Course Finite Volume discretization of PDEs,
University Nice Sophia Antipolis

Exercize : finite volume discretization of a scalar hyperbolic equation with source terms.

Let Ω a bounded polygonal domain of $\mathbb{R}^2$ and u a function from $\Omega \times (0,T)$ to $\mathbb{R}$. We consider the following scalar hyperbolic problem :

$$\begin{cases} \partial_t u(\mathbf{x},t) + \mathrm{div}\Big(f(u(\mathbf{x},t))\mathbf{V}(\mathbf{x})\Big) = h(\mathbf{x})^+ f(c(\mathbf{x})) + h(\mathbf{x})^- f(u(\mathbf{x},t)) & \text{on } \Omega \times (0,T), \\ u(\mathbf{x},t) = d(\mathbf{x}) & \text{on } \partial\Omega^- \times (0,T), \\ u(\mathbf{x},0) = u^0(\mathbf{x}) & \text{on } \Omega, \end{cases}$$

with $u^0 \in L^\infty(\Omega)$, $c \in L^\infty(\Omega^+)$, $d \in L^\infty(\partial\Omega^-)$, $h \in L^\infty(\Omega)$, and $\mathbf{V} \in C^1(\overline{\Omega})$ such that

$$\mathrm{div}(\mathbf{V}) = h \text{ on } \Omega.$$

The function f is assumed non decreasing from $\mathbb{R}$ to $\mathbb{R}$ and to be Lipchitz on bounded sets. The domain Ω^+ is defined by

$$\Omega^+ = \{\mathbf{x} \in \Omega \text{ such that } h(\mathbf{x}) > 0\},$$

and the input boundary $\partial\Omega^-$ by

$$\partial\Omega^- = \{\mathbf{x} \in \partial\Omega \text{ such that } \mathbf{V}\cdot\mathbf{n} < 0\}$$

where $\mathbf{n}$ is the unit normal vector outward to the domain Ω. We have used the notations $h^+(\mathbf{x}) = \max(0, h(\mathbf{x}))$, $h^-(\mathbf{x}) = \min(0, h(\mathbf{x}))$. We recall that $h^+(\mathbf{x}) + h^-(\mathbf{x}) = h(\mathbf{x})$.

(1) We consider a conforming finite volume mesh of Ω such that $\partial\Omega^-$ is the union of a collection of boundary faces. Using the notations of the course, write the finite volume discretization of the hyperbolic equation using an Euler explicit time integration, and the upwind scheme for the fluxes. The initial condition will be approximated in all cells K by

$$u_K^0 = \frac{1}{|K|}\int_K u^0(\mathbf{x})d\mathbf{x}.$$

You will use the notations of the course and in particular

$$V_{K,\sigma} = \int_\sigma \mathbf{V}\cdot\mathbf{n}_{K,\sigma} d\sigma,$$

and

$$d_\sigma = \frac{1}{|\sigma|}\int_\sigma d(\mathbf{x})d\sigma, \quad c_K = \frac{1}{|K|}\int_K c(\mathbf{x})d\mathbf{x}, \quad h_K^+ = \int_K h^+(\mathbf{x})d\mathbf{x}, \quad h_K^- = \int_K h^-(\mathbf{x})d\mathbf{x}.$$

The discretization of the source term $h(\mathbf{x})^+ f(c(\mathbf{x})) + h(\mathbf{x})^- f(u(\mathbf{x},t))$ in the cell K will use only h_K^+, h_K^-, c_K, the unknown u_K taken explicit in time, and the function f.

Correction : Using the Godunov (or upwind) scheme for the approximation of the fluxes combined with the Euler explicit integration scheme we obtain : for all cell $K \in \mathcal{M}$:

$$\begin{aligned} \frac{u_K^n - u_K^{n-1}}{\Delta t^n}|K| &+ \sum_{\sigma=KL \in \mathcal{F}_K \cap \mathcal{F}_{int}} f(u_K^{n-1})(V_{K,\sigma})^+ + f(u_L^{n-1})(V_{K,\sigma})^- \\ &+ \sum_{\sigma \in \mathcal{F}_K \cap \mathcal{F}_{ext}} f(u_K^{n-1})(V_{K,\sigma})^+ + f(d_\sigma)(V_{K,\sigma})^- \\ &= h_K^+ f(c_K) + h_K^- f(u_K^{n-1}). \end{aligned}$$

(2) Using that $\operatorname{div}(\mathbf{V}) = h$ and that f is non decreasing and Lipchitz on bounded sets, derive the CFL condition on the time step in order to obtain the stability of the discretization.

Correction : By integration over the cell $K \in \mathcal{M}$ of the equation $\operatorname{div}(\mathbf{V}) = h$ we obtain that

$$\sum_{\sigma \in \mathcal{F}_K} V_{K,\sigma} = h_K^+ + h_K^-.$$

Multiplying this equation by $f(u_K^{n-1})$ we obtain the equation :

$$\sum_{\sigma \in \mathcal{F}_K} f(u_K^{n-1}) V_{K,\sigma} = f(u_K^{n-1}) h_K^+ + f(u_K^{n-1}) h_K^-.$$

Substracting this equation from the finite volume equation in cell K we obtain that :

$$\begin{aligned}
\frac{u_K^n - u_K^{n-1}}{\Delta t^n}|K| &+ \sum_{\sigma = KL \in \mathcal{F}_K \cap \mathcal{F}_{int}} (V_{K,\sigma})^- (f(u_L^{n-1}) - f(u_K^{n-1})) \\
&+ \sum_{\sigma \in \mathcal{F}_K \cap \mathcal{F}_{ext}} (V_{K,\sigma})^- (f(d_\sigma) - f(u_K^{n-1})) \\
&= h_K^+ (f(c_K) - f(u_K^{n-1})).
\end{aligned}$$

For all $\sigma = KL \in \mathcal{F}_K \cap \mathcal{F}_{int}$ let us set

$$a_{K,\sigma}^{n-1} = \frac{f(u_K^{n-1}) - f(u_L^{n-1})}{u_K^{n-1} - u_L^{n-1}},$$

and for all $\sigma \in \mathcal{F}_K \cap \mathcal{F}_{ext}$

$$a_{K,\sigma}^{n-1} = \frac{f(u_K^{n-1}) - f(d_\sigma)}{u_K^{n-1} - d_\sigma},$$

and for all $K \in \mathcal{M}$

$$b_K^{n-1} = \frac{f(u_K^{n-1}) - f(c_K)}{u_K^{n-1} - c_K}.$$

Let us set

$$I_0 = \Big[\min\Big(\inf_{\mathbf{x} \in \Omega^+} c(\mathbf{x}), \inf_{\mathbf{x} \in \partial\Omega^-} d(\mathbf{x}), \inf_{\mathbf{x} \in \Omega^+} u^0(\mathbf{x})\Big), \max\Big(\sup_{\mathbf{x} \in \Omega^+} c(\mathbf{x}), \sup_{\mathbf{x} \in \partial\Omega^-} d(\mathbf{x}), \sup_{\mathbf{x} \in \Omega^+} u^0(\mathbf{x})\Big)\Big].$$

From the non decreasing and Lipchitz properties of f we deduce that :

$$0 \le a_{K,\sigma}^{n-1} \text{ and } 0 \le b_K^{n-1},$$

for all u_K^{n-1}, u_L^{n-1}, and that

$$0 \le a_{K,\sigma}^{n-1} \le Lip(f) \text{ and } 0 \le b_K^{n-1} \le Lip(f),$$

for all $u_K^{n-1}, u_L^{n-1} \in I_0 \times I_0$, where $Lip(f)$ denotes the Lipchitz constant on $I_0 \times I_0$.

It results that

$$\begin{aligned}
u_K^n =& \Big[1 - \frac{\Delta t^n}{|K|}\Big(h_K^+ b_K^{n-1} + \sum_{\sigma\in\mathcal{F}_K} -(V_{K,\sigma})^- a_{K,\sigma}^{n-1}\Big)\Big] u_K^{n-1}\\
&+ \frac{\Delta t^n}{|K|} \sum_{\sigma=KL\in\mathcal{F}_K\cap\mathcal{F}_{int}} -(V_{K,\sigma})^- a_{K,\sigma}^{n-1} u_L^{n-1}\\
&+ \frac{\Delta t^n}{|K|} \sum_{\sigma\in\mathcal{F}_K\cap\mathcal{F}_{ext}} -(V_{K,\sigma})^- a_{K,\sigma}^{n-1} d_\sigma\\
&+ \frac{\Delta t^n}{|K|} h_K^+ b_K^{n-1} c_K.
\end{aligned}$$

It defines a convex combination iff the following condition is satisfied

$$1 - \frac{\Delta t^n}{|K|}\Big(h_K^+ b_K^{n-1} + \sum_{\sigma\in\mathcal{F}_K} -(V_{K,\sigma})^- a_{K,\sigma}^{n-1}\Big) \geq 0.$$

We deduce the CFL condition on the time steps Δt^n for all $n = 1, \cdots, m$:

$$\Delta t^n \leq \min_{K\in\mathcal{M}} \frac{|K|}{\Big(h_K^+ b_K^{n-1} + \sum_{\sigma\in\mathcal{F}_K} -(V_{K,\sigma})^- a_{K,\sigma}^{n-1}\Big)}.$$

By induction on n, using question (3), we deduce the sufficient condition on the time steps Δt^n for all $n = 1, \cdots, m$:

$$\Delta t^n \leq \frac{1}{Lip(f)} \min_{K\in\mathcal{M}} \frac{|K|}{\Big(h_K^+ + \sum_{\sigma\in\mathcal{F}_K} -(V_{K,\sigma})^-\Big)}.$$

(3) Assuming that the CFL condition is satisfied, write the maximum principle satisfied by the discrete solution u_K^n for all cells K and all time steps n function of c, d and u^0.

Correction : From the convex linear combination it results that

$$\begin{aligned}
&\min\Big(\min_{K\in\mathcal{M}|h_K^+>0} c_K, \min_{\sigma\in\mathcal{F}_{ext}|V_{K,\sigma}<0} d_\sigma, \min_{K\in\mathcal{M}} u_K^0\Big)\\
&\leq u_K^n \leq \max\Big(\max_{K\in\mathcal{M}|h_K^+>0} c_K, \max_{\sigma\in\mathcal{F}_{ext}|V_{K,\sigma}<0} d_\sigma, \max_{K\in\mathcal{M}} u_K^0\Big),
\end{aligned}$$

for all $K \in \mathcal{M}$ and $n = 1, \cdots, m$.