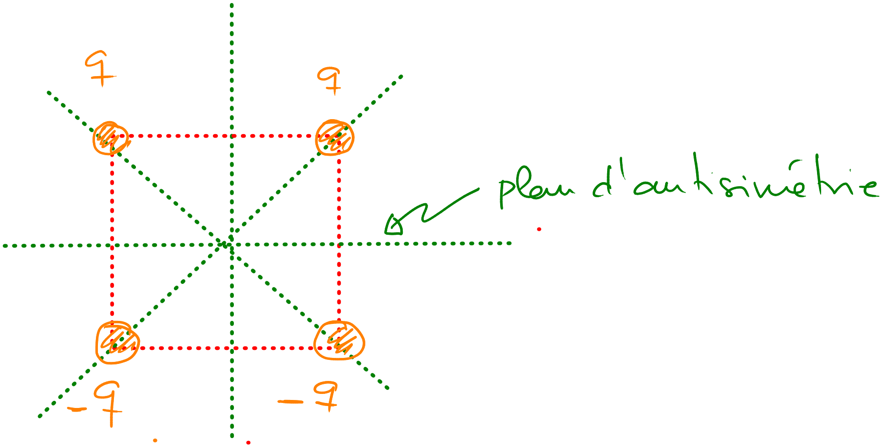
$$\int_{-\infty}^{0} dz' \longrightarrow \int_{\infty}^{0} (-ds') = +\int_{0}^{\infty} ds'$$

$$= k\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \Bigg[\int_{0}^{\infty} dz' \rho(x',y',z') \frac{-z'}{\left[(x-x')^2 + (y-y')^2 + z'^2 \right]}$$

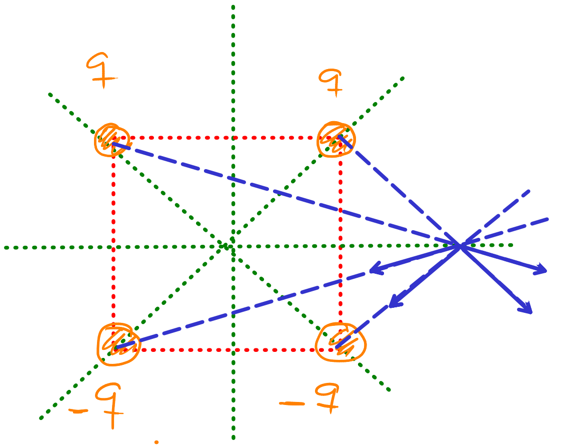
$$+ \int_{0}^{\infty} ds' \underbrace{\rho(x',y',-s')}_{= \rho(x',y',s') \to z'} \frac{+s' \to z'}{\left[(x-x')^2 + (y-y')^2 + \underset{\downarrow\, z'}{s'^2} \right]} \Bigg]^{3/2}$$

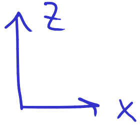
$= \{ s' = z' \} = 0$ ✓

Plan d'antisimétrie



Proposition : champ perpendiculaire au plan de *anti*symétrie





} résultante direction $\hat{z}$

Démonstration mathématique

$$\rho(x,y,z) = -\rho(x,y,-z)$$

On doit démontrer que $E_x(x,y,0) = E_y(x,y,0) = 0$

$$E_x(x,y,0) = k\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{0}^{\infty} dz' \; \rho(x',y',z') \left[\frac{x-x'}{\left[(x-x')^2+(y-y')^2+z'^2\right]^{3/2}} \right.$$

$$\left. - \frac{x-y'}{\left[(x-x')^2+(y-y')^2+z'^2\right]^{3/2}} \right] \equiv 0$$

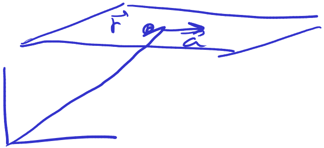
Idem $E_y(x,y,0) = 0$.

Invariance par translation

"Invariance" = la densité de charge est la même si on effectue une translation.

Exemple : plaque infinie.

Mathématiquement : $\rho(\vec{r}+\vec{a}) = \rho(\vec{r})$



$$\vec{E}(\vec{r}+\vec{a}) = k\int_{\mathbb{R}^3} d^3r' \, \rho(\vec{r}') \, \frac{\vec{r}+\vec{a}-\vec{r}'}{|\vec{r}+\vec{a}-\vec{r}'|^3}$$

$$\vec{y}' = \vec{r}' - \vec{a} \Rightarrow \vec{r}' = \vec{y}' + \vec{a}$$

$$d^3y' = d^3r'$$

$$\vec{E}(\vec{r}+\vec{a}) = k\int_{\mathbb{R}^3} d^3y' \, \underbrace{\rho(\vec{y}'+\vec{a})}_{=\rho(\vec{y}')} \, \frac{\vec{r}+\cancel{\vec{a}}-\vec{y}'-\cancel{\vec{a}}}{|\vec{r}+\cancel{\vec{a}}-\vec{y}'-\cancel{\vec{a}}|^3} =$$

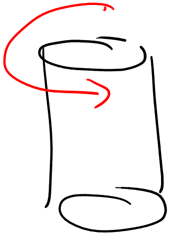
$= \vec{E}(\vec{r})$: le champ aussi invariant par translation.

<u>Commentaire</u> : analogue pour potentiel

$$\text{si } \rho(\vec{r}+\vec{a}) = \rho(\vec{r}) \Rightarrow V(\vec{r}+\vec{a}) = V(\vec{r})$$

Invariance par rotation :

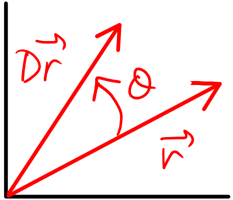
Exemple : cylindre



rotation : "rien ne change"

Mathématiquement :

Rotation :



D : opérateur (matrice de rotation)

En dimension 2 $\quad D(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Rotation inverse $D^{-1}(\theta) = D(-\theta)$

$$D(-\theta) = \begin{pmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$D(\theta)\, D(-\theta) = D(-\theta)\, D(\theta) = \mathbb{1}$$

$\rho(\vec{r})$ invariant par rotation : $\rho(D\vec{r}) = \rho(\vec{r})$

$$\vec{E}(D\vec{r}) = \int d^3r' \, \rho(\vec{r}\,') \, \frac{D\vec{r} - \vec{r}\,'}{|D\vec{r} - \vec{r}\,'|^3}$$

$\vec{y}\,' = D^{-1}\vec{r}$ $\qquad d^3r' = d^3y' \qquad \vec{r}\,' = D\vec{y}\,'$

$$\vec{E}(D\vec{r}) = \int d^3y' \, \underbrace{\rho(D\vec{y}\,')}_{\rho(\vec{y}\,')} \, \frac{D(\vec{r} - \vec{y}\,')}{\underbrace{|D(\vec{r} - \vec{y}\,')|^3}_{|\vec{r} - \vec{y}\,'|^3}} =$$

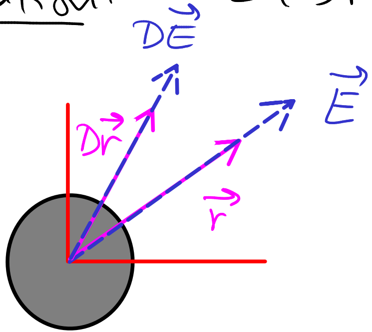
$|\vec{r} - \vec{y}\,'|^3$: rotation ne modifie pas distances.

$$= \int d^3y' \, \rho(\vec{y}') \, \frac{D(\vec{r}-\vec{y}')}{|\vec{r}-\vec{y}'|^3} =$$

$$= D \int d^3y' \, \rho(\vec{y}') \, \frac{\vec{r}-\vec{y}'}{|\vec{r}-\vec{y}'|^3} = D\vec{E}(\vec{r})$$

Donc $\vec{E}(D\vec{r}) = D\vec{E}(\vec{r})$

Signification : $\vec{E}(D\vec{r}) = D\vec{E}(\vec{r})$



Commentaire : pour le potentiel,

$$V(D\vec{r}) = V(\vec{r})$$