On some potential inverse problems

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(Analysis and Inverse problems for Control theory and Signal processing)

From joint work with

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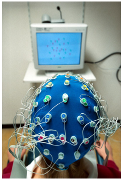
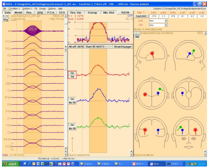
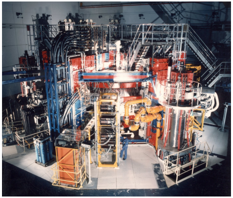
Links between models, problems?

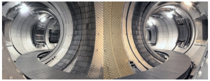


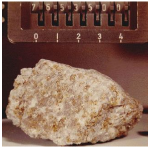
EEG



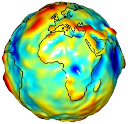


tokamak





magnetized rock



geoid

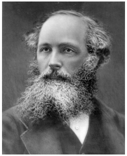
Maxwell equations

Under quasi-static assumptions

- Electroencephalography (EEG), medical engineering, electrical potential
- Magnetic plasma confinment in tokamaks, fusion, magnetic flux
- Rocks magnetization, paleomagnetism, magnetic potential
- Geodesy, geophysics (Newton law), gravitational potential

Inverse problems:
from measurements of a potential (flux, field) outside a domain Ω or on the boundary $\partial\Omega$, recover it, or its singularities, in Ω

EEG: Maxwell $\rightsquigarrow$ conductivity equations



(James Clerk Maxwell)

Electrical field E: $\nabla \times E = 0$ (Faraday)
$\Rightarrow$ $E = -\nabla u$, electrical potential u

Current density J: $\nabla \cdot J = 0$ ($\Leftarrow$ Ampère)

With $J = J^p + \sigma E$ in the head,
σ electrical conductivity, J^p primary cerebral current:

$$\Rightarrow \boxed{\nabla \cdot (\sigma \nabla u) = \nabla \cdot J^p}$$

Operators...

- ∇ is gradient
- $\nabla \cdot$ is divergence (sum of first partial derivatives)
- $\nabla \times$ is curl (rotationnel)
- Δ is Laplace operator (sum of second partial derivatives)

$\Delta u = 0 \Leftrightarrow u$ harmonic function (linked with holomorphic/analytic functions)

EEG: inverse source problem

Being given:

- a model of head $\Omega \subset \mathbb{R}^3$,
- a conductivity function σ — EIT: σ unknown
- measured (approximate) pointwise values on the boundary $\partial\Omega$ of a solution u to

$$\nabla \cdot (\sigma \nabla u) = \nabla \cdot J^p \,, \;\; J^p = \sum_{k=1}^{K} p_k \delta_{C_k} \text{ in } \Omega$$

find the quantity K, locations and moments $C_k \in \Omega$, $p_k \in \mathbb{R}^3$ of sources

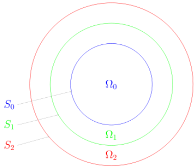
Associated direct problem, properties, ...

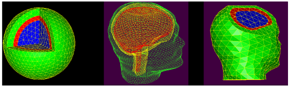
EEG: conductivity $\rightsquigarrow$ Laplace-Poisson equations

Spherical geometry: head Ω made of 3 spherical layers Ω_i (scalp, skull, brain)

σ piecewise constant, equals $\sigma_i > 0$ in Ω_i $(\sigma_0 = 1)$

J^p: pointwise dipolar sources in the brain Ω_0



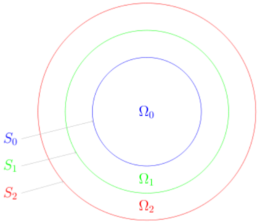


- $\Delta u = 0$ in Ω_2, Ω_1
- $\Delta u = \nabla \cdot J^p = \sum_{k=1}^{K} p_k \cdot \nabla \delta_{C_k}$ in Ω_0

EEG: inverse problems

- Cortical mapping: $\partial\Omega_i = S_i$
  From pointwise measurements of u on part of S_2
  (at electrodes, and $\partial_n u = 0$ on S_2, current flux),
  find u, $\partial_n u$ on S_0 with
  $$\Delta u = 0 \text{ in } \Omega_2, \Omega_1$$

- Source estimation:
  From u, $\partial_n u$ on S_0, find quantity K, locations C_k of sources such that: (and moments p_k)
  $$\Delta u = \sum_{k=1}^{K} p_k \cdot \nabla \delta_{C_k} \text{ in } \Omega_0$$



EEG: 1st cortical mapping step

Data transmission from S_2 to S_0, Cauchy boundary value problem

- representation from boundary data

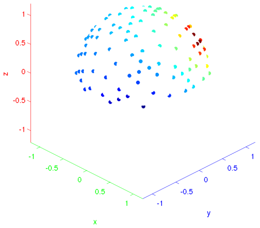
  Green formula, single- and double-layer potentials

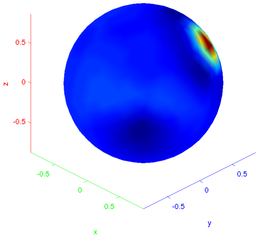
- boundary element methods
- minimizing a regularized quadratic criterion (discrete, at points on S_i)
- software: FindSources3D

best constrained approximation problems, analytic functions, integral criterion

EEG: 1st cortical mapping step

128 electrodes $\rightsquigarrow$ u on S_0, cortex





EEG: 2nd source localization step

From potential and normal current on S_0, localize sources C_k in Ω_0

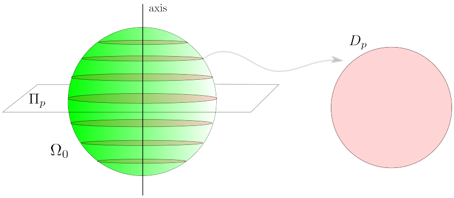
- integral representation

convolution by fundamental solution

$$u(X) \simeq \sum_{k=1}^{K} \frac{< p_k, X - C_k >}{\|X - C_k\|^3}$$

- spherical harmonics expansion of u on S_0
- u on families of parallel planar sections (circles) coincides with a function whose singularities (poles and branchoints) are related to the sources

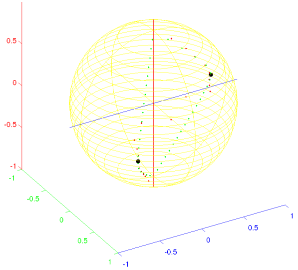
EEG: 2nd source localization step



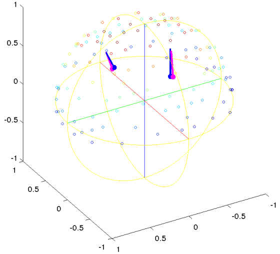
- Fourier expansion
- best quadratic rational approximation on circles APICS team
  $\rightsquigarrow$ planar singularities approx. degree $\rightsquigarrow K$
- clustering the planar singularities
  $\rightsquigarrow$ sources, moments (software: FindSources3D)

EEG: 2nd source localization step

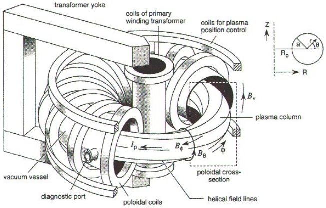
theoretical singularities/
approximating poles/sources

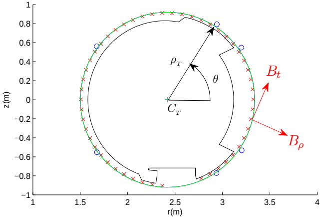


numerical estimation
C_k, p_k from u on S_0



Other problems: plasma shaping





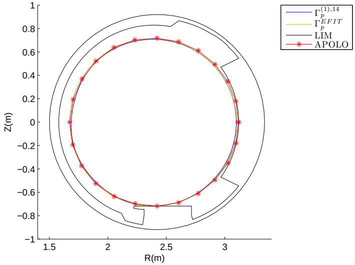
Axi-symmetry, poloidal planar sections, cylindrical coordinates:
Maxwell $\rightsquigarrow$ Laplace (3D) $\rightsquigarrow$ $\nabla \cdot (\sigma \nabla u) = 0$ (2D)
in annular domain (vacuum) between chamber and plasma
u magnetic flux, $\sigma = 1/R$

CEA-IRFM, Tore Supra (WEST)

Inverse problem:
from pointwise measures of magnetic flux, field outside chamber...

Other problems: plasma shaping

… find plasma boundary = level line of u tangent to limitor:

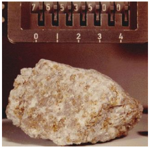


- best quadratic constrained approximation by generalized analytic functions
- expansion on toroidal harmonics basis
- geometrical step (free boundary, Bernoulli)

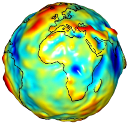
$\rightsquigarrow$ Schödinger equation

Other problems

- Magnetic fields, macroscopic (Maxwell) $\rightsquigarrow \Delta u \simeq \nabla \cdot M$
  IP: magnetization M to be recovered from measures (SQUID microscopy) of magnetic field or scalar potential u



- Geodesy, geophysics (Newton) $\rightsquigarrow \Delta u \simeq \varrho$
  IP: features (anomalies) of Earth density ϱ
  to be recovered from measures of gravitational potential u (or geoid, level surface of u) and other quantities (ground, air, ...)



In conclusion...

Various physical (inverse) problems (Maxwell, Newton equations)

$+$ assumptions lead to similar mathematical issues

Given measures of u, find ϱ, where

$$\Delta u \simeq \varrho \text{ supported in } \Omega \Leftrightarrow u(X) \simeq \iiint_\Omega \frac{\varrho(Y)}{|X - Y|} dY + \text{ harmonic}$$

Use of constructive best constrained approximation techniques for available boundary data, in classes of analytic or rational functions

Well-posed problems, computationally efficient and robust resolution schemes