Waiting times distribution of electrons flowing across mesoscopic conductors

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How much time do I have?
I am afraid you are already dead...
Motivations

Toward a quantum theory of waiting times

Conclusion and Outlook
Outline

1 Motivations

2 Toward a quantum theory of waiting times

3 Conclusion and Outlook

- $s$ is the waiting time between two buses.
- $p(s)$ is the waiting time distribution.
**Mesoscopic conductor:** system size $<$ coherence length. Quantum effects and fluctuations are non negligible.

Examples: ballistic nano-wires, quantum dots, chaotic cavities, disordered wires, nanotubes etc...

**Possible measures of fluctuations.**

- Moments of the current distribution:
  - **Noise:** $S(t) = \langle I(t)I(0)\rangle - \langle I(t)\rangle^2$, \[ \sigma = \sqrt{S(0)} \]
  - Third cumulant $\langle I(t)I(t')I(0)\rangle$, etc...

- Full Counting Statistics (FCS): $P(n, T)$: number of transferred charges over a long time window.

- Waiting time distribution (WTD) $\mathcal{W}(\tau)$
Scattering theory of non-interacting electrons

\[ G = \frac{e^2}{h} \sum_n \tau_n \quad (I = GV) \]

\[ S = \frac{e^3}{h} V \sum_n \tau_n (1 - \tau_n) \] (zero temperature, zero frequency)

Non locality \[ G_{tot}^{-1} \neq G_1^{-1} + G_2^{-1} \]
Waiting time distribution in mesoscopic physics

- **Example in mesoscopic physics**

![Diagram of mesoscopic physics](image)


- Very general concept but rarely studied in mesoscopic physics
Waiting time distribution: why?

- Probe correlations on short time scales
  - Analogy with spectral statistics in regular and chaotic systems
    WTD=Level spacing distribution and FCS=Integrated density of states

- If two events cannot happen at the same time the WTD starts from 0.

- Measure the regularity of a source
- Give access to details that are hidden in other quantities

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WTD in mesoscopic conductors
Quantum optics with single electron

- Splettstoesser et al, PRL 103, 076804 (2009)
- Moskalets et al, PRB 83, 035316 (2011)
- Haack et al, PRB 84, 081303(R) (2011)
- Grenier et al, NJP 13, 093007 (2010)
- Jonckheere et al, PRB 86, 125425 (2012)
Simple examples of waiting time distributions?

Regular process

Uncorrelated process

Correlated process
1 Motivations

2 Toward a quantum theory of waiting times

3 Conclusion and Outlook
Quantum particles are described by wave packets. → Quantum jitter!

Classical measurement → detect a spike.

The WTD probes both the structure of the many-body state and the fluctuations generated by the scatterer.

**Problems!**

- Energy-time Heisenberg inequality.
- The vacuum is not "empty": Fermi sea.
- We have to include the detection process in the theory.
First attempt to solve the problem

**Ideal situation:** $T = 0$, free fermions, no Fermi-Sea, time-independent scatterer, stationary process and ideal detector.

- **Idle time probability** $\Pi(\tau)$: prob to detect nothing in a time slot $\tau$.

\[
\Pi(\tau) = \frac{1}{\langle \tau \rangle} \int_{t_0}^{\infty} dt_0 \left( 1 - \int_{t_0}^{t_0 + \tau} w(t) dt \right)
\]

\[
w(\tau) = \langle \tau \rangle \frac{d^2 \Pi(\tau)}{d\tau^2}
\]

- **Transmission operator** over a finite time window $Q_\tau = \int_0^{\sqrt{F} \tau} |x\rangle\langle x| dx$
  
  - For 1 electron: $\Pi(\tau) = \langle \varphi | 1 - Q_\tau | \varphi \rangle$
  
  - For $N$ electrons: $\Pi(\tau) = \langle \Psi_N | \prod_{1}^{N} [1 - Q_\tau] | \Psi_N \rangle$

If $|\Psi_N\rangle$ is a **Slater determinant** we get the determinant formula:

\[
\Pi^0(\tau) = \det \langle \varphi_n | 1 - Q_\tau | \varphi_m \rangle
\]

We take the $N \to \infty$ limit to mimic a stationary process. Hassler et al PRB 2008.
Single quantum channel

- **Quantum** mechanical time scale $h/eV$.

- FCS in the long time limit: Binomial process with time step $h/eV$ and probability $D$. Levitov and Lesovik JETP 1993. For $D \ll 1$: poissonian statistics (uncorrelated transport).

  - The noise $S(\omega) = \int e^{i\omega t} \langle \delta I(t) \delta I(0) \rangle dt = 0$ at $D = 1$!!! → No access to the structure of the wave function!

The WTD should exhibit the quantum jitter!
Single quantum channel

$D = 0.1, \quad \langle \tau \rangle = \frac{h}{eV D}$

- Almost uncorrelated $\rightarrow$ exponential WTD.
- Pauli exclusion principle $\rightarrow$ hole at $\tau = 0$.
- Quantum oscillations with period $h/eV!!$

- Liquid like correlations due to the strong overlap of the wave packets. Here the particles have to fill the quantum channel.

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WTD in mesoscopic conductors
Single quantum channel

Quantum point contact

$E_F + eV$

$\tau$

$\tau$

$L$

$\omega$ vs $\tau$

$D=0.1$

Numerics

Poisson

$w(\tau)$

$h/eV$

$E_F$

$E_F$

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WTD in mesoscopic conductors
Single quantum channel

$E_F + eV$

Quantum point contact

$\tau$

Numerics

Poisson

$w(\tau)$

$D=0.1$

$D=0.5$

$D=0.1$

$w(\tau)$

$\tau$ [h/eV]

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WTD in mesoscopic conductors
Single quantum channel

$E_F + eV$

Quantum point contact

$\tau$

$h/eV$

$D=0.1$

$D=0.5$

$D=1$

Numerics

Poisson

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WTD in mesoscopic conductors
Single quantum channel

\[ E_F + eV \]

Quantum point contact

\[ \hbar/eV \]

\[ D \]

\[ \tau \]

Average waiting time \( \langle \tau \rangle = \frac{\hbar}{eV D} \).

Crossover from Poisson to Wigner-Dyson (GUE).

\[ p(s) = e^{-s} \quad p(s) = \frac{32}{\pi^2} s^2 e^{-\frac{4}{\pi} s^2} \]

Large fluctuations even at \( D = 1 \).

→ Quantum jitter!
Connection with Random Matrix Theory (RMT)

\[ \mathbf{H} = \begin{pmatrix} H_{11} & \cdots & H_{1N} \\ \vdots & \ddots & \vdots \\ H_{N1} & \cdots & H_{NN} \end{pmatrix} \]

\[ P(\mathbf{H}) \sim \exp[-\text{Tr} \mathbf{V}(\mathbf{H})] \]

\[ P(E_1, \ldots, E_N) \sim \prod_{n>m} |E_n - E_m|^\beta \exp \left[ - \sum_n V(E_n) \right] = |\Psi(E_1, \ldots, E_N)|^2 \]

The level repulsion \((E_n - E_m)^\beta\) depends on symmetries (\(\beta = 1, 2, 4\) for orthogonal, unitary and symplectic ensembles).

\(\Psi(E_1, \ldots, E_N)\) is the ground state of the Calogero-Sutherland Model:

\[ \hat{H} = -\frac{1}{2} \sum_n \frac{\partial^2}{\partial E_n^2} + \frac{\beta}{2} \left( \frac{\beta}{2} - 1 \right) \sum_{n>m} \frac{1}{(E_n - E_m)^2} \]

\(\beta = 2\): free fermions \(\Rightarrow\) mapping between RMT and free fermions in 1D. All the correlation functions are identical. \(E_n \leftrightarrow x_n\)
Lorentzian pulses with $n = 1$


- **Tunable aspect ratio**: liquid to solid crossover.
- $\xi/v_F \ll t_p$: thin peaks reflecting the shape of the wave packet.
- $\xi/v_F \gg t_p$: constant bias limit $eV = h\nu_F/\xi$.
- $D = 1$ is no longer given by RMT.

$R = \frac{\xi}{v_FT_p} = 0.1,\ 1$  
$D = 0.4$

Waiting time distribution as a tool to probe accuracy and correlations.

Quantum theory of WTD for non-interacting electrons. Link with RMT.

Liquid-Solid crossover with Lorentzian pulses.

Open questions: more general systems, arbitrary time dependence, interaction effects, universality classes?
Conclusion and outlook

\[ \begin{pmatrix} H_{11} & \cdots & H_{1N} \\ \vdots & \ddots & \vdots \\ H_{N1} & \cdots & H_{NN} \end{pmatrix} \]

Thank you for your attention!

For further reading see

- Albert et al, PRL 107, 086805 (2011)
Single level system

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WTD in mesoscopic conductors
Let's consider a time independent system with $\Gamma_R = \Gamma_L = \Gamma$

$$\Rightarrow w_{ea}(\Delta t) = w_{ae}(\Delta t) = \Gamma e^{-\Gamma \Delta t}, \quad w_{ee}(\Delta t) = \Gamma^2 \Delta t \ e^{-\Gamma \Delta t}$$

- Absorption and emission are independent events $\Rightarrow$ exponential distribution.
- Simultaneous emissions are prohibited $\Rightarrow$ hole in the WTD.
Master equation approach

Master equation for $P_1(t) = \langle Q(t) \rangle$

$$\partial_t P_1(t) = -[\Gamma_L(t) + \Gamma_R(t)]P_1(t) + \Gamma_L(t)$$

- **Incoming current** $\Rightarrow$ $\langle l_{in}(t) \rangle = \Gamma_L(t)[1 - P_1(t)]$
- **Outgoing current** $\Rightarrow$ $\langle l_{out}(t) \rangle = \Gamma_R(t)P_1(t)$

How to calculate the waiting time distribution?
\( w_{ea}(t, t + \Delta t) \) is proportional to the probability that an electron is absorbed a time \( t \) and emitted at time \( t + \Delta t \).

- Absorption at time \( t \sim |\langle I_{in}(t) \rangle| \)
- **Next emission** at time \( t + \Delta t \sim |\langle I_{out}^s \rangle| = \Gamma_R P_1^s \)

**Master equation for the survival probability**

\[
\partial_t P_1^s(t + \Delta t) = -\Gamma_R (t + \Delta t) P_1^s(t + \Delta t) \quad \text{with} \quad P_1^s(t) = 1
\]

\[
\Rightarrow \quad w_{ea}(t, t + \Delta t) = N |\langle I_{in}(t) \rangle \langle I_{out}^s(t + \Delta t) \rangle| \]

In general the initial time is arbitrary and one should sum over all the possibilities.

\[
\bar{w}_{ea}(\Delta t) = \int_0^T \frac{dt}{T} w_{ea}(t, t + \Delta t)
\]
\[ p(n, t) = \int d\tau_1 \cdots d\tau_n \, w_{ee}(0, \tau_1) \cdots w_{ee}(t - \tau_n, t) \, \delta \left( \sum_{i=1}^{n} \tau_i - t \right). \]

In the long time limit we make the following approximation

\[ p(n, t) \approx \int d\tau_1 \cdots d\tau_n \, \overline{w}_{ee}(\tau_1) \cdots \overline{w}_{ee}(\tau_n) \, \delta \left( \sum_{i=1}^{n} \tau_i - t \right). \]

After some manipulations in Fourier space in the long time limit

\[ \Rightarrow \quad G_{ee}(S(\chi, t)) + i\chi = 0 \]

\[ G_{ee}(z) = \ln \int_{0}^{\infty} d\tau \overline{w}_{ee}(\tau) \, e^{-z\tau} \quad \text{and} \quad S(\chi, t) = \ln \sum_{n=0}^{\infty} p(n, t) \, e^{i\chi n} \]

CGF of waiting times \quad CGF of the FCS

This equation was previously derived in Brandes Ann. Phys. 2008 for time independent systems.
From WTD to FCS

Waiting time distribution

Counting statistics

\[ n = \text{Number of emitted electrons} \]

Equation relating the two cumulant generating functions

\[ \Rightarrow G_{ee}(S(\chi, t)) + i\chi = 0 \]

\[ G_{ee}(z) = \ln \int_0^\infty d\tau w_{ee}(\tau) e^{-z\tau} \quad \text{and} \quad S(\chi, t) = \ln \sum_{n=0}^\infty p(n, t) e^{i\chi n} \]

CGF of waiting times

CGF of the FCS

From this very simple relation one can extract some relations between cumulants

\[ I = \frac{\langle n \rangle}{t} = \frac{1}{\langle \tau \rangle} , \quad F_2 = \frac{\langle n^2 \rangle}{\langle n \rangle} = \frac{\langle \tau^2 \rangle}{\langle \tau \rangle^2} , \]

\[ F_3 = \frac{\langle n^3 \rangle}{\langle n \rangle} = 3 \frac{\langle \tau^2 \rangle^2}{\langle \tau \rangle^4} - \frac{\langle \tau^3 \rangle}{\langle \tau \rangle^3} \]