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CONVERGENCE OF A NUMERICAL SCHEME FOR STRATIGRAPHIC MODELING*

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Abstract. In this paper, we consider a multilithology diffusion model used in the field of stratigraphic basin simulations to simulate large scale depositional transport processes of sediments described as a mixture of L lithologies. This model is a simplified one for which the surficial fluxes are proportional to the slope of the topography and to a lithology fraction with unitary diffusion coefficients.

The main variables of the system are the sediment thickness h, the L surface concentrations c_i^s in lithology i of the sediments at the top of the basin, and the L concentrations c_i in lithology i of the sediments inside the basin. For this simplified model, the sediment thickness decouples from the other unknowns and satisfies a linear parabolic equation. The remaining equations account for the mass conservation of the lithologies, and couple, for each lithology, a first order linear equation for c_i^s with a linear advection equation for c_i for which c_i^s appears as an input boundary condition. For this coupled system, a weak formulation is introduced.

The system is discretized by an implicit time integration and a cell centered finite volume method. This numerical scheme is shown to satisfy stability estimates and to converge, up to a subsequence, to a weak solution of the problem.

Key words. finite volume method, stratigraphic modeling, linear first order equations, convergence analysis, weak formulation

AMS subject classifications. 35M10, 35Q99, 65M12

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1. Introduction.

Recent progress in geosciences, and more especially in seismic- and sequence-stratigraphy, have improved the understanding of sedimentary basins infill. Indeed, the sediment's architecture is the response to complex interactions between the available space created in the basin by sea level variations, tectonic, compaction, the sediment supply (boundary fluxes, sediment production), and the transport of the sediments at the surface of the basin. In order to have a quantified view of this response and to determine the relative influence of each involved process, stratigraphic models have been developed.

Among basin infill models considering the dynamics of sediment transport, authors usually distinguish between fluid-flow and dynamic-slope models (see [14], [15]). The first ones use fluid-flow equations and empirical algorithms to simulate the transport of sediments in the hydrodynamic flow field (see, e.g., [16]). They provide an accurate description of depositional processes for small scales in time and space, but, at larger scale's such as basin scales, they are computationally too expensive.

Dynamic-slope models use mass conservation equations of sediments combined with diffusive transport laws. These laws do not describe each geological process in detail but average over these processes (river transport, creep, slumps, and small

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slides). One can refer to [1], [7], [8], [10], [14], and [17] for a detailed description of these models. The dynamic-slope models have been shown to offer a good description of sedimentation and erosion processes for large time scales (greater than 10^4 y) and basin space scales (greater than 1 km).

We consider here a dynamic-slope model simulating the evolution of a sedimentary basin in which sediments are modeled as a mixture of several lithologies $i = 1, \dots, L$ characterized by different grain size populations. The surficial transport process is a multilithology diffusive model introduced in [14], for which the fluxes are proportional to the slope of the topography and to a lithology fraction c_i^s of the sediments at the surface of the basin (see also [9] and [5]). In what follows, a simplified model is considered for which the diffusion coefficients are taken equal to one. It results that the sediment thickness variable h is decoupled from the other unknowns of the system (i.e., for each lithology, the surface concentration c_i^s and the concentration c_i in lithology i of the sediments in the basin) and satisfies a linear parabolic equation.

The remaining equations accounting for the mass conservation of the lithologies couple, for all $i = 1, \dots, L$, a first order linear equation for the surface concentration variable c_i^s and a linear advection equation for the basin concentration variable c_i for which c_i^s appears as an input boundary condition at the top of the basin. In order to cope with the difficulty of defining the trace of the basin concentration c_i at the top of the basin, an original weak formulation is introduced for this coupled problem.

The system is discretized by an implicit integration in time and a cell centered finite volume scheme in space. The objective of this article is to prove, under Hypothesis 1, the convergence of the approximate solutions for the sediment thickness variable h and for the concentration variables c_i^s, c_i, $i = 1, \dots, L$, up to a subsequence, to a weak solution of problem (2.7) in the sense of Definition 2.1 as the mesh size and time step tend to 0. We state this result in Theorem 3.3 in section 3, after presenting the mathematical model, the weak formulation, and the finite volume scheme.

Regarding the coupling between the parabolic equation for h and the first order linear equations for the variables c_i^s, $i = 1, \dots, L$, our model shares some common features with two phase Darcy flows for which such coupling between an elliptic or parabolic equation and a hyperbolic equation also comes in. The convergence of various numerical schemes for such models have been the subject of several studies. For example, one can refer to [12] for finite differences, to [2] and [3] for mixed and hybrid finite element methods, to [4] for the control volume finite element discretization, and to [19], [18], and [6] for the cell centered finite volume scheme.

The main originality of this work is rather concerned with the coupling between the surface and the basin concentration variables.

The remaining of the paper outlines as follows. The mathematical model and its weak formulation are defined in section 2, and the fully implicit finite volume discretization is derived in section 3. In section 4, stability and error estimates on the discrete solution for the sediment thickness and its time derivative are obtained. Finally, the convergence of the approximate solutions to a weak solution of the problem is proved in section 5.

2. Mathematical model and weak formulation. A basin model specifies the geometry defined by the basin horizontal extension, the position of its base due to vertical tectonics displacements, and the sea level variations. It provides a description of the sediments considered as a mixture of different lithologies such as sand or shale. Finally, it specifies the sediment transport laws and their coupling, as well as the sediment fluxes at the boundary of the basin (boundary conditions).

In this paper, the multilithology diffusion model described in [14], [9], and [5] is studied in a simplified case for which the diffusion coefficients of the lithologies are equal (to one to fix ideas). Also, for the sake of simplicity, the tectonics displacements as well as the sea level variations are not considered in what follows.

The projection of the basin on a reference horizontal plane is considered as a fixed domain $\Omega \subset \mathbb{R}^d$, defining the horizontal extension of the basin, with $d = 1$ for two dimensional basin models and $d = 2$ for three dimensional models.

We denote by h the sediment thickness variable defined on the domain $\mathcal{D} = \Omega \times \mathbb{R}_+^*$ and by $\mathcal{B}$ the domain $\{(x, z, t)$ such that $(x, t) \in \mathcal{D}, z < h(x, t)\}$.

The sediments are modeled as a mixture of L lithologies characterized by their grain size population. Each lithology, $i = 1, \ldots, L$, is considered as an uncompressible material of constant grain density and null porosity. On each point of the basin, the mixture is described by its composition given by the concentrations c_i, defined on $\mathcal{B}$, and such that $c_i \geq 0$ for $i = 1, \ldots, L$, and $\sum_{i=1}^L c_i = 1$.

The model assumes that the sediment fluxes are nonzero only at the surface of the basin (i.e., for $z = h$). The sediments transported by these surficial fluxes, i.e., which are deposited at the surface of the basin in case of sedimentation, or which pass through the surface in case of erosion, are characterized by their concentrations denoted by c_i^s, defined on $\mathcal{D}$, and such that $c_i^s \geq 0$ for $i = 1, \ldots, L$, and $\sum_{i=1}^L c_i^s = 1$.

Since the compaction is not considered, no change in time of the concentration c_i can occur inside the basin. It results that $\partial_t c_i = 0$ on $\mathcal{B}$. The evolution of c_i is governed by the boundary condition at the top of the basin stating that $c_i|_{z=h} = c_i^s$ in the case of sedimentation $\partial_t h > 0$. Let $\mathcal{D}^+$ denote the domain $\{(x, t) \in \mathcal{D}$ such that $\partial_t h(x, t) > 0\}$; then c_i satisfies the conservation equation;

$$\left\{ \begin{aligned} \partial_t c_i &= 0 \quad \text{on } \mathcal{B}, \\ c_i|_{z=h} &= c_i^s \quad \text{on } \mathcal{D}^+. \end{aligned} \right. \tag{2.1}$$

The conservation of the thickness fraction in lithology i

$$\mathcal{M}_i(x, t) = \int_0^{h(x,t)} c_i(x, z, t) dz, \; (x, t) \in \mathcal{D}, \tag{2.2}$$

with $\sum_{i=1}^L \mathcal{M}_i = h$, states that for all $i = 1, \ldots, L$

$$\left\{ \begin{aligned} \partial_t \mathcal{M}_i + \operatorname{div} \mathbf{f}_i &= 0 \quad \text{on } \mathcal{D}, \\ \textstyle\sum_{i=1}^L c_i^s &= 1 \quad \text{on } \mathcal{D}. \end{aligned} \right. \tag{2.3}$$

In the multilithology diffusive model described in [14], the flux $\mathbf{f}_i$ is proportional to the gradient of the topography h and to the concentration c_i^s, with a diffusion coefficient k_i. In what follows, we shall restrict ourselves to the simplified case $k_i = 1$ for all $i = 1, \ldots, L$, i.e., $\mathbf{f}_i := -c_i^s \nabla h$, so that the sediment thickness variable h decouples from the concentrations and satisfies a linear parabolic equation (see (2.6)).

Neumann boundary conditions are imposed to h on $\partial\Omega \times \mathbb{R}_+^*$,

$$\nabla h \cdot \vec{n} = g \text{ on } \partial\Omega \times \mathbb{R}_+^*,$$

with $\vec{n}$ the unit normal vector to $\partial\Omega$, outward to Ω, and Dirichlet boundary conditions are prescribed to the surface concentrations

$$c_i^s = \tilde{c}_i \text{ on } \Sigma^+,$$

with $\Sigma^+ = \{(x, t) \in \partial\Omega \times \mathbb{R}_+^*, g(x, t) > 0\}$, $\tilde{c}_i \geq 0$ for all $i = 1, \ldots, L$, and $\sum_{i=1}^L \tilde{c}_i = 1$.

Initial conditions are prescribed to the sediment thickness such that $h|_{t=0} = h^0$ on Ω, and to the basin concentrations such that $c_i|_{t=0} = c_i^0$ on the domain $\{(x,z), x \in \Omega, z < h^0(x)\}$, with $c_i^0 \geq 0$ for all $i = 1,\dots,L$, and $\sum_{i=1}^L c_i^0 = 1$.

In the following, we shall consider the new coordinate system for which the vertical position of a point in the basin is measured downward from the top of the basin, i.e., given by the change of variable $(x,\xi,t) = (x', h(x',t') - z, t')$. In this coordinate system, let $u_i(x,\xi,t) = c_i(x, h(x,t)-\xi, t)$ on $\Omega \times \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $u_i^0(x,\xi) = c_i^0(x, h^0(x)-\xi, t)$ on $\Omega \times \mathbb{R}_+^*$. Gathering all the equations, we obtain the following multilithology diffusive model:

$$\text{(2.4)} \qquad \text{surface conservations:} \begin{cases} u_i|_{\xi=0}\, \partial_t h + \operatorname{div}(-c_i^s \nabla h) = 0 & \text{on } \mathcal{D}, \\ \sum_{i=1}^L c_i^s = 1 & \text{on } \mathcal{D}, \\ \nabla h \cdot \vec{n}|_{\partial\Omega \times \mathbb{R}_+^*} = g & \text{on } \partial\Omega \times \mathbb{R}_+^*, \\ c_i^s|_{\Sigma^+} = \tilde{c}_i & \text{on } \Sigma^+, \\ h|_{t=0} = h^0 & \text{on } \Omega, \end{cases}$$

$$\text{(2.5)} \qquad \text{column conservations:} \begin{cases} \partial_t u_i + \partial_t h\, \partial_\xi u_i = 0 & \text{on } \Omega \times \mathbb{R}_+^* \times \mathbb{R}_+^*, \\ u_i|_{\xi=0} = c_i^s & \text{on } \mathcal{D}^+, \\ u_i^0|_{t=0} = u_i^0 & \text{on } \Omega \times \mathbb{R}_+^*, \end{cases}$$

where we have taken into account the equality $\partial_t \mathcal{M}_i = u_i|_{\xi=0}\, \partial_t h$ on $\mathcal{D}$ which derives formally from the definition (2.2) and the equation $\partial_t c_i = 0$ on $\mathcal{B}$.

For this simplified model, summing (2.4) over $i = 1,\dots,L$, it appears that the variable h satisfies the parabolic equation

$$\text{(2.6)} \qquad \begin{cases} \partial_t h - \Delta h = 0 & \text{on } \Omega \times \mathbb{R}_+^*, \\ \nabla h \cdot \vec{n}|_{\partial\Omega \times \mathbb{R}_+^*} = g & \text{on } \partial\Omega \times \mathbb{R}_+^*, \\ h_{|t=0} = h^0 & \text{on } \Omega, \end{cases}$$

while the remaining concentration variables (c_i^s, u_i) verify, for each $i = 1,\dots,L$, the system of equations

$$\text{(2.7)} \qquad \begin{cases} u_i|_{\xi=0}\, \partial_t h + \operatorname{div}(-c_i^s \nabla h) = 0 & \text{on } \mathcal{D}, \\ c_i^s|_{\Sigma_+} = \tilde{c}_i & \text{on } \Sigma^+, \\ \partial_t u_i + \partial_t h\, \partial_\xi u_i = 0 & \text{on } \Omega \times \mathbb{R}_+^* \times \mathbb{R}_+^*, \\ u_i|_{\xi=0} = c_i^s & \text{on } \mathcal{D}^+, \\ u_i|_{t=0} = u_i^0 & \text{on } \Omega \times \mathbb{R}_+^*. \end{cases}$$

The sediment thickness variable is decoupled from the concentrations variables and satisfies the linear system (2.6). The solution of this system is then used in problem (2.7), which is linear with respect to the variables c_i^s and u_i.

In what follows, the following assumptions are made on the data.

HYPOTHESIS 1.

(i) *Ω is an open bounded subset of $\mathbb{R}^d$, of class $\mathcal{C}^\infty$,*

(ii) *$h^0 \in \mathcal{C}^2(\bar{\Omega})$,*

(iii) *$g \in \mathcal{C}^1(\partial\Omega \times \mathbb{R}_+) \cap L^2(\partial\Omega \times \mathbb{R}_+)$,*

(iv) *g and h^0 are chosen according to the assumptions of Theorem* 5.3 *of* [11, p. 320] *so that the unique solution h of* (2.6) *is in $C^2(\bar{\Omega} \times [0,T])$ for all $T > 0$,*

(v) *$\tilde{c}_i \in L^\infty(\Sigma^+)$ with $\tilde{c}_i \geq 0$ for $i = 1,\dots,L$, and $\sum_{i=1}^L \tilde{c}_i = 1$,*

(vi) *$u_i^0 \in L^\infty(\Omega \times \mathbb{R}_+^*)$, $u_i^0 \geq 0$ for $i = 1,\dots,L$, and $\sum_{i=1}^L u_i^0 = 1$.*

In the following, we shall denote by $\mathcal{C}_c^\infty(\mathbb{R}^n)$ the space of real valued functions

$$\{\varphi \in \mathcal{C}^\infty(\mathbb{R}^n) \,|\, \mathrm{supp}(\varphi) \text{ bounded in } \mathbb{R}^n\}.$$

To obtain a rigorous mathematical formulation of (2.7), we are looking for weak solutions defined as follows for all $i = 1, \dots, L$.

DEFINITION 2.1. *Let us assume that Hypothesis* 1 *holds and let h denote the solution of problem* (2.6). *Then $(c_i^s, u_i) \in L^\infty(\Omega \times \mathbb{R}_+^*) \times L^\infty(\Omega \times \mathbb{R}_+^* \times \mathbb{R}_+^*)$ is said to be a weak solution of* (2.7) *if it satisfies* (i) *for all $\varphi \in \mathcal{A} = \{v \in \mathcal{C}_c^\infty(\mathbb{R}^{d+2}) \,|\, v(.,0,.) = 0$ on $\mathcal{D} \setminus S^+\}$*

$$\begin{aligned}
&\int_\Omega \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left[\partial_t \varphi(x,\xi,t) + \partial_t h(x,t)\, \partial_\xi \varphi(x,\xi,t)\right] u_i(x,\xi,t)\, dt\, d\xi\, dx \\
&+ \int_\Omega \int_{\mathbb{R}_+} u_i^0(x,\xi)\varphi(x,\xi,0)\, d\xi\, dx + \int_\Omega \int_{\mathbb{R}_+} \partial_t h(x,t) c_i^s(x,t)\varphi(x,0,t)\, dt\, dx = 0,
\end{aligned} \tag{2.8}$$

(ii) *for all $\psi \in \mathcal{A}_0 = \{v \in \mathcal{C}_c^\infty(\mathbb{R}^{d+2}) \,|\, v(.,0,.) = 0$ on $\partial\Omega \times \mathbb{R}_+^* \setminus \Sigma^+\}$*

$$\begin{aligned}
&-\int_\Omega \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left[\partial_t \psi(x,\xi,t) + \partial_t h(x,t)\, \partial_\xi \psi(x,\xi,t)\right] u_i(x,\xi,t)\, dt\, d\xi\, dx \\
&- \int_\Omega \int_{\mathbb{R}_+} u_i^0(x,\xi)\psi(x,\xi,0)\, d\xi\, dx + \int_{\mathbb{R}_+} \left(\int_\Omega c_i^s(x,t)\, \nabla h(x,t) \cdot \nabla \psi(x,0,t)\, dx \right. \\
&\left. - \int_{\partial\Omega} \tilde{c}_i(x,t) g(x,t) \psi(x,0,t) d\gamma(x) \right) dt = 0.
\end{aligned} \tag{2.9}$$

3. Finite volume scheme. The system (2.4)–(2.5) is discretized by a fully implicit time integration and a finite volume method with cell centered variables. We shall consider in what follows admissible meshes according to the following definition.

DEFINITION 3.1 (admissible meshes). *Let Ω be a bounded domain of $\mathbb{R}^d$, $d = 1$ or 2. In the following, $m(.)$ will be used to denote a measure on R^d equal to the Lebesgue measure if $d \geq 1$, and, if $d = 0$, the measure of a point is set to one and the measure of the empty set to zero. An admissible finite volume mesh of Ω for the discretization of problem* (2.4)–(2.5) *is given by a family of "control volumes," denoted by $\mathcal{K}$, which are open disjoint subsets of Ω, and a family of points of Ω, denoted by $\mathcal{P}$, satisfying the following properties:*

(i) *The closure of the union of all the control volumes of $\mathcal{K}$ is $\bar{\Omega}$.*

(ii) *For any κ, $\kappa' \in \mathcal{K}$ with $\kappa \neq \kappa'$, either the $(d-1)$-dimensional measure $m(\bar{\kappa} \cap \bar{\kappa}')$ is null, or it is strictly positive and $\bar{\kappa} \cap \bar{\kappa}'$ is included in a hyperplane of $\mathbb{R}^d$. In the following, we will denote by Σ_{int} the family of subsets σ of Ω contained in hyperplanes of $\mathbb{R}^d$ with strictly positive measures, and such that there exist $\kappa, \kappa' \in \mathcal{K}$ with $m(\bar{\kappa} \cap \bar{\kappa}') > 0$ and $\bar{\sigma} = \bar{\kappa} \cap \bar{\kappa}'$. We shall also denote by $\kappa|\kappa' \in \Sigma_{int}$ the edge between the cells κ and κ'.*

(iii) *The family $\mathcal{P} = (x_\kappa)_{\kappa \in \mathcal{K}}$ is such that $x_\kappa \in \bar{\kappa}$ (for any $\kappa \in \mathcal{K}$), and, if $\sigma = \kappa|\kappa'$, it is assumed that $x_\kappa \neq x_{\kappa'}$ and that the straight line going through x_κ and $x_{\kappa'}$ is orthogonal to the edge $\kappa|\kappa'$.*

(iv) *For any $\kappa \in \mathcal{K}$, there exists a subset Σ_κ of Σ_{int} such that $\partial\kappa \setminus \partial\Omega = \bar{\kappa} \setminus (\kappa \cup \partial\Omega) = \cup_{\sigma \in \Sigma_\kappa} \bar{\sigma}$.*

We shall denote by $(\mathcal{K}, \Sigma_{int}, \mathcal{P})$ this admissible mesh.

Let $(\mathcal{K}, \Sigma_{int}, \mathcal{P})$ be an admissible mesh of Ω in the sense of Definition 3.1. In what follows, $\delta\mathcal{K} = \sup\{\text{diam}(\kappa), \kappa \in \mathcal{K}\}$ will denote the mesh size of $(\mathcal{K}, \Sigma_{int}, \mathcal{P})$, $|\kappa|$ (resp., $|\sigma|$, $|\partial\kappa \cap \partial\Omega|$) is the d-dimensional measure of the cell $m(\kappa)$ (resp., the $(d-1)$-dimensional measure $m(\sigma)$, $m(\partial\kappa \cap \partial\Omega)$), $\mathcal{K}_\kappa$ the set of neighboring cells of κ (excluding κ), $T_{\kappa\kappa'} = T_\sigma$ the transmissibility of the edge $\sigma = \kappa|\kappa'$, defined by $T_{\kappa\kappa'} := \frac{|\sigma|}{d(\kappa,\kappa')}$ with $d(\kappa, \kappa')$ the distance between the points x_κ and $x_{\kappa'}$, $\text{reg}(\mathcal{K})$ the geometrical factor defined by $\text{reg}(\mathcal{K}) = \max_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} \frac{\delta\mathcal{K}}{d(\kappa,\kappa')}$, and $\vec{n}_{\kappa\kappa'}$ the unit normal vector to $\sigma = \kappa|\kappa'$ outward to κ.

We shall also denote by $X(\mathcal{K})$ the set of real valued functions on Ω which are constant over each control volume of the mesh and, for any subset $\mathcal{O}$ of $\mathbb{R}^d$, by $\chi_{\mathcal{O}}$ the function on $\mathbb{R}^d$ equal to one on $\mathcal{O}$ and null elsewhere. Finally, for any function f, let us define $f^+ = \max(f, 0) \geq 0$, $f^- = -\min(f, 0) \geq 0$, such that $f = f^+ - f^-$, and $|f| = f^+ + f^-$.

Following [6], we shall use the discrete seminorm defined as follows.

DEFINITION 3.2 (discrete $\mathcal{H}^1$ seminorm). *Let Ω be an open bounded subset of $\mathbb{R}^d$, $d = 1$ or 2, and $(\mathcal{K}, \Sigma_{int}, \mathcal{P})$ be an admissible finite volume mesh of Ω in the sense of Definition 3.1. For $u \in X(\mathcal{K})$, the discrete $\mathcal{H}^1$ seminorm of u is defined by*

$$|u|_{1,\mathcal{K}} = \left(\sum_{\sigma\in\Sigma_{int}} T_\sigma (D_\sigma u)^2 \right)^{\frac{1}{2}},$$

where u_κ is the value of u in the control volume κ and $D_\sigma u = |u_\kappa - u_{\kappa'}|$ with $\sigma = \kappa|\kappa'$.

Remark 1. Let $(\mathcal{K}, \Sigma_{int}, \mathcal{P})$ be an admissible mesh of Ω in the sense of Definition 3.1 and $|\Omega|$ denote the d-dimensional measure of the domain Ω. Considering the d-dimensional measure of the set of cones of vertex x_κ and base $\sigma \in \Sigma_{int} \cap \partial\kappa$ for all $\kappa \in \mathcal{K}$ and $\sigma \in \Sigma_{int}$, one can prove that

$$\sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} |\sigma|\, d(\kappa, \kappa') \leq d\,|\Omega|. \tag{3.1}$$

The time discretization is denoted by $t^n, n \in \mathbb{N}$, such that $t^0 = 0$ and $\Delta t^{n+1} = t^{n+1} - t^n > 0$. In the following, the superscript n, $n \in \mathbb{N}$, will be used to denote that the variables are considered at time t^n. Assuming that the set $\{\Delta t^n \,|\, n \in \mathbb{N}\}$ is bounded, let Δt denote $\sup\{\Delta t^n \,|\, n \in \mathbb{N}\}$, and, for a given $T > 0$, let $N_{\Delta t}$ be the integer such that $t^{N_{\Delta t}} < T \leq t^{N_{\Delta t}+1}$.

Let us now recall the discretization of (2.4)–(2.5) already introduced in [5]. For all control volumes $\kappa \in \mathcal{K}$, the following initial values are defined:

1. h^0_κ is the initial approximation of h in κ defined by $h^0_\kappa = h^0(x_\kappa)$.
2. $u^0_{i,\kappa}$, for all species i, is the approximation of u^0_i on the cell κ, defined by $u^0_{i,\kappa}(\xi) = \frac{1}{|\kappa|}\int_\kappa u^0_i(x, \xi)\, dx$ for $\xi \in \mathbb{R}^*_+$, and let $c^0_{i,\kappa}$ be defined on $(-\infty, h^0_\kappa)$ by $c^0_{i,\kappa}(z) = u^0_{i,\kappa}(h^0_\kappa - z)$.

We now give a discretization of (2.4)–(2.5) within a given control volume $\kappa \in \mathcal{K}$ between times t^n and t^{n+1}.

Conservation of surface sediments:

$$\begin{aligned} &\frac{\Delta\mathcal{M}^{n+1}_{i,\kappa}}{\Delta t^{n+1}}\, |\kappa| \;+ \sum_{\kappa'\in\mathcal{K}_\kappa} c^{s,n+1}_{i,\kappa\kappa'}\; T_{\kappa\kappa'}(h^{n+1}_\kappa - h^{n+1}_{\kappa'}) \\ &\quad -|\partial\kappa \cap \partial\Omega|\, \tilde{c}^{n+1}_{i,\kappa}\, g^{(+),n+1}_\kappa + |\partial\kappa \cap \partial\Omega|\, c^{s,n+1}_{i,\kappa}\, g^{(-),n+1}_\kappa = 0, \end{aligned} \tag{3.2}$$

$$\sum_{i=1}^{L} c_{i,\kappa}^{s,n+1} = 1. \tag{3.3}$$

Conservation of column sediments:

$$\text{if } h_\kappa^{n+1} \geq h_\kappa^n \begin{cases} \Delta\mathcal{M}_{i,\kappa}^{n+1} = c_{i,\kappa}^{s,n+1}(h_\kappa^{n+1} - h_\kappa^n), \\ c_{i,\kappa}^{n+1}(z) = c_{i,\kappa}^{n}(z), \ z < h_\kappa^n, \\ c_{i,\kappa}^{n+1}(z) = c_{i,\kappa}^{s,n+1}, \ z \in (h_\kappa^n, h_\kappa^{n+1}), \end{cases} \tag{3.4}$$

$$\text{else} \quad \begin{cases} \Delta\mathcal{M}_{i,\kappa}^{n+1} = \int_{h_\kappa^n}^{h_\kappa^{n+1}} c_{i,\kappa}^{n}(z)dz, \\ c_{i,\kappa}^{n+1}(z) = c_{i,\kappa}^{n}(z), \ z < h_\kappa^{n+1}. \end{cases} \tag{3.5}$$

In (3.2)–(3.5), the following notation is used.

1. h_κ^n is the approximation of the sediment thickness h at time t^n in κ.
2. $c_{i,\kappa}^{s,n+1}$ is the approximation of the surface sediment concentration i at time t^{n+1} in κ.
3. The function $c_{i,\kappa}^n$, defined on the column $(-\infty, h_\kappa^n)$. is the approximation of the sediment concentration in lithology i in the column $\{(x,z), x \in \kappa, z < h(x,t^n)\}$ at time t^n.
4. $c_{i,\kappa\kappa'}^{s,n+1}$ is the upstream weighted evaluation of the surface sediment concentration in lithology i at the edge σ between the cells κ and κ' with respect to the sign of $h_\kappa^{n+1} - h_{\kappa'}^{n+1}$:

$$c_{i,\kappa\kappa'}^{s,n+1} = \begin{cases} c_{i,\kappa}^{s,n+1} \text{ if } h_\kappa^{n+1} > h_{\kappa'}^{n+1}, \\ c_{i,\kappa'}^{s,n+1} \text{ otherwise.} \end{cases}$$

5. $g_\kappa^{(+),n+1}$ and $g_\kappa^{(-),n+1}$ are the following approximations of the boundary fluxes g^+ and g^-:

$$g_\kappa^{(+),n+1} = \begin{cases} \frac{1}{\Delta t^{n+1}} \frac{1}{|\partial\kappa\cap\partial\Omega|} \int_{t^n}^{t^{n+1}} \int_{\partial\kappa\cap\partial\Omega} g^+(x,t)\, d\gamma(x) dt & \text{if } |\partial\kappa \cap \partial\Omega| \neq 0, \\ 0 & \text{else,} \end{cases}$$

$$g_\kappa^{(-),n+1} = \begin{cases} \frac{1}{\Delta t^{n+1}} \frac{1}{|\partial\kappa\cap\partial\Omega|} \int_{t^n}^{t^{n+1}} \int_{\partial\kappa\cap\partial\Omega} g^-(x,t)\, d\gamma(x) dt & \text{if } |\partial\kappa \cap \partial\Omega| \neq 0, \\ 0 & \text{else,} \end{cases}$$

and consequently for all $\kappa \in \mathcal{K}$,

$$g_\kappa^{n+1} = \frac{1}{\Delta t^{n+1}} \frac{1}{|\partial\kappa \cap \partial\Omega|} \int_{t^n}^{t^{n+1}} \int_{\partial\kappa\cap\partial\Omega} g(x,t)\, d\gamma(x) dt = g_\kappa^{(+),n+1} - g_\kappa^{(-),n+1}.$$

6. $\tilde{c}_{i,\kappa}^{n+1}$ is the approximation of $\tilde{c}_i$ extended by 0 on $(\partial\Omega \times \mathbb{R}_+^*) \setminus \Sigma^+$:

$$\tilde{c}_{i,\kappa}^{n+1} = \begin{cases} \frac{1}{\Delta t^{n+1}} \frac{1}{|\partial\kappa\cap\partial\Omega|} \int_{t^n}^{t^{n+1}} \int_{\partial\kappa\cap\partial\Omega} \tilde{c}_i(x,t) d\gamma(x) dt & \text{if } |\partial\kappa \cap \partial\Omega| \neq 0, \\ 0 & \text{else,} \end{cases}$$

and it results that $\tilde{c}_{i,\kappa}^{n+1} \in [0,1]$.

Considering the coordinate system $\xi = h_\kappa^n - z$, the function $u_{i,\kappa}^n$ is defined for all $\kappa \in \mathcal{K}$, $n \geq 0$, and $i = 1, \ldots, L$ by

$$u_{i,\kappa}^n(\xi) = c_{i,\kappa}^n(h_\kappa^n - \xi) \text{ for all } \xi \in \mathbb{R}_+^*. \tag{3.6}$$

Let us note that, to obtain a fully discrete scheme, the initial condition $u^0_{i,\kappa}(\xi)$ is projected for each κ on a piecewise constant finite element subspace of $L^\infty(\mathbb{R}^*_+)$. Then, the scheme (3.4)–(3.5) generates a piecewise constant approximation of $u^n_{i,\kappa}(\xi)$ on each cell κ for all $i = 1, \dots, L$, with time-dependent mesh sizes in the direction ξ.

For the sake of simplicity, it is assumed in the remainder of this article that $\Delta t = \Delta t^n$ for all $n \geq 1$, although all the results presented in what follows readily extend to variable time steps.

In sections 4 and 5, we shall prove, for all $n \geq 0$, the existence of solutions $(h^n_\kappa)_{\kappa\in\mathcal{K}}$, $(c^{s,n+1}_{i,\kappa})_{\kappa\in\mathcal{K}}$, $(c^n_{i,\kappa})_{\kappa\in\mathcal{K}}$, and $(u^n_{i,\kappa})_{\kappa\in\mathcal{K}}$, $i = 1, \dots, L$, to problem (3.2)–(3.6). These solutions are unique except for the surface concentration $c^{s,n+1}_{i,\kappa}$ which is arbitrary (such that $\sum_{j=1}^L c^{s,n+1}_{j,\kappa} = 1$) at some degenerate points $(\kappa, n+1)$ for which it is chosen according to Lemma 5.1.

For any admissible mesh $(\mathcal{K}, \Sigma_{int}, \mathcal{P})$ of Ω in the sense of Definition 3.1, any time step $\Delta t > 0$, and $i = 1, \dots, L$, let $h_{\mathcal{K},\Delta t}$, $c^s_{i,\mathcal{K},\Delta t}$ defined on $\Omega \times \mathbb{R}^*_+$ and $u_{i,\mathcal{K},\Delta t}$ defined on $\Omega \times \mathbb{R}^*_+ \times \mathbb{R}^*_+$ denote the functions such that

$$\left\{\begin{aligned} h_{\mathcal{K},\Delta t}(x,t) &= h^{n+1}_\kappa, \\ u_{i,\mathcal{K},\Delta t}(x,\xi,t) &= u^{n+1}_{i,\kappa}(\xi), \\ c^s_{i,\mathcal{K},\Delta t}(x,t) &= c^{s,n+1}_{i,\kappa} \end{aligned}\right. \tag{3.7}$$

for all $x \in \kappa$, $\kappa \in \mathcal{K}$, $t \in (t^n, t^{n+1}]$, $\xi \in \mathbb{R}^*_+$, $n \geq 0$, where h^n_κ, $c^{s,n+1}_{i,\kappa}$, $c^n_{i,\kappa}$ are any given solution of (3.2)–(3.6) chosen according to Lemma 5.1. From Lemma 5.1, the functions $h_{\mathcal{K},\Delta t}$ and $u_{i,\mathcal{K},\Delta t}$ do not depend on the choice of the solution of (3.2)–(3.6).

The aim of this article is then to prove the following theorem.

THEOREM 3.3. *Hypothesis* 1 *is assumed to hold. For all* $m \in \mathbb{N}$, *let* $(\mathcal{K}_m, \Sigma^m_{int}, \mathcal{P}_m)$ *be an admissible mesh of* Ω *in the sense of Definition* 3.1 *and* $\Delta t_m > 0$. *Let us assume that there exists* $\alpha > 0$ *such that* $reg(\mathcal{K}_m) \leq \alpha$ *for all* $m \in \mathbb{N}$, *and that* $\Delta t_m \to 0$, $\frac{\delta\mathcal{K}_m}{\sqrt{\Delta t_m}} \to 0$ *as* $m \to \infty$.

For all $m \in \mathbb{N}$ *and* $i = 1, \dots, L$, *let* $h_{\mathcal{K}_m,\Delta t_m}$, $u_{i,\mathcal{K}_m,\Delta t_m}$ *denote the unique functions defined by* (3.7) *and* $c^s_{i,\mathcal{K}_m,\Delta t_m}$ *be a function defined by* (3.7), *from any solution of* (3.2)–(3.6) *chosen according to Lemma* 5.1 *with* $\mathcal{K} = \mathcal{K}_m$, $\Delta t = \Delta t_m$.

Then, the sequence $(h_{\mathcal{K}_m,\Delta t_m})_{m\in\mathbb{N}}$ *converges to the solution* h *of problem* (2.6) *in* $L^\infty(0,T;L^2(\Omega))$ *for all* $T > 0$, *and there exists a subsequence of* $(\mathcal{K}_m, \Delta t_m)_{m\in\mathbb{N}}$, *still denoted by* $(\mathcal{K}_m, \Delta t_m)_{m\in\mathbb{N}}$, *such that, for all* $i \in \{1, \dots, L\}$, *the subsequence* $(c^s_{i,\mathcal{K}_m,\Delta t_m})_{m\in\mathbb{N}}$ *(resp.,* $(u_{i,\mathcal{K}_m,\Delta t_m})_{m\in\mathbb{N}}$*) converges to a function* c^s_i *in* $L^\infty(\Omega \times \mathbb{R}^*_+)$ *(resp.,* u_i *in* $L^\infty(\Omega \times \mathbb{R}^*_+ \times \mathbb{R}^*_+)$*) for the weak-$\star$ topology. Furthermore, for all* $i \in \{1, \dots, L\}$, *the limit* (c^s_i, u_i) *is a weak solution of problem* (2.7) *in the sense of Definition* 2.1.

This convergence result will be obtained in section 4 for the approximate solution for the sediment thickness and in section 5 for the approximate concentrations.

4. Stability and convergence for the approximate sediment thickness and its time derivative. Summing (3.2) over $i = 1, \dots, L$ yields that for all $n \in \mathbb{N}$, the solution $(h^{n+1}_\kappa)_{\kappa\in\mathcal{K}}$ satisfies the following implicit finite volume discretization of (2.6):

$$|\kappa| \frac{h^{n+1}_\kappa - h^n_\kappa}{\Delta t} + \sum_{\kappa'\in\mathcal{K}_\kappa} T_{\kappa\kappa'}(h^{n+1}_\kappa - h^{n+1}_{\kappa'}) - |\partial\kappa \cap \partial\Omega|\, g^{n+1}_\kappa = 0, \tag{4.1}$$

with $h^0_\kappa = h^0(x_\kappa)$. The proof of existence and uniqueness of the solution $(h^n_\kappa)_{\kappa\in\mathcal{K}}$ for all $n \ge 0$ is classical and can be found, e.g., in [6] for any admissible mesh $(\mathcal{K}, \Sigma_{int}, \mathcal{P})$ of Ω.

The following proposition provides estimates of the error on h and its time derivative. The error estimates on h have already been proved in [6].

Proposition 4.1. *Let us assume that Hypothesis* 1 *holds and let h denote the solution of problem* (2.6)*. Let $(\mathcal{K}, \Sigma_{int}, \mathcal{P})$ be an admissible mesh of Ω in the sense of Definition* 3.1*, $T > 0$, and $\Delta t \in (0,T)$. For all $n \in \{0,\dots,N_{\Delta t}+1\}$, let $(h^n_\kappa)_{\kappa\in\mathcal{K}}$ be the solution of* (4.1) *and $e^n_{\mathcal{K}} \in X(\mathcal{K})$ be defined by $e^n_{\mathcal{K}}(x) = e^n_\kappa = h(x_\kappa, t^n) - h^n_\kappa$ for all $x \in \kappa$, $\kappa \in \mathcal{K}$. Then, there exist D_1, D_2, D_3, and $D_4 > 0$ depending only on $\|\nabla \partial_t h\|_{L^\infty(\Omega\times(0,2T))}$, $\|h\|_{L^\infty(0,2T;W^{2,\infty}(\Omega))}$, T, and Ω such that*

$$\|e^n_{\mathcal{K}}\|^2_{L^2(\Omega)} \le D_1(\Delta t + \delta\mathcal{K})^2 \quad \textit{for all } n \in \{1,\dots,N_{\Delta t}+1\}, \tag{4.2}$$

$$\sum_{n=0}^{N_{\Delta t}} \Delta t\, |e^{n+1}_{\mathcal{K}}|^2_{1,\mathcal{K}} \le D_2\,(\Delta t + \delta\mathcal{K})^2, \tag{4.3}$$

$$\sum_{n=0}^{N_{\Delta t}} \Delta t \left\| \frac{e^{n+1}_{\mathcal{K}} - e^n_{\mathcal{K}}}{\Delta t} \right\|^2_{L^2(\Omega)} \le D_3 \frac{(\delta\mathcal{K} + \Delta t)^2}{\Delta t}, \tag{4.4}$$

$$\begin{aligned} &\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} |\sigma|\, d(\kappa,\kappa') \Bigg(\frac{h^{n+1}_{\kappa'} - h^{n+1}_\kappa}{d(\kappa,\kappa')} \\ &\quad - \frac{1}{\Delta t}\frac{1}{|\sigma|} \int_{t^n}^{t^{n+1}} \int_\sigma \nabla h(x,t)\cdot \vec{n}_{\kappa\kappa'} d\gamma(x)\, dt \Bigg)^2 \le D_4(\Delta t + \delta\mathcal{K})^2. \end{aligned} \tag{4.5}$$

Proof. Integrating (2.6) over the control volume $\kappa \in \mathcal{K}$ and time interval (t^n, t^{n+1}) for all $n \in \{0,\dots,N_{\Delta t}\}$, one obtains

$$\int_{t^n}^{t^{n+1}} \int_\kappa \partial_t h(x,t) dx dt - \int_{t^n}^{t^{n+1}} \int_{\partial\kappa} \nabla h(x,t)\cdot \vec{n}_\kappa d\gamma(x) dt = 0, \tag{4.6}$$

where $\vec{n}_\kappa$ is the normal unit vector to $\partial\kappa$ outward to κ. Subtracting (4.1) from (4.6)/Δt and using the definition of g^{n+1}_κ yield the following equation for the error e^{n+1}_k:

$$|\kappa| \frac{e^{n+1}_\kappa - e^n_\kappa}{\Delta t} + \sum_{\kappa'\in\mathcal{K}_\kappa} T_{\kappa\kappa'}(e^{n+1}_\kappa - e^{n+1}_{\kappa'}) = -|\kappa| P^n_\kappa - \sum_{\sigma\in\Sigma_\kappa} |\sigma| R^n_{\kappa,\sigma} \tag{4.7}$$

with the consistency residuals

$$R^n_{\kappa,\sigma} = \frac{1}{\Delta t}\frac{1}{|\sigma|} \int_{t^n}^{t^{n+1}} \int_\sigma \left[\frac{h(x_{\kappa'}, t^{n+1}) - h(x_\kappa, t^{n+1})}{d(\kappa,\kappa')} - \nabla h(x,t)\cdot \vec{n}_{\kappa\kappa'} \right] d\gamma(x) dt$$

for all $\kappa \in \mathcal{K}$ and $\sigma \in \Sigma_\kappa \cap \Sigma_{\kappa'}$, and

$$P^n_\kappa = \frac{1}{\Delta t}\frac{1}{|\kappa|} \int_{t^n}^{t^{n+1}} \int_\kappa (\partial_t h(x,t) - \partial_t h(x_\kappa, t))\, dx dt \qquad \text{for all } \kappa \in \mathcal{K}.$$

Thanks to the regularity of h, there exists $C_1 > 0$ depending on $\|\nabla \partial_t h\|_{L^\infty(\Omega\times(0,2T))}$ only such that

$$|P^n_\kappa| \le C_1\, \delta\mathcal{K}, \tag{4.8}$$

and $C_2 > 0$ depending only on $\|\partial_t \nabla h\|_{L^\infty(\Omega\times(0,2T))}$, and $\|h\|_{L^\infty(0,2T;W^{2,\infty}(\Omega))}$ such that

$$|R^n_{\kappa,\sigma}| \le C_2\,(\delta\mathcal{K} + \Delta t). \tag{4.9}$$

Then, multiplying (4.7) by e^{n+1}_κ and summing over the cells $\kappa \in \mathcal{K}$ yield the estimate

$$\begin{aligned}&\sum_{\kappa\in\mathcal{K}} |\kappa|(e^{n+1}_\kappa - e^n_\kappa)e^{n+1}_\kappa + \Delta t \sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} T_{\kappa\kappa'}(e^{n+1}_\kappa - e^{n+1}_{\kappa'})^2\\ &\quad = -\Delta t\sum_{\kappa\in\mathcal{K}}|\kappa| P^n_\kappa\, e^{n+1}_\kappa - \Delta t \sum_{\kappa\in\mathcal{K}}\sum_{\sigma\in\Sigma_\kappa}|\sigma| R^n_{\kappa,\sigma}\, e^{n+1}_\kappa.\end{aligned} \tag{4.10}$$

Let us note that $R_{\kappa,\sigma} = -R_{\kappa',\sigma}$ for all $\sigma = \kappa|\kappa' \in \Sigma_{int}$ so that $R_\sigma = |R_{\kappa,\sigma}|$ for $\sigma \in \Sigma_\kappa$ can be defined for all $\sigma \in \Sigma_{int}$. Then, using in (4.10) the equality $(e^{n+1}_\kappa - e^n_\kappa)\,e^{n+1}_\kappa = \frac{1}{2}\big[(e^{n+1}_\kappa)^2 - (e^n_\kappa)^2 + (e^{n+1}_\kappa - e^n_\kappa)^2\big]$, Young's inequality, (3.1), (4.8), and (4.9), we obtain

$$\begin{aligned}&\|e^{n+1}_{\mathcal{K}}\|^2_{L^2(\Omega)} + \Delta t\,|e^{n+1}_{\mathcal{K}}|^2_{1,\mathcal{K}} \le \|e^n_{\mathcal{K}}\|^2_{L^2(\Omega)}\\ &+\Delta t\, C_3\,(\Delta t + \delta\mathcal{K})\,\|e^{n+1}_{\mathcal{K}}\|^2_{L^2(\Omega)} + \Delta t\, C_4\,(\delta\mathcal{K}+\Delta t)^2,\end{aligned} \tag{4.11}$$

with C_3 and C_4 depending only on $\|\nabla\partial_t h\|_{L^\infty(\Omega\times(0,2T))}$, $\|h\|_{L^\infty(0,2T;W^{2,\infty}(\Omega))}$, and Ω. Using the same arguments as in [6], the estimate (4.2) derives from (4.11). Summing (4.11) over $n \in \{0,\dots,N_{\Delta t}\}$ and using inequality (4.2) and the property $e^0_\kappa = 0$ for all $\kappa \in \mathcal{K}$, we obtain inequality (4.3).

Then, (4.3) is equivalent to

$$\begin{aligned}&\sum_{n=0}^{N_{\Delta t}}\Delta t \sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} |\sigma|\, d(\kappa,\kappa')\left(\frac{h^{n+1}_{\kappa'} - h^{n+1}_\kappa}{d(\kappa,\kappa')} - \frac{h(x_{\kappa'},t^{n+1}) - h(x_\kappa,t^{n+1})}{d(\kappa,\kappa')}\right)^2\\ &\qquad\le D_2\,(\Delta t + \delta\mathcal{K})^2.\end{aligned} \tag{4.12}$$

Furthermore,

$$\begin{aligned}&\sum_{n=0}^{N_{\Delta t}}\Delta t \sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} |\sigma|\, d(\kappa,\kappa')\left(\frac{h(x_{\kappa'},t^{n+1}) - h(x_\kappa,t^{n+1})}{d(\kappa,\kappa')}\right.\\ &\qquad\left. - \frac{1}{\Delta t}\frac{1}{|\sigma|}\int_{t^n}^{t^{n+1}}\int_\sigma \nabla h(x,t)\cdot\vec{n}_{\kappa\kappa'}\,d\gamma(x)dt\right)^2\\ &= \sum_{n=0}^{N_{\Delta t}}\Delta t \sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} |\sigma|\, d(\kappa,\kappa')(R^n_\sigma)^2 \;\le\; C_5\,(\delta\mathcal{K}+\Delta t)^2,\end{aligned} \tag{4.13}$$

with C_5 depending on $\|\nabla\partial_t h\|_{L^\infty(\Omega\times(0,2T))}$, $\|h\|_{L^\infty(0,2T;W^{2,\infty}(\Omega))}$, T, and Ω. The estimate (4.5) derives from (4.12) and (4.13).

To prove (4.4), let us multiply (4.7) by $(e^{n+1}_\kappa - e^n_\kappa)/\Delta t$ and sum over $\kappa \in \mathcal{K}$:

$$\begin{aligned}&\Delta t\sum_{\kappa\in\mathcal{K}}|\kappa|\left(\frac{e^{n+1}_\kappa - e^n_\kappa}{\Delta t}\right)^2 + \sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} T_{\kappa\kappa'}(e^{n+1}_\kappa - e^{n+1}_{\kappa'})(e^{n+1}_\kappa - e^{n+1}_{\kappa'} - e^n_\kappa + e^n_{\kappa'})\\ &\quad= -\Delta t\sum_{\kappa\in\mathcal{K}}|\kappa|\,P^n_\kappa\,\frac{e^{n+1}_\kappa - e^n_\kappa}{\Delta t} - \sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} |\sigma| R^n_{\kappa,\sigma}(e^{n+1}_\kappa - e^{n+1}_{\kappa'} - e^n_\kappa + e^n_{\kappa'}).\end{aligned}$$

From $(e_\kappa^{n+1}-e_{\kappa'}^{n+1})(e_\kappa^{n+1}-e_{\kappa'}^{n+1}-e_\kappa^n+e_{\kappa'}^n) = \frac{1}{2}\big[\,(e_\kappa^{n+1}-e_{\kappa'}^{n+1})^2-(e_\kappa^n-e_{\kappa'}^n)^2+(e_\kappa^{n+1}-e_{\kappa'}^{n+1}-e_\kappa^n+e_{\kappa'}^n)^2\,\big]$ and Young's inequality, it results that

$$
\begin{aligned}
&2\Delta t\sum_{\kappa\in\mathcal{K}}|\kappa|\left(\frac{e_\kappa^{n+1}-e_\kappa^n}{\Delta t}\right)^2+\sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}}T_{\kappa\kappa'}(e_\kappa^{n+1}-e_{\kappa'}^{n+1}-e_\kappa^n+e_{\kappa'}^n)^2\\
&\quad+\sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}}T_{\kappa\kappa'}(e_\kappa^{n+1}-e_{\kappa'}^{n+1})^2\le\sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}}T_{\kappa\kappa'}(e_\kappa^{n}-e_{\kappa'}^{n})^2\\
&\quad+\Delta t\sum_{\kappa\in\mathcal{K}}|\kappa|\,(P_\kappa^n)^2+\Delta t\sum_{\kappa\in\mathcal{K}}|\kappa|\left(\frac{e_\kappa^{n+1}-e_\kappa^n}{\Delta t}\right)^2\\
&\quad+\sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}}d(\kappa,\kappa')\,|\sigma|\,(R_\sigma^n)^2+\sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}}T_{\kappa\kappa'}(e_\kappa^{n+1}-e_{\kappa'}^{n+1}-e_\kappa^n+e_{\kappa'}^n)^2.
\end{aligned}
\tag{4.14}
$$

Summing (4.14) for all $n\in\{0,\dots,N_{\Delta t}\}$ and using (4.8), (4.9), (3.1), and the property $e_\kappa^0=0$ for all $\kappa\in\mathcal{K}$, we get

$$
\sum_{n=0}^{N_{\Delta t}}\Delta t\sum_{\kappa\in\mathcal{K}}|\kappa|\left(\frac{e_\kappa^{n+1}-e_\kappa^n}{\Delta t}\right)^2\le C_6\,(\delta\mathcal{K})^2+C_7\,\frac{(\Delta t+\delta\mathcal{K})^2}{\Delta t}
$$

with C_6 and $C_7>0$ depending only on $\|\nabla\partial_t h\|_{L^\infty(\Omega\times(0,2T))}$, $\|h\|_{L^\infty(0,2T;W^{2,\infty}(\Omega))}$, Ω, and T, which proves (4.4). ∎

Remark 2. According to (4.4) given in Proposition 4.1, the discrete time derivative of the error tends to zero with the mesh size and time step under an inverse CFL condition. This condition is due to the fact that the finite volume scheme is implicit in time and that few assumptions have been made on the regularity of h. However, it is possible to get rid of this inverse CFL condition by assuming h much more regular. Such a result can be found in [13].

COROLLARY 1. *Let us assume that Hypothesis* 1 *holds, and let h denote the solution of problem* (2.6). *Let $(\mathcal{K},\Sigma_{int},\mathcal{P})$ be an admissible mesh of Ω in the sense of Definition* 3.1, *$T>0$, $\Delta t\in(0,T)$, and let $\beta>0$ be such that $\delta\mathcal{K}\le\beta\sqrt{\Delta t}$. For all $n\in\{0,\dots,N_{\Delta t}+1\}$, let $(h_\kappa^n)_{\kappa\in\mathcal{K}}$ be the solution of* (4.1), *and let us define $h_\mathcal{K}^n\in X(\mathcal{K})$ (resp., $\delta_t h_\mathcal{K}^n\in X(\mathcal{K})$) by $h_\mathcal{K}^n(x)=h_\kappa^n$ (resp., $\delta_t h_\mathcal{K}^n(x)=\frac{h_\kappa^{n+1}-h_\kappa^n}{\Delta t}$) for $x\in\kappa$, $\kappa\in\mathcal{K}$. Then, there exist $D_5>0$ depending only on $\|h\|_{L^\infty(0,2T;W^{2,\infty}(\Omega))}$, $\|\nabla\partial_t h\|_{L^\infty(\Omega\times(0,2T))}$, Ω, and T and D_6, D_6', $D_6''>0$ depending on $\|\partial_t h\|_{L^\infty(\Omega\times(0,2T))}$, $\|h\|_{L^\infty(0,2T;W^{2,\infty}(\Omega))}$, $\|\nabla\partial_t h\|_{L^\infty(\Omega\times(0,2T))}$, Ω, and T, with D_6 also depending on β, such that*

$$
\sum_{n=0}^{N_{\Delta t}}\Delta t\,|h_\mathcal{K}^{n+1}|_{1,\mathcal{K}}^2\le D_5,
\tag{4.15}
$$

and

$$
\sum_{n=0}^{N_{\Delta t}}\Delta t\,\|\delta_t h_\mathcal{K}^n\|_{L^2(\Omega)}^2\le D_6'+D_6''\,\frac{(\delta\mathcal{K}+\Delta t)^2}{\Delta t}\le D_6.
\tag{4.16}
$$

Proof. The proof is straightforward, using the error estimates (4.3) and (4.4), the regularity of h, and the estimate (3.1). ∎

For any admissible mesh $(\mathcal{K}, \Sigma_{int}, \mathcal{P})$ of Ω in the sense of Definition 3.1 and any time step $\Delta t > 0$, let $(h_\kappa^n)_{\kappa\in\mathcal{K}}$ for all $n \geq 0$ be the solution of (4.1), and let $\delta_t h_{\mathcal{K},\Delta t}$ denote the function defined on $\Omega \times \mathbb{R}_+^*$, such that for all $x \in \kappa$, $\kappa \in \mathcal{K}$, $t \in (t^n, t^{n+1}]$, $n \geq 0$,

$$\delta_t h_{\mathcal{K},\Delta t}(x,t) = \frac{h_\kappa^{n+1} - h_\kappa^n}{\Delta t}. \tag{4.17}$$

PROPOSITION 4.2. *Let us assume that Hypothesis* 1 *holds, and let h denote the solution of problem* (2.6)*. Let us consider a family of admissible discretizations $(\mathcal{K}, \Sigma_{int}, \mathcal{P}, \Delta t)$ of $\Omega \times \mathbb{R}_+^*$, with $(\mathcal{K}, \Sigma_{int}, \mathcal{P})$ an admissible mesh of Ω in the sense of Definition* 3.1 *and $\Delta t > 0$ a time step. For a given discretization $(\mathcal{K}, \Sigma_{int}, \mathcal{P}, \Delta t)$ of this family, let $h_{\mathcal{K},\Delta t}$ (resp., $\delta_t h_{\mathcal{K},\Delta t}$) be the function defined by* (3.7) *(resp., by* (4.17)*) from the solution of* (4.1)*. Then, for all $T > 0$, $h_{\mathcal{K},\Delta t}$ converges to h in $L^\infty(0,T;L^2(\Omega))$ as Δt and $\delta\mathcal{K}$ tend to 0, and $\delta_t h_{\mathcal{K},\Delta t}$ converges to $\partial_t h$ in $L^2(\Omega \times (0,T))$ as Δt, $\delta\mathcal{K}$ and $\frac{\delta\mathcal{K}}{\sqrt{\Delta t}}$ tend to* 0.

Proof. Let $T > 0$, and let $(\mathcal{K}, \Sigma_{int}, \mathcal{P}, \Delta t)$ be an admissible discretization of $\Omega \times \mathbb{R}_+^*$ with $\Delta t < T$. For all $x \in \kappa$, $\kappa \in \mathcal{K}$, and $t \in (t^n, t^{n+1}]$, $n \in \{0, \ldots, N_{\Delta t}\}$, one has

$$\begin{aligned} h(x,t) - h_{\mathcal{K},\Delta t}(x,t) &= (h(x,t) - h(x_\kappa, t^{n+1})) + (h(x_\kappa, t^{n+1}) - h_\kappa^{n+1}) \\ &= (h(x,t) - h(x_\kappa, t^{n+1})) + e_\kappa^{n+1}. \end{aligned}$$

Thus, for all $t \in (t^n, t^{n+1}]$, $n \in \{0, \ldots, N_{\Delta t}\}$,

$$\begin{aligned} &\int_\Omega |h(x,t) - h_{\mathcal{K},\Delta t}(x,t)|^2 dx \\ &\quad \leq 2 \sum_{\kappa\in\mathcal{K}} \left[\int_\kappa |h(x,t) - h(x_\kappa, t^{n+1})|^2 dx + |\kappa|(e_\kappa^{n+1})^2 \right]. \end{aligned} \tag{4.18}$$

Thanks to Proposition 4.1, there exists $C_1 > 0$ depending only on $\|\nabla \partial_t h\|_{L^\infty(\Omega\times(0,2T))}$, $\|h\|_{L^\infty(0,2T;W^{2,\infty}(\Omega))}$, and Ω such that

$$\sum_{\kappa\in\mathcal{K}} |\kappa|(e_\kappa^{n+1})^2 \leq C_1 \big(\Delta t + \delta\mathcal{K}\big)^2 \quad \text{for all } n \in \{0, \ldots, N_{\Delta t}\}. \tag{4.19}$$

Furthermore, thanks to the regularity of h, there exists $C_2 > 0$ depending only on $\|\partial_t h\|_{L^\infty(\Omega\times(0,2T))}$ and $\|\nabla h\|_{L^\infty(\Omega\times(0,2T))}$ such that, for all $x \in \kappa$ and $t \in (t^n, t^{n+1}]$,

$$|h(x,t) - h(x_\kappa, t^{n+1})| \leq C_2 \big(\delta\mathcal{K} + \Delta t\big). \tag{4.20}$$

Then, using (4.19) and (4.20) in (4.18) yields, for all $t \in (0,T)$,

$$\|h(.,t) - h_{\mathcal{K},\Delta t}(.,t)\|_{L^2(\Omega)}^2 \leq C_3 \big(\delta\mathcal{K} + \Delta t\big)^2,$$

and consequently, $\|h - h_{\mathcal{K},\Delta t}\|_{L^\infty(0,T;L^2(\Omega))} \leq C_3' \big(\delta\mathcal{K} + \Delta t\big)$, where C_3, C_3' depend on $\|\nabla \partial_t h\|_{L^\infty(\Omega\times(0,2T))}$, $\|\partial_t h\|_{L^\infty(\Omega\times(0,2T))}$, $\|h\|_{L^\infty(0,2T;W^{2,\infty}(\Omega))}$, and Ω, so that the convergence holds.

Furthermore, for all $x \in \kappa$, $\kappa \in \mathcal{K}$, and $t \in (t^n, t^{n+1}]$, $n \in \{0, \ldots, N_{\Delta t}\}$, one has

$$\partial_t h(x,t) - \delta_t h_{\mathcal{K},\Delta t}(x,t) = \left(\partial_t h(x,t) - \frac{h(x_\kappa, t^{n+1}) - h(x_\kappa, t^n)}{\Delta t} \right) + \frac{e_\kappa^{n+1} - e_\kappa^n}{\Delta t}.$$

Thanks to the regularity of h, there exists a constant $C_4 > 0$ depending only on $\|\partial_t^2 h\|_{L^\infty(\Omega\times(0,2T))}$ and $\|\nabla\partial_t h\|_{L^\infty(\Omega\times(0,2T))}$, such that

$$\left|\frac{h(x_\kappa, t^{n+1}) - h(x_\kappa, t^n)}{\Delta t} - \partial_t h(x,t)\right| \le C_4\big(\delta\mathcal{K} + \Delta t\big),$$

from which, together with (4.4), results

$$\|\partial_t h - \delta_t h_{\mathcal{K},\Delta t}\|^2_{L^2(\Omega\times(0,T))} \le C_5\big(\delta\mathcal{K} + \Delta t\big)^2 + C_6\frac{\big(\delta\mathcal{K} + \Delta t\big)^2}{\Delta t},$$

with C_5 and C_6 depending only on Ω, T, $\|h\|_{W^{2,\infty}(\Omega\times(0,2T))}$. Thus, the convergence of $\delta_t h_{\mathcal{K},\Delta t}$ to $\partial_t h$ in $L^2(\Omega\times(0,T))$ as Δt, $\delta\mathcal{K}$, and $\frac{\delta\mathcal{K}}{\sqrt{\Delta t}} \to 0$ is proved. $\square$

5. Convergence of sequences of approximate concentrations toward a weak solution. We shall first prove the existence of a solution for the concentrations satisfying stability estimates from which the weak-$\star$ convergence, up to a subsequence, of the concentrations in L^∞ is deduced.

Existence, stability, and weak-$\star$ convergence.

LEMMA 5.1. *Let $(\mathcal{K}, \Sigma_{int}, \mathcal{P})$ be an admissible mesh of Ω in the sense of Definition 3.1, $\Delta t > 0$, and, for all $n \in \mathbb{N}$, let $(h^n_\kappa)_{\kappa\in\mathcal{K}}$ be the solution of (4.1). For $i \in \{1,\dots,L\}$ and $n \in \mathbb{N}$, there exists a unique solution $(c^n_{i,\kappa})_{\kappa\in\mathcal{K}}$, and there exists at least one solution $(c^{s,n+1}_{i,\kappa})_{\kappa\in\mathcal{K}}$ to the set of equations (3.2)–(3.5) such that*

$$c^{s,n+1}_{i,\kappa} \in [0,1] \text{ for all } \kappa \in \mathcal{K} \text{ and } n \in \mathbb{N}. \tag{5.1}$$

Furthermore, one has

$$c^n_{i,\kappa}(z) \in [0,1] \text{ for all } \kappa \in \mathcal{K},\ z < h^n_\kappa, \text{ and } n \in \mathbb{N}.$$

Proof. The complete proof can be found in [5]. It is done by induction over $n \in \mathbb{N}^*$ and over the cells $\kappa \in \mathcal{K}$ sorted by decreasing topographical order. For the highest topographical point(s) κ, the fluxes at the edges of the cell κ are either input boundary fluxes or ouput fluxes. Let us consider a control volume $\kappa \in \mathcal{K}$ and a time $n \in \mathbb{N}^*$, and let us assume that the proposition holds for all the previous times t^{l+1}, $0 \le l < n$, and all the lower cells at time t^{n+1}. It results from the induction hypothesis and the upwinding of c^s_i that $c^{s,n+1}_{i,\kappa}$ can be computed explicitly from the lower cell concentrations c^s_i using (3.2), and that the inequality

$$\sum_{\kappa'\in\mathcal{K}_\kappa,\, h^{n+1}_\kappa < h^{n+1}_{\kappa'}} T_{\kappa\kappa'}\, c^{s,n+1}_{i,\kappa\kappa'}\,(h^{n+1}_\kappa - h^{n+1}_{\kappa'}) \le 0 \tag{5.2}$$

holds for all $i = 1,\dots,L$. Let us first assume that $h^{n+1}_\kappa - h^n_\kappa \le 0$ (erosion). It results from the induction hypothesis that

$$c^{s,n+1}_{i,\kappa}\left(\sum_{\kappa'\in\mathcal{K}_\kappa,\, h^{n+1}_\kappa \ge h^{n+1}_{\kappa'}} T_{\kappa\kappa'}\,(h^{n+1}_\kappa - h^{n+1}_{\kappa'}) + |\partial\kappa\cap\partial\Omega|\, g^{(-),n+1}_\kappa\right) \ge 0$$

for all i. In this equation, either the term into brackets is strictly positive for all $i = 1,\dots,L$ and then $c^{s,n+1}_{i,\kappa} \ge 0$, or it vanishes for all i and the point $(\kappa, n+1)$ is a degenerate point in the sense that all the fluxes at the edges of the control volume κ vanish and $h_\kappa^{n+1} = h_\kappa^n$. The concentrations can in that case be chosen arbitrarily such that $\sum_{i=1}^L c_{i,\kappa}^{s,n+1} = 1$. Let us now consider the sedimentation case for which $h_\kappa^{n+1} - h_\kappa^n > 0$. It results from (3.2) and the induction hypothesis that

$$c_{i,\kappa}^{s,n+1}\left(\frac{h_\kappa^{n+1} - h_\kappa^n}{\Delta t^{n+1}}\,|\kappa| + \sum_{\kappa' \in \mathcal{K}_\kappa, h_\kappa^{n+1} \geq h_{\kappa'}^{n+1}} T_{\kappa\kappa'}\,(h_\kappa^{n+1} - h_{\kappa'}^{n+1}) + |\partial\kappa \cap \partial\Omega|\, g_\kappa^{(-),n+1}\right) \geq 0,$$

and hence $c_{i,\kappa}^{s,n+1} \geq 0$ for all $i = 1, \dots, L$. Since $h_\kappa^{n+1} = h_\kappa^n$ for any degenerate point $(\kappa, n+1)$, there exists a unique column concentration $c_{i,\kappa}^{n+1}$ solution of the set of equations (3.2)–(3.5) for each lithology. $\square$

Let us define for all $\kappa \in \mathcal{K}$, $n \in \mathbb{N}$, and $t \in (t^n, t^{n+1}]$ the following interpolation of the discrete sediment thickness:

$$h_\kappa(t) = h_\kappa^n + (t - t^n)\,\frac{h_\kappa^{n+1} - h_\kappa^n}{\Delta t}. \tag{5.3}$$

Then, the discrete solutions $(c_{i,\kappa}^n)_{n\in\mathbb{N}}$, $(u_{i,\kappa}^n)_{n\in\mathbb{N}}$, and $(c_{i,\kappa}^{s,n+1})_{n\in\mathbb{N}}$, given by Lemma 5.1, are extended to $t \in \mathbb{R}_+$ for all $\kappa \in \mathcal{K}$ as follows:

$$\left\{\begin{array}{l} c_{i,\kappa}(z,t) = \begin{cases} c_{i,\kappa}^n(z)\,\chi_{(-\infty,h_\kappa^n]} + c_{i,\kappa}^{s,n+1}\,\chi_{(h_\kappa^n, h_\kappa(t))} & \text{if } h_\kappa^{n+1} \geq h_\kappa^n, \\ c_{i,\kappa}^n(z)\,\chi_{(-\infty,h_\kappa(t))} & \text{otherwise} \end{cases} \\ \qquad\qquad \text{for all } t \in (t^n, t^{n+1}] \text{ and } z < h_\kappa(t), \\ c_{i,\kappa}(z,0) = c_{i,\kappa}^0(z) \text{ for all } z < h_\kappa^0, \end{array}\right. \tag{5.4}$$

$$u_{i,\kappa}(\xi,t) = c_{i,\kappa}(h_\kappa(t) - \xi, t) \text{ for all } t \geq 0 \text{ and } \xi \in \mathbb{R}_+^*, \tag{5.5}$$

$$c_{i,\kappa}^s(t) = c_{i,\kappa}^{s,n+1} \text{ for all } t \in (t^n, t^{n+1}]. \tag{5.6}$$

For any admissible mesh $(\mathcal{K}, \Sigma_{int}, \mathcal{P})$ of Ω in the sense of Definition 3.1 and any time step $\Delta t > 0$, let $\bar{u}_{i,\mathcal{K},\Delta t}$ be defined on $\Omega \times \mathbb{R}_+^* \times \mathbb{R}_+$, and let $c_{i,\mathcal{K},\Delta t}$ be defined on $\{(z,t),\, t \geq 0,\, z < h_\kappa(t)\}$, such that

$$\begin{cases} \bar{u}_{i,\mathcal{K},\Delta t}(x,\xi,t) = u_{i,\kappa}(\xi,t), \\ c_{i,\mathcal{K},\Delta t}(x,z,t) = c_{i,\kappa}(z,t) \end{cases} \tag{5.7}$$

for all $x \in \kappa$, $\kappa \in \mathcal{K}$, $t \geq 0$, $\xi \in \mathbb{R}_+^*$, $z < h_\kappa(t)$.

From Lemma 5.1, the unique functions $c_{i,\mathcal{K},\Delta t}$, $\bar{u}_{i,\mathcal{K},\Delta t}$, $u_{i,\mathcal{K},\Delta t}$ defined by (5.7) and (3.7) and any function $c_{i,\mathcal{K},\Delta t}^s$ defined by (3.7) from any solution of (3.2)–(3.6) chosen according to Lemma 5.1 take their values into the interval $[0,1]$. We deduce the following result.

PROPOSITION 5.2. *For all $m \in \mathbb{N}$, let $(\mathcal{K}_m, \Sigma_{int}^m, \mathcal{P}_m)$ be an admissible mesh of Ω in the sense of Definition* 3.1, *and let $\Delta t_m > 0$. Let us assume that $\Delta t_m \to 0$ and $\delta\mathcal{K}_m \to 0$ as $m \to \infty$.*

For all $m \in \mathbb{N}$ and $i = 1, \dots, L$, let $u_{i,\mathcal{K}_m,\Delta t_m}$ (resp., $\bar{u}_{i,\mathcal{K}_m,\Delta t_m}$) denote the unique function defined by (3.7) *(resp., by* (5.7)*) and $c_{i,\mathcal{K}_m,\Delta t_m}^s$ be a function defined by* (3.7), *from any solution of* (3.2)*–*(3.6) *chosen according to Lemma* 5.1 *with $\mathcal{K} = \mathcal{K}_m$, $\Delta t = \Delta t_m$.*

Then, under Hypothesis 1, *there exists a subsequence of $(\mathcal{K}_m, \Delta t_m)_{m\in\mathbb{N}}$, still denoted by $(\mathcal{K}_m, \Delta t_m)_{m\in\mathbb{N}}$, such that for all $i \in \{1, \dots, L\}$*

(i) *the subsequence* $(c^s_{i,\mathcal{K}_m,\Delta t_m})_{m\in\mathbb{N}}$ *converges to a function* c^s_i *in* $L^\infty(\Omega\times\mathbb{R}^*_+)$ *for the weak-*$\star$ *topology, and*

(ii) *the subsequences* $(u_{i,\mathcal{K}_m,\Delta t_m})_{m\in\mathbb{N}}$ *and* $(\bar u_{i,\mathcal{K}_m,\Delta t_m})_{m\in\mathbb{N}}$ *converge to a function* u_i *in* $L^\infty(\Omega\times\mathbb{R}^*_+\times\mathbb{R}^*_+)$ *for the weak-*$\star$ *topology.*

Proof. For the sake of simplicity, the subscript i is dropped. Thanks to Lemma 5.1, the sequence $(c^s_{\mathcal{K}_m,\Delta t_m})_{m\in\mathbb{N}}$ (resp., $(u_{\mathcal{K}_m,\Delta t_m})_{m\in\mathbb{N}}$ and $(\bar u_{\mathcal{K}_m,\Delta t_m})_{m\in\mathbb{N}}$) is bounded in $L^\infty(\Omega\times\mathbb{R}^*_+)$ (resp., in $L^\infty(\Omega\times\mathbb{R}^*_+\times\mathbb{R}^*_+)$). Then, there exists a subsequence of $(\mathcal{K}_m,\Delta t_m)_{m\in\mathbb{N}}$, still denoted by $(\mathcal{K}_m,\Delta t_m)_{m\in\mathbb{N}}$, such that $(c^s_{\mathcal{K}_m,\Delta t_m})_{m\in\mathbb{N}}$ (resp., $(u_{\mathcal{K}_m,\Delta t_m})_{m\in\mathbb{N}}$ and $(\bar u_{\mathcal{K}_m,\Delta t_m})_{m\in\mathbb{N}}$) converges to c^s (resp., u and u') in $L^\infty(\Omega\times\mathbb{R}^*_+)$ (resp., in $L^\infty(\Omega\times\mathbb{R}^*_+\times\mathbb{R}^*_+)$) for the weak-$\star$ topology. It remains to prove that $u=u'$ in $L^\infty(\Omega\times\mathbb{R}^*_+\times\mathbb{R}^*_+)$.

Using definitions (3.6) and (5.5), for $x\in\kappa$, $\kappa\in\mathcal{K}_m$, and $t\in(t^n,t^{n+1}]$, the functions $\bar u_{\mathcal{K}_m,\Delta t_m}$ and $u_{\mathcal{K}_m,\Delta t_m}$ are related as follows:

$$u^{n+1}_\kappa(\xi)=\begin{cases} u_\kappa(\xi-(h_\kappa(t)-h^n_\kappa),t) & \text{for all } \xi\ge h_\kappa(t)-h^n_\kappa \quad \text{if } h^{n+1}_\kappa\ge h^n_\kappa,\\ u_\kappa(\xi+(h_\kappa(t)-h^{n+1}_\kappa),t) & \text{for all } \xi\ge 0 \quad \text{if } h^{n+1}_\kappa< h^n_\kappa.\end{cases}$$

Let $\varphi\in\mathcal{C}^\infty_c(\Omega\times\mathbb{R}^*_+\times\mathbb{R}^*_+)$ and $T>0$ be such that $\varphi(.,.,t)=0$ for all $t\ge T$. Since the concentrations are bounded in $[0,1]$, it can be shown that

$$\left|\int_\Omega\int_{\mathbb{R}^*_+}\int_{\mathbb{R}^*_+}(\bar u_{\mathcal{K}_m,\Delta t_m}-u_{\mathcal{K}_m,\Delta t_m})\,\varphi(x,\xi,t)\,dx\,d\xi\,dt\right| \le C_1\sum_{n=0}^{N_{\Delta t_m}}\Delta t_m\sum_{\kappa\in\mathcal{K}_m}|\kappa||h^{n+1}_\kappa-h^n_\kappa|,$$

with C_1 depending only on φ, Ω, and T. From the estimate (4.16) it results that

$$\left|\int_\Omega\int_{\mathbb{R}^*_+}\int_{\mathbb{R}^*_+}(\bar u_{\mathcal{K}_m,\Delta t_m}-u_{\mathcal{K}_m,\Delta t_m})\,\varphi(x,\xi,t)\,dx\,d\xi\,dt\right|\to 0 \text{ as } m\to\infty,$$

and $u=u'$ in the space of distributions on $\Omega\times\mathbb{R}^*_+\times\mathbb{R}^*_+$, and hence in $L^\infty(\Omega\times\mathbb{R}^*_+\times\mathbb{R}^*_+)$. ∎

Flux term. The following proposition provides a result of convergence for the flux term appearing in the discretization of the surface conservation equation. It will be used to show that (c^s_i,u_i) satisfies the second equation (2.9) of the weak formulation. The proof of this proposition is an adaptation to the coupling of a parabolic and a hyperbolic equation of the result proved in [6] for the coupling of an elliptic and a hyperbolic equation in the case of a two phase Darcy flow.

PROPOSITION 5.3. *Let us assume that Hypothesis* 1 *holds and let* h *denote the solution of problem* (2.6). *Let us consider a family of admissible discretizations* $(\mathcal{K},\Sigma_{int},\mathcal{P},\Delta t)$ *of* $\Omega\times\mathbb{R}^*_+$, *with* $(\mathcal{K},\Sigma_{int},\mathcal{P})$ *an admissible mesh of* Ω *in the sense of Definition* 3.1 *and* $\Delta t>0$ *a time step. Let us also assume that there exist* α *and* $\beta>0$ *such that, for all discretizations* $(\mathcal{K},\Sigma_{int},\mathcal{P},\Delta t)$ *of this family,* $\delta\mathcal{K}\le\beta\sqrt{\Delta t}$ *and* $reg(\mathcal{K})\le\alpha$. *For any admissible discretization* $(\mathcal{K},\Sigma_{int},\mathcal{P},\Delta t)$, *let* $h_{\mathcal{K},\Delta t}$ *denote the function defined by* (3.7) *from the solution of* (4.1), *and let* $(c^{s,n+1}_{i,\kappa})_{\kappa\in\mathcal{K}_m,n\ge 0}$ *be any solution of* (3.2)–(3.5) *chosen according to Lemma* 5.1. *Let* $T>0$, *then, for all* $\varphi\in\mathcal{A}^s_0=\{v\in\mathcal{C}^\infty_c(\mathbb{R}^{d+1})\,|\,v(x,t)=0$ *on* $\partial\Omega\times\mathbb{R}^*_+\setminus\Sigma^+\}$, *and for all* $i=1,\dots,L$,

$$T_{i,\mathcal{K},\Delta t}\to\int_0^T\left(\int_\Omega c^s_i(x,t)\,\nabla h(x,t)\cdot\nabla\varphi(x,t)\,dx-\int_{\partial\Omega}\tilde c_i(x,t)\,g(x,t)\,\varphi(x,t)\,d\gamma(x)\right)dt$$

as $\Delta t \to 0$, with

$$T_{i,\mathcal{K},\Delta t} = \sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa \in \mathcal{K}} \sum_{\kappa' \in \mathcal{K}_\kappa} T_{\kappa\kappa'} c_{i,\kappa\kappa'}^{s,n+1} \left(h_\kappa^{n+1} - h_{\kappa'}^{n+1}\right) \varphi(x_\kappa, t^{n+1})$$
$$- \sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa \in \mathcal{K}} |\partial\kappa \cap \partial\Omega| \left(g_\kappa^{(+),n+1} \tilde{c}_{i,\kappa}^{n+1} - g_\kappa^{(-),n+1} c_{i,\kappa}^{s,n+1} \right) \varphi(x_\kappa, t^{n+1}).$$

Columns property. The following proposition states that the column concentrations interpolated in time $\bar{u}_{i,\mathcal{K},\Delta t}$, $i = 1, \dots, L$, satisfy in the weak sense a linear advection equation. This property is used in the proof of Theorem 3.3 to show the convergence, up to a subsequence, of the approximate solutions to a solution of the weak formulation (2.7).

PROPOSITION 5.4. *Let us assume that Hypothesis* 1 *holds and let h denote the solution of problem* (2.6). *Let $(\mathcal{K}, \Sigma_{int}, \mathcal{P})$ be an admissible mesh of Ω in the sense of Definition* 3.1, *$T > 0$, and $\Delta t \in (0, T)$.*

Let $h_{\mathcal{K},\Delta t}$, $u_{i,\mathcal{K},\Delta t}$, $i = 1, \dots, L$ (resp., $\delta_t h_{\mathcal{K},\Delta t}$ and $\bar{u}_{i,\mathcal{K},\Delta t}$, $i = 1, \dots, L$), denote the unique functions defined by (3.7) *(resp., by* (4.17) *and* (5.7)*) and $c_{i,\mathcal{K},\Delta t}^s$, $i = 1, \dots, L$, be a function defined by* (3.7), *from any solution of* (3.2)–(3.6) *chosen according to Lemma* 5.1.

Then, for any $\kappa \in \mathcal{K}$ and $i \in \{1, \dots, L\}$, the following hold.

(i) *For all $\varphi \in W_T = \{v \in \mathcal{C}_c^\infty(\mathbb{R}^2) \,|\, v(.,T) = 0$ on $\mathbb{R}\}$,*

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}_+} \left[\partial_t \varphi(\xi, t) + \partial_t h_\kappa(t)\, \partial_\xi \varphi(\xi, t)\right] u_{i,\kappa}(\xi, t)\, d\xi\, dt \\ &\quad + \int_{\mathbb{R}_+} u_{i,\kappa}^0(\xi) \varphi(\xi, 0)\, d\xi + \int_0^T \partial_t h_\kappa(t) u_{i,\kappa}(0, t) \varphi(0, t)\, dt = 0. \end{aligned} \tag{5.8}$$

(ii) *For all $\varphi \in \mathcal{A}_{T,\kappa}^s = \{v \in \mathcal{C}_c^\infty(\mathbb{R}^2) \,|\, v(.,T) = 0$ on $\mathbb{R}$ and $v(0,t) = 0$ for all $t \geq 0$ such that $\partial_t h_\kappa(t) \leq 0\}$,*

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}_+} \left[\partial_t \varphi(\xi, t) + \partial_t h_\kappa(t)\, \partial_\xi \varphi(\xi, t)\right] u_{i,\kappa}(\xi, t)\, d\xi\, dt \\ &\quad + \int_{\mathbb{R}_+} u_{i,\kappa}^0(\xi) \varphi(\xi, 0)\, d\xi + \int_0^T \partial_t h_\kappa(t) c_{i,\kappa}^s(t) \varphi(0, t)\, dt = 0. \end{aligned} \tag{5.9}$$

Proof. Thanks to definition (5.4), $\partial_t c_{i,\kappa}(z, t) = 0$ for all $z \in (-\infty, h_\kappa(t))$ and $t \in (0, T)$. It results that for all $\psi \in W^{1,\infty}(\mathbb{R} \times \mathbb{R}_+)$, compactly supported, one has

$$0 = \int_0^T \int_{-\infty}^{h_\kappa(t)} \partial_t c_{i,\kappa}(z, t) \psi(z, t) dz\, dt = \int_0^T \partial_t \left(\int_{-\infty}^{h_\kappa(t)} c_{i,\kappa}(z, t) \psi(z, t) dz \right) dt$$
$$- \int_0^T \int_{-\infty}^{h_\kappa(t)} c_{i,\kappa}(z, t) \partial_t \psi(z, t) dz\, dt - \int_0^T \partial_t h_\kappa(t) c_{i,\kappa}(h_\kappa(t), t) \psi(h_\kappa(t), t)\, dt,$$

and consequently

$$\begin{aligned} &\int_0^T \int_{-\infty}^{h_\kappa(t)} c_{i,\kappa}(z, t) \partial_t \psi(z, t) dz\, dt + \int_0^T \partial_t h_\kappa(t) c_{i,\kappa}(h_\kappa(t), t) \psi(h_\kappa(t), t)\, dt \\ &\quad - \int_{-\infty}^{h_\kappa(T)} c_{i,\kappa}(z, T) \psi(z, T)\, dz + \int_{-\infty}^{h_\kappa(0)} c_{i,\kappa}^0(z) \psi(z, 0)\, dz = 0. \end{aligned} \tag{5.10}$$

Let φ be in W_T, and let $\psi \in W^{1,\infty}(\mathbb{R} \times \mathbb{R}_+)$ be such that

$$\psi(z,t) = \varphi(h_\kappa(t) - z, t) \quad \forall\, (z,t) \in \mathbb{R} \times \mathbb{R}_+.$$

Considering (5.10) in the new coordinate system $\xi = h_\kappa(t) - z$ and using the property $\varphi(.,T) = 0$, (5.8) is derived. Finally, thanks to the definition of $\mathcal{A}^s_{T,\kappa}$ and since $u_{i,\kappa}(0,t) = c^s_{i,\kappa}(t)$ if $\partial_t h_\kappa(t) > 0$, we obtain (5.9). □

Convergence. We will now prove that the limits $(c_i^s, u_i)_{i=1,\dots,L}$ are solutions of the weak formulation given in Definition 2.1.

LEMMA 5.5. *Let $\mathcal{O}$ be an open bounded subset of $\mathbb{R}^d$, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $L^1(\mathcal{O})$ which converges to f in $L^1(\mathcal{O})$. Let us define, for any $g \in L^1(\mathcal{O})$, $\mathcal{S}_g^+ = \{x \in \mathcal{O} \,|\, g(x) > 0\}$ and $\mathcal{S}_g^- = \{x \in \mathcal{O} \,|\, g(x) \le 0\}$; then*

$$I_n = \int_{\mathcal{O}} f_n \chi_{\mathcal{S}^+_{f_n} \cap \mathcal{S}^-_f} \to 0 \text{ as } n \to \infty, \text{ and } J_n = \int_{\mathcal{O}} f_n \chi_{\mathcal{S}^-_{f_n} \cap \mathcal{S}^+_f} \to 0 \text{ as } n \to \infty.$$

Proof. Note that if $f \in L^1(\mathcal{O})$, then f^+ and f^- belong to $L^1(\mathcal{O})$; thus

$$I_n = \int_{\mathcal{O}} f_n \chi_{\mathcal{S}^+_{f_n}} \chi_{\mathcal{S}^-_f} = \int_{\mathcal{O}} f_n^+ \chi_{\mathcal{S}^-_f} = \int_{\mathcal{O}} (f_n^+ - f^+) \chi_{\mathcal{S}^-_f} + \int_{\mathcal{O}} f^+ \chi_{\mathcal{S}^-_f}.$$

Since $\int_{\mathcal{O}} f^+ \chi_{\mathcal{S}^-_f} = 0$ and $|f_n^+ - f^+| \le |f_n - f|$ on $\mathcal{O}$, we deduce that $I_n \to 0$ as $n \to \infty$. The proof is similar for J_n. □

Let us now prove the convergence result given by Theorem 3.3.

Proof of Theorem 3.3. The convergence of the approximate solutions for the sediment thickness toward the solution of problem (2.6) has already been proved in Proposition 4.2. Let us now show that the limits $(c_i^s, u_i)_{i=1,\dots,L}$ given by Proposition 5.2 satisfy the weak formulation of problem (2.7) in the sense of Definition 2.1.

Let i belong to $\{1,\dots,L\}$ and $\varphi \in \mathcal{A}$. Since $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{d+2})$, there exists $T > 0$ such that, for all $t \ge T$, $\varphi(.,.,t) = 0$. Let $m_0 \in \mathbb{N}$ be such that $\Delta t_{m_0} < T$. For the sake of simplicity, we shall drop the subscript i.

For all $\kappa \in \mathcal{K}_m$, $m \in \mathbb{N}$, note that $\varphi(x_\kappa,.,.) \in W_T$. Applying (5.8) to the test function $\varphi(x_\kappa,.,.)$ and summing this equation over $\kappa \in \mathcal{K}_m$, we get, for any $m \ge m_0$,

$$\begin{aligned} &\underbrace{\sum_{\kappa\in\mathcal{K}_m} |\kappa| \int_{\mathbb{R}_+} \int_0^T \left[\partial_t \varphi(x_\kappa,\xi,t) + \partial_t h_\kappa(t)\, \partial_\xi \varphi(x_\kappa,\xi,t)\right] u_\kappa(\xi,t)\, dt\, d\xi}_{(A_m)} \\ &\qquad + \underbrace{\sum_{\kappa\in\mathcal{K}_m} |\kappa| \int_{\mathbb{R}_+} u^0_\kappa(\xi) \varphi(x_\kappa,\xi,0)\, d\xi}_{(B_m)} \\ &\qquad + \underbrace{\sum_{\kappa\in\mathcal{K}_m} |\kappa| \int_0^T \partial_t h_\kappa(t) u_\kappa(0,t) \varphi(x_\kappa,0,t)\, dt}_{(C_m)} = 0. \end{aligned} \tag{5.11}$$

In this equation, (A_m) is equal to

$$\int_\Omega \int_{\mathbb{R}_+} \int_0^T \left[\partial_t \varphi_{\mathcal{K}_m}(x,\xi,t) + \delta_t h_{\mathcal{K}_m,\Delta t_m}(x,t)\, \partial_\xi \varphi_{\mathcal{K}_m}(x,\xi,t)\right] \bar{u}_{\mathcal{K}_m,\Delta t_m}(x,\xi,t)\, dt\, d\xi\, dx,$$

where $\varphi_{\mathcal{K}_m}(x,\xi,t) = \varphi(x_\kappa,\xi,t)$ for all $x \in \kappa$. Thanks to Proposition 4.2, the sequence of functions $(\delta_t h_{\mathcal{K}_m,\Delta t_m})$ converges strongly to $\partial_t h$ in $L^2(\Omega\times(0,T))$ as $m \to \infty$. Since $\varphi \in \mathcal{A}$, we deduce that the sequence $(\partial_\xi \varphi_{\mathcal{K}_m} \cdot \delta_t h_{\mathcal{K}_m,\Delta t_m})$ converges to $\partial_\xi\varphi \cdot \partial_t h$ in $L^1(\Omega\times\mathbb{R}_+^*\times\mathbb{R}_+^*)$. Since the sequence $(\bar{u}_{\mathcal{K}_m,\Delta t_m})$ converges to u in $L^\infty(\Omega\times\mathbb{R}_+^*\times\mathbb{R}_+^*)$ for the weak-$\star$ topology, we conclude that

$$(5.12) \quad (A_m) \to \int_\Omega\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\big[\partial_t\varphi(x,\xi,t) + \partial_t h(x,t)\,\partial_\xi\varphi(x,\xi,t)\big]u(x,\xi,t)\,dt\,d\xi\,dx \quad \text{as } m\to\infty.$$

Let us define $u^0_{\mathcal{K}_m}$ by $u^0_{\mathcal{K}_m}(x,\xi) = u^0_\kappa(\xi)$ for all $x\in\kappa$, $\kappa\in\mathcal{K}_m$, and $\xi\in\mathbb{R}_+^*$. From Hypothesis 1 on u^0, it results that $u^0_{\mathcal{K}_m}$ converges to u^0 in $L^1(\Omega\times(0,T))$ for all $T>0$, and consequently

$$(5.13) \qquad (B_m) \to \int_\Omega\int_{\mathbb{R}_+} u^0(x,\xi)\varphi(x,\xi,0)\,d\xi\,dx \text{ as } m\to\infty.$$

In (5.11), (C_m) is equal to

$$(C_m) = \int_\Omega\int_0^T \delta_t h_{\mathcal{K}_m,\Delta t_m}(x,t)\bar{u}_{\mathcal{K}_m,\Delta t_m}(x,0,t)\varphi_{\mathcal{K}_m}(x,0,t)\,dt\,dx.$$

Let us introduce the following notation:

$$\begin{aligned}
\mathcal{P}^+_{\mathcal{K}_m} &= \{(x,t)\in\Omega\times(0,T)\,|\,\delta_t h_{\mathcal{K}_m,\Delta t_m}(x,t) > 0\},\\
\mathcal{P}^-_{\mathcal{K}_m} &= \{(x,t)\in\Omega\times(0,T)\,|\,\delta_t h_{\mathcal{K}_m,\Delta t_m}(x,t) \le 0\},\\
\mathcal{P}^+ &= \{(x,t)\in\Omega\times(0,T)\,|\,\partial_t h(x,t) > 0\},\\
\mathcal{P}^- &= \{(x,t)\in\Omega\times(0,T)\,|\,\partial_t h(x,t) \le 0\}.
\end{aligned}$$

Noticing that $\mathcal{P}^+_{\mathcal{K}_m} = (\mathcal{P}^+\setminus(\mathcal{P}^+\cap\mathcal{P}^-_{\mathcal{K}_m}))\cup(\mathcal{P}^+_{\mathcal{K}_m}\cap\mathcal{P}^-)$ and $\mathcal{P}^-_{\mathcal{K}_m} = (\mathcal{P}^-\setminus(\mathcal{P}^-\cap\mathcal{P}^+_{\mathcal{K}_m}))\cup(\mathcal{P}^-_{\mathcal{K}_m}\cap\mathcal{P}^+)$, one has

$$\begin{aligned}
(C_m) = &\int_\Omega\int_0^T \delta_t h_{\mathcal{K}_m,\Delta t_m}(x,t)\,c^s_{\mathcal{K}_m,\Delta t_m}(x,t)\,\varphi_{\mathcal{K}_m}(x,0,t)\\
&\qquad\cdot[\chi_{\mathcal{P}^+} - \chi_{\mathcal{P}^+\cap\mathcal{P}^-_{\mathcal{K}_m}} + \chi_{\mathcal{P}^+_{\mathcal{K}_m}\cap\mathcal{P}^-}]\,dt\,dx\\
&+\int_\Omega\int_0^T \delta_t h_{\mathcal{K}_m,\Delta t_m}(x,t)\,\bar{u}_{\mathcal{K}_m,\Delta t_m}(x,0,t)\,\varphi_{\mathcal{K}_m}(x,0,t)\\
&\qquad\cdot[\chi_{\mathcal{P}^-} - \chi_{\mathcal{P}^-\cap\mathcal{P}^+_{\mathcal{K}_m}} + \chi_{\mathcal{P}^-_{\mathcal{K}_m}\cap\mathcal{P}^+}]\,dt\,dx.
\end{aligned}$$

Since the functions $c^s_{\mathcal{K}_m,\Delta t_m}(x,t)$, $\bar{u}_{\mathcal{K}_m,\Delta t_m}(x,0,t)$, and $\varphi_{\mathcal{K}_m}(x,0,t)$ are bounded on $\Omega\times(0,T)$ and $(\delta_t h_{\mathcal{K}_m,\Delta t_m})$ converges to $\partial_t h$ in $L^2(\Omega\times(0,T))$, Lemma 5.5 applied to the sequence $(\delta_t h_{\mathcal{K}_m,\Delta t_m})_{m\in\mathbb{N}}$ yields

$$\int_\Omega\int_0^T \delta_t h_{\mathcal{K}_m,\Delta t_m}(x,t)\,c^s_{\mathcal{K}_m,\Delta t_m}(x,t)\,\varphi_{\mathcal{K}_m}(x,0,t)[-\chi_{\mathcal{P}^+\cap\mathcal{P}^-_{\mathcal{K}_m}} + \chi_{\mathcal{P}^+_{\mathcal{K}_m}\cap\mathcal{P}^-}]\,dt\,dx \to 0,$$

$$\int_\Omega\int_0^T \delta_t h_{\mathcal{K}_m,\Delta t_m}(x,t)\,\bar{u}_{\mathcal{K}_m,\Delta t_m}(x,0,t)\varphi_{\mathcal{K}_m}(x,0,t)[-\chi_{\mathcal{P}^-\cap\mathcal{P}^+_{\mathcal{K}_m}} + \chi_{\mathcal{P}^-_{\mathcal{K}_m}\cap\mathcal{P}^+}]\,dt\,dx \to 0$$

as $m\to\infty$. Furthermore, $\varphi\in\mathcal{A}$, so that the sequence $(\varphi_{\mathcal{K}_m}(.,0,.)\,\delta_t h_{\mathcal{K}_m,\Delta t_m})$ converges to $\varphi(.,0,.)\,\partial_t h$ in $L^1(\Omega\times(0,T))$. As the sequence $(c^s_{\mathcal{K}_m,\Delta t_m})$ converges to

c^s in $L^\infty(\Omega \times \mathbb{R}_+^*)$ for the weak-$\star$ topology, we conclude that

$$\int_\Omega \int_0^T \delta_t h_{\mathcal{K}_m,\Delta t_m}(x,t) c^s_{\mathcal{K}_m,\Delta t_m}(x,t) \varphi_{\mathcal{K}_m}(x,0,t)\, \chi_{\mathcal{P}_+}\, dt\, dx \to \int_\Omega \int_0^T \partial_t h(x,t) c^s(x,t) \varphi(x,0,t)\, \chi_{\mathcal{P}_+}\, dt\, dx \text{ as } m \to \infty.$$

On $\chi_{\mathcal{P}_-}$, by definition, one has $\varphi(x,0,t) = 0$. Since $\bar{u}_{\mathcal{K}_m,\Delta t_m}(x,0,t)$ is bounded and the sequence $(\varphi_{\mathcal{K}_m}(.,0,.)\, \delta_t h_{\mathcal{K}_m,\Delta t_m})$ converges to $\varphi(.,0,.)\, \partial_t h$ in $L^1(\Omega \times (0,T))$, we obtain

$$\int_\Omega \int_0^T \delta_t h_{\mathcal{K}_m,\Delta t_m}(x,t)\, \bar{u}_{\mathcal{K}_m,\Delta t_m}(x,0,t)\, \varphi_{\mathcal{K}_m}(x,0,t)\, \chi_{\mathcal{P}_-}\, dt\, dx \to 0 \text{ as } m \to \infty,$$

and finally

$$(C_m) \to \int_\Omega \int_0^T \partial_t h(x,t) c^s(x,t) \varphi(x,0,t) dt\, dx = \int_\Omega \int_{\mathbb{R}_+} \partial_t h(x,t) c^s(x,t) \varphi(x,0,t)\, dt\, dx$$

as $m \to \infty$. Then (c_i^s, u_i) satisfy the first part (2.8) of the weak formulation.

Let $\varphi \in \mathcal{A}_0$. Since $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{d+2})$, there exists $T > 0$ such that $\varphi(.,.,t) = 0$ for all $t \geq T$. Let $m_0 \in \mathbb{N}$ be such that $\Delta t_{m_0} < T$.

Multiplying the scheme (3.2) by $\varphi(x_\kappa, 0, t^{n+1})$ and summing over $\kappa \in \mathcal{K}_m$ and $n \in \{0, \dots, N_{\Delta t_m}\}$, one obtains, for any $m \geq m_0$,

$$\underbrace{\sum_{n=0}^{N_{\Delta t_m}} \sum_{\kappa \in \mathcal{K}_m} |\kappa| \Delta \mathcal{M}_\kappa^{n+1} \varphi(x_\kappa, 0, t^{n+1})}_{(1_m)}$$
$$+ \underbrace{\sum_{n=0}^{N_{\Delta t_m}} \Delta t_m \sum_{\kappa \in \mathcal{K}_m} \sum_{\kappa' \in \mathcal{K}_\kappa} c_{\kappa\kappa'}^{s,n+1} T_{\kappa\kappa'} (h_\kappa^{n+1} - h_{\kappa'}^{n+1}) \varphi(x_\kappa, 0, t^{n+1})}_{(2_m)}$$
$$\underbrace{- \sum_{n=0}^{N_{\Delta t_m}} \Delta t_m \sum_{\kappa \in \mathcal{K}_m} |\partial\kappa \cap \partial\Omega| \left(\tilde{c}_\kappa^{n+1} g_\kappa^{(+),n+1} - c_\kappa^{s,n+1} g_\kappa^{(-),n+1} \right) \varphi(x_\kappa, 0, t^{n+1})}_{(3_m)} = 0.$$

Since $\varphi(.,0,.) \in \mathcal{A}_0^s$, Proposition 5.3 with $\mathcal{K} = \mathcal{K}_m$ and $\Delta t = \Delta t_m$ states that (2_m)+(3_m) converges to

$$\begin{aligned} A &= \int_0^T \left(\int_\Omega c^s(x,t)\, \nabla h(x,t) \cdot \nabla \varphi(x,0,t)\, dx - \int_{\partial\Omega} \tilde{c}(x,t) g(x,t) \varphi(x,0,t) d\gamma(x) \right) dt \\ &= \int_{\mathbb{R}_+} \left(\int_\Omega c^s(x,t)\, \nabla h(x,t) \cdot \nabla \varphi(x,0,t)\, dx - \int_{\partial\Omega} \tilde{c}(x,t) g(x,t) \varphi(x,0,t) d\gamma(x) \right) dt, \end{aligned}$$

as $m \to \infty$. Let us now prove the convergence of

$$A'_m = -(1_m) = - \sum_{n=0}^{N_{\Delta t_m}} \sum_{\kappa \in \mathcal{K}_m} |\kappa| \Delta \mathcal{M}_\kappa^{n+1} \varphi(x_\kappa, 0, t^{n+1})$$

toward

$$B = \int_\Omega \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left[\partial_t \varphi(x,\xi,t) + \partial_t h(x,t)\, \partial_\xi \varphi(x,\xi,t)\right] u(x,\xi,t)\, dt\, d\xi\, dx \\ + \int_\Omega \int_{\mathbb{R}_+} u^0(x,\xi)\varphi(x,\xi,0)\, d\xi\, dx$$

as $m \to \infty$. From (5.12) and (5.13), we have, for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{d+2}) \supset \mathcal{A}_0$,

$$B'_m = \sum_{\kappa \in \mathcal{K}_m} |\kappa| \int_{\mathbb{R}_+} \int_0^T \left[\partial_t \varphi(x_\kappa,\xi,t) + \partial_t h_\kappa(t)\, \partial_\xi \varphi(x_\kappa,\xi,t)\right] u_\kappa(\xi,t)\, dt\, d\xi \\ + \sum_{\kappa \in \mathcal{K}_m} |\kappa| \int_{\mathbb{R}_+} u_\kappa^0(\xi)\varphi(x_\kappa,\xi,0)\, d\xi \to B \text{ as } m \to \infty,$$

and, from (5.11), $B'_m = -\sum_{\kappa \in \mathcal{K}_m} |\kappa| \int_0^T \partial_t h_\kappa(t) u_\kappa(0,t)\varphi(x_\kappa,0,t)\, dt$. Hence, it will suffice to show that $|A'_m - B'_m| \to 0$ as $m \to \infty$.

For given $\kappa \in \mathcal{K}_m$ and $n \in \{0,\dots,N_{\Delta t_m}\}$, let us recall that

$$\Delta \mathcal{M}_\kappa^{n+1} = \begin{cases} \int_{h_\kappa^n}^{h_\kappa^{n+1}} c_\kappa^{n+1}(z) dz & \text{if } h_\kappa^{n+1} \geq h_\kappa^n, \\ \int_{h_\kappa^n}^{h_\kappa^{n+1}} c_\kappa^n(z)\, dz & \text{if } h_\kappa^{n+1} < h_\kappa^n. \end{cases}$$

Considering the change of coordinates $z = h_\kappa(t)$ in these integrals, one can show that, in both the sedimentation ($h_\kappa^{n+1} \geq h_\kappa^n$) and erosion ($h_\kappa^{n+1} < h_\kappa^n$) cases, one has

$$\Delta \mathcal{M}_\kappa^{n+1} = \int_{t^n}^{t^{n+1}} c_\kappa(h_\kappa(t),t)\partial_t h_\kappa(t)\, dt = \int_{t^n}^{t^{n+1}} u_\kappa(0,t)\partial_t h_\kappa(t)\, dt.$$

Substituting this equality in the definition of A'_m leads to

$$B'_m - A'_m = \sum_{\kappa \in \mathcal{K}_m} |\kappa| \int_T^{t^{N_{\Delta t_m}+1}} \bar{u}_{\mathcal{K}_m,\Delta t_m}(x,0,t)\, \delta_t h_{\mathcal{K}_m,\Delta t_m}(x,t)\varphi(x_\kappa,0,t) dt \\ - \sum_{\kappa \in \mathcal{K}_m} |\kappa| \sum_{n=0}^{N_{\Delta t_m}} \int_{t^n}^{t^{n+1}} \bar{u}_{\mathcal{K}_m,\Delta t_m}(x,0,t) \delta_t h_{\mathcal{K}_m,\Delta t_m}(x,t)[\varphi(x_\kappa,0,t^{n+1}) - \varphi(x_\kappa,0,t)]\, dt.$$

Thanks to the regularity of φ, there exists $C_1 > 0$, depending only on φ, such that $|\varphi(x_\kappa,0,t^{n+1}) - \varphi(x_\kappa,0,t)| \leq C_1\, \Delta t_m$ for all $t \in [t^n, t^{n+1}]$. Since the function $\delta_t h_{\mathcal{K}_m,\Delta t_m}$ is uniformly bounded in $L^2(\Omega \times (0, t^{N_{\Delta t_m}+1}))$, and $\bar{u}_{\mathcal{K}_m,\Delta t_m} \in [0,1]$, and $|t^{N_{\Delta t_m}+1} - T| < \Delta t_m$, the convergence of $|A'_m - B'_m|$ to 0 as $m \to \infty$ is obtained, which ends the proof of the theorem. ∎

6. Conclusion. In this article, a fully implicit finite volume discretization of the multilithology stratigraphic model is considered in the simplified case for which the diffusion coefficients of all the lithologies are equal.

In such a case, the sediment thickness variable decouples from the other variables and satisfies a parabolic equation. A weak formulation has been defined for the remaining surface and basin concentration variables in order to cope with the difficulty to define the trace of the basin concentrations at the top of the basin. Then, the main result of this article is the convergence, up to a subsequence, of the discrete sediment thickness in $L^\infty(0,T;L^2(\Omega))$ and of the discrete concentrations in the L^∞ weak-$\star$ topology to a weak solution.

In particular, this proves the existence of at least one solution to the weak formulation for the coupled problem. The uniqueness of such a solution, and hence the full convergence of the discrete solutions, will be obtained in a forthcoming paper.

Appendix. Proof of Proposition 5.3.

To prove Proposition 5.3, the following weak-BV estimate will be used. It is an extension to the coupling of a parabolic and a hyperbolic equation of the result proved in [6] for the coupling of an elliptic and a hyperbolic equation in the case of a two phase Darcy flow.

LEMMA A.1. *Let us assume that Hypothesis* 1 *holds, and let* h *denote the solution of problem* (2.6). *Let* $i \in \{1,\dots,L\}$ $(\mathcal{K},\Sigma_{int},\mathcal{P})$ *be an admissible mesh of* Ω *in the sense of Definition* 3.1, $T>0$, *and* $\Delta t \in (0,T)$. *Let* $\alpha>0$ *be such that* $reg(\mathcal{K}) \le \alpha$ *and* $\beta>0$ *be such that* $\delta\mathcal{K} \le \beta\sqrt{\Delta t}$. *Then, there exists* $H>0$, *depending only on* T, Ω, $\|h\|_{W^{2,\infty}(\Omega\times(0,2T))}$, $\|g\|_{L^2(\partial\Omega\times\mathbb{R}_+)}$, β, *and* α, *such that the following inequality holds:*

$$\begin{aligned}(\text{A.1})\qquad &\sum_{n=0}^{N_{\Delta t}}\Delta t\sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} T_{\kappa\kappa'}\,|h_\kappa^{n+1}-h_{\kappa'}^{n+1}|\,|c_{i,\kappa}^{s,n+1}-c_{i,\kappa'}^{s,n+1}|\\ &\quad+\sum_{n=0}^{N_{\Delta t}}\Delta t\sum_{\kappa\in\mathcal{K}}|\partial\kappa\cap\partial\Omega|\,|c_{i,\kappa}^{s,n+1}-\tilde{c}_{i,\kappa}^{n+1}|\,g_\kappa^{(+),n+1}\le\frac{H}{\sqrt{\delta\mathcal{K}}}.\end{aligned}$$

Proof. Let i belong to the set $\{1,\dots,L\}$. Again, the subscript i will be dropped in the proof, and c_i^s will be denoted by c. Multiplying (3.2) by c_κ^{n+1} and summing over $\kappa\in\mathcal{K}$ and $n\in\{0,\dots,N_{\Delta t}\}$ yield that

$$\begin{aligned}(\text{A.2})\qquad &\sum_{n=0}^{N_{\Delta t}}\sum_{\kappa\in\mathcal{K}}|\kappa|c_\kappa^{*,n+1}\,c_\kappa^{n+1}(h_\kappa^{n+1}-h_\kappa^n)\\ &\quad+\sum_{n=0}^{N_{\Delta t}}\Delta t\sum_{\kappa\in\mathcal{K}}\sum_{\kappa'\in\mathcal{K}_\kappa}T_{\kappa\kappa'}c_{\kappa\kappa'}^{n+1}\,c_\kappa^{n+1}(h_\kappa^{n+1}-h_{\kappa'}^{n+1})\\ &\quad-\sum_{n=0}^{N_{\Delta t}}\Delta t\sum_{\kappa\in\mathcal{K}}|\partial\kappa\cap\partial\Omega|\,g_\kappa^{(+),n+1}\tilde{c}_\kappa^{n+1}c_\kappa^{n+1}\\ &\quad+\sum_{n=0}^{N_{\Delta t}}\Delta t\sum_{\kappa\in\mathcal{K}}|\partial\kappa\cap\partial\Omega|\,g_\kappa^{(-),n+1}(c_\kappa^{n+1})^2=0,\end{aligned}$$

where $c_\kappa^{*,n+1}$ is defined by $c_\kappa^{*,n+1}(h_\kappa^{n+1}-h_\kappa^n)=\Delta\mathcal{M}_\kappa^{n+1}$, such that $c_\kappa^{*,n+1}\in[0,1]$.

The upstream evaluation of the surface concentrations at the edges of the control volumes implies that, for all $\kappa\in\mathcal{K}$,

$$\begin{aligned}\sum_{\kappa'\in\mathcal{K}_\kappa}T_{\kappa\kappa'}c_{\kappa\kappa'}^{n+1}\,c_\kappa^{n+1}(h_\kappa^{n+1}-h_{\kappa'}^{n+1})&=\sum_{\kappa'\in\mathcal{K}_\kappa}T_{\kappa\kappa'}(c_\kappa^{n+1})^2(h_\kappa^{n+1}-h_{\kappa'}^{n+1})^+\\ &\quad-\sum_{\kappa'\in\mathcal{K}_\kappa}T_{\kappa\kappa'}c_{\kappa'}^{n+1}\,c_\kappa^{n+1}(h_\kappa^{n+1}-h_{\kappa'}^{n+1})^-.\end{aligned}$$

Therefore, since $(h_\kappa-h_{\kappa'})^+=(h_{\kappa'}-h_\kappa)^-$, one has

$$\begin{aligned}&\sum_{\kappa\in\mathcal{K}}\sum_{\kappa'\in\mathcal{K}_\kappa}T_{\kappa\kappa'}c_{\kappa\kappa'}^{n+1}\,c_\kappa^{n+1}(h_\kappa^{n+1}-h_{\kappa'}^{n+1})\\ &\quad=\sum_{\kappa\in\mathcal{K}}\sum_{\kappa'\in\mathcal{K}_\kappa}T_{\kappa\kappa'}((c_\kappa^{n+1})^2-c_{\kappa'}^{n+1}\,c_\kappa^{n+1})(h_\kappa^{n+1}-h_{\kappa'}^{n+1})^+.\end{aligned}$$

Then, using the equalities $(c_\kappa)^2 - c_\kappa\, c_{\kappa'} = \frac{1}{2}(c_\kappa - c_{\kappa'})^2 + \frac{1}{2}\big((c_\kappa)^2 - (c_{\kappa'})^2\big)$, $(h_\kappa - h_{\kappa'})^+ = (h_{\kappa'} - h_\kappa)^-$, and $(h_\kappa - h_{\kappa'}) = (h_\kappa - h_{\kappa'})^+ - (h_\kappa - h_{\kappa'})^-$ leads to the following successive equalities:

$$
\begin{aligned}
&\sum_{\kappa\in\mathcal{K}}\sum_{\kappa'\in\mathcal{K}_\kappa} T_{\kappa\kappa'} c_{\kappa\kappa'}^{n+1}\, c_\kappa^{n+1}(h_\kappa^{n+1} - h_{\kappa'}^{n+1})\\
&\quad = \frac{1}{2}\sum_{\kappa\in\mathcal{K}}\sum_{\kappa'\in\mathcal{K}_\kappa} T_{\kappa\kappa'}(c_\kappa^{n+1} - c_{\kappa'}^{n+1})^2 (h_\kappa^{n+1} - h_{\kappa'}^{n+1})^+\\
&\quad + \frac{1}{2}\sum_{\kappa\in\mathcal{K}}\sum_{\kappa'\in\mathcal{K}_\kappa} T_{\kappa\kappa'}(c_\kappa^{n+1})^2 (h_\kappa^{n+1} - h_{\kappa'}^{n+1})^+\\
&\quad - \frac{1}{2}\sum_{\kappa\in\mathcal{K}}\sum_{\kappa'\in\mathcal{K}_\kappa} T_{\kappa\kappa'}(c_{\kappa'}^{n+1})^2 (h_\kappa^{n+1} - h_{\kappa'}^{n+1})^+\\
&\quad = \frac{1}{2}\sum_{\kappa\in\mathcal{K}}\sum_{\kappa'\in\mathcal{K}_\kappa} T_{\kappa\kappa'}(c_\kappa^{n+1} - c_{\kappa'}^{n+1})^2 (h_\kappa^{n+1} - h_{\kappa'}^{n+1})^+\\
&\quad + \frac{1}{2}\sum_{\kappa\in\mathcal{K}}\sum_{\kappa'\in\mathcal{K}_\kappa} T_{\kappa\kappa'}(c_\kappa^{n+1})^2 (h_\kappa^{n+1} - h_{\kappa'}^{n+1}).
\end{aligned}
\tag{A.3}
$$

Furthermore, summing (3.2) over $i \in \{1, \dots, L\}$, we obtain, for all $\kappa \in \mathcal{K}$ and $n \in \{0, \dots, N_{\Delta t}\}$,

$$
|\kappa|(h_\kappa^{n+1} - h_\kappa^n) + \Delta t \sum_{\kappa'\in\mathcal{K}_\kappa} T_{\kappa\kappa'}(h_\kappa^{n+1} - h_{\kappa'}^{n+1}) - \Delta t\, |\partial\kappa \cap \partial\Omega|\, g_\kappa^{n+1} = 0. \tag{A.4}
$$

Multiplying (A.4) by $(c_\kappa^{n+1})^2$ and summing over $\kappa \in \mathcal{K}$ gives in (A.3)

$$
\begin{aligned}
&\sum_{\kappa\in\mathcal{K}}\sum_{\kappa'\in\mathcal{K}_\kappa} T_{\kappa\kappa'} c_{\kappa\kappa'}^{n+1}\, c_\kappa^{n+1}(h_\kappa^{n+1} - h_{\kappa'}^{n+1}) = -\frac{1}{2}\sum_{\kappa\in\mathcal{K}} |\kappa| (c_\kappa^{n+1})^2 \frac{h_\kappa^{n+1} - h_\kappa^n}{\Delta t}\\
&+ \frac{1}{2}\sum_{\kappa\in\mathcal{K}} |\partial\kappa \cap \partial\Omega|\, (c_\kappa^{n+1})^2 g_\kappa^{n+1} + \frac{1}{2}\sum_{\kappa\in\mathcal{K}}\sum_{\kappa'\in\mathcal{K}_\kappa} T_{\kappa\kappa'}(c_\kappa^{n+1} - c_{\kappa'}^{n+1})^2 (h_\kappa^{n+1} - h_{\kappa'}^{n+1})^+,
\end{aligned}
$$

which finally results in the equality

$$
\begin{aligned}
&\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}}\sum_{\kappa'\in\mathcal{K}_\kappa} T_{\kappa\kappa'} c_{\kappa\kappa'}^{n+1}\, c_\kappa^{n+1}(h_\kappa^{n+1} - h_{\kappa'}^{n+1})\\
&\qquad - \sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}} |\partial\kappa \cap \partial\Omega|\, g_\kappa^{(+),n+1} \tilde{c}_\kappa^{n+1} c_\kappa^{n+1}\\
&\qquad + \sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}} |\partial\kappa \cap \partial\Omega|\, g_\kappa^{(-),n+1} (c_\kappa^{n+1})^2\\
&\quad = \frac{1}{2}\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} T_{\kappa\kappa'}(c_\kappa^{n+1} - c_{\kappa'}^{n+1})^2 |h_\kappa^{n+1} - h_{\kappa'}^{n+1}|\\
&\qquad + \frac{1}{2}\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}} |\partial\kappa \cap \partial\Omega|\, g_\kappa^{(+),n+1}(c_\kappa^{n+1} - \tilde{c}_\kappa^{n+1})^2 - \frac{1}{2}\sum_{n=0}^{N_{\Delta t}} \sum_{\kappa\in\mathcal{K}} |\kappa| (c_\kappa^{n+1})^2 (h_\kappa^{n+1} - h_\kappa^n)\\
&\qquad + \frac{1}{2}\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}} (|\partial\kappa \cap \partial\Omega|\, g_\kappa^{(-),n+1}(c_\kappa^{n+1})^2 - |\partial\kappa \cap \partial\Omega|\, g_\kappa^{(+),n+1}(\tilde{c}_\kappa^{n+1})^2).
\end{aligned}
$$

Using this last result in (A.2), together with $g_\kappa^{(-),n+1} \geq 0$ for all $\kappa \in \mathcal{K}$ and $n \in \{0,\dots,N_{\Delta t}\}$, one obtains the estimate

$$\begin{aligned}
&\frac{1}{2}\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} T_{\kappa\kappa'}(c_\kappa^{n+1}-c_{\kappa'}^{n+1})^2|h_\kappa^{n+1}-h_{\kappa'}^{n+1}| \\
&\quad+\frac{1}{2}\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}} |\partial\kappa\cap\partial\Omega|\, g_\kappa^{(+),n+1}(c_\kappa^{n+1}-\tilde{c}_\kappa^{n+1})^2 \\
&\quad\leq \frac{1}{2}\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}} |\kappa|\big[(c_\kappa^{n+1})^2 - 2\,c_\kappa^{*,n+1}\,c_\kappa^{n+1}\big]\frac{h_\kappa^{n+1}-h_\kappa^n}{\Delta t} \\
&\quad+\frac{1}{2}\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}} |\partial\kappa\cap\partial\Omega| g_\kappa^{(+),n+1}(\tilde{c}_\kappa^{n+1})^2.
\end{aligned} \tag{A.5}$$

Noticing that, according to Corollary 1,

$$\begin{aligned}
&\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}} |\kappa|\big[(c_\kappa^{n+1})^2 - 2\,c_\kappa^{*,n+1}\,c_\kappa^{n+1}\big]\frac{h_\kappa^{n+1}-h_\kappa^n}{\Delta t} \\
&\quad\leq C_1(T,\Omega)\left(\sum_{n=0}^{N_{\Delta t}} \Delta t \|\delta_t h_{\mathcal{K}}^n\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \leq C_1(T,\Omega) D_6
\end{aligned} \tag{A.6}$$

and

$$\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}} |\partial\kappa\cap\partial\Omega|\, g_\kappa^{(+),n+1}\,(\tilde{c}_\kappa^{n+1})^2 \leq C_2(\Omega,T)\,\|g^+\|_{L^2(\partial\Omega\times\mathbb{R}_+)}, \tag{A.7}$$

we deduce from (A.5), (A.6), and (A.7) the estimate

$$\begin{aligned}
&\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} T_{\kappa\kappa'}(c_\kappa^{n+1}-c_{\kappa'}^{n+1})^2|h_\kappa^{n+1}-h_{\kappa'}^{n+1}| \\
&\quad+\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}} |\partial\kappa\cap\partial\Omega|\, g_\kappa^{(+),n+1}(c_\kappa^{n+1}-\tilde{c}_\kappa^{n+1})^2 \leq C_1\sqrt{D_6} + C_2\,\|g^+\|_{L^2(\partial\Omega\times\mathbb{R}_+)}.
\end{aligned}$$

Finally, the Cauchy–Schwarz inequality yields

$$\begin{aligned}
&\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} T_{\kappa\kappa'}|c_\kappa^{n+1}-c_{\kappa'}^{n+1}||h_\kappa^{n+1}-h_{\kappa'}^{n+1}| \\
&\quad+\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}} |\partial\kappa\cap\partial\Omega||c_\kappa^{n+1}-\tilde{c}_\kappa^{n+1}|\, g_\kappa^{(+),n+1} \\
&\quad\leq \Bigg(\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} T_{\kappa\kappa'}(c_\kappa^{n+1}-c_{\kappa'}^{n+1})^2|h_\kappa^{n+1}-h_{\kappa'}^{n+1}| \\
&\quad+\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}} |\partial\kappa\cap\partial\Omega|(c_\kappa^{n+1}-\tilde{c}_\kappa^{n+1})^2 g_\kappa^{(+),n+1}\Bigg)^{\frac{1}{2}} \\
&\quad\left(\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} T_{\kappa\kappa'}|h_\kappa^{n+1}-h_{\kappa'}^{n+1}| + \sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}} |\partial\kappa\cap\partial\Omega|\, g_\kappa^{(+),n+1}\right)^{\frac{1}{2}}.
\end{aligned}$$

The term

$$\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} T_{\kappa\kappa'} |h_\kappa^{n+1} - h_{\kappa'}^{n+1}| \le \left(\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} T_{\kappa\kappa'} \right)^{\frac12} \left(\sum_{n=0}^{N_{\Delta t}} \Delta t |h_{\mathcal{K}}^{n+1}|_{1,\mathcal{K}}^2 \right)^{\frac12}$$

is estimated by Corollary 1 and the following bound from (3.1):

$$\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} T_{\kappa\kappa'} \le \left(\sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} |\sigma|\, d(\kappa,\kappa') \right) \frac{2T\,\alpha^2}{\delta\mathcal{K}^2} \le \frac{2Td\,\alpha^2\,|\Omega|}{\delta\mathcal{K}^2}. \tag{A.8}$$

We conclude from estimates similar to (A.7) that the inequality (A.1) holds. $\square$

Proof of Proposition 5.3. Let i belong to the set $\{1,\dots,L\}$, and let $(\mathcal{K},\Sigma_{int},\mathcal{P},\Delta t)$ be an admissible discretization of $\Omega\times\mathbb{R}_+^*$ with $\Delta t < T$. For all $\kappa\in\mathcal{K}$, $x\in\partial\kappa\cap\partial\Omega$, $t\in(t^n,t^{n+1}]$, $n\ge 0$, let us define

$$\tilde c_{i,\mathcal{K},\Delta t}(x,t) = \tilde c_{i,\kappa}^{n+1}.$$

Throughout this proof we shall now drop the subscript i and use the simplified notation $c_i^s = c$.

Let us define the auxiliary expression E_3 by

$$E_3 = \sum_{n=0}^{N_{\Delta t}} \int_{t^n}^{t^{n+1}} \left(\int_\Omega c_{\mathcal{K},\Delta t}(x,t)\,\nabla h(x,t)\cdot\nabla\varphi(x,t^{n+1})\,dx \right.$$
$$\left. - \int_{\partial\Omega} \tilde c_{\mathcal{K},\Delta t}(x,t)\,g(x,t)\,\varphi(x,t^{n+1})\,d\gamma(x) \right) dt.$$

From the L^∞ weak-$\star$ convergence of $c_{\mathcal{K},\Delta t}$ to c and $\tilde c_{\mathcal{K},\Delta t}$ to $\tilde c$ as Δt and $\delta\mathcal{K}\to 0$, and their boundedness, it results that

$$E_3 \to \int_0^T \left(\int_\Omega c(x,t)\,\nabla h(x,t)\cdot\nabla\varphi(x,t)dx - \int_{\partial\Omega} \tilde c(x,t) g(x,t)\varphi(x,t)d\gamma(x) \right) dt$$

as $\Delta t\to 0$.

Multiplying (2.6) by $\varphi(x,t^{n+1})$ and integrating it over the time interval (t^n,t^{n+1}) and cell κ yield

$$\int_{t^n}^{t^{n+1}} \int_\kappa \partial_t h(x,t)\,\varphi(x,t^{n+1})\,dx\,dt - \int_{t^n}^{t^{n+1}} \int_\kappa \Delta h(x,t)\,\varphi(x,t^{n+1})\,dx\,dt = 0.$$

Since $\varphi\in\mathcal{C}_c^\infty(\mathbb{R}^{d+1})$, one obtains

$$\int_{t^n}^{t^{n+1}} \int_\kappa \nabla h(x,t)\cdot\nabla\varphi(x,t^{n+1})\,dx\,dt = -\int_{t^n}^{t^{n+1}} \int_\kappa \partial_t h(x,t)\,\varphi(x,t^{n+1})\,dx\,dt$$
$$+ \int_{t^n}^{t^{n+1}} \int_{\partial\kappa} \nabla h(x,t)\cdot\vec n_\kappa\,\varphi(x,t^{n+1})d\gamma(x)\,dt,$$

where $\vec{n}_\kappa$ is the normal unit vector to $\partial\kappa$ outward to κ. Thus, one has

$$\begin{aligned} E_3 \quad &= \sum_{n=0}^{N_{\Delta t}} \sum_{\substack{\sigma\in\Sigma_{int} \\ \sigma=\kappa|\kappa'}} (c_\kappa^{n+1} - c_{\kappa'}^{n+1}) \int_{t^n}^{t^{n+1}} \int_\sigma \nabla h(x,t)\cdot \vec{n}_{\kappa\kappa'}\, \varphi(x,t^{n+1})\, d\gamma(x)\, dt \\ &+ \sum_{n=0}^{N_{\Delta t}} \sum_{\kappa\in\mathcal{K}} \int_{t^n}^{t^{n+1}} \int_{\partial\kappa\cap\partial\Omega} (c_\kappa^{n+1} - \tilde{c}_\kappa^{n+1})\, g(x,t)\, \varphi(x,t^{n+1})\, d\gamma(x)\, dt \\ &- \sum_{n=0}^{N_{\Delta t}} \sum_{\kappa\in\mathcal{K}} \int_{t^n}^{t^{n+1}} \int_\kappa c_\kappa^{n+1}\, \partial_t h(x,t)\, \varphi(x,t^{n+1})\, dx\, dt. \end{aligned}$$

Defining the second auxiliary expression E_2 by

$$\begin{aligned} E_2 \quad &= -\sum_{n=0}^{N_{\Delta t}} \sum_{\kappa\in\mathcal{K}} |\kappa| c_\kappa^{n+1} (h_\kappa^{n+1} - h_\kappa^n)\, \varphi(x_\kappa, t^{n+1}) \\ &+ \sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\substack{\sigma\in\Sigma_{int} \\ \sigma=\kappa|\kappa'}} (c_\kappa^{n+1} - c_{\kappa'}^{n+1}) T_{\kappa\kappa'} (h_{\kappa'}^{n+1} - h_\kappa^{n+1}) \frac{1}{|\sigma|} \int_\sigma \varphi(x,t^{n+1})\, d\gamma(x) \\ &+ \sum_{n=0}^{N_{\Delta t}} \sum_{\kappa\in\mathcal{K}} \int_{t^n}^{t^{n+1}} \int_{\partial\kappa\cap\partial\Omega} (c_\kappa^{n+1} - \tilde{c}_\kappa^{n+1})\, g(x,t)\, \varphi(x,t^{n+1})\, d\gamma(x)\, dt, \end{aligned}$$

we have

$$\begin{aligned} E_3 - E_2 = &-\sum_{n=0}^{N_{\Delta t}} \sum_{\kappa\in\mathcal{K}} c_\kappa^{n+1} \int_{t^n}^{t^{n+1}} \int_\kappa \Big[\partial_t h(x,t)\, \varphi(x,t^{n+1}) \\ &- \frac{(h_\kappa^{n+1} - h_\kappa^n)}{\Delta t}\, \varphi(x_\kappa, t^{n+1}) \Big]\, dx\, dt + \sum_{n=0}^{N_{\Delta t}} \sum_{\substack{\sigma\in\Sigma_{int} \\ \sigma=\kappa|\kappa'}} (c_\kappa^{n+1} - c_{\kappa'}^{n+1}) \\ &\cdot \int_{t^n}^{t^{n+1}} \int_\sigma \left[\nabla h(x,t)\cdot \vec{n}_{\kappa\kappa'} - \frac{h_{\kappa'}^{n+1} - h_\kappa^{n+1}}{d(\kappa,\kappa')} \right] \varphi(x,t^{n+1}) d\gamma(x)\, dt. \end{aligned}$$

Multiplying (2.6) by $\varphi(x_\kappa, t^{n+1})$ and c_κ^{n+1} and integrating it over the time interval (t^n, t^{n+1}) and cell κ yield

$$\begin{aligned} &\sum_{n=0}^{N_{\Delta t}} \sum_{\kappa\in\mathcal{K}} \int_{t^n}^{t^{n+1}} \int_\kappa c_\kappa^{n+1}\, \partial_t h(x,t)\, \varphi(x_\kappa, t^{n+1})\, dx\, dt \\ &\quad - \sum_{n=0}^{N_{\Delta t}} \sum_{\kappa\in\mathcal{K}} \int_{t^n}^{t^{n+1}} \int_{\partial\kappa} c_\kappa^{n+1}\, \nabla h(x,t)\cdot \vec{n}_\kappa\, \varphi(x_\kappa, t^{n+1})\, d\gamma(x)\, dt = 0. \end{aligned} \tag{A.9}$$

Similarly, multiplying (4.1) by c_κ^{n+1} and $\varphi(x_\kappa, t^{n+1})$ and summing the result over $\kappa\in\mathcal{K}$ and $n\in\{0,\ldots,N_{\Delta t}\}$, we obtain

$$\begin{aligned} &\sum_{n=0}^{N_{\Delta t}} \sum_{\kappa\in\mathcal{K}} |\kappa| c_\kappa^{n+1} (h_\kappa^{n+1} - h_\kappa^n) \varphi(x_\kappa, t^{n+1}) \\ &\quad + \sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}} \sum_{\kappa'\in\mathcal{K}_\kappa} T_{\kappa\kappa'} c_\kappa^{n+1} (h_\kappa^{n+1} - h_{\kappa'}^{n+1}) \varphi(x_\kappa, t^{n+1}) \\ &\quad - \sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}} |\partial\kappa\cap\partial\Omega|\, g_\kappa^{n+1} c_\kappa^{n+1} \varphi(x_\kappa, t^{n+1}) = 0. \end{aligned} \tag{A.10}$$

Then, $E_3 - E_2 +$ (A.9) $-$ (A.10) yields the equality

$$E_3 - E_2 = \sum_{n=0}^{N_{\Delta t}} \sum_{\kappa \in \mathcal{K}} c_\kappa^{n+1} \int_{t^n}^{t^{n+1}} \int_\kappa \partial_t h(x,t) \left[\varphi(x_\kappa, t^{n+1}) - \varphi(x, t^{n+1})\right] dx\, dt$$
$$+ \sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa \in \mathcal{K}} c_\kappa^{n+1} \sum_{\sigma \in \Sigma_\kappa \cap \Sigma_{\kappa'}} \int_\sigma \left[\frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \nabla h(x,t) \cdot \vec{n}_{\kappa\kappa'}\, dt - \frac{h_{\kappa'}^{n+1} - h_\kappa^{n+1}}{d(\kappa, \kappa')}\right]$$
$$\cdot \left[\varphi(x, t^{n+1}) - \varphi(x_\kappa, t^{n+1})\right] d\gamma(x).$$

Since φ is regular, there exists $C_1 > 0$ depending only on φ such that, for all $\kappa \in \mathcal{K}$ and $x \in \kappa$,

$$|\varphi(x, t^{n+1}) - \varphi(x_\kappa, t^{n+1})| \le C_1\, \delta\mathcal{K}. \tag{A.11}$$

Thanks to the regularity of h, there exists $C_2 > 0$ depending only on $\|h\|_{L^\infty(0,2T;W^{2,\infty}(\Omega))}$, such that, for all $\kappa \in \mathcal{K}$, $\sigma \in \Sigma_\kappa$, $x \in \sigma$, and $t \in (0, 2T)$,

$$\left| \frac{1}{|\sigma|} \int_\sigma \nabla h(u,t) \cdot \vec{n}_\kappa\, d\gamma(u) - \nabla h(x,t) \cdot \vec{n}_\kappa \right| \le C_2\, \delta\mathcal{K}.$$

Thus, the following estimate is derived:

$$|E_3 - E_2| \le C_3\, \delta\mathcal{K}\, \|\partial_t h\|_{L^\infty(\Omega \times [0,2T])} + 2C_1\, \delta\mathcal{K} \sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\substack{\sigma \in \Sigma_{int} \\ \sigma = \kappa|\kappa'}} \Bigg(|\sigma| C_2 \delta\mathcal{K}$$
$$+ \left| T_{\kappa\kappa'} (h_{\kappa'}^{n+1} - h_\kappa^{n+1}) - \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \int_\sigma \nabla h(u,t) \cdot \vec{n}_{\kappa\kappa'}\, d\gamma(u)\, dt \right| \Bigg).$$

The last term in this estimate is bounded using Cauchy–Schwarz inequality as follows:

$$\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\substack{\sigma \in \Sigma_{int} \\ \sigma = \kappa|\kappa'}} \left| T_{\kappa\kappa'} (h_{\kappa'}^{n+1} - h_\kappa^{n+1}) - \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \int_\sigma \nabla h(u,t) \cdot \vec{n}_{\kappa\kappa'}\, d\gamma(u)\, dt \right|$$
$$\le \left[\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\substack{\sigma \in \Sigma_{int} \\ \sigma = \kappa|\kappa'}} T_{\kappa\kappa'} \right]^{\frac{1}{2}} \left[\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\substack{\sigma \in \Sigma_{int} \\ \sigma = \kappa|\kappa'}} d(\kappa, \kappa')\, |\sigma| \right.$$
$$\left. \left(\frac{h_{\kappa'}^{n+1} - h_\kappa^{n+1}}{d(\kappa, \kappa')} - \frac{1}{\Delta t} \frac{1}{|\sigma|} \int_{t^n}^{t^{n+1}} \int_\sigma \nabla h(u,t) \cdot \vec{n}_{\kappa\kappa'}\, d\gamma(u)\, dt \right)^2 \right]^{\frac{1}{2}}.$$

Finally, using (A.8), (4.5), and the bound $\sum_{\sigma \in \Sigma_{int}} |\sigma| \le \frac{d\, \alpha\, |\Omega|}{\delta\mathcal{K}}$, we obtain that

$$E_3 - E_2 \to 0 \text{ as } \Delta t \to 0.$$

It remains only to prove that $E_2 - E \to 0$ as $\Delta t \to 0$. Removing (A.10) from E yields

$$\begin{aligned}E = &\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}} \sum_{\kappa'\in\mathcal{K}_\kappa} \underbrace{T_{\kappa\kappa'}(c_{\kappa\kappa'}^{n+1} - c_\kappa^{n+1})\,(h_\kappa^{n+1} - h_{\kappa'}^{n+1})\,\varphi(x_\kappa, t^{n+1})}_{(F)}\\
&+ \sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}} |\partial\kappa \cap \partial\Omega|\, g_\kappa^{(+),n+1}(c_\kappa^{n+1} - \tilde{c}_\kappa^{n+1})\,\varphi(x_\kappa, t^{n+1})\\
&- \sum_{n=0}^{N_{\Delta t}} \sum_{\kappa\in\mathcal{K}} |\kappa| c_\kappa^{n+1}(h_\kappa^{n+1} - h_\kappa^n)\,\varphi(x_\kappa, t^{n+1}).\end{aligned}$$

Thanks to the upstream evaluation of the concentrations at the edges, (F) vanishes if $h_\kappa^{n+1} \geq h_{\kappa'}^{n+1}$. In the opposite case, it is equal to $T_{\kappa\kappa'}(c_{k'}^{n+1} - c_\kappa^{n+1})\,(h_\kappa^{n+1} - h_{\kappa'}^{n+1})\,\varphi(x_\kappa, t^{n+1})$, and thus

$$\begin{aligned}E = &\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} T_{\kappa\kappa'}(c_{\kappa'}^{n+1} - c_\kappa^{n+1})\,(h_\kappa^{n+1} - h_{\kappa'}^{n+1})\,\varphi(x_{\kappa\kappa'}, t^{n+1})\\
&+ \sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa\in\mathcal{K}} |\partial\kappa \cap \partial\Omega|\, g_\kappa^{(+),n+1}(c_\kappa^{n+1} - \tilde{c}_\kappa^{n+1})\,\varphi(x_\kappa, t^{n+1})\\
&- \sum_{n=0}^{N_{\Delta t}} \sum_{\kappa\in\mathcal{K}} |\kappa| c_\kappa^{n+1}(h_\kappa^{n+1} - h_\kappa^n)\,\varphi(x_\kappa, t^{n+1}),\end{aligned}$$

with

$$x_{\kappa\kappa'} = \begin{cases} x_\kappa & \text{if } h_\kappa \leq h_{\kappa'},\\ x_{\kappa'} & \text{otherwise.}\end{cases}$$

Therefore, $E_2 - E$ writes

$$\begin{aligned}E_2 - E = &\sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\substack{\sigma\in\Sigma_{int}\\ \sigma=\kappa|\kappa'}} T_{\kappa\kappa'}(c_{\kappa'}^{n+1} - c_\kappa^{n+1})\,(h_\kappa^{n+1} - h_{\kappa'}^{n+1})\\
&\left[\frac{1}{|\sigma|}\int_\sigma \varphi(x, t^{n+1})\,d\gamma(x) - \varphi(x_{\kappa\kappa'}, t^{n+1})\right] + \sum_{n=0}^{N_{\Delta t}} \sum_{\kappa\in\mathcal{K}} (c_\kappa^{n+1} - \tilde{c}_\kappa^{n+1})\\
&\int_{t^n}^{t^{n+1}} \int_{\partial\kappa\cap\partial\Omega} \left[g(x,t)\,\varphi(x, t^{n+1}) - g_\kappa^{(+),n+1}\,\varphi(x_\kappa, t^{n+1})\right] d\gamma(x)\,dt.\end{aligned}$$

Thanks to the regularity of φ, there exists $C_3 > 0$ depending only on φ such that

$$\left|\frac{1}{|\sigma|}\int_\sigma \varphi(x, t^{n+1})\,d\gamma(x) - \varphi(x_{\kappa\kappa'}, t^{n+1})\right| \leq C_3\,\delta\mathcal{K}. \tag{A.12}$$

Furthermore, since $\varphi \in \mathcal{A}_0^s$, one has

$$\int_{\partial\kappa\cap\partial\Omega} g(x, t^{n+1})\,\varphi(x, t^{n+1}) d\gamma(x) = \int_{\partial\kappa\cap\partial\Omega} g^+(x, t^{n+1})\,\varphi(x, t^{n+1}) d\gamma(x).$$

Finally, inequalities (A.11) and (A.12) and the definition of $g_\kappa^{(+),n+1}$ give the estimate

$$|E_2 - E| \leq C_3 \, \delta\mathcal{K} \sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\substack{\sigma \in \Sigma_{int} \\ \sigma = \kappa|\kappa'}} T_{\kappa\kappa'} |c_{\kappa'}^{n+1} - c_\kappa^{n+1}| \, |h_\kappa^{n+1} - h_{\kappa'}^{n+1}|$$

$$+ C_1 \, \delta\mathcal{K} \sum_{n=0}^{N_{\Delta t}} \Delta t \sum_{\kappa \in \mathcal{K}} |\partial\kappa \cap \partial\Omega| |c_\kappa^{n+1} - \tilde{c}_\kappa^{n+1}| g_\kappa^{(+),n+1}.$$

It results from Lemma A.1 that $|E_2 - E| \leq C_4 \, \delta\mathcal{K} \, \frac{H}{\sqrt{\delta\mathcal{K}}}$, which ends the proof. $\square$

REFERENCES

[1] R. Anderson and N. Humphrey, *Interaction of weathering and transport processes in the evolution of arid landscapes*, in Quantitative Dynamics Stratigraphy, T. Cross, ed., Prentice–Hall, 1989, Englewood Cliffs, NJ, pp. 349–361.

[2] T. Arbogast, M. F. Wheeler, and N.-Y. Zhang, *A nonlinear mixed finite element method for a degenerate parabolic equation arising in flow in porous media*, SIAM J. Numer. Anal., 33 (1996), pp. 1669–1687.

[3] G. Chavent and J. Jaffré, *Mathematical Models and Finite Elements for Reservoir Simulation*, North–Holland, Amsterdam, 1986.

[4] R. Eymard and T. Gallouët, *Convergence d'un schéma de type éléments finis—volumes finis pour un système couplé elliptique-hyperbolique*, M2AN Math. Model. Numer. Anal., 27 (1993), pp. 843–891.

[5] R. Eymard, T. Gallouët, D. Granjeon, R. Masson, and Q. Tran, *Multi-lithology stratigraphic model under maximum erosion rate constraint*, Internat. J. Numer. Methods Engrg., 60 (2004), pp. 527–548.

[6] R. Eymard, T. Gallouët, and R. Herbin, *The Finite Volume Method*, in Handbook of Numerical Analysis, P. Ciarlet and J. Lions, eds., Handb. Numer. Anal. 7, North–Holland, Amsterdam, 2000, pp. 715–1022.

[7] P. Flemings and T. Jordan, *A synthetic stratigraphic model of foreland basin development*, J. Geophysical Research, 94 (1989), pp. 3851–3866.

[8] D. Granjeon, *Modélisation Stratigraphique Déterministe: Conception et Application d'un Modèle Diffusif 3D Multilithologique*, Ph.D. thesis, Géosciences Rennes, Rennes, France, 1997.

[9] D. Granjeon and P. Joseph, *Concepts and applications of a 3D multiple lithology, diffusive model in stratigraphic modelling*, in Numerical Experiments in Stratigraphy, J. Harbaugh and al., eds., Society for Sedimentary Geology Special Publication 62, R. W. Dalrymple, Editor of Special Publications, Tulsa, OK, 1999, pp. 197–210.

[10] P. Kenyon and D. Turcotte, *Morphology of a delta prograding by bulk sediment transport*, Geological Society of America Bulletin, 96 (1985), pp. 1457–1465.

[11] O. Ladyzenskaja, V. Solonnikov, and N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monogr. 23, AMS, Providence, RI, 1968.

[12] D. Peaceman, *Fundamentals of Numerical Reservoirs Simulation*, Elsevier Science, Amsterdam, 1991.

[13] N. Ramarosy, *Application de la Méthode des Volumes Finis à des Problèmes d'Environnement et de Traitement d'Image*, Ph.D. thesis, Université de Paris-Sud, Paris, France, 1999.

[14] J. Rivenaes, *Application of a dual-lithology, depth-dependent diffusion equation in stratigraphic simulation*, Basin Research, 4 (1992), pp. 133–146.

[15] J. Rivenaes, *Impact of sediment transport efficiency on large-scale sequence architecture: Results from stratigraphic computer simulation*, Basin Research, 9 (1997), pp. 91–105.

[16] D. Tetzlaff and J. Harbaugh, *Simulating Clastic Sedimentation*, Van Norstrand Reinhold, New York, 1989.

[17] G. Tucker and R. Slingerland, *Erosional dynamics, flexural isostasy, and long-lived escarpments: A numerical modeling study*, J. Geophysical Research, 99 (1994), pp. 12,229–12,243.

[18] S. Verdière and M. Vignal, *Numerical and theoretical study of a dual mesh method using finite volume schemes for two phase flow problems in porous media*, Numer. Math., 80 (1998), pp. 601–639.

[19] M. Vignal, *Convergence of finite volume scheme for a system of an elliptic equation and a hyperbolic equation*, M2AN Math. Model. Numer., 30 (1996), pp. 841–872.