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Domain decomposition and splitting methods for Mortar mixed finite element approximations to parabolic equations

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Summary. We introduce in this article a new domain decomposition algorithm for parabolic problems that combines Mortar Mixed Finite Element methods for the space discretization with operator splitting schemes for the time discretization. The main advantage of this method is to be fully parallel. The algorithm is proven to be unconditionally stable and a convergence result in $\mathcal{O}(\Delta t/h^{\frac{1}{2}})$ is presented.

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1 Introduction

Mixed Finite Element (MFE) methods have become popular for the numerical simulation of single phase flow in porous media due to their good approximation of the flux variable and their local and global mass conservation properties. In many situations such as flow around wells or through conductive faults, the complexity of the geometry, the heterogeneities of the media, or the singularities of the data may require the use of flexible meshes including hybrid meshes or local refinements to capture the spatial behavior of the solution. In that case, non-overlapping domain decomposition techniques with Mortar elements at the interfaces of the decomposition have

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proven to be efficient since they enable to define the grids independently in the subdomain regions (see [GW88], [Yot96]), [ACWY00]).

On the other hand, the time behavior of the solution may also warrant the use of different time steps in the different subdomains.

The idea of the domain decomposition method for parabolic problems introduced in this paper, is to combine Mortar Mixed Finite Element methods for the space discretization with operator splitting techniques for the time discretization in order (1) to obtain a fully parallel algorithm and (2) to be able to use flexible meshes and local time stepping in the subdomains.

Most domain decomposition algorithms for parabolic problems involve, at each time step, the solution of an elliptic problem, using classical domain decomposition iterative algorithms for elliptic equations. The present domain decomposition approach takes advantage of the parabolic structure of the problem to obtain, through operator splitting, a non-iterative method in the sense that the subdomain problems are solved only once at each time step. Other related non-iterative domain decomposition and splitting methods for parabolic problems can be found in [MPRW98], [Cho68], and [Dry91], and the references therein. A similar idea to combine domain decomposition and operator splitting techniques is also presented in [Lio89], [GLT89]. The main originality of our method is to allow, by construction, non-matching grids at the interfaces of the domain decomposition.

Throughout this paper, we consider a bounded domain $\Omega \subset \mathbb{R}^d$ with boundary Γ and the parabolic equation

$$\begin{cases} \partial_t p + \nabla \cdot u = f, \ u = -K\nabla p \text{ in } \Omega, \\ p = g \text{ on } \Gamma, \ p|_{t=0} = p_0, \end{cases} \tag{1.1}$$

where K is a symmetric matrix, positive definite uniformly in $\overline{\Omega}$.

Mixed and Hybrid Finite Element Methods are described in a large number of publications and we refer to [Tho77], [TR91], [BF91] and the references therein for their detailed description. The Mortar Mixed Finite Element (MMFE) discretization of equation (1.1) is a partially hybridized version of the Mixed Finite Element method. Lagrange multipliers, playing the role of an interface pressure, are introduced on the skeleton of the domain decomposition to enforce the weak continuity of the normal fluxes at the interfaces of the decomposition.

This formulation has been first considered for elliptic problems in [GW88] in the case of matching grids at the interfaces, and extended in [Yot96], [ACWY00] to the case of non-matching grids at the interfaces between the subdomains.

In this paper, we focus on the time discretization of the MMFE semi-discrete approximation of (1.1), using operator splitting techniques. The first step is to eliminate the pressure unknown in order to derive an equivalent flux formulation, which formally appears as a mixed formulation for the flux variable and the time derivative of the interface pressure variable. Then, we formally apply to this mixed formulation a projection scheme introduced by Chorin [CL96] for the Navier-Stokes equations and analysed in [She92], and [GQ98], and also in [BA98] in its more accurate incremental version. In the Mortar MFE framework, the projection scheme decouples the system of equations into two steps: (i) advance in time for a fixed interface pressure, and (ii) projection of the new flux on the subspace of weakly continuous fluxes, and computation of the new interface pressure.

The main advantage of the projection scheme is that the prediction step (i) can be solved in a fully parallel way on each subdomain independently, while the projection step (ii) reduces to the solution of an interface problem which can be efficiently preconditioned. In addition, for the simplest Raviart-Thomas mixed finite elements (RT_0 MFE), provided that a mass condensation is performed in the neighborhood of the skeleton, the interface problem further reduces to a diagonal system in the nodal basis and is readily solved.

The rest of the paper is organized as follows. Section 2 recalls the framework of the MMFE method as described in [Yot96], [ACWY00], and introduces the equivalent flux formulation. Section 3 analyses the fully discrete incremental and non-incremental schemes. The stability of the incremental and non-incremental schemes is studied in Sect. 3.1, applying the techniques developped for Navier Stokes equations to the MMFE flux formulation. Error estimates are derived in Sect. 3.2. It is shown that the convergence is obtained if the time step is of smaller order than $h^{1/2}$, where h stands for the mesh size. This dependence on h of the convergence rate appears as the price to pay to obtain a fully parallel algorithm. Finally, in Sect. 3.3, these results are tested on a two-dimensional example.
Notation: for two positive functions $A(v)$ and $B(v)$, the notation $A \lesssim B$ means that there exists a constant C, independent of the various parameters, such that for all v one has $A(v) \le CB(v)$.

2 Mixed finite element domain decomposition method

Let us consider a domain decomposition of Ω into N non-overlapping subdomains $\Omega_i, i = 1, \dots, N$ such that $\Omega_i \cap \Omega_j = \emptyset$ for all $i \neq j$, and $\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i$.

Let us define $\Gamma_i := \partial\Omega_i / \Gamma$, and $I := \{\{i,j\}$ s.t. $i \neq j$ and $\text{mes}_{d-1}\, \partial \Omega_i \cap \partial\Omega_j \neq 0\}$, where we do not distinguish $\{i,j\}$ and $\{j,i\}$. We denote by $\Gamma_{i,j} := \partial\Omega_i \cap \partial\Omega_j$ the interface between two subdomains for $\{i,j\} \in I$, and by $\gamma := \bigcup_{\{i,j\}\in I} \Gamma_{i,j}$, the skeleton of the domain decomposition.

On each subdomain Ω_i, we introduce the function spaces $M_i := L^2(\Omega_i)$ and

$$V_i = H(\Omega_i; \text{div}) := \{v \in M_i^d \text{ s.t. } \nabla \cdot v \in M_i\},$$

endowed with their usual norms denoted by $\|q_i\|_{0,i}$ and $\|v_i\|_{V_i} := \Big(\|v_i\|_{0,i}^2 + \|\nabla \cdot v_i\|_{0,i}^2\Big)^{1/2}$, respectively. On the domain Ω, we define the product spaces

$$M := \bigoplus_{i=1}^{N} M_i = L^2(\Omega) \text{ and } V := \bigoplus_{i=1}^{N} V_i,$$

endowed with their Hilbertian product norms $\|q\|_0$ and $\|v\|_V$, respectively. The $L^2(\Omega)^d$ norm is denoted by $\|\cdot\|_0$.

In the non-overlapping domain decomposition framework, the smoothness assumptions on the solution will be as usual measured in the broken norms $\|\cdot\|_{\mathcal{H}^r(\Omega)}$ related to the product spaces

$$\mathcal{H}^r(\Omega) := \bigoplus_{i=1}^{N} H^r(\Omega_i), \ r \geq 0.$$

On the skeleton γ, we define the norm

$$\|\mu\|_{\frac{1}{2},\gamma} := \sup_{v \in V} \frac{\sum_{i=1}^{N} \int_{\Gamma_i} (v \cdot n_i)\mu d\gamma}{\|v\|_V}$$

and we shall denote by $H^{\frac{1}{2}}(\gamma)$, the subspace of $L^2(\gamma)$ of functions μ such that $\|\mu\|_{\frac{1}{2},\gamma} < \infty$.

We consider on the domain decomposition $(\Omega_i)_{i=1,\dots,N}$, a Mortar Mixed Finite Element (MMFE) discretization of (1.1), introduced in [GW88] for matching grids, and extended in [Yot96], [ACWY00] to the case of non-matching grids at the interfaces between the subdomains Ω_i. In that case, a so called Mortar space $\Lambda_h \subset L^2(\gamma)$ is introduced on the skeleton γ. Then, equation (1.1) is discretized on each subdomain by a Mixed Finite Element Method, and the matching at the interfaces is forced in a weak sense through the continuity of the orthogonal projection on Λ_h of the normal fluxes defined on either side of $\Gamma_{i,j}$.

Let $\mathcal{T}_{i,h}$ be a quasi-uniform family of meshes of Ω_i. We consider, on these grids, MFE approximation spaces $V_{i,h} \subset V_i$, $M_{i,h} \subset M_i$ of order $k+1$. that can be either the Raviart-Thomas or Brezzi-Douglas-Fortin, or Brezzi-Douglas-Fortin-Marini mixed finite elements of order $k+1$, denoted respectively by RT_k, BDF_k, and BDFM_k (see [Tho77], [TR91] or [BF91] for their description). In addition we shall assume in the sequel that $\nabla \cdot V_{i,h} \subset M_{i,h}$.

On the domain Ω, we define the product spaces

$$M_h := \bigoplus_{i=1}^{N} M_{i,h} \subset M \text{ and } V_h := \bigoplus_{i=1}^{N} V_{i,h} \subset V.$$

The dual space of V_h is denoted by V_h' and endowed with the dual norm $\|\cdot\|_{V_h'}$. The dual space of M_h, denoted by M_h', will be implicitely identified with M_h. We shall denote by $\langle \cdot, \cdot \rangle$ the duality pairing.

The choice of the Mortar space Λ_h is described and discussed in [Yot96]. Let $\mathcal{T}_{i,j,h}$, $\{i,j\} \in I$ be a quasi-uniform family of meshes on $\Gamma_{i,j}$ and $\Lambda_{i,j.h}$ a finite element space on $\mathcal{T}_{i,j,h}$, either continuous or discontinuous, and of order $k+2$. The Mortar space on the skeleton γ is the product space

$$\Lambda_h := \bigoplus_{\{i,j\}\in I} \Lambda_{i,j,h} \subset L^2(\gamma).$$

Remark 2.1 When considering matching grids at the interfaces $\Gamma_{i,j}$, the natural choices for the meshes $\mathcal{T}_{i,j,h}$ and the spaces $\Lambda_{i,j.h}$ are respectively $\mathcal{T}_{i,h}|_{\Gamma_{i,j}} = \mathcal{T}_{j,h}|_{\Gamma_{i,j}}$, and the FE spaces $V_{i,h}.n|_{\Gamma_{i,j}} = V_{j,h}.n|_{\Gamma_{i,j}}$ of order $k+1$. In the case of non-matching grids, the order of approximation $k+2$ is justified to preserve the optimal order of approximation $k+1$ of the MFE discretization (see [Yot96] or the proof of the error estimates in Sect. 3.2).

In order to write the MMFE variational formulation of (1.1), we define the operators $S_h, A_h : V_h \to V_h'$, $B_h^t : \Lambda_h \to V_h'$, $\text{div}_h : V_h \to M_h'$, $T_h^t : H^{1/2}(\Gamma) \to V_h'$ such that for all $v_h = (v_{i,h})_{i=1,\dots,N}$, $w_h = (w_{i,h})_{i=1,\dots,N} \in V_h$, $q_h = (q_{i,h})_{i=1,\dots,N} \in M_h$, $\mu_h \in \Lambda_h$, $\varphi \in H^{1/2}(\Gamma)$:

$$\begin{aligned}
\langle S_h v_h, w_h \rangle &:= \sum_{i=1}^{N} \int_{\Omega_i} K^{-1} v_{i,h} \cdot w_{i,h} dx, \\
\langle A_h v_h, w_h \rangle &:= \sum_{i=1}^{N} \int_{\Omega_i} (\nabla \cdot v_{i,h})(\nabla \cdot w_{i,h}) dx, \\
\langle \text{div}_h v_h, q_h \rangle &:= \sum_{i=1}^{N} \int_{\Omega_i} (\nabla \cdot v_{i,h}) q_{i,h} dx, \\
\langle B_h^t \mu_h, v_h \rangle &:= \sum_{i=1}^{N} \int_{\Gamma_i} \mu_h (v_{i,h} \cdot n_i) d\gamma, \\
\langle T_h^t \varphi, v_h \rangle &:= \int_{\Gamma} \varphi (v_h \cdot n) d\sigma.
\end{aligned} \tag{2.1}$$

We shall also use the notations i_{V_h} and i_{M_h} for the continuous embeddings from V_h to V and from M_h to M respectively. Then, the MMFE spatial discretization of (1.1) is to find $(p_h, u_h, p_{\gamma,h}) \in M_h \times V_h \times \Lambda_h$ such that

(2.2)
$$\begin{cases} \partial_t p_h + \mathrm{div}_h u_h = i^t_{M_h} f, \\ S_h u_h = \mathrm{div}^t_h p_h - B^t_h p_{\gamma,h} - T^t_h g, \\ B_h u_h = 0, \\ p_h|_{t=0} = p_{0,h}. \end{cases}$$

The stationnary MMFE approximation (2.2) is analysed in [Yot96] and [ACWY00]. In order to obtain a well posed problem, one has to assume that the Mortar space Λ_h satisfies a compatibility condition with the normal trace on γ of V_h. Let us define the subspace of V_h:

$$W_h := \{v_h \in V_h \text{ s.t. } B_h v_h = 0\}\,.$$

The compatibility condition ensures in particular that the operator B^t_h is injective as well as that the property

$$\{q_h, \text{ s.t. } \langle \mathrm{div}_h v_h, q_h\rangle = 0, \text{ for all } v_h \in W_h\} = \{0\},$$

is satisfied which all together guarantees existence and uniqueness of the solution. For the convenience of the reader, this condition is reproduced in Hypothesis 2.1 below.

Hypothesis 2.1 *Let $Q_{i,h}$ be the orthogonal projector from $L^2(\Gamma_i)$ onto $V_{i,h} \cdot n_i|_{\Gamma_i}$. Then, we assume the following stability condition to hold uniformly in h:*

$$\|\mu_h\|_{L^2(\Gamma_{i,j})} \lesssim \|Q_{i,h}\mu_h\|_{L^2(\Gamma_{i,j})} + \|Q_{j,h}\mu_h\|_{L^2(\Gamma_{i,j})}, \textit{ for all } \mu_h \in \Lambda_h.$$

Under this assumption, a projector $\Pi_h : V \to W_h$ is built in [Yot96] which satisfies the following error estimates:

(2.3)
$$\begin{aligned} &\langle \nabla\cdot(\Pi_h u - u), q_h\rangle = 0, \text{ for all } q_h \in M_h, \\ &\|\nabla\cdot(\Pi_h u - u)\|_0 \lesssim h^r \|\nabla\cdot u\|_{\mathcal{H}^r(\Omega)}, \; 1 \le r \le k+1, \\ &\|\Pi_h u - u\|_0 \lesssim h^r \|u\|_{\mathcal{H}^r(\Omega)^d}, \; 1 \le r \le k+1. \end{aligned}$$

2.1 An equivalent flux formulation

As a preliminary step toward the time discretization by an operator splitting technique, it is useful to introduce an equivalent flux formulation of (2.2) obtained by elimination of the discrete pressure unknown in (2.2). This formulation will also be crucial to analyse the stability and the error estimates of our method.

Proposition 2.1 *Let us define* $\lambda_h := \partial_t p_{\gamma,h}$ *and* $g_0 := g|_{t=0}$. *Then problem (2.2) has the following equivalent flux formulation:*

$$\begin{cases} S_h\partial_t u_h + A_h u_h + B_h^t\lambda_h + T_h^t\partial_t g = \mathrm{div}_h^t f, \\ B_h u_h = 0, \\ u_h|_{t=0} = u_h^0, \end{cases} \tag{2.4}$$

given the initialization

$$\begin{cases} S_h u_h^0 = \mathrm{div}_h^t p_{0,h} - B_h^t p_{\gamma,h}^0 - T_h^t g_0, \\ B_h u_h^0 = 0, \end{cases} \tag{2.5}$$

and the pressure equations

$$\begin{cases} \partial_t p_h + \mathrm{div}_h u_h = i_{M_h}^t f, \\ \partial_t p_{\gamma,h} = \lambda_h, \\ p_h|_{t=0} = p_{0,h},\ p_{\gamma,h}|_{t=0} = p_{\gamma,h}^0. \end{cases} \tag{2.6}$$

Proof. the proof relies on elementary algebra using the assumption on the MFE spaces that $\nabla\cdot V_h \subset M_h$, and assuming enough regularity on the solution.

3 Time discretization by projection schemes

The flux formulation (2.4) has the structure of a discrete Stokes problem. The idea of the time discretization by operator splitting is then to apply to the flux formulation (2.4) a projection scheme closely related to a scheme introduced by Chorin in [CL96] and analysed in [Ran92] in the framework of the Navier-Stokes equations.

In the framework of the MMFE method, the projection scheme splits the system (2.4) into two successive steps: (i) advance in time with $\lambda_h = 0$, and (ii) project the flux orthogonally (with respect to the scalar product $\langle S_h\cdot,\cdot\rangle$) onto W_h. We have then:

$$(i)\ \ S_h\frac{\tilde u_h^{n+1} - u_h^n}{\Delta t} + A_h\tilde u_h^{n+1} + T_h^t\frac{g^{n+1} - g^n}{\Delta t} = \mathrm{div}_h^t f^{n+1}, \tag{3.1}$$

$$(ii)\ \begin{cases} S_h\dfrac{u_h^{n+1} - \tilde u_h^{n+1}}{\Delta t} + B_h^t\lambda_h^{n+1} = 0, \\ B_h u_h^{n+1} = 0, \end{cases} \tag{3.2}$$

The pressures p_h^n et $p_{\gamma,h}^n$ are recovered by a discrete integration in time of the equations

$$\begin{cases} \dfrac{p_h^{n+1} - p_h^n}{\Delta t} + \mathrm{div}_h\tilde u_h^{n+1} = i_{M_h}^t f^{n+1},\ p_h^0 = p_{0,h}, \\ \dfrac{p_{\gamma,h}^{n+1} - p_{\gamma,h}^n}{\Delta t} = \lambda_h^{n+1},\ p_{\gamma,h}^0 \text{ given by (2.5)}, \end{cases} \tag{3.3}$$

and the initial flux u_h^0 is defined by (2.5).

As for the semi-discrete formulation, the space-time discretization (3.1)-(3.2)-(3.3) admits an equivalent mixed pressure-flux formulation which, from elementary algebra, writes:

$$(i)\begin{cases} \dfrac{p_h^{n+1}-p_h^n}{\Delta t} + \mathrm{div}_h \tilde{u}_h^{n+1} = i_{M_h}^t f^{n+1}, \\ S_h \tilde{u}_h^{n+1} = \mathrm{div}_h^t p_h^{n+1} - B_h^t p_{\gamma,h}^n - T_h^t g^{n+1}, \end{cases} \tag{3.4}$$

$$(ii)\begin{cases} S_h u_h^{n+1} = \mathrm{div}_h^t p_h^{n+1} - B_h^t p_{\gamma,h}^{n+1} - T_h^t g^{n+1}, \\ B_h u_h^{n+1} = 0, \end{cases} \tag{3.5}$$

given $p_h^0 := p_{0,h}$ and $p_{\gamma,h}^0$ defined by equation (2.5). From (3.4), we note that step (i) corresponds to the explicit extrapolation of the interface pressure, i.e. $p_{\gamma,h}^{n+1} \simeq p_{\gamma,h}^n$.

There exists an incremental version of Chorin's projection scheme, which is known to be more accurate in time (see [She92] or [GQ98]). Applied to the MMFE flux formulation (2.4), the incremental projection scheme splits the system (2.4) into two steps: (i) advance in time with λ_h given by the previous time step, (ii) orthogonal projection (with respect to the scalar product $\langle S_h\cdot,\cdot\rangle$) of the flux onto W_h, and update of λ_h.

$$\begin{aligned}(i)\ \ S_h \frac{\tilde{u}_h^{n+1}-u_h^n}{\Delta t} + A_h \tilde{u}_h^{n+1} + B_h^t \lambda_h^n + T_h^t \frac{g^{n+1}-g^n}{\Delta t} \\ = \mathrm{div}_h^t f^{n+1},\end{aligned} \tag{3.6}$$

$$(ii)\begin{cases} S_h \dfrac{u_h^{n+1}-\tilde{u}_h^{n+1}}{\Delta t} + B_h^t(\lambda_h^{n+1}-\lambda_h^n) = 0, \\ B_h u_h^{n+1} = 0, \end{cases} \tag{3.7}$$

Remark 3.1 The initialization of the flux is still given by equation (2.5). Compared with the non-incremental projection scheme, in addition the incremental scheme requires an approximation $\lambda_h^0 \in \Lambda_h$ of $\lambda|_{t=0}$. To obtain first order accuracy in time, we shall see that it will suffice to set $\lambda_h^0 = 0$. However, in order to expect second order accuracy, a first order accurate approximation of λ_h^0 has to be obtained by calculating one time step of the fully coupled system with a second order accurate time discretization.

The pressures p_h^n and $p_{\gamma,h}^n$ are again recovered by a discrete integration in time of equations (3.3).

It can be easily checked that the equivalent mixed pressure-flux formulation of (3.6)-(3.7)-(3.3) corresponds, at step (i), to a second order linear extrapolation in time of the interface pressure, i.e. $p^{n+1}_{\gamma,h} \simeq 2p^n_{\gamma,h} - p^{n-1}_{\gamma,h}$, rather than to the first order extrapolation $p^{n+1}_{\gamma,h} \simeq p^n_{\gamma,h}$ obtained for the non-incremental scheme. We have then:

$$(3.8)\quad (i) \begin{cases} \dfrac{p_h^{n+1} - p_h^n}{\Delta t} + \mathrm{div}_h \tilde{u}_h^{n+1} = i^t_{M_h} f^{n+1}, \\ S_h \tilde{u}_h^{n+1} = \mathrm{div}^t_h p_h^{n+1} - B^t_h(2p^n_{\gamma,h} - p^{n-1}_{\gamma,h}) - T^t_h g^{n+1}, \end{cases}$$

$$(3.9)\qquad (ii) \begin{cases} S_h u_h^{n+1} = \mathrm{div}^t_h p_h^{n+1} - B^t_h p^{n+1}_{\gamma,h} - T^t_h g^{n+1}, \\ B_h u_h^{n+1} = 0, \end{cases}$$

with $p^0_h := p_{0,h}$ and $p^{-1}_{\gamma,h} := p^0_{\gamma,h} - \Delta t \lambda^0_h$.

The main advantage of the projection scheme is that the prediction step (i) can be solved in a fully parallel way on each subdomain independently, while the projection step (ii) reduces to solve the interface problem related to the operator $B_h S_h^{-1} B_h^t$.

Let us restrict ourselves to the assumption that only RT_0 mixed finite elements are used in the neighborhood of the skeleton γ. Then, a mass condensation of the matrix representing the operator S_h in the canonical basis can be performed, preserving the order of approximation of the discretization. It follows then that the interface operator matrix in the canonical basis of Λ_h is diagonal and can be readily inverted in $\mathcal{O}(N_{\Lambda_h})$ operations where N_{Λ_h} is the dimension of Λ_h.

More generally, the interface problem can be efficiently solved by a conjugate gradient algorithm preconditioned by the approximate interface matrix obtained by mass condensation of S_h in the neighborhood of γ.

3.1 Stability analysis of the projection scheme

Let $Z_h := B_h S_h^{-1} B_h^t$, from Λ_h to Λ'_h, denote the interface operator related to the projection step (ii). Extending the definition (2.1) of B^t_h to $L^2(\gamma)$, Z_h also operates from $L^2(\gamma)$ to $L^2(\gamma)$, and we shall keep the same notations for these two operators for simplicity. Then, for any $\mu \in L^2(\gamma)$, we set $\|\mu\|_{Z_h} := \langle Z_h \mu, \mu \rangle^{\frac{1}{2}}$, which defines a semi-norm on $L^2(\gamma)$ and a norm on Λ_h from Hypothesis 2.1.

Let I_h denote the Riesz operator from V_h to V'_h. We also need to define the semi-norm on $L^2(\gamma)$ (norm on Λ_h) related to the interface operator $B_h(A_h + I_h)^{-1} B_h^t$:

$$\|B^t_h \mu\|_{V'_h} := \sup_{v_h \in V_h} \frac{\sum_{i=1}^N \int_{\Gamma_i} (v_h \cdot n_i) \mu d\gamma}{\|v_h\|_V} = \langle B_h(A_h + I_h)^{-1} B^t_h \mu, \mu \rangle^{\frac{1}{2}}.$$

Finally, for $u \in L^2(\Omega)^d$, we denote by $\|u\|_S$, the hilbertian norm $\left(\int_\Omega K^{-1} u\cdot udx\right)^{\frac{1}{2}}$.

The stability analysis of the incremental scheme is carried out in its equivalent flux formulation (3.6)-(3.7)-(3.3) in order to avoid having to deal with the three step equations (3.8)-(3.9). It is then formally similar to the analysis performed for Navier Stokes equations (see [She92], [GQ98] and also [BA98]) with necessary adaptations to the framework of domain decomposition and MMFE.

Theorem 3.1 *Let $t_n := n\Delta t$, and assume that $\partial_t g \in L^2(0, t_m; H^{\frac{1}{2}}(\Gamma))$, $\sum_{n=0}^{m-1} \Delta t\|f^{n+1}\|_0^2 \lesssim 1$, then the incremental projection scheme (3.6)-(3.7)-(3.3) or (3.8)-(3.9) is unconditionally stable in the sense that for all $\Delta t \leq 1$ one has*

$$
\begin{cases}
\|u_h^m\|_S^2 + \Delta t^2\|\lambda_h^m\|_{Z_h}^2 + \sum_{n=0}^{m-1} \Delta t\|\nabla \cdot \tilde{u}_h^{n+1}\|_0^2 \\
\quad \lesssim \|u_h^0\|_S^2 + \Delta t^2\|\lambda_h^0\|_{Z_h}^2 + \sum_{n=0}^{m-1} \Delta t\|f^{n+1}\|_0^2 \\
\quad + \int_0^{t_m} \|\partial_t g(s)\|_{H^{\frac{1}{2}}(\Gamma)}^2 ds, \\
\|p_h^m\|_0^2 \lesssim \|p_{0,h}\|_0^2 + \sum_{n=0}^{m-1} \Delta t\|\nabla \cdot \tilde{u}_h^{n+1}\|_0^2 \\
\quad + \sum_{n=0}^{m-1} \Delta t\|f^{n+1}\|_0^2, \\
\|B_h^t p_{\gamma,h}^m\|_{V_h'} \lesssim \|u_h^m\|_0 + \|p_h^m\|_0 + \|g^m\|_{H^{\frac{1}{2}}(\Gamma)},
\end{cases}
\tag{3.10}
$$

with constants independent of h, N, and Δt.

Proof. Considering the duality pairing of (3.6) with $\tilde{u}_h^{n+1}$, we obtain for all $\delta > 0$

$$
\begin{aligned}
&\|\tilde{u}_h^{n+1}\|_S^2 + \|\tilde{u}_h^{n+1} - u_h^n\|_S^2 - \|u_h^n\|_S^2 + 2\Delta t\|\nabla \cdot \tilde{u}_h^{n+1}\|_0^2 \\
&\quad +2\Delta t\langle B_h^t \lambda_h^n, \tilde{u}_h^{n+1}\rangle \\
&\quad \leq \delta\Delta t\Big(\|\nabla \cdot \tilde{u}_h^{n+1}\|_0^2 + \|\tilde{u}_h^{n+1} - u_h^n\|_S^2 + \|u_h^n\|_S^2\Big) \\
&\qquad + c_\delta\left(\Delta t\|f^{n+1}\|_0^2 + \int_{t_n}^{t_{n+1}} \|\partial_t g(s)\|_{H^{\frac{1}{2}}(\Gamma)}^2 ds\right),
\end{aligned}
\tag{3.11}
$$

with c_δ independent of h, Δt, and N. To control $2\Delta t\langle B_h^t \lambda_h^n, \tilde{u}_h^{n+1}\rangle$, we consider the equations (3.7). First, u_h^{n+1} is the orthogonal projection of

$\tilde{u}_h^{n+1}$ on W_h with respect to the scalar product defined by S_h, hence

$$\|u_h^{n+1}\|_S^2 + \|u_h^{n+1} - \tilde{u}_h^{n+1}\|_S^2 - \|\tilde{u}_h^{n+1}\|_S^2 = 0. \tag{3.12}$$

Then, taking the duality pairing of (3.7) with successively $\Delta t^2 S_h^{-1} B_h^t \lambda_h^n$ and $u_h^{n+1} - \tilde{u}_h^{n+1}$, we obtain the relation:

$$\begin{aligned} &-\|u_h^{n+1} - \tilde{u}_h^{n+1}\|_S^2 - 2\Delta t \langle B_h^t \lambda_h^n, \tilde{u}_h^{n+1}\rangle \\ &+\Delta t^2 \|\lambda_h^{n+1}\|_{Z_h}^2 - \Delta t^2 \|\lambda_h^n\|_{Z_h}^2 = 0. \end{aligned} \tag{3.13}$$

Adding (3.11), (3.12), (3.13), and summing up the resulting inequalities from $n = 0$ to $n = m-1$ for $\delta = 1$, we obtain the estimate

$$\begin{aligned} \|u_h^m\|_S^2 + \Delta t^2 \|\lambda_h^m\|_{Z_h}^2 + \sum_{n=0}^{m-1} \Delta t \|\nabla \cdot \tilde{u}_h^{n+1}\|_0^2 \lesssim \|u_h^0\|_S^2 + \Delta t^2 \|\lambda_h^0\|_{Z_h}^2 \\ + \sum_{n=0}^{m-1} \Delta t \|u_h^n\|_S^2 + \sum_{n=0}^{m-1} \Delta t \|f^{n+1}\|_0^2 + \int_0^{t_m} \|\partial_t g(s)\|_{H^{\frac{1}{2}}(\Gamma)}^2 ds. \end{aligned}$$

The flux stability result in (3.10) is then a direct application of the Gromwall's Lemma (see [HR90]).

For the pressure stability, we take the scalar product of the first equation in (3.3) with p_h^{n+1} and apply the Cauchy Schwarz Inequality to obtain:

$$\|p_h^{n+1}\|_0 \leq \|p_h^n\|_0 + \Delta t \|\nabla \cdot \tilde{u}_h^{n+1}\|_0 + \Delta t \|f^{n+1}\|_0.$$

Summing up these inequalities from $n = 0$ to $n = m-1$, we obtain the second stability estimate in (3.10).

Finally, the interface pressure stability is readily obtained from equation (3.9). □

The stability analysis of the non-incremental scheme is carried out in a similar way also using the flux formulation.

Theorem 3.2 *Assume $\partial_t g \in L^2(0, t_m; H^{\frac{1}{2}}(\Gamma))$, $\sum_{n=0}^{m-1} \Delta t \|f^{n+1}\|_0^2 \lesssim 1$, then the incremental projection scheme (3.1)-(3.2)-(3.3) or (3.4)-(3.5) is unconditionally stable in the sense that for all $\Delta t \leq 1$ one has*

$$
(3.14)\quad \begin{cases}
\|u_h^m\|_0^2 + \Delta t \sum_{n=0}^{m-1} \Delta t \|\lambda_h^{n+1}\|_{Z_h}^2 + \sum_{n=0}^{m-1} \Delta t \|\nabla \cdot \tilde{u}_h^{n+1}\|_0^2 \\
\quad \lesssim \|u_h^0\|_0^2 + \sum_{n=0}^{m-1} \Delta t \|f^{n+1}\|_0^2 + \int_0^{t_m} \|\partial_t g(s)\|_{H^{\frac{1}{2}}(\Gamma)}^2 ds, \\
\|p_h^m\|_0^2 \lesssim \|p_{0,h}\|_0^2 + \sum_{n=0}^{m-1} \Delta t \|\nabla \cdot \tilde{u}_h^{n+1}\|_0^2 \\
\quad + \sum_{n=0}^{m-1} \Delta t \|f^{n+1}\|_0^2, \\
\|B_h^t p_{\gamma,h}^m\|_{V_h'} \lesssim \|u_h^m\|_0 + \|p_h^m\|_0 + \|g^m\|_{H^{\frac{1}{2}}(\Gamma)},
\end{cases}
$$

with constants independent of h, N, and Δt.

3.2 Error estimates

Let $(u, p) \in C^0(0, t_m; H(\Omega; \mathrm{div})) \times C^0(0, t_m; M)$ denote the weak solution of (1.1). We shall use the notations $t_n = n\Delta t$, and $u^n := u(t_n), p^n := p(t_n)$, $\lambda^n := \lambda(t_n)$, $p_\gamma^n := p_\gamma(t_n)$. We consider the orthogonal projector from M onto M_h denoted by ρ_h, and the orthogonal projector from $L^2(\gamma)$ onto Λ_h denoted by $\mathcal{R}_h$. Then, we define the discrete errors $e_{u,h}^n := \Pi_h u^n - u_h^n$, $\tilde{e}_{u,h}^n := \Pi_h u^n - u_h^n$, $\varepsilon_h^n := \mathcal{R}_h \lambda^n - \lambda_h^n$, $e_{p,h}^n := \rho_h p^n - p_h^n$, and $e_{\gamma,h}^n := \mathcal{R}_h p_\gamma^n - p_{\gamma,h}^n$.

3.2.1 Incremental scheme. The error analysis of the incremental scheme is done in its flux formulation with the assumption that both the pressure p and $\partial_t p$ are globally in $H^1(\Omega)$ in order to define the interface pressure $p_\gamma := p|_\gamma$ and its derivative $\lambda := \partial_t p|_\gamma = \partial_t p_\gamma$ in $H^{1/2}(\gamma)$.

Again, the error estimates are obtained by extension of the analysis in [She92] or [GQ98]) for Navier Stokes equations to the framework of domain decomposition and MMFE.

Theorem 3.3 *Assuming Hypothesis 2.1 and* $(u, p) \in C^0(0, t_m; H(\Omega; \mathrm{div}))$ $\times C^0(0, t_m; M)$*,* $p \in C^1(0, t_m; H^1(\Omega))$*, the incremental scheme (3.6)-(3.7)-(3.3) or (3.8)-(3.9) satisfies the error estimates:*

$$\|e^m_{u,h}\|^2_S + \Delta t^2\|\varepsilon^m_h\|^2_{Z_h} + \sum_{n=0}^{m-1}\Delta t\|\nabla\cdot\tilde{e}^{n+1}_{u,h}\|^2_0 \lesssim \|e^0_{u,h}\|^2_S + \Delta t^2\|\varepsilon^0_h\|^2_{Z_h}$$
$$+\Delta t^2\int_0^{t_m}\|\mathcal{R}_h\partial_t\lambda(s)\|^2_{Z_h}ds + \Delta t^2\int_0^{t_m}\|\partial_{t^2}g(s)\|^2_{H^{\frac{1}{2}}(\Gamma)}ds$$
$$+\Delta t^2\int_0^{t_m}\|\partial_{t^2}u(s)\|^2_{V'_h}ds + \int_0^{t_m}\|(\Pi_h - I)\partial_t u(s)\|^2_{V'_h}ds \tag{3.15}$$
$$+\sum_{n=0}^{m-1}\Delta t\sum_{i=1}^{N}h^{-1}\|(\mathcal{R}_h - I)\lambda^{n+1}\|^2_{L^2(\Gamma_i)},$$
$$\|e^m_{p,h}\|^2_0 \lesssim \|e^0_{p,h}\|^2_0 + \sum_{n=0}^{m-1}\Delta t\|\nabla\cdot\tilde{e}^{n+1}_{u,h}\|^2_0 + \Delta t^2\int_0^{t_m}\|\partial_{t^2}p(s)\|^2_0 ds,$$

$$\|B^t_h e^m_{\gamma,h}\|_{V'_h} \lesssim \|p^m - p^m_h\|_0 + \|u^m - u^m_h\|_0 + \left(\sum_{i=1}^{N}h^{-1}\|(\mathcal{R}_h - I)p^m_\gamma\|^2_{L^2(\Gamma_i)}\right)^{\frac{1}{2}},$$

for all $\Delta t \le 1$ *and with constants independent of* h, Δt, *and* N.

Proof. From our regularity assumptions, the solution (u,p) verifies on each subdomain Ω_i, $i = 1,\dots,N$, for all $q_h \in M_h$ and $v_h \in V_h$

$$\int_{\Omega_i}(\partial_t p)q_h dx + \int_{\Omega_i}(\nabla\cdot u)q_h = \int_{\Omega_i} f q_h,$$

$$\int_{\Omega_i}K^{-1}u\cdot v_h dx = \int_{\Omega_i}(\nabla\cdot v_h)p - \int_{\Gamma_i}p_\gamma(v_h\cdot n_i)d\gamma - \int_{\partial\Omega_i\cap\Gamma}g(v_h\cdot n)d\sigma, \tag{3.16}$$

$$\int_{\Omega_i}K^{-1}\partial_t u\cdot v_h dx = \int_{\Omega_i}(\nabla\cdot v_h)\partial_t p - \int_{\Gamma_i}\lambda(v_h\cdot n_i)d\gamma - \int_{\partial\Omega_i\cap\Gamma}\partial_t g(v_h\cdot n)d\sigma.$$

Setting $q_h = \nabla\cdot v_h$ in (3.16), and combining the above equations we obtain:

$$\begin{aligned} &S_h u = \mathrm{div}^t_h p - B^t_h p_\gamma - T^t_h g,\\ &S_h\partial_t u + A_h u + B^t_h\lambda + T^t_h\partial_t g = \mathrm{div}^t_h f,\\ &i^t_{M_h}\partial_t p + \mathrm{div}_h u = i^t_{M_h} f, \end{aligned} \tag{3.17}$$

where, for the sake of conciseness, we have implicitely extended the operators S_h, A_h, div_h, B_h to the space V, B_h^t to $H^{1/2}(\gamma)$, and div_h^t to M. On the other hand, since $u \in H(\Omega; \text{div})$, we have $B_h u = 0$ and $B_h \Pi_h u = 0$. Note that, from the identification of M_h and M_h', we can also identify $i_{M_h}^t$ with the orthogonal projector ρ_h.

Combining (3.6)-(3.7)-(3.3)-(3.8)-(3.9) with (3.18) taken at time t_{n+1}, we obtain the equations governing the errors $e_{u,h}^n$,$\tilde{e}_{u,h}^n$, $e_{p,h}^n$, and ε_h^n, $e_{\gamma,h}^n$:

$$\begin{cases} S_h \dfrac{\tilde{e}_{u,h}^{n+1} - e_{u,h}^n}{\Delta t} + A_h \tilde{e}_{u,h}^{n+1} + B_h^t(\varepsilon_h^n + \mathcal{R}_h \Delta\lambda^{n+1}) = R_u^{n+1}, \\ \dfrac{e_{p,h}^{n+1} - e_{p,h}^n}{\Delta t} + \text{div}_h \tilde{e}_{u,h}^{n+1} = R_p^{n+1}, \end{cases} \tag{3.18}$$

$$\begin{cases} S_h \dfrac{e_{u,h}^{n+1} - \tilde{e}_{u,h}^{n+1}}{\Delta t} + B_h^t(\varepsilon_h^{n+1} - \varepsilon_h^n - \mathcal{R}_h \Delta\lambda^{n+1}) = 0, \\ B_h e_{u,h}^{n+1} = 0, \end{cases} \tag{3.19}$$

$$\begin{aligned} B_h^t e_{\gamma,h}^{n+1} &= -S_h(u^{n+1} - u_h^{n+1}) + \text{div}_h^t(p^{n+1} - p_h^{n+1}) \\ &\quad + B_h^t(\mathcal{R}_h - I)p_\gamma^{n+1}, \end{aligned} \tag{3.20}$$

where $\Delta\lambda^{n+1} = \lambda^{n+1} - \lambda^n$, and

$$\begin{aligned} R_u^{n+1} &:= -\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (s - t_n) S_h \partial_{t^2} u(s) ds \\ &\quad + \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} S_h(\Pi_h - I)\partial_t u(s) ds \\ &\quad + B_h^t(\mathcal{R}_h - I)\lambda^{n+1} + \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (s - t_n) T_h^t \partial_{t^2} g(s) ds, \\ R_p^{n+1} &:= -\frac{1}{\Delta t} i_{M_h}^t \int_{t_n}^{t_{n+1}} (s - t_n)\partial_{t^2} p(s) ds. \end{aligned} \tag{3.21}$$

Using (3.18)-(3.19)-(3.20), we proceed as in the proof of Theorem 3.1 to obtain the estimates:

$$\begin{aligned} &\|e_{u,h}^{n+1}\|_S^2 + \|\tilde{e}_{u,h}^{n+1} - e_{u,h}^n\|_S^2 + \Delta t^2 \|\varepsilon_h^{n+1}\|_{Z_h}^2 + 2\Delta t \|\nabla \cdot \tilde{e}_{u,h}^{n+1}\|_0^2 \\ &\quad \le \|e_{u,h}^n\|_S^2 + \Delta t^2 \|\varepsilon_h^n \\ &\qquad + \mathcal{R}_h \Delta\lambda^{n+1}\|_{Z_h}^2 + 2\Delta t \langle R_u^{n+1}, \tilde{e}_{u,h}^{n+1}\rangle, \\ &\|e_{p,h}^{n+1}\|_0 \le \|e_{p,h}^n\|_0 + \Delta t \|\nabla \cdot \tilde{e}_{u,h}^{n+1}\|_0 + \Delta t \|R_p^{n+1}\|_0, \\ &\|B_h^t e_{\gamma,h}^{n+1}\|_{V_h'} \lesssim \|u^{n+1} - u_h^{n+1}\|_0 + \|p^{n+1} - p_h^{n+1}\|_0 \\ &\qquad + \|B_h^t(\mathcal{R}_h - I)p_\gamma^{n+1}\|_{V_h'}. \end{aligned} \tag{3.22}$$

It remains to estimate $\langle R_u^{n+1}, \tilde{e}_{u,h}^{n+1}\rangle$, $\|R_p^{n+1}\|_0$, $\|B_h^t(\mathcal{R}_h - I)p_\gamma^{n+1}\|_{V_h'}$, and $\|\varepsilon_h^n + \mathcal{R}_h \Delta\lambda^{n+1}\|_{Z_h}^2$.

Applying the inverse inequality $\|v_h \cdot n\|_{L^2(\Gamma_i)} \lesssim h^{-1/2}\|v_h\|_{L^2(\Omega_i)^d}$ for all $v_h \in V_h$, we obtain

$$\|B_h^t(\mathcal{R}_h - I)p_\gamma^{n+1}\|_{V_h'} \lesssim \left(\sum_{i=1}^N h^{-1}\|(\mathcal{R}_h - I)p_\gamma^{n+1}\|_{L^2(\Gamma_i)}^2\right)^{\frac{1}{2}},$$

which proves the interface pressure error estimate in (3.15). Similarly, for all $\delta > 0$ (and c_δ independent of h, Δt, and N), one has

$$\begin{aligned}
&\langle B_h^t(\mathcal{R}_h - I)\lambda^{n+1}, \tilde{e}_{u,h}^{n+1}\rangle \\
&\qquad \lesssim \sum_{i=1}^N h^{-1/2}\|(\mathcal{R}_h - I)\lambda^{n+1}\|_{L^2(\Gamma_i)}\|\tilde{e}_{u,h}^{n+1}\|_{L^2(\Omega_i)^d} \\
&\qquad \le \delta\Big(\|\tilde{e}_{u,h}^{n+1} - e_{u,h}^n\|_S^2 + \|e_{u,h}^n\|_S^2\Big) \\
&\qquad\quad + c_\delta \sum_{i=1}^N h^{-1}\|(\mathcal{R}_h - I)\lambda^{n+1}\|_{L^2(\Gamma_i)}^2.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\langle R_u^{n+1}, \tilde{e}_{u,h}^{n+1}\rangle \\
&\quad \le \delta\Big(\|\tilde{e}_{u,h}^{n+1} - e_{u,h}^n\|_S^2 + \|e_{u,h}^n + \|\nabla\cdot\tilde{e}_{u,h}^{n+1}\|_0^2\Big) \\
&\quad\quad + c_\delta\left(\sum_{i=1}^N h^{-1}\|(\mathcal{R}_h - I)\lambda^{n+1}\|_{L^2(\Gamma_i)}^2\right. \\
&\quad\quad + \Delta t\int_{t_n}^{t_{n+1}}\|\partial_{t^2}u(s)\|_{V'}^2 ds + \Delta t\int_{t_n}^{t_{n+1}}\|\partial_{t^2}g(s)\|_{H^{\frac{1}{2}}(\Gamma)}^2 ds \\
&\quad\quad \left. + \int_{t_n}^{t_{n+1}}\|(\Pi_h - I)\partial_t u(s)\|_0^2 ds\right).
\end{aligned} \tag{3.23}$$

On the over hand, for all $\delta > 0$, there exists c_δ independent of h, Δt, and N, such that

$$\begin{aligned}
\|\varepsilon_h^n + \mathcal{R}_h\Delta\lambda^{n+1}\|_{Z_h}^2 &\le (1+\delta\Delta t)\|\varepsilon_h^n\|_{Z_h}^2 \\
&\quad + c_\delta\int_{t_n}^{t_{n+1}}\|\mathcal{R}_h\partial_t\lambda(s)\|_{Z_h}^2 ds,
\end{aligned} \tag{3.24}$$

and the pressure residual $\|R_p^{n+1}\|_0$ satisfies the bound $\|R_p^{n+1}\|_0 \lesssim \int_{t_n}^{t_{n+1}}\|\partial_{t^2}p(s)\|_0 ds$.

Summing up each of the first two inequalities in (3.22) from $n = 0$ to $n = m - 1$, taking into account the above bounds with $\delta = 1/2$, and applying Gromwall's Lemma leads to the proposed error estimates. □

In order to derive, from Theorem 3.3, the order of convergence of the method, we need an estimate for the norm $\|\cdot\|_{Z_h}$, given by the following lemma.

Lemma 3.1 *For all $\mu \in L^2(\gamma)$, $\|\mu\|_{Z_h} \lesssim h^{-\frac{1}{2}}\|\mu\|_{L^2(\gamma)}$ with a constant independent of h and N.*

Proof. from the definition of Z_h, we obtain

$$\langle Z_h\mu, \mu\rangle = \langle S_h(S_h^{-1}B_h^t\mu), (S_h^{-1}B_h^t\mu)\rangle = \sup_{v_h\in V_h} \frac{\langle B_h^t\mu, v_h\rangle^2}{\langle S_h v_h, v_h\rangle}$$
$$= \sup_{v_h\in V_h} \frac{(\sum_{i=1}^N \int_{\Gamma_i}(v_h\cdot n_i)\mu d\gamma)^2}{\langle S_h v_h, v_h\rangle}.$$

The lemma easily results from the inverse inequality $\|v_h\cdot n\|_{L^2(\Gamma_i)} \lesssim h^{-1/2}$ $\|v_h\|_{L^2(\Omega_i)^d}$ for all $v_h \in V_h$. □

This dependency of the semi-norm $\|\cdot\|_{Z_h}$ on the discretization parameter h results in a reduction of the convergence order of the method. This is a major difference to the case of Navier Stokes equations for which the semi-norm $\|\cdot\|_{Z_h}$ is uniformly bounded by the H^1 norm.

Let us choose $p_{0,h} := \rho_h p_0$, from the previous Lemma and Theorem 3.3, we obtain the following error estimates.

Theorem 3.4 *Let $(u, p) \in C^0(0, t_m; H(\Omega; \mathrm{div})) \times C^0(0, t_m; M)$, be the weak solution of (1.1) such that $p \in C^1(0, t_m; H^1(\Omega))$. For $1 \le r \le k+1$ and $u \in H^1(0, t_m; \mathcal{H}^r(\Omega)^d)$, $\partial_{t^2}u \in L^2(0, t_m; V')$, $\partial_t\lambda \in L^2(0, t_m; L^2(\gamma))$, $\partial_{t^2}g \in L^2(0, t_m; H^{\frac{1}{2}}(\Gamma))$, $p \in W^{1,\infty}(0, t_m; \mathcal{H}^{r+1}(\Omega))$, $\partial_{t^2}p \in L^2(0, t_m; L^2(\Omega))$, $\sum_{n=0}^{m-1}\Delta t\|\nabla\cdot u^{n+1}\|^2_{\mathcal{H}^r(\Omega)} \lesssim 1$, the solution of the incremental scheme (3.6)-(3.7)-(3.3) or (3.8)-(3.9) satisfies*

$$\|u^m - u_h^m\|_0 + \|p^m - p_h^m\|_0 + \|B_h^t(p_\gamma^m - p_{\gamma,h}^m)\|_{V_h'}$$
$$+\Big(\sum_{n=0}^{m-1}\Delta t\|\nabla\cdot(u^{n+1} - \tilde{u}_h^{n+1})\|_0^2\Big)^{\frac{1}{2}}$$
$$\lesssim \Delta t(1 + h^{-\frac{1}{2}}) + h^r, \tag{3.25}$$

with constants independent of h, Δt, and possibly depending on N at most like $N^{1/d}$. In order to obtain these estimations it suffices to choose $\lambda_h^0 = 0$.

Proof. From Theorem 3.3, we need to estimate the right hand sides of (3.15). From Lemma 3.1, we obtain

$$\Delta t^2 \int_0^{t_m} \|\mathcal{R}_h \partial_t \lambda(s)\|_{Z_h}^2 ds \lesssim h^{-1} \Delta t^2 \int_0^{t_m} \|\partial_t \lambda(s)\|_0^2 ds.$$

Similarly, assuming that λ_h^0 is chosen so that $\|\lambda_h^0\|_0 \lesssim 1$, then $\Delta t^2 \|\varepsilon_h^0\|_{Z_h}^2 \lesssim \Delta t^2 h^{-1}$. From the definition of the initial flux (2.5), and the choice $p_{0,h} := \rho_h p_0$, we have

$$\|e_{u,h}^0\|_0 \lesssim \|(\Pi_h - I)u^0\|_0 \lesssim h^r \|u^0\|_{\mathcal{H}^r(\Omega)}.$$

To estimate the projection errors at the interfaces Γ_i, we use the assumption that the order of approximation of Λ_h is $k+2$, so that for all $0 \le r \le k+1$:

$$\begin{aligned}
\sum_{i=1}^N h^{-1} \|(\mathcal{R}_h - I)p_\gamma\|_{L^2(\Gamma_i)}^2 &\lesssim \sum_{i=1}^N h^{2r} \|p_\gamma\|_{H^{r+\frac{1}{2}}(\Gamma_i)}^2 \\
&\lesssim h^{2r} \|p\|_{\mathcal{H}^{r+1}(\Omega)}^2, \\
\sum_{i=1}^N h^{-1} \|(\mathcal{R}_h - I)\lambda\|_{L^2(\Gamma_i)}^2 &\lesssim \sum_{i=1}^N h^{2r} \|\lambda\|_{H^{r+\frac{1}{2}}(\Gamma_i)}^2 \\
&\lesssim h^{2r} \|\partial_t p\|_{\mathcal{H}^{r+1}(\Omega)}^2,
\end{aligned}$$

where on each subdomain Ω_i, we have applied the trace theorem between $H^{r+\frac{1}{2}}(\Gamma_i)$ and $H^{r+1}(\Omega_i)$, hence with a constant possibly depending on N like $N^{1/d}$. The remaining terms in (3.15) are easily estimated using the smoothness assumptions, and the error estimates (2.3) for the projector Π_h as well as classical error estimates for the orthogonal projector ρ_h onto M_h. □

Remark 3.2 Although the scheme is unconditionally stable independently of both h and N, the convergence is only obtained if the condition $\Delta t \lesssim h^{1/2}$ holds true. This is the price to pay to obtain a fully parallel domain decomposition algorithm.

3.2.2 The non-incremental scheme. The above error analysis based on the flux formulation readily carries over to the non-incremental scheme.

Theorem 3.5 *Assuming Hypothesis 2.1 and* $(u,p) \in C^0(0,t_m; H(\Omega;\text{div})) \times C^0(0,t_m; M)$, $p \in C^1(0,t_m; H^1(\Omega))$, *the non-incremental scheme (3.1)-*

(3.2)-(3.3) or (3.4)-(3.5) satisfies the error estimates:

$$\|e^m_{u,h}\|^2_S + \Delta t \sum_{n=0}^{m-1} \Delta t \|\varepsilon^{n+1}_h\|^2_{Z_h} + \sum_{n=0}^{m-1} \Delta t \|\nabla \cdot \tilde{e}^{n+1}_{u,h}\|^2_0 \lesssim \|e^0_{u,h}\|^2_S$$
$$+ \Delta t \sum_{n=0}^{m-1} \Delta t \|\mathcal{R}_h \lambda^{n+1}\|^2_{Z_h} + \Delta t^2 \int_0^{t_m} \|\partial_{t^2} g(s)\|^2_{H^{\frac{1}{2}}(\Gamma)} ds$$
$$+ \Delta t^2 \int_0^{t_m} \|\partial_{t^2} u(s)\|^2_{V'_h} ds + \int_0^{t_m} \|(\Pi_h - I)\partial_t u(s)\|^2_{V'_h} ds$$
$$+ \sum_{n=0}^{m-1} \Delta t \sum_{i=1}^{N} h^{-1} \|(\mathcal{R}_h - I)\lambda^{n+1}\|^2_{L^2(\Gamma_i)} , \tag{3.26}$$

$$\|e^m_{p,h}\|^2_0 \lesssim \|e^0_{p,h}\|^2_0 + \sum_{n=0}^{m-1} \Delta t \|\nabla \cdot \tilde{e}^{n+1}_{u,h}\|^2_0 + \Delta t^2 \int_0^{t_m} \|\partial_{t^2} p(s)\|^2_0 ds,$$

$$\|B^t_h e^m_{\gamma,h}\|_{V'_h} \lesssim \|p^m - p^m_h\|_0 + \|u^m - u^m_h\|_0 + \left(\sum_{i=1}^{N} h^{-1} \|(\mathcal{R}_h - I) p^m_\gamma\|^2_{L^2(\Gamma_i)} \right)^{\frac{1}{2}} ,$$

with constants independent of h, Δt, and N.

Theorem 3.6 *Let $(u,p) \in C^0(0,t_m;H(\Omega;\mathrm{div})) \times C^0(0,t_m;M)$ be the weak solution of (1.1) such that $p \in C^1(0,t_m;H^1(\Omega))$. For $1 \le r \le k+1$ and $u \in H^1(0,t_m;\mathcal{H}^r(\Omega)^d)$, $\partial_{t^2} u \in L^2(0,t_m;V')$, $\partial_{t^2} g \in L^2(0,t_m;H^{\frac{1}{2}}(\Gamma))$, $p \in W^{1,\infty}(0,t_m;\mathcal{H}^{r+1}(\Omega))$, $\partial_{t^2} p \in L^2(0,t_m;L^2(\Omega))$, $\sum_{n=0}^{m-1} \Delta t \|\nabla \cdot u^{n+1}\|^2_{\mathcal{H}^r(\Omega)} \lesssim 1$, the solution of the incremental scheme (3.1)-(3.2)-(3.3) or (3.4)-(3.5) satisfies*

$$\|u^m - u^m_h\|_0 + \|p^m - p^m_h\|_0 + \|B^t_h(p^m_\gamma - p^m_{\gamma,h})\|_{V'_h}$$
$$+ \Big(\sum_{n=0}^{m-1} \Delta t \|\nabla \cdot (u^{n+1} - \tilde{u}^{n+1}_h)\|^2_0 \Big)^{\frac{1}{2}}$$
$$\lesssim \Delta t^{\frac{1}{2}} (1 + h^{-\frac{1}{2}}) + h^r, \tag{3.27}$$

with constants independent of h, Δt, and possibly depending on N at most like $N^{1/d}$.

In order to avoid having to resort to the assumption $\partial_t p \in C^0(0,t_m;H^1(\Omega))$, another error analysis can be carried out directly from the mixed pressure-flux formulation (3.4)-(3.5).

Theorem 3.7 *Assuming Hypothesis 2.1 and* $(u,p) \in C^0(0,t_m;H(\Omega;\mathrm{div})) \times C^0(0,t_m;M)$*,* $p \in C^0(0,t_m;H^1(\Omega))$*, the non-incremental scheme (3.1)-(3.2)-(3.3) or (3.4)-(3.5) satisfies the error estimates:*

$$
\begin{aligned}
\|e^m_{p,h}\|_0^2 + \Delta t\|e^m_{\gamma,h}\|^2_{Z_h} + \sum_{n=0}^{m-1} \Delta t\Big(\|\tilde{e}^{n+1}_{u,h}\|^2_S + \|e^{n+1}_{u,h}\|^2_S\Big) \\
\lesssim\ \|e^0_{p,h}\|_0^2 + \Delta t\|e^0_{\gamma,h}\|^2_{Z_h} \\
+ \Delta t\int_0^{t_m} \|\mathcal{R}_h\partial_t p_\gamma(s)\|^2_{Z_h} ds + \Delta t^2\int_0^{t_m}\|\partial_{t^2}p(s)\|_0^2 ds \\
+ \sum_{n=0}^{m-1} \Delta t\|(\Pi_h - I)u^{n+1}\|_0^2 \\
+ \sum_{n=0}^{m-1}\Delta t\sum_{i=1}^{N} h^{-1}\|(\mathcal{R}_h - I)p^{n+1}_\gamma\|^2_{L^2(\Gamma_i)},
\end{aligned}
\tag{3.28}
$$

with constants independent of h*,* Δt*, and* N*.*

Theorem 3.8 *Let* $(u,p) \in C^0(0,t_m;H(\Omega;\mathrm{div})) \times C^0(0,t_m;M)$ *be the weak solution of (1.1) such that* $p \in C^0(0,t_m;H^1(\Omega))$*. For* $1 \le r \le k+1$ *and* $u \in L^\infty(0,t_m;\mathcal{H}^r(\Omega)^d)$*,* $p \in L^\infty(0,t_m;\mathcal{H}^{r+1}(\Omega))$*,* $\partial_{t^2}p \in L^2(0,t_m;L^2(\Omega))$*,* $\partial_t p_\gamma \in L^2(0,t_m;L^2(\gamma))$*, the solution of the incremental scheme (3.1)-(3.2)-(3.3) or (3.4)-(3.5) satisfies*

$$
\begin{aligned}
\|p^m - p^m_h\|_0 + \left(\Delta t\sum_{n=0}^{m-1}\|u^{n+1} - \tilde{u}^{n+1}_h\|_0^2 + \|u^{n+1} - u^{n+1}_h\|_0^2\right)^{\frac12} \\
\lesssim\ \Delta t^{\frac12}(1 + h^{-\frac12}) + h^r,
\end{aligned}
\tag{3.29}
$$

with constants independent of h*,* Δt*, and possibly depending on* N *at most like* $N^{1/d}$*.*

Proof of Theorem 3.7. Combining (3.4)-(3.5) with (3.18) taken at time t_{n+1}, we obtain the equations governing the errors $e^n_{u,h}$,$\tilde{e}^n_{u,h}$, $e^n_{p,h}$, and $e^n_{\gamma,h}$:

$$
\begin{cases}
\dfrac{e^{n+1}_{p,h} - e^n_{p,h}}{\Delta t} + \mathrm{div}_h\tilde{e}^{n+1}_{u,h} = R^{n+1}_p, \\
S_h\tilde{e}^{n+1}_{u,h} = \mathrm{div}^t_h e^{n+1}_{p,h} - B^t_h\varphi^n_{\gamma,h} + S_h(\Pi_h - I)u^{n+1} \\
\qquad + B^t_h(\mathcal{R}_h - I)p^{n+1}_\gamma,
\end{cases}
\tag{3.30}
$$

$$
\begin{cases}
S_h(e^{n+1}_{u,h} - \tilde{e}^{n+1}_{u,h}) + B^t_h(e^{n+1}_{\gamma,h} - \varphi^n_{\gamma,h}) = 0, \\
B_h e^{n+1}_{u,h} = 0,
\end{cases}
\tag{3.31}
$$

with $\varphi^n_{\gamma,h} := e^n_{\gamma,h} + \mathcal{R}_h \Delta p^{n+1}_\gamma$. Adding the scalar product of the first equation in (3.30) with $e^{n+1}_{p,h}$ with the duality pairing of the second equation in (3.30) with $\tilde{e}^{n+1}_{u,h}$, we obtain

$$
\begin{aligned}
&\|e^{n+1}_{p,h}\|_0^2 + \|e^{n+1}_{p,h} - e^n_{p,h}\|_0^2 - \|e^n_{p,h}\|_0^2 + 2\Delta t \|\tilde{e}^{n+1}_{u,h}\|_S^2 \\
&\qquad + 2\Delta t \langle B_h^t \varphi^n_{\gamma,h}, \tilde{e}^{n+1}_{u,h} \rangle \\
&\quad \le 2\Delta t \langle e^{n+1}_{p,h}, R^{n+1}_p \rangle + 2\Delta t \langle S_h(\Pi_h - I) u^{n+1}, \tilde{e}^{n+1}_{u,h} \rangle \\
&\qquad + 2\Delta t \langle B_h^t (\mathcal{R}_h - I) p^{n+1}_\gamma, \tilde{e}^{n+1}_{u,h} \rangle.
\end{aligned} \tag{3.32}
$$

As for the proof of Theorem 3.3, we use both equations in (3.31) to derive the relations

$$
\begin{cases}
\|e^{n+1}_{u,h}\|_S + \|e^{n+1}_{u,h} - \tilde{e}^{n+1}_{u,h}\|_S + \|\tilde{e}^{n+1}_{u,h}\|_S = 0, \\
-\|e^{n+1}_{u,h} - \tilde{e}^{n+1}_{u,h}\|_S - 2\langle B_h^t \varphi^n_{\gamma,h}, \tilde{e}^{n+1}_{u,h} \rangle \\
\qquad + \|e^{n+1}_{\gamma,h}\|_{Z_h}^2 - \|\varphi^n_{\gamma,h}\|_{Z_h}^2 = 0.
\end{cases} \tag{3.33}
$$

Multiplying equations (3.33) by Δt and adding them to inequality (3.32), we obtain

$$
\begin{aligned}
&\|e^{n+1}_{p,h}\|_0^2 + \|e^{n+1}_{p,h} - e^n_{p,h}\|_0^2 - \|e^n_{p,h}\|_0^2 \\
&\qquad + \Delta t (\|\tilde{e}^{n+1}_{u,h}\|_S^2 + \|e^{n+1}_{u,h}\|_S^2) \\
&\qquad + \Delta t \|e^{n+1}_{\gamma,h}\|_{Z_h}^2 - \Delta t \|\varphi^n_{\gamma,h}\|_{Z_h}^2 \\
&\quad \le 2\Delta t \langle e^{n+1}_{p,h}, R^{n+1}_p \rangle + 2\Delta t \langle S_h(\Pi_h - I) u^{n+1}, \tilde{e}^{n+1}_{u,h} \rangle \\
&\qquad + 2\Delta t \langle B_h^t (\mathcal{R}_h - I) p^{n+1}_\gamma, \tilde{e}^{n+1}_{u,h} \rangle.
\end{aligned} \tag{3.34}
$$

The rest of the proof follows the lines of the proof of Theorem 3.3 □

3.3 Numerical example

Let us consider equation (1.1) over the two dimensional domain $\Omega = (0,2) \times (0,1)$ for $K = 1$ and with exact solution $p(x,y,t) = \Big(x^2y^3 + \cos(\frac{\pi}{2}xy)\Big) \cos\frac{\pi t}{2}$. Furthermore, let Ω be split into two subdomains $\Omega_1 = (0,1) \times (0,1)$ and $\Omega_2 = (1,2) \times (0,1)$.

This problem is discretized on a Cartesian uniform mesh of step h in both directions, using RT_0 MFE with mass condensation (i.e. a finite volume scheme). The time discretization is uniform with time step Δt.

Figure 1 shows the convergence history of the error $p^n_h - p^n$ in $l^\infty(L^2(\Omega))$ norm for two different time discretizations: the incremental projection

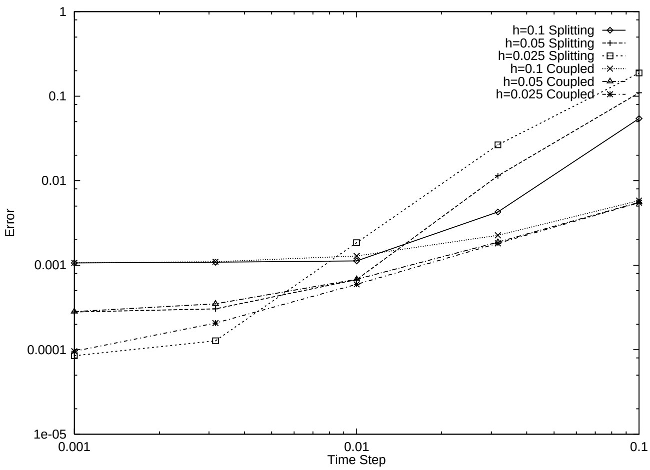

Fig. 1. Convergence history of the error $p_h^n - p^n$ in the $l^\infty(L^2(\Omega))$ norm: incremental scheme (splitting) and 1st order coupled scheme (coupled) for $h = 0.1, 0.05, 0.025$

scheme (3.8)-(3.9), and the first order backward Euler fully coupled discretization (coupled scheme).

From the numerical results displayed Fig. 1, we deduce that the error of the time discretization behaves like $\min(\Delta t/h^{1/2}, \Delta t^2/h) + \Delta t$ for the incremental projection scheme, which is better than the predicted result of order $\Delta t/h^{1/2} + \Delta t$.

This result suggests that the error is the sum of the error produced by the coupled scheme and the splitting error (i.e. the difference between the coupled scheme and the projection scheme solutions) of order $\min(\Delta t/h^{1/2}, \Delta t^2/h)$ for the incremental version.

Assuming these convergence estimates (which still remain to be proven), a convergence of order h is obtained for the incremental scheme if $\Delta t = \mathcal{O}(h)$.

4 Conclusion

The method introduced in this paper combines the Mortar Mixed Finite Element domain decomposition spatial discretization with projection schemes for the time discretization, in order to obtain a fully parallel algorithm for parabolic equations. In addition, this method enables the use of hybrid meshes and local time stepping.

Although the scheme is shown to be unconditionally stable, the convergence is obtained only if the condition $\Delta t \lesssim h^{1/2}$ holds true (for the incremental version). This is the price to pay to decouple the interface problem from the computation of the subdomain solutions.

This strategy has proven to be efficient to solve single phase Darcy flow problems around 2D wells and faults with strong heterogeneities, and we refer to [Gai00] for the numerical tests.

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