Finite Volume Schemes for multi-phase flow simulation on near well grids

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Objective

Study of Finite Volume Schemes on 3D near well multi-phase flow simulations.

Outline

- Applications and Difficulties
- 3D Near Well Grids
- Finite Volume Schemes
- Numerical Experiments
- Conclusion
Multi-phase Flow Simulation on Near Well Grids

Applications
- Reservoir Simulation
- CO₂ geological storage

Difficulties
- Singular Pressure Distribution
- Well-Radius << Reservoir Dimension
- Deviated Well
- Anisotropy
3D Near-well model

- Exponentially refined radial mesh
- Unstructured mesh with only hexahedra
- Hybrid mesh with hexahedra, tetrahedra and pyramids

Discretization on complex 3D general meshes

⇒ MultiPoint Flux Approximation (MPFA) Finite Volume Schemes
Finite Volume Scheme
Model Problem

Model Equation

Find \( u \) (potential) in \( H^1_0(\Omega) \) such that:

\[
-\nabla \cdot (\Lambda \nabla u) = f \quad \text{in} \quad \Omega,
\]

\[
u = 0 \quad \text{on} \quad \partial \Omega
\]

\( \Omega \) : bounded polygonal domain of \( \mathbb{R}^d \)

\( \Lambda \) : symmetric positive definite tensor field

\( f \) : function of \( L^2(\Omega) \)
Finite Volume Scheme

 Flux Formulation

$$\mathcal{I}_h : \text{ set of cells } K$$

$$\mathcal{V}_h : \text{ space of piecewise constant functions on } \mathcal{I}_h$$

- $$F_{K,\sigma}(u_h) \approx \int_{\sigma} \nabla u \cdot n_{K,\sigma} \text{ linearly}$$
- Conservativity: $$F_{K,\sigma}(u_h) + F_{L,\sigma}(u_h) = 0, \sigma = K|L$$

Find $$u_h \in \mathcal{V}_h$$ \[ - \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u_h) = \int_K f \quad \forall K \in \mathcal{I}_h \]
Finite Volume Scheme
Discrete Variational Formulation

For all $u_h, v_h \in V_h$, let

$$a_h(u_h, v_h) = \sum_{\sigma = \mathcal{E}_K \cap \mathcal{E}_L \in \mathcal{E}_h} F_{K,\sigma}(u_h)(v_L - v_K) - \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^b} F_{K,\sigma}(u_h)v_K$$

The finite volume scheme is equivalent to

Find $u_h \in V_h$ | $a_h(u_h, v_h) = \int_{\Omega} f v_h \quad \forall v_h \in V_h$
Sample of MPFA
Finite Volume Schemes

O scheme  [Aavatsmark et al., 1996, Edwards and Rogers, 1998]

L scheme  [Aavatsmark, 2007]

G scheme  inspired by the L scheme  [Agélas et al., 2010a]
  → Subcell gradients satisfying continuity conditions
  → Subfluxes $F^G_{L,\sigma}$
  → Convex linear combination $F_{K,\sigma} = \sum \theta^G_\sigma F^G_{K,\sigma}$
  → Choose the $\theta^G_\sigma$ to enhance the coercivity and remove singularities
The GradCell scheme uses a discrete variational formulation

Non symmetric discrete variational formulation based on two cellwise constant gradients and residuals for stabilization

\[ a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} m_K \Lambda_K (\nabla_h u_h)_K \cdot (\tilde{\nabla}_h v_h)_K + \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \frac{m_\sigma}{d_{K,\sigma}} R_{K,\sigma}(u_h) R_{K,\sigma}(v_h) \]

\[(\nabla_h v_h)_K = \frac{1}{m_K} \sum_{\sigma \in \mathcal{E}_K} m_\sigma (l_{K,\sigma}(v_h) - v_K) n_{K,\sigma} \]

\[(\tilde{\nabla}_h v_h)_K = \frac{1}{m_K} \sum_{\sigma \in \mathcal{E}_K} m_\sigma (\gamma_{\sigma}(v_h) - v_K) n_{K,\sigma} \]
The GradCell scheme has a compact stencil

Fluxes are derived from the bilinear form.

\[ a_h(u_h, v_h) = \sum_{\sigma = E_K \cap E_L \in E_h} F_{K,\sigma}(u_h)(v_L - v_K) - \sum_{K \in T_h} \sum_{\sigma \in E_K \cap E_h^b} F_{K,\sigma}(u_h)v_K \]

Fluxes \( F_{K,\sigma}(u_h) \) only between cells sharing a face

The stencil is compact: neighbours of the neighbours

For topologically cartesian grids: 13 cells in 2D, 21 cells in 3D

[Agélas et al., 2010b]
Outcome on Symmetry and Sparsity properties

Fact

■ Previous schemes: compact but non symmetric $\Rightarrow$ conditionnal coercivity

■ If GradCell symmetric $\Rightarrow$ large stencil (81 cells in 3D)

Difficult to combine both properties

Wish

■ Symmetric unconditionally coercive scheme

■ Sparse stencil: 9 points in 2D and 27 points in 3D on topologically Cartesian meshes
SUSHI combining smart ideas...

**Scheme Using Stabilization and Harmonic Interfaces**

- O scheme ideas: subcell gradients \( (\nabla_h u)_K^s \) and subfaces unknowns \( u_\sigma^s \)
- A symmetric variational bilinear form for coercivity
- Weak and consistent subcell gradient for convergence
- Two point harmonic interpolation at the faces
SUSHI
Nice harmonic interpolation formula

Find a point $y_\sigma$ and a coefficient $\alpha$ with...

- a linear two point interpolation...
- ... exact on piecewise linear functions,
- normal flux and potential continuity.

Harmonic point $y_\sigma$

Harmonic interpolation

$$u(y_\sigma) = \alpha \, u(x_K) + (1 - \alpha) \, u(x_L)$$

[Agélas et al., 2009]
How SUSHI uses the interpolation formula?

At each face $\sigma$, choose the harmonic point $y_{\sigma}$

Subcell $K_s$ around a vertex $s$

$K_s = (x_K, y_{\sigma}, s, y_{\sigma'}, x_K)$
\[(\nabla_h u)_{K_s} = \frac{1}{m_{K_s}} \left( m^s_{\sigma} (u^s_{\sigma} - u_K) n_{K,\sigma} \right. \]
\[+ m^s_{\sigma'} (u^s_{\sigma'} - u_K) n_{K,\sigma'} \]
\[+ m_\delta (u_\delta - u_K) n_{K_s,\delta} \]
\[+ m_{\delta'} (u_{\delta'} - u_K) n_{K_s,\delta'} \) \]
SUSHI
A symmetric formulation

Symmetric discrete variational formulation

$$a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \sum_{s \in \mathcal{V}_K} \left( m_{K,s} \Lambda_K (\nabla_h u_h)_K^s \cdot (\nabla_h v_h)_K^s + \frac{m_{K,s}}{(d_{K,\sigma})^2} R_{K,\sigma}^s (u_h) R_{K,\sigma}^s (v_h) \right)$$

Subfluxes

$$a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \sum_{s \in \mathcal{V}_K} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} F_{K,\sigma}^s (u_h) (v^s - v_K)$$

$$F_{K,\sigma}^s(u_h) + F_{L,\sigma}^s(u_h) = 0$$
Problem studied

- Single phase flow
- Anisotropy of the tensor permeability $\Lambda$
- Slanted well

$\Rightarrow$ Analytical solution [Aavatsmark and Klausen, 2003]

Numerical study

- O, L, G, GradCell and SUSHI schemes
- Hexahedra and Hybrid mesh families
- $\Lambda = \text{diag}(1,1,\frac{1}{20})$
Hexahedral mesh family
$L^2$ pressure error

- O Scheme
- G scheme
- GradCell
- L scheme
- SUSHI

**Mesh size $h$**

**Nonzero elements in the linear system**

- O and L schemes have the same behavior
Hybrid mesh family \(l^2\) pressure error

- GradCell stencil \(\approx 4\) times smaller than O scheme
- L scheme fails but not the more flexible G scheme
Two-phase flow ($w$-$g$) near well simulation

Injection of gaseous CO$_2$ miscible in a reservoir full of water

- Two-component

  \[
  \begin{array}{c|cc}
  \text{H}_2\text{O} & (w) \\
  \text{CO}_2 & (w-g)
  \end{array}
  \]

- Thermodynamic equilibrium defined by the solubility $\bar{C}$

  \[
  \begin{cases}
  (w-g) : & C^w_{\text{CO}_2} < \bar{C} & S_g = 0 \\
  (g) : & C^w_{\text{CO}_2} = \bar{C} & S_g > 0
  \end{cases}
  \]

Test the O scheme on both types of meshes
Total Mass of CO$_2$ function of time

GOE = Grid Orientation Effect
Mass of CO$_2$ in phase gas function of time

Cell size affect the oscillations
Conclusion

- SUSHI scheme exhibits very promising results thanks to its unconditional coercivity

- Hybrid meshes show drawbacks of the schemes:
  - O scheme has a stencil $\approx$ 4 times bigger than GradCell
  - L scheme fails but not the more flexible G scheme

- Two-phase flow numerical solution is sensitive to GOE and size of the cells


