Convergence of Finite Volume MPFA O type Schemes for Heterogeneous Anisotropic Diffusion Problems on General Meshes

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Abstract

In this paper we prove the convergence of the finite volume MultiPoint Flux Approximation (MPFA) O scheme for anisotropic and heterogeneous diffusion problems, under a local coercivity condition which can be easily checked numerically. Our framework is based on a discrete hybrid variational formulation which generalizes the usual construction of the MPFA O scheme. The novel feature of our framework is that it holds for general polygonal and polyhedral meshes as well as for L^∞ diffusion coefficients, which is essential in many practical applications.

Key words : Finite Volume, MPFA, Convergence Analysis, Diffusion Equation, Full Tensor, Anisotropy, Heterogeneities, General Meshes

1 Introduction

In this paper, we consider the second order elliptic equation

$$\begin{cases} \mathrm{div}(-\Lambda\ \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open bounded connected polygonal subset of $\mathbb{R}^d$, $d \in \mathbb{N}^*$, and $f \in L^2(\Omega)$. It is assumed in the following that Λ is a measurable function from Ω to the set of square d-dimensional matrices $\mathcal{M}_d(\mathbb{R})$ such that for a.e. (almost every) $x \in \Omega$, $\Lambda(x)$ is symmetric and its eigenvalues are in the interval $[\alpha(x), \beta(x)]$ with $\alpha, \beta \in L^\infty(\Omega)$, and $0 < \alpha_0 \le \alpha(x) \le \beta(x) \le \beta_0$. It results that there exists a unique weak solution to (1) in $H_0^1(\Omega)$ denoted by $\bar{u}$ in the following of this paper.

The MultiPoint Flux Approximation (MPFA) O method is a cell centered finite volume discretization of such second order elliptic equations described for example in [1] and [8]. It is a widely used scheme in the oil industry for the discretization of diffusion fluxes in multiphase Darcy porous media flow models (see for example [13], [14], and [18]).

Let σ be any interior face of the mesh shared by the two cells K and L, and $\mathbf{n}_{K,\sigma}$ its normal vector outward K. Cell centered finite volume schemes use the cell unknowns u_M for each cell M of the mesh as degrees of freedom. They aim to build conservative approximations $F_{K,\sigma}$ of the fluxes $-\int_\sigma \Lambda\nabla u \cdot \mathbf{n}_{K,\sigma} d\sigma$ as linear combinations of the cell unknowns u_M using neighbouring cells M of the cells K or L. The fluxes are conservative in the sense that $F_{K,\sigma} + F_{L,\sigma} = 0$.

The main assets of the MPFA O scheme are to derive a consistent approximation of the fluxes on general meshes, and to be adapted to discontinuous anisotropic diffusion coefficients in the sense that it reproduces cellwise linear solutions for cellwise constant diffusion tensors. For that purpose, its construction uses in addition to the cell unknowns, the intermediate subface unknowns u_σ^s for each face (edge in 2D) σ of the mesh and each vertex s of the face σ. Roughly speaking, assuming that each vertex s of any cell K is shared by exactly d faces of the cell K, subfluxes $F_{K,\sigma}^s$ are built using a cellwise constant diffusion coefficient and a linear approximation of u on each cell K shared by s. Then, the intermediate unknowns are eliminated by the flux continuity equations on each face around the vertex s, and the approximate flux $F_{K,\sigma}$ is the sum of the subfluxes over the vertices of the face σ. A generalization of this construction is proposed in [13] for general polyhedral meshes.

Recent papers have studied the convergence of the MPFA O scheme. In [17], [3], [15], the convergence of the scheme is obtained on quadrilateral meshes. The proofs are based on equivalences of the MPFA O scheme to mixed finite element methods using specific quadrature rules. The convergence of the scheme is obtained provided that a square d-dimensional matrix defined locally for each cell and each vertex of the cell, depending both on the distortion of cell and on the cell diffusion tensor, is uniformly positive definite. This analysis confirms the numerical experiments showing that the coercivity and convergence of the scheme is lost in the cases of strong distortion of the mesh and/or anisotropy of the diffusion tensor.

The first convergence proof of the MPFA O scheme on general polygonal and polyhedral meshes is introduced in [6]. The convergence analysis holds for fairly general meshes in 2D and 3D, for diffusion tensors with minimal regularity including discontinuous diffusion coefficients which are essential in oil industry applications, and for minimal regularity assumptions on the solution. Moreover, it covers the all family of MPFA O schemes for arbitrary choices of the cell centers, of the so called continuity points, and of the subfaces.

A different approach is presented in [20] based on symmetric and non symmetric mimetic finite difference schemes using subfaces unknowns. The symmetric version of this scheme has also been independently introduced in [19] in two dimensions. As shown in [16] which develops a similar analysis, the non symmetric version of this mimetic finite difference scheme matches with the MPFA O scheme family. Error estimates are derived in [20] on general polygonal and polyhedral meshes under a local coercivity criteria and for piecewise regular diffusion tensors.

In [6], it is assumed that for each cell κ and each vertex s of the cell, the number of faces of the cell κ sharing the vertex s is equal to the space dimension d. This paper presents a generalization of the MPFA O scheme to polyhedral meshes non satisfying this latter assumption and extends the convergence analysis presented in [6]. It also details the proofs only sketched in [6].

In this paper, following [6], a discrete hybrid variational formulation is introduced using the framework described in [12], [11]. It involves the definition of two piecewise constant gradients and stability terms using residuals of the second gradient. The first gradient has a weak convergence property and is fixed in the construction. The second one is assumed to be consistent in the sense that it is exact on linear functions. For usual meshes such that each vertex of any cell K is shared by exactly d faces of the cell K, the stability terms are vanishing and our discrete variational formulation will be shown to be equivalent to the usual MPFA O scheme.

Moreover, it provides a generalization of the O scheme on more general polyhedral cells.

A sufficient local condition for the coercivity of the scheme is derived which will yield existence, and uniqueness of the solution. Under this coercivity condition, and a uniform stability assumption for the consistent gradient, the convergence of the scheme including the case of L^∞ diffusion coefficients can be proved.

This paper is outlined as follows. Section 2 describes the discrete framework including the definition of the finite volume discretization of the domain, the degrees of freedom and the discrete function spaces with their associated inner products and norms. Section 3 is devoted to the definition of a general framework for MPFA O type schemes based on a hybrid variational formulation and the definition of two piecewise constant gradients. Section 4 proves the well-posedness of the finite volume scheme under a sufficient coercivity condition involving computations local to each node of the mesh and depending on the geometry and on the diffusion tensor anisotropy. The convergence of the scheme is proved under the above coercivity assumption, usual shape regularity assumptions, and a uniform stability assumption for the consistent gradient in section 5 for L^∞ diffusion tensor. In section 6, two examples of construction of the consistent gradient are discussed. The first construction allows us to derive a stronger but simpler coercivity condition involving the coercivity of a d-dimensional matrix for each vertex s of each cell K. On the other hand this construction does not hold for non-matching meshes. The second example is based on the consistent gradient introduced in [13]. Section 7 is devoted to numerical examples in 2D and 3D.

Notations: In the following, for any vectors $x, y \in \mathbb{R}^d$, we will denote by $x \cdot y$ their dot product $\sum_{i=1}^d x_i\ y_i$, and by $|x|$ the norm $\sqrt{x \cdot x}$. The notations $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ will stand for the maximum and minimum eigenvalues of any given square symmetric matrix M. For any matrix A, we denote by $|A|$ its norm defined by $\sup_{x\in\mathbb{R}^d} \frac{|Ax|}{|x|} = \sqrt{\lambda_{\max}(A^t A)}$.

2 Discrete framework

2.1 The Finite Volume discretization of the domain Ω

For polygonal bounded subdomains Ω of $\mathbb{R}^d$, $d \in \mathbb{N}^\star$, the following definition of the finite volume discretization covers fairly general polygonal meshes either conforming or non-conforming (see Figure 1 for a 2D example).

Definition 2.1 (Admissible finite volume discretization) Let Ω be an open bounded subset of $\mathbb{R}^d$, with $d \in \mathbb{N}^\star$, and $\partial\Omega = \overline{\Omega}\setminus\Omega$ its boundary. An admissible finite volume discretization of Ω, denoted by $\mathcal{D}$, is given by $\mathcal{D} = (\mathcal{T}, \mathcal{E}, \mathcal{P}, \mathcal{V})$, where:

- $\mathcal{T}$ is a finite family of non-empty connected open disjoint subsets of Ω (the "cells") such that $\overline{\Omega} = \cup_{K\in\mathcal{T}}\overline{K}$. For any $K \in \mathcal{T}$, let $\partial K = \overline{K} \setminus K$ be the boundary of K and $\mathrm{m}_K > 0$ denote the d-dimensional measure (named volume in the following) of K.

- $\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "faces" of the mesh), such that, for all $\sigma \in \mathcal{E}$, σ is a non-empty closed subset of a hyperplane of $\mathbb{R}^d$, which has a $(d-1)$-dimensional measure (named surface in the following) $\mathrm{m}_\sigma > 0$. We assume that, for all $K \in \mathcal{T}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \cup_{\sigma\in\mathcal{E}_K}\sigma$. We then denote by $\mathcal{T}_\sigma$ the set $\{K \in \mathcal{T} \,|\, \sigma \in \mathcal{E}_K\}$. It is assumed that, for all $\sigma \in \mathcal{E}$, either $\mathcal{T}_\sigma$ has exactly one element and then $\sigma \subset \partial\Omega$ (boundary face) or $\mathcal{T}_\sigma$ has exactly two elements (interior face). For all $\sigma \in \mathcal{E}$, we denote by x_σ the center of gravity of σ

- $\mathcal{P}$ is a family of points of Ω indexed by $\mathcal{T}$ ("the cell centers"), denoted by $\mathcal{P} = (x_K)_{K\in\mathcal{T}}$, such that $x_K \in K$ and K is star-shaped with respect to x_K.

- $\mathcal{V}$ is a family of points ("the vertices of the mesh"), such that for any $K \in \mathcal{T}$, for all subset H_K of $\mathcal{E}_K$ with $\mathrm{Card}(H_K) \geq d$, then $\cap_{\sigma\in H_K}\sigma = \emptyset$ or $\cap_{\sigma\in H_K}\sigma = s$ where $s \in \mathcal{V}$. For all $s \in \mathcal{V}$, we denote by $\mathcal{E}_s$ the set $\{\sigma \in \mathcal{E} \,|\, s \in \sigma\}$ and by $\mathcal{T}_s$ the set $\{K \in \mathcal{T} \,|\, s \in \overline{K}\}$. For all $K \in \mathcal{T}$, the set $\mathcal{V}_K$ stands for $\{s \in \mathcal{V} \,|\, s \in \overline{K}\}$, and for all $\sigma \in \mathcal{E}$ the set $\{s \in \mathcal{V} \,|\, s \in \sigma\}$ is denoted by $\mathcal{V}_\sigma$.

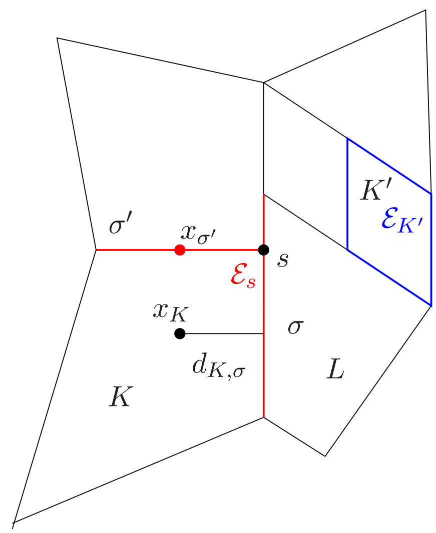


Figure 1: Example of an admissible finite volume discretization and notations: cells K, L, and K', faces σ and σ', vertex s, cell center x_K of the cell K, center of gravity $x_{\sigma'}$ of the face σ', distance $d_{K,\sigma}$ from the cell center x_K to the face σ, set $\mathcal{T}_\sigma = \{K, L\}$ of cells sharing the face σ, set $\mathcal{E}_s$ of faces sharing the vertex s, set $\mathcal{E}_{K'}$ of faces of the cell K'.

The following notations are used. The size of the discretization is defined by

$$h_{\mathcal{D}} = \sup\{\text{diam}(K), K \in \mathcal{T}\}.$$

For all $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, we denote by $\mathbf{n}_{K,\sigma}$ the unit vector normal to σ outward to K, and by $d_{K,\sigma}$ the Euclidean distance between x_K and σ.

The set of interior (resp. boundary) faces is denoted by $\mathcal{E}_{\text{int}}$ (resp. $\mathcal{E}_{\text{ext}}$), defined by $\mathcal{E}_{\text{int}} = \{\sigma \in \mathcal{E} \,|\, \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E} \,|\, \sigma \subset \partial\Omega\}$).

Shape regularity of the mesh: it will be measured by the following parameters:

$$\text{CardFace}(\mathcal{D}) = \max_{K \in \mathcal{T}_s, s \in \mathcal{V}} \text{Card}(\mathcal{E}_K \cap \mathcal{E}_s), \tag{2}$$

$$\text{RegulCell}(\mathcal{D}) = \min_{\sigma \in \mathcal{E}_K, K \in \mathcal{T}} \left\{ \frac{d_{K,\sigma}}{\text{diam}(K)} \right\}, \tag{3}$$

$$\text{RegulKL}(\mathcal{D}) = \min_{\sigma \in \mathcal{E}_{\text{int}}, \mathcal{T}_\sigma = \{K,L\}} \left\{ \frac{\min(d_{K,\sigma}, d_{L,\sigma})}{\max(d_{K,\sigma}, d_{L,\sigma})} \right\}. \tag{4}$$

In the convergence analysis of the finite volume scheme, the parameters RegulCell($\mathcal{D}$) and RegulKL($\mathcal{D}$) will be assumed to be uniformly bounded from below, and the parameter CardFace($\mathcal{D}$) to be uniformly bounded from above.

In particular assuming that CardFace($\mathcal{D}$) is uniformly bounded amounts to requiring that the number of faces sharing a node remains bounded as the mesh is refined. The uniform bound on RegulCell($\mathcal{D}$) ensures that the cell centers are uniformly away from the cell boundary, whereas the uniform bound on RegulKL($\mathcal{D}$)

implies roughly speaking that the cell size is smoothly varying across the mesh.

Parameters of the MPFA O finite volume scheme: in addition to the choice of the cell centers satisfying the above assumptions, the construction of the MPFA O scheme involves two families of parameters defined on the set $\{(\sigma, s) \,|\, s \in \mathcal{V}_\sigma, \sigma \in \mathcal{E}\}$.

The first family of non-negative reals $(\mathrm{m}_\sigma^s)_{s\in\mathcal{V}_\sigma,\sigma\in\mathcal{E}}$ defines the distribution of the surface m_σ of each face σ to the face vertices $s \in \mathcal{V}_\sigma$ such that $\mathrm{m}_\sigma = \sum_{s\in\mathcal{V}_\sigma} \mathrm{m}_\sigma^s$. It results that the volume of each cell $K \in \mathcal{T}$ is also distributed to the vertices of the cell according to the subvolumes m_K^s, $s \in \mathcal{V}_K$ defined by

$$\mathrm{m}_K^s = \frac{1}{d} \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \mathrm{m}_\sigma^s d_{K,\sigma}, \tag{5}$$

and which satisfy $\mathrm{m}_K = \sum_{s\in\mathcal{V}_K} \mathrm{m}_K^s$ for all $K \in \mathcal{T}$.

The second family is the set of the so called continuity points $(x_\sigma^s)_{\sigma\in\mathcal{E}_s,s\in\mathcal{V}}$ such that $x_\sigma^s \in \sigma$. On each continuity point x_σ^s, the intermediate unknown u_σ^s is defined which will be used together with the cell unknowns u_K, $K \in \mathcal{T}$ for the construction of the finite volume scheme in the next section.

2.2 Discrete functional framework

The MPFA O scheme is a cell centered finite volume scheme with main degrees of freedom the cell unknowns u_K on each cell K of the mesh $\mathcal{T}$. The following definition introduces the space of piecewise constant functions on each cell K of the mesh.

Definition 2.2 Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, with $d \in \mathbb{N}^\star$. Let $\mathcal{D} = (\mathcal{T}, \mathcal{E}, \mathcal{P}, \mathcal{V})$ be an admissible finite volume discretization of Ω in the sense of Definition 2.1. We denote by $H_\mathcal{T}(\Omega) \subset L^2(\Omega)$ the set of all functions $u \in L^2(\Omega)$ such that, for all $K \in \mathcal{T}$, there exists some real value denoted by $u_K \in \mathbb{R}$ such that $u(x) = u_K$ for a.e. $x \in K$.

Then, for all $\sigma \in \mathcal{E}$, let us define $\gamma_\sigma u$ such that

$$\left\{ \begin{aligned} \gamma_\sigma u &= 0 \quad \text{for all } \sigma \in \mathcal{E}_{\mathrm{ext}}, \\ \frac{\gamma_\sigma u - u_K}{d_{K,\sigma}} + \frac{\gamma_\sigma u - u_L}{d_{L,\sigma}} &= 0 \quad \text{for all } \sigma \in \mathcal{E}_{\mathrm{int}} \text{ with } \mathcal{T}_\sigma = \{K, L\}. \end{aligned} \right. \tag{6}$$

The space $H_\mathcal{T}(\Omega)$ is equipped with the Euclidean structure defined by the inner product

$$[v, w]_\mathcal{T} = \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \frac{\mathrm{m}_\sigma}{d_{K,\sigma}} (\gamma_\sigma v - v_K)(\gamma_\sigma w - w_K), \tag{7}$$

and the associated norm

$$\|v\|_\mathcal{T} = \left([v, v]_\mathcal{T}\right)^{1/2},$$

for all $(v, w) \in (H_\mathcal{T}(\Omega))^2$.

The construction of the scheme uses additional degrees of freedom u_σ^s for each vertex s of the face σ and each face σ. These subface unknowns will be locally eliminated as linear combinations of the neighbouring cell unknowns using the flux continuity equations. In our approach the finite volume scheme will be derived in section 3 from a hybrid variational formulation defined on the space $\mathcal{H}_\mathcal{D}$ spanned by the cell and subface unknowns and introduced below.

Definition 2.3 Let us define the discrete function space $\mathcal{H}_\mathcal{D}$ as the set of all $((u_K)_{K\in\mathcal{T}}, (u_\sigma^s)_{\sigma\in\mathcal{E}_s, s\in\mathcal{V}})$, $u_K \in \mathbb{R}$, $K \in \mathcal{T}$, $u_\sigma^s \in \mathbb{R}$, $\sigma \in \mathcal{E}_s$, $s \in \mathcal{V}$ such that $u_\sigma^s = 0$ for all $\sigma \in \mathcal{E}_{\text{ext}}$. It is equipped with the Euclidean structure defined by the inner product

$$[v, w]_\mathcal{D} = \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \sum_{s\in\mathcal{V}_\sigma} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2} (v_\sigma^s - v_K)(w_\sigma^s - w_K), \tag{8}$$

and the associated norm

$$\|v\|_\mathcal{D} = \left([v, v]_\mathcal{D}\right)^{1/2},$$

for all $(v, w) \in (\mathcal{H}_\mathcal{D})^2$.

The projection operator $P_\mathcal{T}$ from $\mathcal{H}_\mathcal{D}$ to $H_\mathcal{T}(\Omega)$ is defined for all $u \in \mathcal{H}_\mathcal{D}$ by $(P_\mathcal{T} u)_K = u_K$ for all $K \in \mathcal{T}$. Note that, from definition (6) of $\gamma_\sigma u$, we have

$$\frac{(\gamma_\sigma u - u_K)^2}{d_{K,\sigma}} + \frac{(\gamma_\sigma u - u_L)^2}{d_{L,\sigma}} = \min_{u_\sigma^s \in \mathbb{R}} \left(\frac{(u_\sigma^s - u_K)^2}{d_{K,\sigma}} + \frac{(u_\sigma^s - u_L)^2}{d_{L,\sigma}} \right),$$

for all $\sigma \in \mathcal{E}_{\text{int}}$, $\mathcal{T}_\sigma = \{K, L\}$. Since from (5) we have $\frac{\mathrm{m}_\sigma^s}{\mathrm{d}_{K,\sigma}} \le d\, \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2}$ for all $s \in \mathcal{V}_\sigma$, $\sigma \in \mathcal{E}_K$, $K \in \mathcal{T}$, it implies that

$$\|P_\mathcal{T} u\|_\mathcal{T} \le \sqrt{d}\, \|u\|_\mathcal{D}, \text{ for all } u \in \mathcal{H}_\mathcal{D}. \tag{9}$$

Denoting by $C_0(\overline{\Omega})$ the set of continuous functions which vanish on $\partial\Omega$, we define the interpolation operator $P_\mathcal{D} : C_0(\overline{\Omega}) \to \mathcal{H}_\mathcal{D}$ by $(P_\mathcal{D}\varphi)_K = \varphi(x_K)$, $K \in \mathcal{T}$, and $(P_\mathcal{D}\varphi)_\sigma^s = \varphi(x_\sigma^s)$, $s \in \mathcal{V}_\sigma$, $\sigma \in \mathcal{E}$, for all $\varphi \in C_0(\overline{\Omega})$.

Let us now recall the following lemma:

Lemma 2.4 (Discrete Sobolev Inequality) Let Ω be an open bounded subset of $\mathbb{R}^d$, with $d \in \mathbb{N}^*$, and $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2.1. Then, there exists a constant $C_{\text{sob}} > 0$, depending only on d, Ω, RegulCell($\mathcal{D}$), and RegulKL($\mathcal{D}$) such that for all $q \in [2, +\infty)$, if $d = 2$, and $q \in [2, 2d/(d-2)]$ if $d > 2$, we have

$$\|u\|_{L^q(\Omega)} \le q\, C_{\text{sob}}\, \|u\|_\mathcal{T}, \tag{10}$$

for any $u \in H_\mathcal{T}(\Omega)$.

Proof The proof is given in [10]. □

3 The Finite Volume Scheme

The definition of the finite volume scheme is based on a hybrid variational formulation on the space $\mathcal{H}_\mathcal{D}$ using the construction of two discrete gradients for each cell K of the mesh and each vertex s of the cell. The first gradient defined by

$$(\widetilde{\nabla}_\mathcal{D} u)_K^s \quad = \quad \frac{1}{\mathrm{m}_K^s} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} \mathrm{m}_\sigma^s (u_\sigma^s - u_K) \mathbf{n}_{K,\sigma}, \tag{11}$$

is built to have a weak convergence property stated in Lemma 5.6, once averaged for each cell K over its vertices $s \in \mathcal{V}_K$ with the weights m_K^s. The second gradient is defined by

$$(\overline{\nabla}_\mathcal{D} u)_K^s \quad = \quad \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} (u_\sigma^s - u_K)\; g_{K,\sigma}^s, \tag{12}$$

where the vectors $g_{K,\sigma}^s \in \mathbb{R}^d$ are given for all $\sigma \in \mathcal{E}_K \cap \mathcal{E}_s$. The gradient $(\overline{\nabla}_\mathcal{D} u)_K^s$ is built to be consistent in the sense that it is exact for linear functions. More precisely, the vectors $g_{K,\sigma}^s$, $\sigma \in \mathcal{E}_K \cap \mathcal{E}_s$ are assumed to satisfy the following hypothesis:

Hypothesis 1 [consistency of the gradient] For all $K \in \mathcal{T}$, $s \in \mathcal{V}_K$, the vectors $g_{K,\sigma}^s$, $\sigma \in \mathcal{E}_K \cap \mathcal{E}_s$ are such that for all vectors $v \in \mathbb{R}^d$ we have

$$\sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} v \cdot (x_\sigma^s - x_K)\; g_{K,\sigma}^s = v. \tag{13}$$

Let us now define the bilinear form $a_\mathcal{D}$ on $\mathcal{H}_\mathcal{D} \times \mathcal{H}_\mathcal{D}$ by

$$\begin{aligned} a_\mathcal{D}(u,v) = \sum_{K \in \mathcal{T}} \sum_{s \in \mathcal{V}_K} \Big(& \mathrm{m}_K^s (\overline{\nabla}_\mathcal{D} u)_K^s \cdot \Lambda_K (\widetilde{\nabla}_\mathcal{D} v)_K^s \\ + \;& \alpha_K^s \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2} R_{K,\sigma}^s(u) R_{K,\sigma}^s(v) \Big) \end{aligned} \tag{14}$$

for all $(u,v) \in \mathcal{H}_\mathcal{D} \times \mathcal{H}_\mathcal{D}$, with

$$\Lambda_K = \frac{1}{\mathrm{m}_K} \int_K \Lambda(x) dx,$$

for all $K \in \mathcal{T}$. In (14), the residual functions $R_{K,\sigma}^s$ are defined for all $u \in \mathcal{H}_\mathcal{D}$, $\sigma \in \mathcal{E}_K \cap \mathcal{E}_s$, $s \in \mathcal{V}_K$, $K \in \mathcal{T}$, by

$$R_{K,\sigma}^s(u) = u_\sigma^s - u_K - (\overline{\nabla}_\mathcal{D} u)_K^s \cdot (x_\sigma^s - x_K), \tag{15}$$

and the parameters α_K^s are real such that

$$\mu_0 \le \alpha_K^s \le \gamma_0 \tag{16}$$

for all $s \in \mathcal{V}_K$, $K \in \mathcal{T}$ with $\mu_0 > 0$ and $\gamma_0 > 0$. Note that instead of the scalar parameter α_K^s, we could have considered a more general positive definite matrix D_K^s

of size $\text{Card}(\mathcal{E}_K \cap \mathcal{E}_s)$ such that $\mu_0 \; I \leq D_K^s \leq \gamma_0 \; I$. The subsequent analysis will readily extends to this more general framework but we keep to the scalar term for the sake of simplicity in the notations.

The discretization of (1) on $\mathcal{D}$ is defined by the following discrete hybrid variational formulation: find $u_{\mathcal{D}} \in \mathcal{H}_{\mathcal{D}}$ such that

$$a_{\mathcal{D}}(u_{\mathcal{D}}, v) = \int_{\Omega} f(x) P_{\mathcal{T}} v(x) dx \quad \text{for all } v \in \mathcal{H}_{\mathcal{D}}. \tag{17}$$

For all $u \in \mathcal{H}_{\mathcal{D}}$, let us introduce the following subfluxes $F_{K,\sigma}^s(u)$ defined for all $s \in \mathcal{V}_\sigma$, $\sigma \in \mathcal{E}_K$, $K \in \mathcal{T}$ by

$$\begin{aligned} F_{K,\sigma}^s(u) \;&= -\mathrm{m}_\sigma^s \Lambda_K (\overline{\nabla}_{\mathcal{D}} u)_K^s \cdot \mathbf{n}_{K,\sigma} \\ &- \alpha_K^s \; \mathrm{m}_K^s \left(\frac{R_{K,\sigma}^s(u)}{(d_{K,\sigma})^2} - g_{K,\sigma}^s \cdot \sum_{\sigma' \in \mathcal{E}_K \cap \mathcal{E}_s} \frac{R_{K,\sigma'}^s(u)}{(d_{K,\sigma'})^2} (x_{\sigma'}^s - x_K) \right), \end{aligned} \tag{18}$$

in such a way that

$$a_{\mathcal{D}}(u, v) = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \sum_{s \in \mathcal{V}_\sigma} F_{K,\sigma}^s(u)(v_K - v_\sigma^s), \tag{19}$$

for all $(u, v) \in \mathcal{H}_{\mathcal{D}} \times \mathcal{H}_{\mathcal{D}}$. Then, it is easily shown from (19) that the variational formulation (17) is equivalent to the following hybrid finite volume scheme: find $u_{\mathcal{D}} \in \mathcal{H}_{\mathcal{D}}$ such that

$$\left\{ \begin{aligned} \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u_{\mathcal{D}}) &= \int_K f(x) dx && \text{for all } K \in \mathcal{T}, \\ F_{K,\sigma}(u_{\mathcal{D}}) &= \sum_{s \in \mathcal{V}_\sigma} F_{K,\sigma}^s(u_{\mathcal{D}}) && \text{for all } \sigma \in \mathcal{E}_K, K \in \mathcal{T}, \\ F_{K,\sigma}^s(u_{\mathcal{D}}) &= -F_{L,\sigma}^s(u_{\mathcal{D}}) && \text{for all } s \in \mathcal{V}_\sigma, \sigma \in \mathcal{E}_{\text{int}}, \, \mathcal{T}_\sigma = \{K, L\}. \end{aligned} \right. \tag{20}$$

From definition (18) of the subfluxes $F_{K,\sigma}^s(u)$, we can compute the coefficients $(T_K^s)_{\sigma,\sigma'}$, $\sigma' \in \mathcal{E}_s \cap \mathcal{E}_K$ such that

$$F_{K,\sigma}^s(u) = \sum_{\sigma' \in \mathcal{E}_s \cap \mathcal{E}_K} (T_K^s)_{\sigma,\sigma'} (u_K - u_{\sigma'}^s), \tag{21}$$

for all $s \in \mathcal{V}_\sigma$, $\sigma \in \mathcal{E}_K$, $K \in \mathcal{T}$ and $u \in \mathcal{H}_{\mathcal{D}}$. It results that around each vertex $s \in \mathcal{V}$, the face unknowns $(u_\sigma^s)_{\sigma \in \mathcal{E}_s}$ can be eliminated in terms of the $(u_K)_{K \in \mathcal{T}_s}$ assuming the well-posedness of the linear system

$$\left\{ \begin{aligned} F_{K,\sigma}^s(u_{\mathcal{D}}) + F_{L,\sigma}^s(u_{\mathcal{D}}) &= 0 \quad \text{for all } \sigma \in \mathcal{E}_s \cap \mathcal{E}_{\text{int}} \text{ with } \mathcal{T}_\sigma = \{K, L\}, \\ u_\sigma^s &= 0 \quad \text{for all } \sigma \in \mathcal{E}_s \cap \mathcal{E}_{\text{ext}}. \end{aligned} \right. \tag{22}$$

Then, the hybrid finite volume scheme reduces to the cell centered finite volume scheme: find $u_{\mathcal{T}} \in H_{\mathcal{T}}(\Omega)$ such that for all $K \in \mathcal{T}$

$$\sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{int}}, \mathcal{T}_\sigma = \{K, L\}} F_{K,L}(u_{\mathcal{T}}) \; + \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}} F_\sigma(u_{\mathcal{T}}) = \int_K f(x) dx, \tag{23}$$

where the inner fluxes $F_{K,L}(u_{\mathcal{T}})$, $\mathcal{T}_\sigma = \{K, L\}$, $\sigma \in \mathcal{E}_{\rm int}$, and the boundary fluxes $F_\sigma(u_{\mathcal{T}})$, $\sigma \in \mathcal{E}_{\rm ext}$, are linear combinations of the cell unknowns $(u_{\mathcal{T}})_M$ with $M \in \bigcup_{s\in\mathcal{V}_\sigma} \mathcal{T}_s$.

The well-posedness of the hybrid finite volume scheme (20), of the local linear systems (22), and of the cell centered scheme (23) is shown in the next section to result from the coercivity of the bilinear form $a_{\mathcal{D}}$ which will hold assuming a local coercivity assumption as stated in Proposition 4.1.

3.1 Equivalence with the usual MPFA O scheme

The MPFA O scheme described in [1] and [8] is defined for polygonal and polyedral meshes such that for all cells K and all vertices s of K, the cardinal of $\mathcal{E}_K \cap \mathcal{E}_s$ denoted by q_K^s is equal to the space dimension d, and such that the set of vectors $(x_\sigma^s - x_K)_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s}$ spans $\mathbb{R}^d$. Note that in two dimensions $d = 2$, the first condition $q_K^s = d = 2$ is always true, but it is no longer the case in three dimensions for which q_K^s can be larger than $d = 3$.

For such meshes, the MPFA O scheme from [1] or [8] is precisely defined by the hybrid finite volume formulation (20) using subfluxes $F_{K,\sigma}^s(u)$ given by

$$-\mathrm{m}_\sigma^s \Lambda_K (\nabla_{\mathcal{D}}^{MPFA} u)_K^s \cdot \mathbf{n}_{K,\sigma}$$

where the gradient $(\nabla_{\mathcal{D}}^{MPFA} u)_K^s$ is the gradient of the unique linear function defined by its $d + 1$ values u_K at point x_K and u_σ^s at points x_σ^s, $\sigma \in \mathcal{E}_K \cap \mathcal{E}_s$.

In such cases, the equivalence between our hybrid finite volume scheme (20) and the MPFA O scheme defined in [1] and [8] readily results from the following lemma stating that $(\overline{\nabla}_{\mathcal{D}} u)_K^s = (\nabla_{\mathcal{D}}^{MPFA} u)_K^s$, and that $R_{K,\sigma}^s(u) = 0$ for all $u \in \mathcal{H}_{\mathcal{D}}$.

LEMMA 3.1 Let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2.1, and let $K \in \mathcal{T}$, $s \in \mathcal{V}_K$ be such that $q_K^s = d$ and such that the set of d vectors $(x_\sigma^s - x_K)_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s}$ spans $\mathbb{R}^d$. Let us consider a discrete gradient $(\overline{\nabla}_{\mathcal{D}} u)_K^s$ given by (12) and satisfying the consistency hypothesis 1. Then, for all $u \in \mathcal{H}_{\mathcal{D}}$, the discrete gradient $(\overline{\nabla}_{\mathcal{D}} u)_K^s$ is the gradient of the unique linear function defined by its $d + 1$ values u_K at point x_K and u_σ^s at points x_σ^s, $\sigma \in \mathcal{E}_K \cap \mathcal{E}_s$, and the residuals $R_{K,\sigma}^s(u)$ vanish for all $\sigma \in \mathcal{E}_K \cap \mathcal{E}_s$.

Proof Let us denote by $(\bar{g}_{K,\sigma}^s)_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s}$ the biorthogonal basis of the basis $(x_\sigma^s - x_K)_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s}$ of $\mathbb{R}^d$. It is uniquely defined by the equations $\bar{g}_{K,\sigma}^s \cdot (x_{\sigma'}^s - x_K) = \delta_{\sigma,\sigma'}$ for all $\sigma, \sigma' \in \mathcal{E}_K \cap \mathcal{E}_s$. Setting $v = \bar{g}_{K,\sigma}^s$ in (13) shows that $g_{K,\sigma}^s = \bar{g}_{K,\sigma}^s$ for all $\sigma \in \mathcal{E}_K \cap \mathcal{E}_s$ and the gradient $\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} (u_\sigma^s - u_K)\bar{g}_{K,\sigma}^s$ is the unique gradient satisfying the consistency hypothesis 1. Let $u \in \mathcal{H}_{\mathcal{D}}$ be given and let φ be the unique linear function defined by its $d+1$ values u_K at point x_K and u_σ^s at points x_σ^s, $\sigma \in \mathcal{E}_K \cap \mathcal{E}_s$. We have by definition $\nabla\varphi \cdot (x_\sigma^s - x_K) = u_\sigma^s - u_K$. Hence setting $v = \nabla\varphi$ in (13) it results that

$$\nabla\varphi = \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \nabla\varphi \cdot (x_\sigma^s - x_K)\bar{g}_{K,\sigma}^s = \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} (u_\sigma^s - u_K)\bar{g}_{K,\sigma}^s,$$

which proves the first part of the lemma. The second part results from the equation

$$\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} R^s_{K,\sigma}(u)\ \bar{g}^s_{K,\sigma} = 0,$$

for all $u \in \mathcal{H}_\mathcal{D}$. □

For cells such that $q^s_K > d$, they are several ways to define a gradient $(\overline{\nabla}_\mathcal{D} u)^s_K = (u^s_\sigma - u_K)\ g^s_{K,\sigma}$ satisfying the consistency hypothesis 1. In such cases, the residuals $R^s_{K,\sigma}(u)$, $\sigma \in \mathcal{E}_K \cap \mathcal{E}_s$ satisfying the relation $\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} R^s_{K,\sigma}(u)\ g^s_{K,\sigma} = 0$ for all $u \in \mathcal{H}_\mathcal{D}$ do not a priori vanish since the family $g^s_{K,\sigma}$, $\sigma \in \mathcal{E}_K \cap \mathcal{E}_s$ is not free. They play the role of stabilization terms in the hybrid variational formulation (20) as shown in the following example. For $d = 3$, let us consider two pyramids K and L sharing a triangular face σ, and let $s \in \sigma$ denote the top of the two pyramids. We can easily build two consistent gradients $(\overline{\nabla}_\mathcal{D} u)^s_K$ and $(\overline{\nabla}_\mathcal{D} u)^s_L$ such that $g^s_{K,\sigma} = g^s_{L,\sigma} = 0$. Then, the residuals $R^s_{K,\sigma'}(u)$ and $R^s_{K,\sigma''}(u)$ vanish except for $\sigma' = \sigma'' = \sigma$. In this example, it is clear that only the residual terms in (20) can ensure the well-posedness of the system since the discrete gradients (12) do not depend on u^s_σ.

4 Well-posedness of the finite volume scheme

The well-posedness of the hybrid finite volume scheme (20) and the cell centered finite volume scheme (23) will be derived from the coercivity of the bilinear form $a_\mathcal{D}$. This coercivity property depends on the finite volume discretization $\mathcal{D}$, on the diffusion tensor Λ, and on the parameters of the finite volume scheme. In the following, we shall make the stronger assumption that the coercivity holds locally around each vertex s of the mesh. For a given discretization and diffusion tensor, this assumption can easily be checked numerically computing the eigenvalues of a small linear system of size $2 \times \mathrm{Card}(\mathcal{T}_s)$ for each vertex $s \in \mathcal{V}$.

In practical numerical experiments, for a proper choice of the consistent gradient (see section 6), the singularity of the linear system has never been observed for polygonal and polyhedral meshes. Nevertheless, as exhibited in subsection 7.3, negative eigenvalues of the bilinear form can occur breaking the coercivity of the bilinear form and the stability of the scheme.

Let s be a given vertex in $\mathcal{V}$, and let $\mathcal{H}^s_\mathcal{D}$ be the subspace of

$$\{u^s_\sigma \in \mathbb{R}, u_K \in \mathbb{R},\ K \in \mathcal{T}_s, \sigma \in \mathcal{E}_K \cap \mathcal{E}_s\}$$

such that $u^s_\sigma = 0$ for all $\sigma \in \mathcal{E}_{\text{ext}}$. The space $\mathcal{H}_{\mathcal{D}^s}$ is endowed with the semi-norm

$$\|u\|_{\mathcal{D}^s} = \left(\sum_{K\in\mathcal{T}_s} \sum_{\sigma\in\mathcal{E}_s\cap\mathcal{E}_K} \frac{\mathrm{m}^s_K}{(d_{K,\sigma})^2} (u^s_\sigma - u_K)^2 \right)^{\frac{1}{2}}. \tag{24}$$

Let us also denote by $a_{\mathcal{D}^s}$ the bilinear form defined by

$$a_{\mathcal{D}^s}(u,v) = \sum_{K\in\mathcal{T}_s} \Big(\mathrm{m}_K^s (\overline{\nabla}_{\mathcal{D}} u)_K^s \cdot \Lambda_K (\widetilde{\nabla}_{\mathcal{D}} v)_K^s + \alpha_K^s \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2} R_{K,\sigma}^s(u) R_{K,\sigma}^s(v) \Big), \tag{25}$$

for all $u,v \in \mathcal{H}_{\mathcal{D}^s}$, where we have used the canonical injection from $\mathcal{H}_{\mathcal{D}^s}$ to $\mathcal{H}_{\mathcal{D}}$ to define the residual and the gradient functions on $\mathcal{H}_{\mathcal{D}^s}$.

Let us now introduce the following local coercivity criterion

$$\mathrm{coernode}(\mathcal{D},\Lambda) = \min_{s\in\mathcal{V}} \inf_{\{u\in\mathcal{H}_{\mathcal{D}^s} \,|\, \|u\|_{\mathcal{D}^s}=1\}} a_{\mathcal{D}^s}(u,u). \tag{26}$$

It will be used to check the coercivity of the bilinear form $a_{\mathcal{D}}$ as stated in the following proposition:

PROPOSITION 4.1 Let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2.1, and let us assume that there exists $\theta_{\mathcal{D}} > 0$ such that $\mathrm{coernode}(\mathcal{D},\Lambda) \geq \theta_{\mathcal{D}}$. Then, the bilinear form $a_{\mathcal{D}}$ is coercive in the sense that for all $u \in \mathcal{H}_{\mathcal{D}}$ we have

$$a_{\mathcal{D}}(u,u) \geq \theta_{\mathcal{D}} \|u\|_{\mathcal{D}}^2. \tag{27}$$

Proof From the definition (14) of the bilinear form, we have for any $u \in \mathcal{H}_{\mathcal{D}}$ that

$$a_{\mathcal{D}}(u,u) = \sum_{K\in\mathcal{T}} \sum_{s\in\mathcal{V}_K} \Big(\mathrm{m}_K^s (\overline{\nabla}_{\mathcal{D}} u)_K^s \cdot \Lambda_K (\widetilde{\nabla}_{\mathcal{D}} u)_K^s + \alpha_K^s \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2} R_{K,\sigma}^s(u) R_{K,\sigma}^s(u) \Big).$$

Permuting the first two sums leads to

$$a_{\mathcal{D}}(u,u) = \sum_{s\in\mathcal{V}} \sum_{K\in\mathcal{T}_s} \Big(\mathrm{m}_K^s (\overline{\nabla}_{\mathcal{D}} u)_K^s \cdot \Lambda_K (\widetilde{\nabla}_{\mathcal{D}} u)_K^s + \alpha_K^s \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2} R_{K,\sigma}^s(u) R_{K,\sigma}^s(u) \Big).$$

It results from (25) that

$$a_{\mathcal{D}}(u,u) = \sum_{s\in\mathcal{V}} a_{\mathcal{D}_s}(u,u). \tag{28}$$

Similarly, one has from (24) and (8) that

$$\|u\|_{\mathcal{D}}^2 = \sum_{s\in\mathcal{V}} \|u\|_{\mathcal{D}_s}^2. \tag{29}$$

Let $s \in \mathcal{V}$ and let us assume that $\|u\|_{D^s} = 0$. From the definition (24) of the semi norm it implies that $u_K = u_\sigma^s$ for all $K \in \mathcal{T}_s$, $\sigma \in \mathcal{E}_s \cap \mathcal{E}_K$. Then, from the definitions of the discrete gradients (11) and (12), it results that for all $K \in \mathcal{T}_s$ one has $(\widetilde{\nabla}_{\mathcal{D}} u)^s_K = 0$, and $(\overline{\nabla}_{\mathcal{D}} u)^s_K = 0$. Also, from the definition of the residuals (15), we deduce that for all $K \in \mathcal{T}_s$, $\sigma \in \mathcal{E}_s \cap \mathcal{E}_K$ one has $R^s_{K,\sigma}(u) = 0$. Therefore, for any $s \in \mathcal{V}$ we have shown that

$$\|u\|_{D^s} = 0 \text{ implies } a_{\mathcal{D}_s}(u,u) = 0.$$

For any $s \in \mathcal{V}$, if $\|u\|_{D^s} \neq 0$, one has by the assumption on coernode$(\mathcal{D},\Lambda)$ that

$$\begin{aligned} a_{\mathcal{D}_s}(u,u) &= a_{\mathcal{D}_s}\left(\frac{u}{\|u\|_{D^s}}, \frac{u}{\|u\|_{D^s}}\right) \|u\|^2_{D^s} \\ &\geq \theta_{\mathcal{D}} \, \|u\|^2_{D^s}. \end{aligned}$$

From the previous remark, the same inequality still holds for $\|u\|_{D^s} = 0$ and hence for any u which together with (28) and (29) conclude the proof of the lemma. □
The following propositions state the well-posedness of the hybrid and cell centered finite volume schemes under the local coercivity assumption.

Proposition 4.2 [Estimate on the solution of the hybrid finite volume scheme] Let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2.1, and let us assume that there exists a real $\theta_{\mathcal{D}} > 0$ such that coernode$(\mathcal{D},\Lambda) \geq \theta_{\mathcal{D}}$. Then, there exists a unique solution $u_{\mathcal{D}} \in \mathcal{H}_{\mathcal{D}}$ of the hybrid finite volume scheme (20) which satisfies the estimate

$$\|u_{\mathcal{D}}\|_{\mathcal{D}} \leq 2 \, \frac{C_{\text{sob}}}{\theta_{\mathcal{D}}} \|f\|_{L^2(\Omega)}, \tag{30}$$

where the constant C_{sob} is given by Lemma 2.4.

Proof Thanks to Proposition 4.1, for any solution $u \in \mathcal{H}_{\mathcal{D}}$ of (20), we have

$$\theta_{\mathcal{D}} \|u\|^2_{\mathcal{D}} \leq a_{\mathcal{D}}(u,u) = \int_\Omega f(x) P_{\mathcal{T}} u(x) dx. \tag{31}$$

On the other hand, using (10) and (9), we have for all $u \in \mathcal{H}_{\mathcal{D}}$

$$\begin{aligned} \int_\Omega f(x) P_{\mathcal{T}} u(x) dx &\leq \|f\|_{L^2(\Omega)} \|P_{\mathcal{T}} u\|_{L^2(\Omega)} &(32)\\ &\leq 2 \, \|f\|_{L^2(\Omega)} \, C_{\text{sob}} \|u\|_{\mathcal{D}}, &(33) \end{aligned}$$

which proves the bound (30) for any solution $u_{\mathcal{D}} \in \mathcal{H}_{\mathcal{D}}$ of (20). Since (20) is a square linear system, it also proves the uniqueness and existence of the solution of (20). □

Corollary 4.3 [Estimate on the solution of the cell centered finite volume scheme] Let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2.1, and let us assume that there exists a real $\theta_{\mathcal{D}} > 0$ such that coernode$(\mathcal{D},\Lambda) \geq \theta_{\mathcal{D}}$. Then, for each vertex $s \in \mathcal{V}$, the linear system (22) is non-singular, and there exists a unique solution $u_{\mathcal{T}}$ to the cell centered finite volume scheme (23) equal to $P_{\mathcal{T}}(u_{\mathcal{D}})$ where $u_{\mathcal{D}}$ is the unique solution of the hybrid finite volume scheme (20). Moreover, the solution $u_{\mathcal{T}}$ satisfies the bound

$$\|u_{\mathcal{T}}\|_{\mathcal{T}} \leq 2\sqrt{d} \, \frac{C_{\text{sob}}}{\theta_{\mathcal{D}}} \|f\|_{L^2(\Omega)}. \tag{34}$$

Proof Let $s \in \mathcal{V}$, and let V^s, be the subspace of $\mathcal{H}_\mathcal{D}$ such that

$$V^s = \{v \in \mathcal{H}_\mathcal{D} \mid P_\mathcal{T} v = 0,\ v_\sigma^{s'} = 0 \text{ for all } s' \neq s, \sigma \in \mathcal{E}_{s'}\},$$

and let P_{V^s}, be the canonical orthogonal projector onto V^s. Let us also identify $H_\mathcal{T}(\Omega)$ with the subspace $\{v \in \mathcal{H}_\mathcal{D} \mid v_\sigma^{s'} = 0 \text{ for all } s' \in \mathcal{V}, \sigma \in \mathcal{E}_{s'}\}$ of $\mathcal{H}_\mathcal{D}$. Then, we have for all $v^s \in V^s$, $u \in \mathcal{H}_\mathcal{D}$

$$a_\mathcal{D}(u, v^s) = a_\mathcal{D}(P_{V^s}(u) + P_\mathcal{T} u, v^s) = \sum_{\sigma \in \mathcal{E}_s \cap \mathcal{E}_{\rm int},\ \mathcal{T}_\sigma = \{K,L\}} - \left(F_{K,\sigma}^s(u) + F_{L,\sigma}^s(u)\right)\ v_\sigma^s,$$

and the linear system (22) is equivalent to: given $P_\mathcal{T} u_\mathcal{D}$, find $u^s \in V^s$, such that $a_\mathcal{D}(u^s + P_\mathcal{T} u_\mathcal{D}, v^s) = 0$ for all $v^s \in V^s$. The non-singularity of this system results from the coercivity of the bilinear form $a_\mathcal{D}$. Hence, from Proposition 4.2, there exists a unique solution $u_\mathcal{T}$ to the cell centered finite volume scheme and this solution verifies $u_\mathcal{T} = P_\mathcal{T}(u_\mathcal{D})$ where $u_\mathcal{D}$ is the unique solution of the hybrid finite volume scheme (20). From (9) the solution $u_\mathcal{T}$ satisfies the estimate (34). □

5 Convergence Analysis

Let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2.1. It is always assumed in the following that the local coercivity assumption $\text{coernode}(\mathcal{D}, \Lambda) \geq \theta_\mathcal{D} > 0$. is satisfied, which ensures that there exists a unique solution $u_\mathcal{D} \in \mathcal{H}_\mathcal{D}$ to (20).

Let us introduce the following notation for a given finite volume discretization and a given construction of the gradients $(\overline{\nabla}_\mathcal{D} u)_K^s$:

$$\text{RegulGrad}(\mathcal{D}) = \max_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s, K \in \mathcal{T}_s, s \in \mathcal{V}} |g_{K,\sigma}^s| \text{diam}(K). \tag{35}$$

The proof of convergence uses piecewise constant reconstructions of the discrete gradients (11) and (12) defined as follows. For all $K \in \mathcal{T}$, let us choose arbitrarily a family $(K_s)_{s \in \mathcal{V}_K}$ of non empty connected open disjoint subsets of K such that the volume of K_s is equal to m_K^s and $\overline{K} = \cup_{s \in \mathcal{V}_K} \overline{K_s}$.

For all $u \in \mathcal{H}_\mathcal{D}$, let us denote by $\widetilde{\nabla}_\mathcal{D} u \in L^2(\Omega)^d$ the function

$$\widetilde{\nabla}_\mathcal{D} u(x) = (\widetilde{\nabla}_\mathcal{D} u)_K^s, \quad \text{for a.e. } x \in K_s, \tag{36}$$

and by $\overline{\nabla}_\mathcal{D} u \in L^2(\Omega)^d$ the function

$$\overline{\nabla}_\mathcal{D} u(x) = (\overline{\nabla}_\mathcal{D} u)_K^s, \quad \text{for a.e. } x \in K_s. \tag{37}$$

We shall also use an averaging of the discrete gradients over each cell $K \in \mathcal{T}$. For all $u \in \mathcal{H}_\mathcal{D}$, let $\nabla_\mathcal{D} u \in L^2(\Omega)^d$ be the function defined for a.e. $x \in K$ by

$$(\nabla_\mathcal{D} u)_K = \frac{1}{\text{m}_K} \sum_{s \in \mathcal{V}_K} \text{m}_K^s (\widetilde{\nabla}_\mathcal{D} u)_K^s = \frac{1}{\text{m}_K} \sum_{\sigma \in \mathcal{E}_K} \sum_{s \in \mathcal{V}_\sigma} \text{m}_\sigma^s (u_\sigma^s - u_K) \mathbf{n}_{K,\sigma}. \tag{38}$$

This gradient is shown in [12] to satisfy a weak convergence property as stated below in Lemma 5.6. Similarly, let $\widehat{\nabla}_{\mathcal{D}} u \in L^2(\Omega)^d$ be the function defined by

$$\widehat{\nabla}_{\mathcal{D}} u(x) = (\widehat{\nabla}_{\mathcal{D}} u)_K, \quad \text{for a.e. } x \in K, \tag{39}$$

with

$$(\widehat{\nabla}_{\mathcal{D}} u)_K = \frac{1}{\mathrm{m}_K} \sum_{s \in \mathcal{V}_K} \mathrm{m}_K^s (\overline{\nabla}_{\mathcal{D}} u)_K^s. \tag{40}$$

In the subsequent of this section, we shall prove the following theorem.

THEOREM 5.1 [Convergence of the scheme] Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, with $d \in \mathbb{N}^*$. Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of admissible discretizations in the sense of Definition 2.1, such that $h_{\mathcal{D}_n} \to 0$ as $n \to \infty$. It is assumed that hypothesis 1 holds and that there exist $\theta > 0$, $\gamma \geq 0$, $\beta > 0$, $\eta > 0$, and $M \in \mathbb{N}$ with $\mathrm{coernode}(\mathcal{D}_n, \Lambda) \geq \theta$, $\mathrm{RegulGrad}(\mathcal{D}_n) \leq \gamma$, $\mathrm{CardFace}(\mathcal{D}_n) \leq M$, $\mathrm{RegulKL}(\mathcal{D}_n) \geq \eta$, and $\mathrm{RegulCell}(\mathcal{D}_n) \geq \beta$ for all $n \in \mathbb{N}$. Then, there exists for all $n \in \mathbb{N}$ a unique solution $u_{\mathcal{D}_n} \in \mathcal{H}_{\mathcal{D}_n}$ to (20), and the sequence $P_{\mathcal{T}} u_{\mathcal{D}_n}$, $n \in \mathbb{N}$ converges to the weak solution $\bar{u}$ of (1) in $L^q(\Omega)$, for all $q \in [1, +\infty)$ if $d = 2$ and all $q \in [1, 2d/(d-2))$ if $d > 2$. Moreover, the sequence $\overline{\nabla}_{\mathcal{D}_n} u_{\mathcal{D}_n}$, $n \in \mathbb{N}$ converges to $\nabla \bar{u}$ in $L^2(\Omega)^d$.

The proof of this theorem involves a series of lemmae listed in the following sketch of the proof.

- A uniform stability estimate in $\mathcal{H}_{\mathcal{D}_n}$ of the discrete solutions $u_{\mathcal{D}_n}$, $n \in \mathbb{N}$ is readily obtained by Proposition 4.2 stated in the previous section and from the assumption that there exist $\theta > 0$ such that $\mathrm{coernode}(\mathcal{D}_n, \Lambda) \geq \theta$ for all $n \in \mathbb{N}$.

- Stability estimates of the gradient and residual functions will be derived in Lemmae 5.2 and 5.3.

- The consistency of the discrete gradients $(\overline{\nabla}_{\mathcal{D}}(P_{\mathcal{D}}\varphi))_K^s$, and of the residual functions $R_{K,\sigma}^s(P_{\mathcal{D}}\varphi)$ for smooth compactly supported functions φ is derived respectively in Lemmae 5.4 and 5.5.

- Using the stability estimate of $u_{\mathcal{D}_n}$ in $\mathcal{H}_{\mathcal{D}_n}$ we can apply the Discrete Rellich Theorem already proved in [12] and recalled in Lemma 5.6. It results that there exist a function $\widetilde{u} \in H_0^1(\Omega)$ and a subsequence of $n \in \mathbb{N}$, still denoted by $n \in \mathbb{N}$ for simplicity, such that $P_{\mathcal{T}} u_{\mathcal{D}_n}$, $n \in \mathbb{N}$ converges to $\widetilde{u} \in H_0^1(\Omega)$ in $L^q(\Omega)$ for all $q \in [1, +\infty)$ if $d = 2$ and all $q \in [1, 2d/(d-2))$ if $d > 2$, and such that the gradient $\nabla_{\mathcal{D}} u_n$, $n \in \mathbb{N}$ weakly converges to $\nabla \widetilde{u}$ in $L^2(\Omega)^d$.

- The core of the proof is derived in Lemma 5.7 which proves the convergence in $L^2(\Omega)^d$ up to a subsequence of the gradient functions $\overline{\nabla}_{\mathcal{D}_n} u_{\mathcal{D}_n}$ and $\widehat{\nabla}_{\mathcal{D}_n} u_{\mathcal{D}_n}$, $n \in \mathbb{N}$ to $\nabla \widetilde{u}$. The proof uses the coercivity of $a_{\mathcal{D}}$, and Lemmae 5.4, 5.5, 5.2 and 5.3.

- To complete the proof of Theorem 5.1 it is then shown that $\widetilde{u}$ is the unique weak solution $\bar{u}$ of (1) by passing to the limit in the discrete hybrid variational formulation (17).

Lemma 5.2 [Estimate of the gradient functions] Let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2.1. Then, for all $u \in \mathcal{H}_\mathcal{D}$ we have the bounds

$$\|\widetilde{\nabla}_\mathcal{D} u\|_{L^2(\Omega)^d} \leq \sqrt{d}\ \|u\|_\mathcal{D}, \tag{41}$$

$$\|\nabla_\mathcal{D} u\|_{L^2(\Omega)^d} \leq \|\widetilde{\nabla}_\mathcal{D} u\|_{L^2(\Omega)^d}, \tag{42}$$

$$\|\overline{\nabla}_\mathcal{D} u\|_{L^2(\Omega)^d} \leq \mathrm{CardFace}(\mathcal{D})^{1/2}\ \mathrm{RegulGrad}(\mathcal{D})\ \|u\|_\mathcal{D}, \tag{43}$$

$$\|\widehat{\nabla}_\mathcal{D} u\|_{L^2(\Omega)^d} \leq \|\overline{\nabla}_\mathcal{D} u\|_{L^2(\Omega)^d}. \tag{44}$$

Proof The first bound is proved using definition (5) of m_K^s, Definition 2.3 of the norm in $\mathcal{H}_\mathcal{D}$ as well as the Cauchy Schwarz inequality. The third bound is derived using Cauchy Schwarz inequality and the definitions of RegulGrad($\mathcal{D}$) and CardFace($\mathcal{D}$) as follows:

$$\begin{aligned}
\|\overline{\nabla}_\mathcal{D} u\|^2_{L^2(\Omega)^d} &= \sum_{K\in\mathcal{T}}\sum_{s\in\mathcal{V}_K} \mathrm{m}_K^s \left(\sum_{\sigma\in\mathcal{E}_s\cap\mathcal{E}_K} (u_\sigma^s - u_K) g_{K,\sigma}^s\right)^2 \\
&\leq \sum_{K\in\mathcal{T}}\sum_{s\in\mathcal{V}_K} \left(\sum_{\sigma\in\mathcal{E}_s\cap\mathcal{E}_K} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2}(u_\sigma^s - u_K)^2\right)\left(\sum_{\sigma\in\mathcal{E}_s\cap\mathcal{E}_K} |g_{K,\sigma}^s|^2 (d_{K,\sigma})^2\right) \\
&\leq \mathrm{CardFace}(\mathcal{D})\ \mathrm{RegulGrad}(\mathcal{D})^2\ \|u\|^2_\mathcal{D}.
\end{aligned}$$

The two remaining bounds readily derive from the above definitions of the gradient functions and the convexity of the function $x \to x^2$. □

Lemma 5.3 [Estimate of the residual function] Let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2.1. Then, there exists a real $C > 0$ which depends only on RegulCell($\mathcal{D}$), CardFace($\mathcal{D}$), RegulGrad($\mathcal{D}$) such that for all $u \in \mathcal{H}_\mathcal{D}$ we have the estimate

$$\sum_{K\in\mathcal{T}}\sum_{s\in\mathcal{V}_K}\left(\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2} R_{K,\sigma}^s(u)^2\right) \leq C\|u\|^2_\mathcal{D}.$$

Proof Using the estimate $(a-b)^2 \leq 2(a^2+b^2)$ for all $(a,b) \in \mathbb{R}\times\mathbb{R}$ in the definition of the residual $R_{K,\sigma}^s(u) = (u_\sigma^s - u_K) - (\overline{\nabla}_\mathcal{D} u)_K^s \cdot (x_\sigma^s - x_K)$, we obtain the bound

$$\begin{aligned}
\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2} R_{K,\sigma}^s(u)^2 \leq\ & 2\ (\overline{\nabla}_\mathcal{D} u)_K^s \cdot A_K^s (\overline{\nabla}_\mathcal{D} u)_K^s \\
&+ 2 \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2}(u_\sigma^s - u_K)^2,
\end{aligned} \tag{45}$$

where the square matrix A_K^s is defined by

$$A_K^s = \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2}(x_\sigma^s - x_K)(x_\sigma^s - x_K)^t, \tag{46}$$

and satisfies the bound

$$|A_K^s| \le \mathrm{m}_K^s \frac{\mathrm{CardFace}(\mathcal{D})}{\mathrm{RegulCell}(\mathcal{D})^2}. \tag{47}$$

Using the bound

$$\mathrm{m}_K^s |(\overline{\nabla}_{\mathcal{D}} u)_K^s|^2 \le \mathrm{CardFace}(\mathcal{D})\ \mathrm{RegulGrad}(\mathcal{D})^2 \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2}(u_\sigma^s - u_K)^2,$$

combined with (45), and (47), we obtain the following estimate

$$\begin{aligned}&\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2} R_{K,\sigma}^s(u)^2 \\ &\le 2\left(1+\left(\frac{\mathrm{CardFace}(\mathcal{D})\ \mathrm{RegulGrad}(\mathcal{D})}{\mathrm{RegulCell}(\mathcal{D})}\right)^2\right) \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2}(u_\sigma^s - u_K)^2,\end{aligned}$$

which completes the proof. □

Lemma 5.4 [Consistency of the discrete gradients] Let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2.1, and let us assume that hypothesis 1 holds. Let φ be a given function in $C_c^\infty(\Omega)$. Then, there exists M_φ depending only on φ, such that for all $s\in\mathcal{V}_K$, $K\in\mathcal{T}$,

$$|(\overline{\nabla}_{\mathcal{D}} P_{\mathcal{D}}\varphi)_K^s - \nabla\varphi(x_K)| \le M_\varphi\ \mathrm{CardFace}(\mathcal{D})\ \mathrm{RegulGrad}(\mathcal{D})\ \mathrm{diam}(K),$$

and

$$|(\widehat{\nabla}_{\mathcal{D}} P_{\mathcal{D}}\varphi)_K - \nabla\varphi(x_K)| \le M_\varphi\ \mathrm{CardFace}(\mathcal{D})\ \mathrm{RegulGrad}(\mathcal{D})\ \mathrm{diam}(K).$$

Proof Let $K\in\mathcal{T}$, $s\in\mathcal{V}_K$, $\varphi\in C_c^\infty(\Omega)$ be given. For all $\sigma\in\mathcal{E}_s\cap\mathcal{E}_K$, let us set $\epsilon_{K,\sigma}^s = \varphi(x_\sigma^s) - \varphi(x_K) - \nabla\varphi(x_K)\cdot(x_\sigma^s - x_K)$. Since $\varphi\in C_c^\infty(\Omega)$, there exists a real $M_\varphi > 0$ depending only on φ such that $|\epsilon_{K,\sigma}^s| \le M_\varphi |x_\sigma^s - x_K|^2$. From hypothesis 1, we have

$$(\overline{\nabla}_{\mathcal{D}} P_{\mathcal{D}}\varphi)_K^s - \nabla\varphi(x_K) = \sum_{\sigma\in\mathcal{E}_s\cap\mathcal{E}_K} \epsilon_{K,\sigma}^s\ g_{K,\sigma}^s,$$

which ends the proof from the definitions of CardFace($\mathcal{D}$) and RegulGrad($\mathcal{D}$). □

Lemma 5.5 [Consistency of the residual functions] Let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2.1, and let us assume that hypothesis 1 holds. Let φ be a given function in $C_c^\infty(\Omega)$. Then, there exists a real $C>0$ depending only on φ, RegulCell($\mathcal{D}$), RegulGrad($\mathcal{D}$), CardFace($\mathcal{D}$), and Ω, such that

$$\sum_{K\in\mathcal{T}} \sum_{s\in\mathcal{V}_K} \left(\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2} (R_{K,\sigma}^s(P_{\mathcal{D}}\varphi))^2 \right) \le C\ h_{\mathcal{D}}^2.$$

Proof Let $K \in \mathcal{T}$, $s \in \mathcal{V}_K$, $\sigma \in \mathcal{E}_s \cap \mathcal{E}_K$, $\varphi \in C_c^\infty(\Omega)$ be given. For all $\sigma \in \mathcal{E}_s \cap \mathcal{E}_K$, let us set $\epsilon^s_{K,\sigma} = \varphi(x^s_\sigma) - \varphi(x_K) - \nabla\varphi(x_K) \cdot (x^s_\sigma - x_K)$. Since $\varphi \in C_c^\infty(\Omega)$, there exists a real $M_\varphi > 0$ already introduced in Lemma 5.4 and depending only on φ such that $|\epsilon^s_{K,\sigma}| \leq M_\varphi |x^s_\sigma - x_K|^2$. From the definition of the residual function we have

$$R^s_{K,\sigma}(P_{\mathcal{D}}\varphi) = \epsilon^s_{K,\sigma} - \left((\overline{\nabla}_{\mathcal{D}} P_{\mathcal{D}}\varphi)^s_K - \nabla\varphi(x_K)\right) \cdot (x^s_\sigma - x_K).$$

We deduce from Lemma 5.4, and the definition of RegulCell($\mathcal{D}$) that

$$\frac{\mathrm{m}^s_K}{(d_{K,\sigma})^2}(R^s_{K,\sigma}(P_{\mathcal{D}}\varphi))^2 \leq \mathrm{m}^s_K M_\varphi^2 \left(\frac{1 + \mathrm{CardFace}(\mathcal{D})\ \mathrm{RegulGrad}(\mathcal{D})}{\mathrm{RegulCell}(\mathcal{D})}\right)^2 h^2_{\mathcal{D}},$$

from which we deduce the estimate

$$\sum_{K\in\mathcal{T}} \sum_{s\in\mathcal{V}_K} \left(\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \frac{\mathrm{m}^s_K}{(d_{K,\sigma})^2}(R^s_{K,\sigma}(P_{\mathcal{D}}\varphi))^2 \right) \leq \mathrm{m}(\Omega) M_\varphi^2 \left(\frac{1 + \mathrm{CardFace}(\mathcal{D})\ \mathrm{RegulGrad}(\mathcal{D})}{\mathrm{RegulCell}(\mathcal{D})}\right)^2 h^2_{\mathcal{D}},$$

which concludes the proof. □

LEMMA 5.6 [Discrete Rellich theorem] Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, with $d \in \mathbb{N}^*$. Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of admissible discretizations such that $h_{\mathcal{D}_n} \to 0$ as $n \to \infty$, and let $u_n \in \mathcal{H}_{\mathcal{D}_n}$ be such that there exists $C > 0$ with $\|u_n\|_{\mathcal{D}_n} \leq C$ for all $n \in \mathbb{N}$. Then, there exist a subsequence, still denoted by $n \in \mathbb{N}$ for simplicity, and a function $\widetilde{u} \in H^1_0(\Omega)$, such that $P_{\mathcal{T}} u_n$ converges in $L^q(\Omega)$ to $\widetilde{u}$ for all $q \in [1, \infty)$ if $d = 2$ else if $d > 2$, $q \in [1, 2d/(d-2)]$ and such that the gradient $\nabla_{\mathcal{D}} u_n$ weakly converges to $\nabla\widetilde{u}$ in $L^2(\Omega)^d$.

Proof The proof uses the same arguments as in [12]. □

LEMMA 5.7 [Strong convergence of the discrete gradients] Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, with $d \in \mathbb{N}^*$. Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of admissible discretizations in the sense of Definition 2.1, such that $h_{\mathcal{D}_n} \to 0$ as $n \to \infty$. It is assumed that hypothesis 1 holds and that there exist $\theta > 0$, $\gamma \geq 0$, $\beta > 0$, $\eta > 0$, and $M \in \mathbb{N}$ with coernode($\mathcal{D}_n, \Lambda$) $\geq \theta$, RegulGrad($\mathcal{D}_n$) $\leq \gamma$, CardFace($\mathcal{D}_n$) $\leq M$, RegulKL($\mathcal{D}_n$) $\geq \eta$, and RegulCell($\mathcal{D}_n$) $\geq \beta$ for all $n \in \mathbb{N}$. Then, there exist for all $n \in \mathbb{N}$ a unique solution $u_{\mathcal{D}_n} \in \mathcal{H}_{\mathcal{D}_n}$ to (20), and a function $\tilde{u} \in H^1_0(\Omega)$ such that $P_{\mathcal{T}} u_{\mathcal{D}_n}$ converges up to a subsequence to $\tilde{u}$ in $L^q(\Omega)$, for all $q \in [1, +\infty)$ if $d = 2$ and all $q \in [1, 2d/(d-2))$ if $d > 2$, as $h_{\mathcal{D}} \to 0$. Moreover, the gradients $\overline{\nabla}_{\mathcal{D}_n} u_{\mathcal{D}_n}$ and $\widehat{\nabla}_{\mathcal{D}_n} u_{\mathcal{D}_n}$ converge strongly up to a subsequence to $\nabla\tilde{u}$ in $L^2(\Omega)^d$.

Proof Thanks to Proposition 4.2 and Lemma 5.6, there exist a subsequence still denoted by $n \in \mathbb{N}$ for conveniency, and a function $\tilde{u} \in H^1_0(\Omega)$ such that $P_{\mathcal{T}} u_{\mathcal{D}_n} \to \tilde{u}$ in $L^q(\Omega)$, for all $q \in [1, +\infty)$ if $d = 2$ and all $q \in [1, 2d/(d-2))$ if $d > 2$, and such that $\nabla_{\mathcal{D}} u_{\mathcal{D}_n}$ converges weakly to $\nabla\tilde{u}$ in $L^2(\Omega)^d$ as $n \to \infty$. It remains to prove that the gradients $\overline{\nabla}_{\mathcal{D}_n} u_{\mathcal{D}_n}$ and $\widehat{\nabla}_{\mathcal{D}_n} u_{\mathcal{D}_n}$ converge strongly to $\nabla\tilde{u}$ in $L^2(\Omega)^d$. For the sake of simplicity in the notations, the subscript $n \in \mathbb{N}$, will be dropped in the remaining of the proof.

Let us first prove that $I_{\mathcal{D}} = \int_\Omega (\overline{\nabla}_{\mathcal{D}} u_{\mathcal{D}}(x) - \nabla \tilde{u}(x))^2 \, dx$ tends to zero as $h_{\mathcal{D}} \to 0$. Let φ be a given function in $C_c^\infty(\Omega)$ and let us bound $I_{\mathcal{D}}$ as follows

$$I_{\mathcal{D}} \le 3 \; (T_{\mathcal{D}}^1 + T_{\mathcal{D}}^2 + T_{\mathcal{D}}^3),$$

with

$$T_{\mathcal{D}}^1 = \int_\Omega (\overline{\nabla}_{\mathcal{D}}(u_{\mathcal{D}} - P_{\mathcal{D}}\varphi)(x))^2 \; dx,$$

$$T_{\mathcal{D}}^2 = \int_\Omega (\overline{\nabla}_{\mathcal{D}} P_{\mathcal{D}}\varphi(x) - \nabla\varphi(x))^2 \; dx,$$

and

$$T_{\mathcal{D}}^3 = \int_\Omega (\nabla\varphi(x) - \nabla\tilde{u}(x))^2 \; dx.$$

Using the coercivity of the bilinear form $a_{\mathcal{D}}$ and the stability of the gradient function $\overline{\nabla}_{\mathcal{D}} u$ stated in Proposition 4.1 and Lemma 5.2 respectively, the first term $T_{\mathcal{D}}^1$ satisfies the following upper bounds

$$\begin{aligned} T_{\mathcal{D}}^1 &\le \frac{\gamma^2 \; M}{\theta} \; a_{\mathcal{D}}(u_{\mathcal{D}} - P_{\mathcal{D}}\varphi, u_{\mathcal{D}} - P_{\mathcal{D}}\varphi) \\ &\le \frac{\gamma^2 \; M}{\theta} \; \Big(a_{\mathcal{D}}(u_{\mathcal{D}}, u_{\mathcal{D}}) - a_{\mathcal{D}}(u_{\mathcal{D}}, P_{\mathcal{D}}\varphi) \\ &\qquad - a_{\mathcal{D}}(P_{\mathcal{D}}\varphi, u_{\mathcal{D}}) + a_{\mathcal{D}}(P_{\mathcal{D}}\varphi, P_{\mathcal{D}}\varphi) \Big). \end{aligned} \tag{48}$$

As $u_{\mathcal{D}}$ is the solution of (17), we deduce that $a_{\mathcal{D}}(u_{\mathcal{D}}, u_{\mathcal{D}}) = \int_\Omega f(x) P_{\mathcal{T}} u_{\mathcal{D}}(x) dx$ and $a_{\mathcal{D}}(u_{\mathcal{D}}, P_{\mathcal{D}}\varphi) = \int_\Omega f(x) P_{\mathcal{T}}(P_{\mathcal{D}}\varphi)(x) dx$. It results that

$$\begin{aligned} \lim_{h_{\mathcal{D}} \to 0} a_{\mathcal{D}}(u_{\mathcal{D}}, u_{\mathcal{D}}) &= \int_\Omega f(x) \tilde{u}(x) dx, \\ \lim_{h_{\mathcal{D}} \to 0} a_{\mathcal{D}}(u_{\mathcal{D}}, P_{\mathcal{D}}\varphi) &= \int_\Omega f(x) \varphi(x) dx. \end{aligned} \tag{49}$$

Next, let us split the term $a_{\mathcal{D}}(P_{\mathcal{D}}\varphi, u_{\mathcal{D}})$ into the following three terms

$$a_{\mathcal{D}}(P_{\mathcal{D}}\varphi, u_{\mathcal{D}}) = L_{\mathcal{D}}^1 + L_{\mathcal{D}}^2 + L_{\mathcal{D}}^3,$$

with

$$\begin{aligned} L_{\mathcal{D}}^1 &= \sum_{K \in \mathcal{T}} \sum_{s \in \mathcal{V}_K} \Big(\mathrm{m}_K^s \left((\overline{\nabla}_{\mathcal{D}} P_{\mathcal{D}}\varphi)_K^s - \nabla\varphi(x_K) \right) \cdot \Lambda_K (\widetilde{\nabla}_{\mathcal{D}} u_{\mathcal{D}})_K^s \Big), \\ L_{\mathcal{D}}^2 &= \sum_{K \in \mathcal{T}} \mathrm{m}_K \nabla\varphi(x_K) \cdot \Lambda_K (\nabla_{\mathcal{D}} u_{\mathcal{D}})_K, \\ L_{\mathcal{D}}^3 &= \sum_{K \in \mathcal{T}} \sum_{s \in \mathcal{V}_K} \left(\alpha_K^s \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2} R_{K,\sigma}^s (P_{\mathcal{D}}\varphi) R_{K,\sigma}^s (u_{\mathcal{D}}) \right). \end{aligned}$$

Thanks to the Cauchy-Schwarz inequality, and Lemmae 5.4 and 5.2, we obtain the following bounds

$$\begin{aligned} L^1_{\mathcal{D}} &\leq \beta_0 \left(\sum_{K\in\mathcal{T}} \sum_{s\in\mathcal{V}_K} \mathrm{m}^s_K \left((\overline{\nabla}_{\mathcal{D}} P_{\mathcal{D}}\varphi)^s_K - \nabla\varphi(x_K) \right)^2 \right)^{\frac{1}{2}} \\ &\qquad\qquad \left(\sum_{K\in\mathcal{T}} \sum_{s\in\mathcal{V}_K} \mathrm{m}^s_K \left((\widetilde{\nabla}_{\mathcal{D}} u_{\mathcal{D}})^s_K \right)^2 \right)^{\frac{1}{2}}, \\ &\leq C \, \|u_{\mathcal{D}}\|_{\mathcal{D}} \, h_{\mathcal{D}}, \end{aligned}$$

with a real C depending only on β_0, d, γ, M, Ω, and φ. Thanks to (30) and the fact that $\theta_{\mathcal{D}} \geq \theta > 0$, it results that $\lim\limits_{h_{\mathcal{D}}\to 0} L^1_{\mathcal{D}} = 0$.

The second term rewrites

$$L^2_{\mathcal{D}} = \sum_{K\in\mathcal{T}} \mathrm{m}_K \int_K \Lambda(x)\nabla\varphi(x_K) \cdot (\nabla_{\mathcal{D}} u_{\mathcal{D}})_K \, dx.$$

Since the gradient $\nabla_{\mathcal{D}} u_{\mathcal{D}}$ converges weakly to $\nabla\tilde{u}$ in $L^2(\Omega)^d$, and the function $x \to \Lambda(x)\nabla\varphi(x_K)$ for all $x \in K$, $K \in \mathcal{T}$, converges strongly to $\Lambda\nabla\varphi$ in $L^2(\Omega)^d$ as $h_{\mathcal{D}} \to 0$, we deduce that $\lim\limits_{h_{\mathcal{D}}\to 0} L^2_{\mathcal{D}} = \int_\Omega \nabla\tilde{u}(x) \cdot \Lambda(x)\nabla\varphi(x) \, dx$.

Using the assumption (16) on the coefficients α^s_K as well as Lemmae 5.5 and 5.3 leads to $\lim\limits_{h_{\mathcal{D}}\to 0} L^3_{\mathcal{D}} = 0$, and all together it is proved that

$$\lim_{h_{\mathcal{D}}\to 0} a_{\mathcal{D}}(P_{\mathcal{D}}\varphi, u_{\mathcal{D}}) = \int_\Omega \nabla\tilde{u}(x) \cdot \Lambda(x)\nabla\varphi(x) dx. \tag{50}$$

From Lemma 5.6 and since $\lim_{h_{\mathcal{D}}\to 0} P_{\mathcal{D}}\varphi = \varphi$ in $L^2(\Omega)$, the gradient $\nabla_{\mathcal{D}} P_{\mathcal{D}}\varphi$ converges weakly in $L^2(\Omega)^d$ to $\nabla\varphi$ as $h_{\mathcal{D}} \to 0$. It results that the same type of arguments as above can be used to prove that

$$\lim_{h_{\mathcal{D}}\to 0} a_{\mathcal{D}}(P_{\mathcal{D}}\varphi, P_{\mathcal{D}}\varphi) = \int_\Omega \nabla\varphi(x) \cdot \Lambda(x)\nabla\varphi(x) dx. \tag{51}$$

Summing the limits (49),(50), and (51) in (48) , we obtain that

$$\begin{aligned} \lim_{h_{\mathcal{D}}\to 0} T^1_{\mathcal{D}} \leq \frac{\gamma^2 \, M}{\theta} \Bigg(\int_\Omega f(x) \, & \Big(\tilde{u}(x) - \varphi(x)\Big) \\ & + \int_\Omega \nabla\Big(\varphi(x) - \tilde{u}(x)\Big) \cdot \Lambda(x)\nabla\varphi(x) dx \Bigg). \end{aligned}$$

Thanks to Lemma 5.4, it is clear that $\lim\limits_{h_{\mathcal{D}}\to 0} T^2_{\mathcal{D}} = 0$. Then, using the density of $C^\infty_c(\Omega)$ in $H^1_0(\Omega)$ we can show that $\lim\limits_{h_{\mathcal{D}}\to 0} I_{\mathcal{D}} = 0$, which proves the convergence in $L^2(\Omega)^d$ of the gradient $\overline{\nabla}_{\mathcal{D}} u_{\mathcal{D}}$.

Thanks to Lemmae 5.2 and 5.4, the previous proof is readily adapted to prove the convergence in $L^2(\Omega)^d$ of the gradient $\widehat{\nabla}_{\mathcal{D}} u_{\mathcal{D}}$, which completes the proof of the lemma. □

Proof of Theorem 5.1

Thanks to Lemma 5.7, there exists $\tilde{u} \in H_0^1(\Omega)$, and a subsequence still denoted by $n \in \mathbb{N}$ for conveniency, such that $P_{\mathcal{T}} u_{\mathcal{D}_n}$ converges to $\tilde{u}$ in $L^q(\Omega)$, for all $q \in [1, +\infty)$ if $d = 2$ and all $q \in [1, 2d/(d-2))$ if $d > 2$. Moreover, $\widehat{\nabla}_{\mathcal{D}_n} u_{\mathcal{D}_n}$ and $\overline{\nabla}_{\mathcal{D}_n} u_{\mathcal{D}_n}$ converge to $\nabla \tilde{u}$ in $L^2(\Omega)^d$. In the remaining, we shall prove that $\tilde{u}$ is a weak solution of (1) which will complete the proof from the uniqueness of the weak solution. For the sake of simplicity in the notations, the subscript $n \in \mathbb{N}$, will be dropped in the remaining of the proof.

Let φ be a given function in $C_c^\infty(\Omega)$ and let us set $v = P_{\mathcal{D}}\varphi \in \mathcal{H}_{\mathcal{D}}$ in the variational formulation (17):

$$a_{\mathcal{D}}(u_{\mathcal{D}}, P_{\mathcal{D}}\varphi) = \int_\Omega f(x) P_{\mathcal{T}}(P_{\mathcal{D}}\varphi)(x)dx. \tag{52}$$

Let us now split the expression of $a_{\mathcal{D}}(u_{\mathcal{D}}, P_{\mathcal{D}}\varphi)$ into the following three terms

$$a_{\mathcal{D}}(u_{\mathcal{D}}, P_{\mathcal{D}}\varphi) = T_{\mathcal{D}}^1 + T_{\mathcal{D}}^2 + T_{\mathcal{D}}^3,$$

with

$$\begin{aligned}
T_{\mathcal{D}}^1 &= \sum_{K \in \mathcal{T}} \sum_{s \in \mathcal{V}_K} \left(\mathrm{m}_K^s ((\overline{\nabla}_{\mathcal{D}} u_{\mathcal{D}})_K^s - (\widehat{\nabla}_{\mathcal{D}} u_{\mathcal{D}})_K) \cdot \Lambda_K (\widetilde{\nabla}_{\mathcal{D}} P_{\mathcal{D}}\varphi)_K^s \right), \\
T_{\mathcal{D}}^2 &= \sum_{K \in \mathcal{T}} \mathrm{m}_K (\widehat{\nabla}_{\mathcal{D}} u_{\mathcal{D}})_K \cdot \Lambda_K (\nabla_{\mathcal{D}} P_{\mathcal{D}}\varphi)_K, \\
T_{\mathcal{D}}^3 &= \sum_{K \in \mathcal{T}} \sum_{s \in \mathcal{V}_K} \left(\alpha_K^s \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2} R_{K,\sigma}^s (P_{\mathcal{D}}\varphi) R_{K,\sigma}^s (u_{\mathcal{D}}) \right).
\end{aligned}$$

Thanks to the Cauchy-Schwarz inequality, our assumption on Λ, and Lemma 5.2, the following bounds hold:

$$\begin{aligned}
|T_{\mathcal{D}}^1| &\le \beta_0 \|\overline{\nabla}_{\mathcal{D}} u_{\mathcal{D}} - \widehat{\nabla}_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)} \, \|\widetilde{\nabla}_{\mathcal{D}} P_{\mathcal{D}}\varphi\|_{L^2(\Omega)}, \\
&\le \beta_0 \sqrt{d} \, \|\overline{\nabla}_{\mathcal{D}} u_{\mathcal{D}} - \widehat{\nabla}_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)} \, \|P_{\mathcal{D}}\varphi\|_{\mathcal{D}}.
\end{aligned}$$

From the estimate

$$\|P_{\mathcal{D}}\varphi\|_{\mathcal{D}} \le \frac{(\mathrm{CardFace}(\mathcal{D}) \ \mathrm{m}(\Omega))^{1/2}}{\mathrm{RegulCell}(\mathcal{D})} \sup_{x \in \Omega} |\nabla \varphi(x)|,$$

and Lemma 5.7, it results that $\lim_{h_{\mathcal{D}} \to 0} T_{\mathcal{D}}^1 = 0$.

Let us now consider $T_{\mathcal{D}}^2 = \int_\Omega \widehat{\nabla}_{\mathcal{D}} u_{\mathcal{D}}(x) \cdot \Lambda(x) (\nabla_{\mathcal{D}} P_{\mathcal{D}}\varphi)(x) dx$. It has been shown in the above proof of Lemma 5.7 that $\nabla_{\mathcal{D}} P_{\mathcal{D}}\varphi$ converges weakly in $L^2(\Omega)^d$ to $\nabla\varphi$ as $h_{\mathcal{D}} \to 0$. Then, we obtain the following limit of $T_{\mathcal{D}}^2$ as $h_{\mathcal{D}} \to 0$:

$$\lim_{h_{\mathcal{D}} \to 0} T_{\mathcal{D}}^2 = \int_\Omega \nabla \tilde{u}(x) \cdot \Lambda(x) \nabla \varphi(x) dx.$$

Using the assumption (16) on the coefficients α_K^s as well as Lemmae 5.5 and 5.3, we obtain that

$$\lim_{h_{\mathcal{D}}\to 0} T_{\mathcal{D}}^3 = 0.$$

All together, on the one hand, we have

$$\lim_{h_{\mathcal{D}}\to 0} \int_\Omega f(x) P_{\mathcal{T}}(P_{\mathcal{D}}\varphi)(x)dx = \int_\Omega f(x)\varphi(x)dx.$$

On the other hand, we have

$$\lim_{h_{\mathcal{D}}\to 0} a_{\mathcal{D}}(u_{\mathcal{D}}, P_{\mathcal{D}}\varphi) = \int_\Omega \nabla\tilde{u}(x)\cdot\Lambda(x)\nabla\varphi(x)dx.$$

Then, using (52), we conclude that

$$\int_\Omega \nabla\tilde{u}(x)\cdot\Lambda(x)\nabla\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx$$

which completes the proof of Theorem 5.1.

6 Two examples of construction of the gradient (12)

From Lemma 3.1, there is only one way to build a gradient (12) satisfying the consistency hypothesis 1 when the cardinal q_K^s of $\mathcal{E}_K \cap \mathcal{E}_s$ is equal to d. On the other hand, when $q_K^s > d$ there are many ways to build such gradient. Two examples are given in the two subsections below.

6.1 First construction

For all $K \in \mathcal{T}$ and $s \in \mathcal{V}_K$, let us define the square d-dimensional matrix B_K^s by

$$B_K^s = \frac{1}{m_K^s} \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \mathrm{m}_\sigma^s \mathbf{n}_{K,\sigma}(x_\sigma^s - x_K)^t. \tag{53}$$

The gradient (12) is defined by

$$B_K^s g_{K,\sigma}^s = \frac{\mathrm{m}_\sigma^s}{\mathrm{m}_K^s}\mathbf{n}_{K,\sigma}, \tag{54}$$

for all $\sigma \in \mathcal{E}_s \cap \mathcal{E}_K$, i.e.

$$(\overline{\nabla}_{\mathcal{D}} u)_K^s \quad = \quad (B_K^s)^{-1}(\widetilde{\nabla}_{\mathcal{D}} u)_K^s, \tag{55}$$

assuming that the matrix B_K^s is non-singular. If q_K^s is equal to the space dimension d, and the set of vectors $(x_\sigma^s - x_K)_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s}$ spans $\mathbb{R}^d$, the matrix B_K^s is non-singular iff the set of vectors $(\mathbf{n}_{K,\sigma})_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s}$ spans also $\mathbb{R}^d$. For more general meshes, the non-singularity of B_K^s will be shown in subsection 6.1.1 to result from a stronger assumption (56) ensuring also the coercivity of the scheme. Note however that if the set of vectors $(\mathbf{n}_{K,\sigma})_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s}$ does not span $\mathbb{R}^d$, as it may be the case for non-matching meshes, the matrix B_K^s is singular and the present construction does not apply. This case will be taken into account in the second example.

Assuming that B_K^s is non-singular, we can easily check that the consistency hypothesis 1 is satisfied.

6.1.1 Coercivity and convergence of the finite volume scheme

The main advantage of this construction is that a simple condition can be derived which ensures the non-singularity of the matrices B_K^s, the coercivity condition $\text{coernode}(\mathcal{D},\Lambda) \geq \theta_{\mathcal{D}}$ as well as an upper bound for the parameter $\text{RegulGrad}(\mathcal{D})$ involved in the stability of the gradient function $\overline{\nabla}_{\mathcal{D}} u$.

This condition imposes the following non-negative lower bound

$$\text{coercell}(\mathcal{D},\Lambda) \geq \bar{\theta}_{\mathcal{D}} > 0, \tag{56}$$

on the coercivity parameter defined by

$$\text{coercell}(\mathcal{D},\Lambda) = \min_{K\in\mathcal{T}, s\in\mathcal{V}_K} \lambda_{\min}\left(\frac{\Lambda_K B_K^s + (\Lambda_K B_K^s)^t}{2}\right). \tag{57}$$

It can be easily computed for any given finite volume discretization $\mathcal{D}$ and diffusion tensor Λ.

The condition (56) ensures that the matrices B_K^s (53) defining the discrete gradients (55) are non-singular for all $s \in \mathcal{V}_K$, $K \in \mathcal{T}$ as stated in Lemma 6.2. To prove this result, we first need to state the following lemma.

LEMMA 6.1 Let $A \in \mathcal{M}_d(\mathbb{R})$ such that $\lambda_{\min}(A + A^t) > 0$, then A is a non-singular matrix and satisfies the estimate

$$|A^{-1}| \leq \frac{8}{3}\,\frac{1}{\lambda_{\min}(A+A^t)}$$

Proof We readily have $A \neq 0$. Let us consider the following estimates

$$\begin{aligned} |rA - I_d|^2 &= |(rA - I_d)^t(rA - I_d)| = |(I_d - r(A^t + A)) + r^2 A^t A|, \\ &\leq |I_d - r(A^t + A)| + |r^2 A^t A| = |I_d - r(A^t + A)| + r^2|A|^2. \end{aligned}$$

Choosing in the following $r = \dfrac{\lambda_{\min}(A+A^t)}{4|A|^2}$ ensures that all the eigenvalues of the symmetric matrix $I_d - r(A^t + A)$ are positive, and we have $|I_d - r(A^t + A)| = 1 - r\lambda_{\min}(A + A^t)$. Hence, we have proved the estimate

$$|rA - I_d|^2 \leq 1 - 3\left(\frac{\lambda_{\min}(A+A^t)}{4|A|}\right)^2.$$

It results that $|rA - I_d| < 1$. Then, setting $rA = I_d + (rA - I_d)$ we can obtain that rA is a non-singular matrix and that the following estimates hold

$$\begin{aligned} |(rA)^{-1}| &\leq \frac{1}{1 - |rA - I_d|} = \frac{1 + |rA - I_d|}{1 - |rA - I_d|^2} \\ &\leq \frac{2}{1 - |rA - I_d|^2} \\ &\leq \frac{2}{3}\left(\frac{4|A|}{\lambda_{\min}(A+A^t)}\right)^2, \end{aligned}$$

which concludes the proof. □

Lemma 6.2 Let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2.1 such that there exists a real $\bar{\theta}_{\mathcal{D}} > 0$ with $\text{coercell}(\mathcal{D}, \Lambda) \geq \bar{\theta}_{\mathcal{D}}$, then for all $s \in \mathcal{V}_K$, $K \in \mathcal{T}$, the matrix B_K^s is non-singular, and its norm satisfies the following estimate

$$|(B_K^s)^{-1}| \leq \frac{4\beta_0}{3\bar{\theta}_{\mathcal{D}}}. \tag{58}$$

Proof From the assumption one has

$$\lambda_{\min}(\Lambda_K B_K^s + (\Lambda_K B_K^s)^t) \geq 2 \text{ coercell}(\mathcal{D}, \Lambda) \geq 2\ \bar{\theta}_{\mathcal{D}} > 0.$$

We deduce from Lemma 6.1 that the matrix $\Lambda_K B_K^s$ is non-singular as well as the matrix B_K^s. Still from Lemma 6.1, we have the estimate

$$|(\Lambda_K B_K^s)^{-1}| \leq \frac{4}{3\bar{\theta}_{\mathcal{D}}},$$

which concludes the proof from the bound $|\Lambda_K| \leq \beta_0$. □

The following Lemmae 6.3 and 6.4 state respectively that the condition (56) provides an upper bound for the parameter $\text{RegulGrad}(\mathcal{D})$ and that it ensures the coercivity condition $\text{coernode}(\mathcal{D}, \Lambda) \geq \theta_{\mathcal{D}}$.

Lemma 6.3 Let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2.1 such that there exists a real $\bar{\theta}_{\mathcal{D}} > 0$ with $\text{coercell}(\mathcal{D}, \Lambda) \geq \bar{\theta}_{\mathcal{D}}$. Then, we have the estimate

$$\text{RegulGrad}(\mathcal{D}) \leq \frac{4\ \beta_0\ d}{3\ \bar{\theta}_{\mathcal{D}}\ \text{RegulCell}(\mathcal{D})}.$$

Proof The estimate derives from the definition (35) of $\text{RegulGrad}(\mathcal{D})$, from (54), from Lemma 6.2, and from the definitions (5) of m_K^s, and (3) of $\text{RegulCell}(\mathcal{D})$. □

Proposition 6.4 [coercivity of the scheme] Let $\mathcal{D}$ be an admissible discretization in the sense of Definition 2.1 such that there exists a real $\bar{\theta}_{\mathcal{D}} > 0$ with $\text{coercell}(\mathcal{D}, \Lambda) \geq \bar{\theta}_{\mathcal{D}}$. Then, setting $\theta_{\mathcal{D}} = \frac{1}{2} \min\left(\mu_0, \frac{\text{RegulCell}(\mathcal{D})^2\ \bar{\theta}_{\mathcal{D}}}{\text{CardFace}(\mathcal{D})}\right)$, we have the lower bound $\text{coernode}(\mathcal{D}, \Lambda) \geq \theta_{\mathcal{D}}$ and hence the coercivity of the bilinear form $a_{\mathcal{D}}$

$$a_{\mathcal{D}}(u, u) \geq \theta_{\mathcal{D}} \|u\|_{\mathcal{D}}^2, \tag{59}$$

for all $u \in \mathcal{H}_{\mathcal{D}}$.

Proof Let s be a given vertex of $\mathcal{V}$. From the definition (25) of the bilinear form $a_{\mathcal{D}^s}$, and from formula (55), we have for all $u \in \mathcal{H}_{\mathcal{D}^s}$

$$\begin{aligned} a_{\mathcal{D}^s}(u, u) = \sum_{K \in \mathcal{T}_s} \Bigg(& \mathrm{m}_K^s (\overline{\nabla}_{\mathcal{D}} u)_K^s \cdot \frac{\Lambda_K B_K^s + (B_K^s)^t \Lambda_K}{2} (\overline{\nabla}_{\mathcal{D}} u)_K^s \\ & + \alpha_K^s \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2} R_{K,\sigma}^s(u)^2 \Bigg). \end{aligned} \tag{60}$$

Using the following inequality

$$\mu(a-b)^2 \geq \frac{1}{2}\min(\mu,\lambda)a^2 - \lambda b^2, \text{ for all } (\mu,\lambda)\in(\mathbb{R}_+)^2, (a,b)\in\mathbb{R}^2$$

with $\mu = \alpha_K^s$, $a = u_\sigma^s - u_K$, $b = (\overline{\nabla}_{\mathcal{D}} u)_K^s \cdot (x_\sigma^s - x_K)$ and $\lambda = \rho_K^s$, we obtain for all $\rho_K^s \geq 0$ the lower bound

$$\begin{aligned} \alpha_K^s \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2} R_{K,\sigma}^s(u)^2 \geq \\ \frac{1}{2}\min(\rho_K^s,\alpha_K^s) \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2}(u_\sigma^s - u_K)^2 - \rho_K^s (\overline{\nabla}_{\mathcal{D}} u)_K^s \cdot A_K^s (\overline{\nabla}_{\mathcal{D}} u)_K^s, \end{aligned} \tag{61}$$

where the symmetric matrix A_K^s is defined by (46) and satisfies the upper bound (47). Let us choose ρ_K^s such that

$$\rho_K^s = \sup\left\{\rho\in\mathbb{R},\ \mathrm{m}_K^s\,\frac{\Lambda_K B_K^s + (\Lambda_K B_K^s)^t}{2} - \rho A_K^s \geq 0\right\}. \tag{62}$$

Using the upper bound (47), and the local coercivity assumption (56), (57), we can prove that ρ_K^s defined by (62) satisfies the lower bound

$$\rho_K^s \geq \frac{\mathrm{RegulCell}(\mathcal{D})^2\ \bar{\theta}_{\mathcal{D}}}{\mathrm{CardFace}(\mathcal{D})}, \tag{63}$$

for all $s\in\mathcal{V}_K$, $K\in\mathcal{T}$. Using (60), (61), (62), (63), and (16), we obtain the lower bound

$$a_{\mathcal{D}^s}(u,u) \geq \frac{1}{2}\min\left(\mu_0, \frac{\mathrm{RegulCell}(\mathcal{D})^2\ \bar{\theta}_{\mathcal{D}}}{\mathrm{CardFace}(\mathcal{D})}\right)\ \|u\|_{\mathcal{D}}^2, \tag{64}$$

for all $u\in\mathcal{H}_{\mathcal{D}^s}$ which concludes the proof. □

From Proposition 6.4, Lemma 6.3, and Theorem 5.1 we can state the following theorem showing the convergence of the finite volume scheme under the coercivity condition (56).

Theorem 6.5 [Convergence of the scheme] Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, with $d\in\mathbb{N}^*$. Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of admissible discretizations in the sense of Definition 2.1, such that $h_{\mathcal{D}_n}\to 0$ as $n\to\infty$. It is assumed that there exist $\bar{\theta}>0$, $\beta>0$, $\eta>0$, and $M\in\mathbb{N}$ with $\mathrm{coercell}(\mathcal{D}_n,\Lambda)\geq\bar{\theta}$, $\mathrm{CardFace}(\mathcal{D}_n)\leq M$, $\mathrm{RegulKL}(\mathcal{D}_n)\geq\eta$, and $\mathrm{RegulCell}(\mathcal{D}_n)\geq\beta$ for all $n\in\mathbb{N}$. Then, there exists for all $n\in\mathbb{N}$ a unique solution $u_{\mathcal{D}_n}\in\mathcal{H}_{\mathcal{D}_n}$ to (20), and the sequence $P_{\mathcal{T}} u_{\mathcal{D}_n}$, $n\in\mathbb{N}$ converges to the weak solution $\bar{u}$ of (1) in $L^q(\Omega)$, for all $q\in[1,+\infty)$ if $d=2$ and all $q\in[1,2d/(d-2))$ if $d>2$. Moreover, the sequence $\overline{\nabla}_{\mathcal{D}_n} u_{\mathcal{D}_n}$, $n\in\mathbb{N}$ converges to $\nabla\bar{u}$ in $L^2(\Omega)^d$.

6.2 Second construction

This second finite volume scheme uses the construction of the gradient $(\overline{\nabla}_{\mathcal{D}} u)^s_K$ introduced in [13] for $d = 2$ and 3. Compared with the previous approach, its main advantage is to cover the case of non-matching or locally refined grids for which the set of vectors $(\mathbf{n}_{K,\sigma})_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s}$ may not span $\mathbb{R}^d$.

For each $\sigma \in \mathcal{E}$, let us denote by $\mathcal{E}^s_{K,\sigma}$ the subset of $\mathcal{E}_s \cap \mathcal{E}_K$ of cardinality d defined as follows for $d = 2$ and $d = 3$. For $d = 2$, let us set $\mathcal{E}^s_{K,\sigma} = \mathcal{E}_s \cap \mathcal{E}_K$. For $d = 3$, let e_1 and e_2 be the two edges of the face σ intersecting the vertex s, and σ_1 and σ_2 be the two faces of $\mathcal{E}_s \cap \mathcal{E}_K$ sharing respectively the edge e_1 and e_2 with the face σ. Then, we set $\mathcal{E}^s_{K,\sigma} = \{\sigma, \sigma_1, \sigma_2\}$.

For all $K \in \mathcal{T}$ and $s \in \mathcal{V}_K$, the gradient $(\overline{\nabla}_{\mathcal{D}} u)^s_K$ is defined by

$$(\overline{\nabla}_{\mathcal{D}} u)^s_K = \sum_{\sigma \in \mathcal{E}_s \cap \mathcal{E}_K} \frac{\mathrm{m}^s_\sigma \, d_{K,\sigma}}{d \, \mathrm{m}^s_K} \sum_{\sigma' \in \mathcal{E}^s_{K,\sigma}} (u^s_{\sigma'} - u_K) \, g^s_{K,\sigma,\sigma'},$$

where $\{g^s_{K,\sigma,\sigma'}, \, \sigma' \in \mathcal{E}^s_{K,\sigma}\}$ is the biorthogonal basis of $\{(x^s_{\sigma'} - x_K), \, \sigma' \in \mathcal{E}^s_{K,\sigma}\}$ such that

$$(x^s_{\sigma'} - x_K) \cdot g^s_{K,\sigma,\sigma''} = \delta_{\sigma',\sigma''}$$

for all $\sigma', \sigma'' \in \mathcal{E}^s_{K,\sigma}$, assuming that the set of vectors $(x^s_{\sigma'} - x_K), \, \sigma' \in \mathcal{E}^s_{K,\sigma}$ is free. Note that by construction, $\sum_{\sigma' \in \mathcal{E}^s_{K,\sigma}} v \cdot (x^s_{\sigma'} - x_K) \, g^s_{K,\sigma,\sigma'} = v$ for any vector $v \in \mathbb{R}^d$. It results that the gradient $(\overline{\nabla}_{\mathcal{D}} u)^s_K$ is consistent in the sense of hypothesis 1.

The upper bound of the parameter RegulGrad$(\mathcal{D})$ is controlled in two dimensions by the minimum angle between the two vectors $(x^s_{\sigma'} - x_K), \sigma' \in \mathcal{E}_s \cap \mathcal{E}_K$. In three dimensions it is controlled by the minimum angles between a vector of $\{(x^s_{\sigma'} - x_K), \sigma' \in \mathcal{E}^s_{K,\sigma}\}$ and the two remaining ones. These minimum angles should not tend to zero.

From Lemma 3.1, this second approach is equivalent to the MPFA O scheme described in [1] and [8] as soon as q^s_K is equal to the space dimension d for all cells K and all vertices s of the cell K. It is always the case in two dimensions $d = 2$. If in addition the set of vectors $(\mathbf{n}_{K,\sigma})_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s}$ spans $\mathbb{R}^d$, then both the first and second constructions are equivalent to the MPFA O scheme [1] and [8].

The coercivity condition coernode$(\mathcal{D}, \Lambda) \geq \theta_{\mathcal{D}}$ has to be checked numerically. The stronger but simpler condition coercell$(\mathcal{D}, \Lambda) \geq \bar{\theta}_{\mathcal{D}}$ can also be used when both constructions match. As exhibited in subsection 7.3, this latter condition is less sharp but it is cheaper to compute.

7 Numerical tests

There are many papers investigating the numerical convergence properties of the MPFA O scheme. For example, let us refer to [2] for quadrilateral grids in two and three dimensions, and to [9] in two dimensions with discontinuous diffusion coefficients. Also in [7], the MPFA O scheme is compared on challenging two dimensional anisotropic test cases with two unconditionally symmetric coercive finite volume schemes which exhibit a more robust convergence but at the expense of a much larger stencil.

Let us first discuss the coercivity condition (56) on a few particular remarkable cases.

7.1 Symmetry and unconditional coercivity for two families of meshes

In this section we consider arbitrary positive definite tensors Λ and two remarkable families of meshes for which the symmetry and the unconditional coercivity of the finite volume scheme can be achieved for a proper choice of the cell centers x_K, of the continuity points x_σ^s, and of the subsurfaces m_σ^s.

Parallelogram (in 2D) and parallelepiped (in 3D) meshes define the first family, and triangular (in 2D) and tetrahedral (in 3D) meshes the second family. For both families, x_K is the center of gravity of the cell and m_σ^s is set to $\frac{m_\sigma}{\mathrm{Card}(\mathcal{V}_\sigma)}$. For the first family, the continuity points x_σ^s are the center of gravity of the face σ for all the vertices $s \in \mathcal{V}_\sigma$. For triangles, x_σ^s is the barycenter of the two vertices of the edge σ with weights 2/3 at the vertex s and 1/3 at the second vertex of the edge σ. For tetrahedra, x_σ^s is the barycenter of the three vertices of the face σ with weights 1/2 at the vertex s and 1/4 at the two remaining vertices of the face σ.

In all those cases we will show that the local $d \times d$ matrix B_K^s defined in (53) is equal to the identity matrix I. Recalling that $(\overline{\nabla}_\mathcal{D} u)_K^s = (B_K^s)^{-1}(\widetilde{\nabla}_\mathcal{D} u)_K^s$ from (55), the symmetry of the finite volume scheme will follow, as well as its unconditional coercivity for any tensor Λ resulting from the coercivity sufficient condition

$$\min_{K\in\mathcal{T}, s\in\mathcal{V}_K} \lambda_{\min}\left(\frac{\Lambda_K B_K^s + (\Lambda_K B_K^s)^t}{2}\right) \geq \bar{\theta}_\mathcal{D} > 0,$$

(see (57)) and Proposition 6.4.

Proof: For both families of meshes the cardinal q_K^s of $\mathcal{E}_K \cap \mathcal{E}_s$ is equal to d and we can assume that the set of d vectors $(x_\sigma^s - x_K)_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s}$ spans $\mathbb{R}^d$ since otherwise the matrix B_K^s would be singular. Then, from Lemma 3.1, there is a unique consistent gradient (12). It results from the proof of Lemma 3.1 and subsection 6.1 that the matrix B_K^s is equal to I if and only if

$$\frac{\mathrm{m}_\sigma^s}{\mathrm{m}_K^s}\vec{n}_{K,\sigma} \cdot (x_{\sigma'}^s - x_K) = \delta_{\sigma,\sigma'} \qquad (65)$$

for all $\sigma, \sigma' \in \mathcal{E}_K \cap \mathcal{E}_s$.

In other words we must check (i) that $m_K^s = m_\sigma^s d_{K,\sigma}$ for each face $\sigma \in \mathcal{E}_K \cap \mathcal{E}_s$ and (ii) that the line $x_K x_\sigma^s$ is parallel to the face σ' for all $\sigma, \sigma' \in \mathcal{E}_K \cap \mathcal{E}_s$ with $\sigma \neq \sigma'$.

For the first family of meshes, properties (i) and (ii) are readily checked. For triangles, (i) results from the fact that the center of gravity x_K is the intersection of the midlines and (ii) is easily checked (see Figure 2 (b)). For tetrahedra, let us check (i) and (ii) using barycentric coordinates. Let A,B,C,D be the ordered vertices of the tetrahedron (see Figure 2 (a)), and let us consider to fix ideas $s = A$. In barycentric coordinates we have $x_K = (1/4, 1/4, 1/4, 1/4)$, $x^s_{ABC} = (1/2, 1/4, 1/4, 0)$, $x^s_{ACD} = (1/2, 0, 1/4, 1/4)$, and $x^s_{ABD} = (1/2, 1/4, 0, 1/4)$. Hence $x_K x^s_{ABC} = -1/4AD$, $x_K x^s_{ACD} = -1/4AB$, and $x_K x^s_{ABD} = -1/4AC$ which proves (ii). To prove (i) it suffices to remark that

$$\mathrm{Det}(x_K A, x_K B, x_K C) = \mathrm{Det}(x_K A, x_K C, x_K D) = -\mathrm{Det}(x_K A, x_K B, x_K D).$$

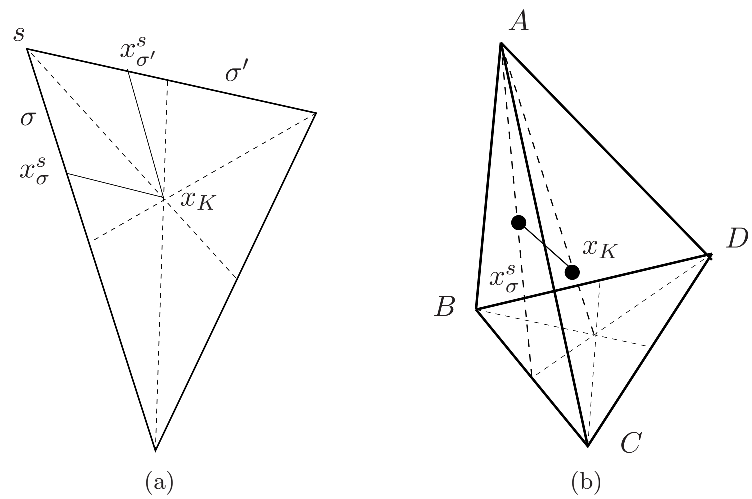


Figure 2: (a) Choice of the continuity points x^s_σ and $x^s_{\sigma'}$ at the vertex s and of the cell center x_K for a triangle. (b) Center of gravity x_K and continuity point x^s_σ for a tetrahedron $ABCD$ with $\sigma = ABC$ and $s = A$.

7.2 Study of the local coercivity criteria for $\Lambda = I$ in 2D

Let us now consider the case $d = 2$ with $\Lambda = I$, and let σ_1 and σ_2 be the two edges shared by a given vertex s of a given cell K. For $\sigma = \sigma_1, \sigma_2$, we assume that the continuity point x^s_σ is the center of gravity x_σ of the edge σ and that $m^s_\sigma = |x_\sigma - s|$. Then, the condition $\lambda_{\min}(B^s_K + (B^s_K)^t) \geq 2\theta$ is equivalent to $|x_{\sigma_1} - x_{\sigma_2}|\, |\overrightarrow{sx_{\sigma_1}} - \overrightarrow{x_{\sigma_2}x_K}| \leq 2(1-\theta)m^s_K$. For example, the trapezoidal mesh shown in Figure 1 satisfies the coercivity condition (56) if and only if $\frac{b-a}{h} \leq (1-\theta)\frac{3a+b}{(b^2+h^2)^{1/2}}$ which exhibits the lack of robustness of the MPFA O scheme for distorted quadrangular meshes.

Next, let us discuss the sharpness of the coercivity criteria on a two dimensional example.

7.3 Sharpness of the coercivity criteria

We solve the anisotropic diffusion test case introduced in [19] on a family of skewed quadrangular meshes of the domain $\Omega = (0,1)^2$ of size $n_x \times n_x$ with $n_x = 20, 40, 80, 160$.

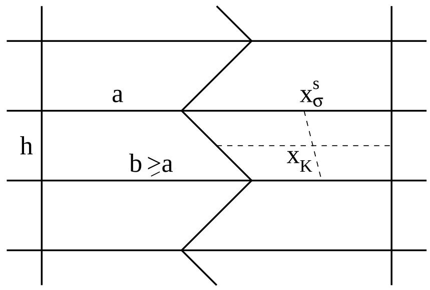


Figure 3: Example of a trapezoidal mesh.

The exact solution and the expression for the permeability coefficient are given below:

$$u = \sin(\pi x)\sin(\pi y), \quad K = \frac{1}{x^2+y^2}\begin{bmatrix} \delta x^2 + y^2 & (\delta-1)xy \\ (\delta-1)xy & x^2+\delta y^2 \end{bmatrix}. \tag{66}$$

We shall understand that Dirichlet boundary conditions are given on each boundary edge $\sigma \in \mathcal{E}_{\text{int}}$ by $u(x_\sigma^s)$, $s \in \mathcal{V}_\sigma$, and that the forcing term is equal to $-\nabla \cdot (K\nabla u)$. The parameter δ is in fact the ratio between the minimum and the maximum eigenvalue of K.

The continuity points x_σ^s are the center of gravity of the edge σ and $m_\sigma^s = m_\sigma/2$ for all $s \in \mathcal{V}_\sigma$, $\sigma \in \mathcal{E}$, and the cell center is the isobarycenter of its four vertices.

The mesh $n_x = 20$ is plotted in Figure 4 as well as the convergence of the MPFA O scheme for different values of δ. We note that the convergence seems to be broken for $\delta = 0.001$.

In Table 1 the sharpness of the two criteria of coercivity coercell$(\mathcal{D}, \Lambda)$, and coernode$(\mathcal{D}, \Lambda)$ are assessed. For that purpose, we also compute the smallest eigenvalue coerschurmesh$(\mathcal{D}, \Lambda)$ of the symmetric part of the cell centered scheme matrix, as well as coerschurnode$(\mathcal{D}, \Lambda)$, the smallest non-zero eigenvalue of the symmetric part of all the cell centered scheme submatrices around each vertex s of the mesh.

We note in Table 1 that the positivity criteria coercell$(\mathcal{D}, \Lambda)\geq 0$ as well as coernode$(\mathcal{D}, \Lambda)\geq 0$ are more restrictive than the positivity of the cell centered scheme around each vertex coerschurnode$(\mathcal{D}, \Lambda)\geq 0$ which is a sufficient condition for the positivity of the cell centered finite volume scheme but not for the positivity of the hybrid finite volume scheme. From Table 1 and Figure 4, the convergence of the MPFA O scheme seems to be more closely related to the coercivity of the cell centered scheme. We refer to [5] for a general convergence analysis of finite volume schemes based on a cell centered coercivity condition which can apply to the MPFA O scheme.

7.4 Numerical examples on 3D meshes

7.4.1 Randomly distorted Cartesian meshes

Let us consider a family of uniform Cartesian meshes of the domain $\Omega = [0,1]^3$ of step size h. A distorsion of size $\frac{h}{3}$ in a random direction is applied on each node of the Cartesian meshes as exhibited in Figure 5.

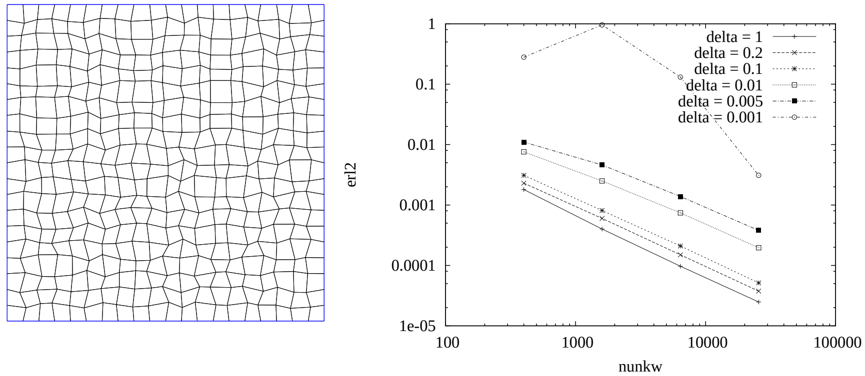


Figure 4: Mesh of size $n_x = 20$, and convergence of the L^2 error (erl2) for the MPFA O scheme for different values of δ (nunkw denotes the number of cells n_x^2).

criterion/mesh	$n_x = 10$	$n_x = 40$	$n_x = 80$	$n_x = 160$
coercell$(\mathcal{D}, \Lambda) \geq 0$	0.1	0.14	0.17	0.18
coernode$(\mathcal{D}, \Lambda) \geq 0$	0.06	0.09	0.09	0.11
coerschurnode$(\mathcal{D}, \Lambda) \geq 0$	0.012	0.014	0.016	0.02
coerschurmesh$(\mathcal{D}, \Lambda) \geq 0$	0.0055	0.0058	0.0068	0.014

Table 1: Approximate smallest value of δ for which the coercivity criterion is positive for the different meshes and the various criteria.

The right hand side and the Dirichlet boundary condition are such that the exact solution is given by $u(x, y, z) = sin(\pi x)sin(\pi y)sin(\pi z)$. Two differents diffusion tensors Λ_1=diag(1,1,100) and Λ_2=diag(1,1,1000) are considered. Table 2 below exhibits the errors measured in discrete L_2 norm between the exact solution and the approximate solutions both for the potential and the normal fluxes, obtained on four meshes of step sizes $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$. It clearly shows the good convergence of the O scheme for an anisotropic ratio of 100 while for a larger anisotropic ratio of 1000 the O scheme no longer converges. This is due to the loss of coercivity of the O scheme when a large anisotropic ratio is combined with a distorsion of the mesh. Note that a sparse direct solver has been used in this latter case rather than an iterative solver in order to check that the solution of the linear system was correct.

h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
Λ_1 potential	8.04e-02	2.30e-02	5.31e-03	1.38e-03
Λ_2 potential	9.70e-01	1.85e-01	8.92e-01	9.02e-01
Λ_1 fluxes	5.79e-02	2.38e-02	1.02e-02	5.0e-03
Λ_2 fluxes	8.98e-02	3.40e-01	1.29e-01	6.48e-01

Table 2: Errors of the potential u and of the normal fluxes measured in discrete L^2 norm for randomly distorted Cartesian meshes.

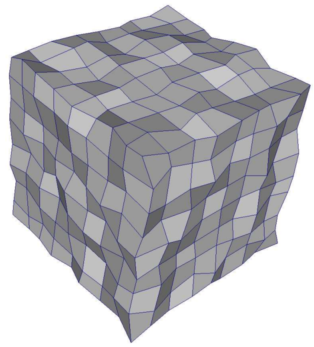
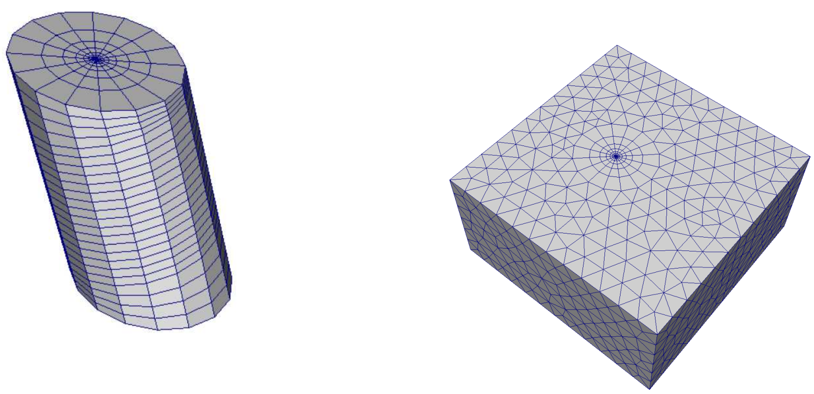


Figure 5: Randomly distorted Cartesian mesh.

7.4.2 Hybrid Near-well meshes

In order to test the MPFA O scheme on 3D meshes with more than 3 faces joining a given vertex on a given cell, we consider in the following a nearwell single phase Darcy flow model arising in reservoir or CO2 storage simulations.

An analytical solution of the single phase Darcy flow equation around a straight deviated well of fixed radius in an infinite domain is described in [4] for a fixed diagonal anisotropic diffusion tensor Λ. The diagonal elements of Λ are denoted by Λ_x, Λ_y and Λ_z.



(a) Exponentially refined radial mesh

(b) Hybrid mesh with hexahedra, tetrahedra and pyramids

Figure 6: Near-well meshes

The mesh is radial around the well axis and exponentially refined down to the well radius as can be seen in Figure 6. This radial local refinement is then matched with the reservoir domain using both tetrahedra and pyramids.

Let us set $\Lambda_x = \Lambda_y = \tau\Lambda_z$. We shall consider two anisotropy ratios $\tau = 5$

and $\tau = 20$. The discrete equation is solved with Dirichlet boundary conditions given by the analytical solution both at the wellbore boundary and at the outer boundary. Table 3 exhibits the good convergence of the O scheme for both anisotropy ratios showing the good behavior of the O scheme on unstructured grids with mild anisotropy.

	mesh 1	mesh 2	mesh 3	mesh 4
Number of cells	11 766	14 468	19 872	29 772
$\tau = 5$ potential	7.50e-03	2.90e-03	1.21e-03	6.50e-04
$\tau = 20$ potential	9.09e-03	3.49e-03	1.47e-03	7.88e-04
$\tau = 5$ fluxes	2.17e-03	8.98e-04	4.52e-04	2.96e-04
$\tau = 20$ fluxes	2.14e-03	9.04e-04	4.87e-04	3.42e-04
	mesh 5	mesh 6	mesh 7	mesh 8
Number of cells	49 139	77 599	124 768	218 970
$\tau = 5$ potential	3.49e-04	2.19e-04	1.44e-04	9.21e-05
$\tau = 20$ potential	4.20e-04	2.63e-04	1.74e-04	1.11e-04
$\tau = 5$ fluxes	2.08e-04	1.61e-04	1.26e-04	9.85e-05
$\tau = 20$ fluxes	2.54e-04	2.03e-04	1.64e-04	1.31e-04

Table 3: Errors of the potential u and of the normal fluxes measured in discrete L_2 norm for a family of refined hybrid near well meshes.

8 Conclusion

This article defines a framework for MPFA O type finite volume schemes which generalizes the construction described in [1] and [8]. This framework uses a hybrid variational formulation involving a weak and a consistent piecewise constant gradients, as well as residual terms for the stabilization of the scheme. For meshes such that for all cells K and all vertices s of K, the cardinal of $\mathcal{E}_K \cap \mathcal{E}_s$ is equal to the space dimension d, our approach is shown to be equivalent to the usual MPFA O scheme. A local coercivity assumption is made ensuring the coercivity of the hybrid variational formulation. Under this coercivity assumption, the well-posedness of the scheme is derived and the convergence of the scheme is proved covering the case of L^∞ diffusion coefficients. Numerical tests performed for three dimensional structured and unstructured meshes exhibit the good convergence of the MPFA O scheme for mild anisotropy and its limits for diffusion problems combining a distorsion of the (typically hexahedral) mesh with large anisotropy.

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