Discontinuous Galerkin methods

Alexandre Ern

Université Paris-Est, CERMICS, ENPC

Journées numériques, Nice, 17 mai 2016
Introduction

- dG methods were introduced more than 40 years ago
- They have sparked extensive interest in the scientific computing and applied math communities

![Graph showing dG-related publications/year (Mathscinet)]
A brief historical perspective

- **Elliptic PDEs**
  - boundary penalty methods [Nitsche 71]
  - interior penalty methods [Babuška 73, Douglas & Dupont 75, Baker 77, Wheeler 78, Arnold 82]

- **First-order PDEs**
  - neutron transport simulation [Reed & Hill 73] (steady, linear)
  - CV analysis [Lesaint & Raviart 74, Johnson & Pitkäranta 86]
  - time-dependent conservation laws [Cockburn & Shu 89-]

- **Friedrichs systems**
  - linear PDE systems with symmetry and $L^2$-positivity properties
  - unify mixed elliptic and first-order PDEs [AE & Guermond, 06-]
Motivations

- Discontinuous Galerkin (dG) methods can be viewed as
  - finite element methods with discontinuous discrete functions
  - finite volume methods with more than one DOF per mesh cell

- Possible motivations to consider dG methods
  - flexibility in the choice of basis functions
  - general meshes: non-matching interfaces, polyhedral cells
  - local discrete formulation using fluxes and local test functions (in particular, for strongly-contrasted material properties)
  - block-diagonal mass matrices for time-stepping
  - easily amenable to variable polynomial order, local time-stepping
Outline

▷ Part I: Diffusion

▷ Part II: Diffusion-advection-reaction


See also forthcoming book on FEM [AE & Guermond 16]
Diffusion

- Discrete setting
- Laplacian
- Variable diffusion
Discrete setting

- dG methods accommodate fairly general meshes
  - polyhedral cells (with various shapes)
  - nonmatching contact between adjacent cells (hanging nodes)

- Mesh indexed by $h$ (e.g., maximal meshsize); CV analysis as $h \to 0$

- Given a mesh $\mathcal{T}_h$ of a domain $\Omega$, examples of discrete spaces are the broken polynomial spaces ($k \geq 0$)

$$\mathbb{P}_d^k(\mathcal{T}_h) = \{ v_h \in L^\infty(\Omega) \mid v_h|_T \in \mathbb{P}_d^k(T), \ \forall T \in \mathcal{T}_h \}$$
Faces, mean-values, and jumps

- **Interface** \( F_h^i \ni F = \partial T_1 \cap \partial T_2 \)
  - oriented by unit normal \( n_F \) from \( T_1 \) to \( T_2 \) (fixed once and for all)
  - mean-values and jumps at interfaces (\( v_i := v|_{T_i}, \ i \in \{1,2\} \))
    \[
    \{ v \} = \frac{1}{2}(v_1 + v_2) \\
    [v] = v_1 - v_2
    \]

- **Boundary face** \( F_h^b \ni F = \partial T \cap \partial \Omega \), \( n_F \) pointing outward \( \Omega \)
  \[
  \{ v \} = [v] = v|_T
  \]

- Mesh faces are collected in the set \( F_h = F_h^i \cup F_h^b \)
**Important algebraic identity**

- Crucial when integrating by parts cellwise
- For pcw. smooth functions $a$ (vector-valued) and $b$ (scalar-valued)

\[
\sum_{T \in \mathcal{T}_h} \int_T \nabla \cdot (ab) = \sum_{T \in \mathcal{T}_h} \int_{\partial T} (ab) \cdot n_T \quad \text{(outward unit normal to } T)\\
= \sum_{F \in \mathcal{F}_h^i} \int_F \llbracket ab \rrbracket \cdot n_F + \sum_{F \in \mathcal{F}_h^b} \int_F (ab) \cdot n_F \quad (n_F = n_{T_1} = -n_{T_2})\\
= \sum_{F \in \mathcal{F}_h^i} \int_F (\llbracket a \rrbracket \llbracket b \rrbracket + \llbracket a \rrbracket \llbracket b \rrbracket) \cdot n_F + \sum_{F \in \mathcal{F}_h^b} \int_F (\llbracket a \rrbracket \llbracket b \rrbracket) \cdot n_F\\
= \sum_{F \in \mathcal{F}_h^i} \int_F \llbracket a \rrbracket \llbracket b \rrbracket \cdot n_F + \sum_{F \in \mathcal{F}_h^i} \int_F \llbracket a \rrbracket \llbracket b \rrbracket \cdot n_F
\]
Some basic facts from functional analysis

- Broken Sobolev spaces, e.g.,
  
  \[ H^1(T_h) := \{ v \in L^2(\Omega) \mid v|_T \in H^1(T), \forall T \in T_h \} \]

- Broken gradient (defined cellwise) \( \nabla_h : H^1(T_h) \to [L^2(\Omega)]^d \)
  \[
  (\nabla_h v)|_T = \nabla(v|_T) \quad \forall T \in T_h
  \]
  We have \( \nabla_h v = \nabla v \) if \( v \in H^1(\Omega) \)

- A function \( v \in H^1(T_h) \) belongs to \( H^1(\Omega) \) if and only if
  \[
  [v] = 0 \quad \forall F \in F_h^i
  \]
  (distributional argument)
Regularity of a mesh sequence $\{T_h\}_{h>0}$

- Described by means of a matching simplicial submesh
  - shape-regular in the usual sense of Ciarlet
  - local meshsize comparable to that of $T_h$

- Geometric properties resulting from mesh regularity
  - $\#(\text{subsimplices})$ of $T \in T_h$ is uniformly bounded
  - $\#(\text{faces})$ of $T \in T_h$ is uniformly bounded
  - $h_{T_1} \sim h_F \sim h_{T_2}$
Introduction

1. Analysis tools

- **Local inverse inequality** \( \forall v_h \in \mathbb{P}_d^k(T), \forall T \in \mathcal{T}_h, \)

  \[ \| \nabla v_h \|_{[L^2(T)]^d} \leq C_{\text{inv}} h_T^{-1} \| v_h \|_{L^2(T)} \]

- Markov brothers’ inequality in \( L^\infty(-1, 1) \) (1890)
- \( C_{\text{inv}} \sim k^2 \) [Schwab 98]; \( C_{\text{inv}} \) computable from eigenvalue pb.

- **Multiplicative trace inequality** \( \forall v \in H^1(T), \forall T \in \mathcal{T}_h, \)

  \[ \| v \|_{L^2(\partial T)} \leq C_{\text{mtr}} \left( h_T^{-\frac{1}{2}} \| v \|_{L^2(T)} + \| v \|_{L^2(T)}^{\frac{1}{2}} \| \nabla v \|_{[L^2(T)]^d}^{\frac{1}{2}} \right) \]

- lowest-order Raviart–Thomas functions and divergence formula
  [Carstensen & Funken 00; Stephansen 07; Di Pietro & AE 12]
- in a polyhedral cell, carve a sub-simplex from each triangular
  sub-face with height \( \sim h_T \) (allows for some face degeneration)

- **Discrete trace inequality** \( \forall v_h \in \mathbb{P}_d^k(T), \forall T \in \mathcal{T}_h, \)

  \[ \| v_h \|_{L^2(\partial T)} \leq C_{\text{dtr}} h_T^{-\frac{1}{2}} \| v_h \|_{L^2(T)} \]

- follows from LI and MT inequalities; \( C_{\text{dtr}} \sim k \)

2. Discontinuous Galerkin methods
Polynomial approximation in polyhedral cells

- $L^2$-orthogonal projection $\pi^k_T : L^2(T) \to \mathbb{P}_d^k(T)$

\[(\pi^k_T(v) - v, q)_{L^2(T)} = 0 \quad \forall q \in \mathbb{P}_d^k(T)\]

- Poincaré–Steklov inequality $\forall v \in H^1(T), \forall T \in T_h,$

\[\|v - \pi^0_h(v)\|_{L^2(T)} \leq C_{PS} h_T \|\nabla v\|_{[L^2(T)]^d}\]

  - $\pi^0_h(v)$ is the mean-value of $v$ over $T$
  - $C_{PS} = \pi^{-1}$ for convex $T$ (Poincaré (1894) [eigenvalue pb], Steklov (1897) $[d = 1]$, Payne & Weinberger (60) $[d = 2]$, Bebendorf (03) $[d \geq 3]$)
  - For non-convex $T$, uniform bound on $C_{PS}$ using simplicial sub-cells and MT inequality [AE & Guermond 16]

- PS inequality can be bootstrapped using Bramble–Hilbert polynomial to $|v - \pi^k_T(v)|_{H^m(T)} \leq C_{app} h_T^{k+1-m} |v|_{H^{k+1}(T)}$ for all $0 \leq m \leq k + 1$

- See also [Dupont & Scott 80] for alternate proof using averaged Taylor polynomials
Most useful properties

- $\forall \nu \in H^{k+1}(T), \forall T \in \mathcal{T}_h,$

$$
\|\nu - \pi^k_T \nu\|_{L^2(T)} \leq C_{\text{app}} h^{k+1}_T |\nu|_{H^{k+1}(T)}
$$

$$
\|\nabla (\nu - \pi^k_T \nu)\|_{[L^2(T)]^d} \leq C_{\text{app}} h^k_T |\nu|_{H^{k+1}(T)}
$$

$$
\|\nu - \pi^k_T \nu\|_{L^2(\partial T)} \leq C_{\text{app}} h^{k+\frac{1}{2}}_T |\nu|_{H^{k+1}(T)}
$$

- bounds extend to fractional Sobolev regularity [AE & Guermond 16]

- Global $L^2$-orth. projection $\pi^k_h : L^2(\Omega) \rightarrow P^k_d(\mathcal{T}_h)$ is assembled cellwise

$$
\pi^k_h(\nu)|_T = \pi^k_T(\nu|_T) \quad \forall T \in \mathcal{T}_h
$$

(global mass matrix is block-diagonal)
The Laplacian

- Let \( f \in L^2(\Omega) \); seek \( u : \Omega \rightarrow \mathbb{R} \) s.t. \(-\Delta u = f\) in \( \Omega \) and \( u|_{\partial\Omega} = 0 \)

- Weak formulation: \( u \in V := H^1_0(\Omega) \) s.t.

\[
a(u, w) := \int_\Omega \nabla u \cdot \nabla w = \int_\Omega f w =: \ell(w) \quad \forall w \in V
\]

- The exact solution satisfies

\[
[u] = 0 \quad \forall F \in \mathcal{F}_h = \mathcal{F}^i_h \cup \mathcal{F}^b_h
\]

- Other BC’s (Neumann, Robin) can be considered as well
Normal flux

- Physically, the normal component of the diffusive flux $\sigma := -\nabla u$ is continuous across interfaces.

- What is the mathematical meaning of $[\sigma] \cdot n_F = 0$ for $F \in F_h$?

- If $\sigma \in [L^p(\Omega)]^d$, $p > 2$, and $\nabla \cdot \sigma \in L^2(\Omega)$ then

  $$\sigma|_T \cdot n_F \in W^{-\frac{1}{p}, p}(F) \quad \forall T \in T_h, \ \forall F \subset \partial T$$

  - this holds provided $u \in H^{1+s}(\Omega)$, $s > 0$, and $\Delta u \in L^2(\Omega)$

- If $\sigma \in [H^s(\Omega)]^d$, $s > \frac{1}{2}$, then $\sigma|_{\partial T} \in [L^2(\partial T)]^d$

  - this holds provided $u \in H^{1+s}(\Omega)$, $s > \frac{1}{2}$

- Elliptic regularity theory shows that on a polyhedron, $u \in H^{1+s}(\Omega)$, $s > \frac{1}{2}$, and $s = 1$ if $\Omega$ is convex.
Symmetric Interior Penalty

- **Discrete space** $V_h := \mathbb{P}_d^k(T_h)$, $k \geq 1$

- **Seek** $u_h \in V_h$ s.t. $a_h(u_h, w_h) = \ell(w_h)$, $\forall w_h \in V_h$, with

$$a_h(v_h, w_h) := \int_\Omega \nabla_h v_h \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h} \int_F \{ \nabla_h v_h \} \cdot n_F \{ w_h \}$$

**consistency**

$$- \sum_{F \in \mathcal{F}_h} \int_F \{v_h\} \{ \nabla_h w_h \} \cdot n_F + \sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F \{v_h\} \{ w_h \}$$

**symmetry**

**penalty**

- **Main properties of $a_h$**
  - **strong consistency**: $a_h(u, w_h) = \ell(w_h)$, $\forall w_h \in V_h$
  - **coercivity on $V_h$ if $\eta$ is large enough**
Step-by-step derivation

- **Starting point:** Use broken gradient in exact bilinear form
  \[
a_h^{(0)}(v_h, w_h) := \int_\Omega \nabla_h v_h \cdot \nabla_h w_h
\]

- **Restore consistency** \((a_h^{(0)}(u, w_h) = \ell(w_h) + \sum_{F \in \mathcal{F}_h} \int_F \{\nabla u\} \cdot n_F [w_h])\)
  \[
a_h^{(1)}(v_h, w_h) := \int_\Omega \nabla_h v_h \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h} \int_F \{\nabla_h v_h\} \cdot n_F [w_h]
\]

- **Restore symmetry in a consistent way**
  \[
a_h^{(2)}(v_h, w_h) := \int_\Omega \nabla_h v_h \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h} \int_F \{\nabla_h v_h\} \cdot n_F [w_h] - \sum_{F \in \mathcal{F}_h} \int_F [v_h] \{\nabla h w_h\} \cdot n_F
\]

- **Achieve coercivity by penalizing jumps** [Arnold 82]
Stability

- **dG norm**: broken gradient plus jump seminorm

\[
\| \nu_h \|_{dG}^2 := \| \nabla_h \nu_h \|_{L^2(\Omega)}^2 + | \nu_h |^2, \quad | \nu_h |^2 = \sum_{F \in \mathcal{F}_h} \frac{1}{h_F} \| [ \nu_h ] \|_{L^2(F)}^2
\]

- \( \| \cdot \|_{dG} \) is a norm on \( V_h \) (direct verification)
  - discrete Sobolev inequality \( \| \nu_h \|_{L^q(\Omega)} \leq \sigma_q \| \nu_h \|_{dG}, \forall \nu_h \in V_h \), with \( q \in [1, \frac{2d}{d-2}] \) if \( d \geq 3 \) and \( q \in [1, \infty) \) if \( d = 2 \)
  - see [Brenner 03] (for \( q = 2 \)), [Eymard, Gallouët & Herbin 10] (for FV and general \( q \)), [Di Pietro & AE 10] (for general \( q, k \))

- If \( \eta > C_{dtr}^2 N_\partial \), where
  - \( C_{dtr} \) results from discrete trace inequality (recall \( C_{dtr} \sim k \))
  - \( N_\partial \) is the maximum number of faces a mesh cell can have

then \( \exists C_{sta} > 0 \) s.t. \( a_h(\nu_h, \nu_h) \geq C_{sta} \| \nu_h \|_{dG}^2, \forall \nu_h \in V_h \)
Algebraic realization

- SPD stiffness matrix
- Compact stencil (only neighbors in the sense of faces)
Error analysis: Boundedness

- Approximation error \( (u - u_h) \) is in \( V_b = (H^{1+s}(\Omega) \cap V) + V_h, \ s > \frac{1}{2} \)

- Boundedness: \( a_h(v, w_h) \leq C_{\text{bnd}} \|v\|_{dG,\#} \|w_h\|_{dG}, \ \forall (v, w_h) \in V_b \times V_h \)

\[
\|v\|_{dG,\#}^2 := \|v\|_{dG}^2 + \sum_{T \in T_h} h_T \|\nabla v \cdot n_T\|_{L^2(\partial T)}^2
\]

- The two norms are equivalent on \( V_h \)

\[
\|v_h\|_{dG} \leq \|v_h\|_{dG,\#} \leq C_{\#} \|v_h\|_{dG} \ \ \ \forall v_h \in V_h
\]
Error analysis: Second Strang’s Lemma

- Optimal error estimate in $\| \cdot \|_{dG,\#}$-norm

\[
\| u - u_h \|_{dG,\#} \leq C \inf_{y_h \in V_h} \| u - y_h \|_{dG,\#}
\]

- Let $y_h \in V_h$; coercivity, consistency, and boundedness imply

\[
\| u_h - y_h \|_{dG,\#} \leq C_{\#} \| u_h - y_h \|_{dG}
\]
\[
\leq C_{\#} C_{\text{sta}}^{-1} \sup_{w_h \in V_h \setminus \{0\}} \frac{a_h(u_h - y_h, w_h)}{\| w_h \|_{dG}}
\]
\[
= C_{\#} C_{\text{sta}}^{-1} \sup_{w_h \in V_h \setminus \{0\}} \frac{a_h(u - y_h, w_h)}{\| w_h \|_{dG}}
\]
\[
\leq C_{\#} C_{\text{sta}}^{-1} C_{\text{bnd}} \| u - y_h \|_{dG,\#}
\]

and use the triangle inequality

\[
\| u - u_h \|_{dG,\#} \leq (1 + C_{\#} C_{\text{sta}}^{-1} C_{\text{bnd}}) \| u - y_h \|_{dG,\#}
\]
Convergence rates

- Assume exact solution $u$ is smooth enough

- Using polynomial approximation properties in dG spaces yields

$$
\|u - u_h\|_{dG,h} \leq C \left( \sum_{T \in \mathcal{T}_h} h_T^{2k} |u|_{H^{k+1}(T)}^2 \right)^{1/2}
$$

- Assuming full elliptic regularity pickup, Aubin–Nitsche’s duality argument leads to

$$
\|u - u_h\|_{L^2(\Omega)} \leq C h \left( \sum_{T \in \mathcal{T}_h} h_T^{2k} |u|_{H^{k+1}(T)}^2 \right)^{1/2}
$$
Two side-excursions

- Lifting the jumps
- Mixed dG methods
Lifting the jumps I

- **Local lifting** Let \( l \geq 0, F \in \mathcal{F}_h; \ r^l_F : L^1(F) \to [\mathbb{P}_d^l(T_h)]^d \) is s.t.

\[
\int_{\Omega} r^l_F(\varphi) \cdot \tau_h = \int_F \{\{\tau_h\}\} \cdot n_F \varphi \quad \forall \tau_h \in [\mathbb{P}_d^l(T_h)]^d
\]

- \( r^l_F \) is vector-valued, collinear to \( n_F \)
- the support of \( r^l_F \) reduces to the (one or two) mesh cells sharing \( F \)
- \( r^l_F \) is easy to compute (invert local mass matrix)
- see [Bassi, Rebay et al 97], [Brezzi et al 00]

- Penalizing local liftings of jumps instead of jumps yields coercivity for \( \eta > N_{\partial} \) with the same stencil, \( l \in \{k - 1, k\} \)

\[
a_h(v_h, w_h) := \int_{\Omega} \nabla_h v_h \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h} \int_F \{\{\nabla_h v_h\}\} \cdot n_F [w_h]
\]

\[
- \sum_{F \in \mathcal{F}_h} \int_F [v_h] \{\{\nabla_h w_h\}\} \cdot n_F + \sum_{F \in \mathcal{F}_h} \eta \int_{\Omega} r^l_F([v_h]) \cdot r^l_F([w_h])
\]
Lifting the jumps II

- **Global lifting of jumps**: For all \( v \in H^1(\mathcal{T}_h) \),

\[
R_h^l(\mathbb{V}) := \sum_{F \in \mathcal{F}_h} r_F^l(\mathbb{V}) \in [P^l_d(\mathcal{T}_h)]^d
\]

- **Discrete gradient** \( G_h^l : H^1(\mathcal{T}_h) \to [L^2(\Omega)]^d \) s.t.

\[
G_h^l(v) := \nabla_h v - R_h^l(\mathbb{V})
\]

- Discrete gradients are important tools in **nonlinear problems**
  - asymptotic consistency: Let \((v_h)_{h>0}\) be a sequence in \((V_h)_{h>0}\) bounded in the \(\|\cdot\|_{dG}\)-norm. Then, \(\exists v \in H_0^1(\Omega)\) s.t. as \(h \to 0\), up to subseq., \(v_h \to v\) strongly in \(L^2(\Omega)\) and for all \(l \geq 0\), \(G_h^l(v_h) \rightharpoonup \nabla v\) weakly in \([L^2(\Omega)]^d\) [Di Pietro & AE ’10]
Lifting the jumps III

- **Local formulation with numerical fluxes (FV viewpoint)**
  - Let $T \in \mathcal{T}_h$ with faces collected in $\mathcal{F}_T$, let $\xi \in \mathbb{P}_d^k(T)$
  - For the exact solution
    \[
    \int_T \nabla u \cdot \nabla \xi + \sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F \Phi_F(u) \xi = \int_T f \xi
    \]
    with $\epsilon_{T,F} = n_T \cdot n_F$ and exact flux $\Phi_F(u) = -\nabla u \cdot n_F$
  - For the discrete solution ($l \in \{k - 1, k\}$)
    \[
    \int_T (\nabla u_h - R^l_h([u_h])) \cdot \nabla \xi + \sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F \phi_F(u_h) \xi = \int_T f \xi
    \]
    with numerical flux $\phi_F(u_h) = -\{\nabla_h u_h\} \cdot n_F + \frac{\eta}{h_F} [u_h]$
Mixed dG methods I

- Mixed formulation: $\sigma + \nabla u = 0$ and $\nabla \cdot \sigma = f$ in $\Omega$

- Mixed dG method: Find $u_h \in P^k_d(T_h)$, $\sigma_h \in [P^k_d(T_h)]^d$ (equal-order) s.t.

$$\int_T \sigma_h \cdot \zeta - \int_T u_h \nabla \cdot \zeta + \sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F \hat{u}_F (\zeta \cdot n_F) = 0 \quad \forall \zeta \in [P^k_d(T)]^d$$

$$- \int_T \sigma_h \cdot \nabla \xi + \sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F (\hat{\sigma}_F \cdot n_F) \xi = \int_T f \xi \quad \forall \xi \in P^k_d(T)$$

for all $T \in \mathcal{T}_h$, with numerical fluxes $\hat{u}_F$ and $\hat{\sigma}_F$

- $\sigma_h$ can be eliminated locally whenever $\hat{u}_F$ does not depend on $\sigma_h$

- See [Arnold, Brezzi, Cockburn, and Marini 02] for a unified analysis of dG methods based on numerical fluxes
Mixed dG methods II

- Numerical fluxes for SIP

\[
\hat{u}_F = \begin{cases} 
\{ u_h \} & \forall F \in \mathcal{F}_h^i \\
0 & \forall F \in \mathcal{F}_h^b
\end{cases}
\]

\[
\hat{\sigma}_F = -\{ \nabla_h u_h \} + \eta h_F^{-1} \llbracket u_h \rrbracket n_F \quad \forall F \in \mathcal{F}_h
\]

- Numerical fluxes for LDG (Local dG [Cockburn & Shu 98])

\[
\hat{u}_F \text{ as for SIP and } \hat{\sigma}_F = \{ \sigma_h \} + \eta h_F^{-1} \llbracket u_h \rrbracket n_F
\]

- \( \sigma_h \) can be eliminated locally

- main advantage: discrete coercivity for \( \eta > 0 \) (e.g. \( \eta = 1 \))

- drawback: \textbf{larger stencil} (neighbors of neighbors)

- stencil reduction [Castillo, Cockburn, Perugia, and Schötzau 00]
Mixed dG methods III

- **Two-field approach** [AE & Guermond 06]

\[
\hat{u}_F = \begin{cases} 
\{ u_h \} + \eta_\sigma \llbracket \sigma_h \rrbracket \cdot n_F & \forall F \in F^i_h \\
0 & \forall F \in F^b_h 
\end{cases}
\]

\[
\hat{\sigma}_F = \{ \sigma_h \} + \eta_u \llbracket u_h \rrbracket n_F & \forall F \in F_h
\]

- **Drawback** \( \sigma_h \) cannot be eliminated

- **Advantages**
  - a simple choice for penalty is \( \eta_u = \eta_\sigma = 1 \)
  - the choice \( k = 0 \) is possible
  - quasi-optimal estimate on the diffusive flux
Mixed dG methods IV

- **Hybridizable dG (HDG) methods** introduce interface DOFs
  - [Cockburn, Gopalakrishnan, and Lazarov 09]
  - see also [Causin and Sacco 05], [Droniou and Eymard 06]

- Skeletal discrete space $\Lambda_h := \bigoplus_{F \in \mathcal{F}_h} \mathbb{P}_d^k(F)$

- Discrete unknowns $(\sigma_h, u_h, \lambda_h) \in \Sigma_h \times U_h \times \Lambda_h$
  - $\sigma_h$ and $u_h$ can be **eliminated locally**
  - global problem in $\lambda_h \in \Lambda_h$ with **compact stencil**

- A new viewpoint emerged recently: **Hybrid High-Order (HHO) methods**
  - introduced in [Di Pietro & AE 15], [Di Pietro, AE & Lemaire 14]
  - see tomorrow’s lecture!
Variable diffusion

- Seek $u : \Omega \rightarrow \mathbb{R}$ s.t. $-\nabla \cdot (\kappa \nabla u) = f$ in $\Omega$ and $u|_{\partial \Omega} = 0$

- Weak formulation: For $f \in L^2(\Omega)$, seek $u \in V := H^1_0(\Omega)$ s.t.

$$a(u, v) := \int_{\Omega} \kappa \nabla u \cdot \nabla v = \int_{\Omega} fv \quad \forall v \in V$$

- $\kappa$ is scalar-valued, bounded, and uniformly positive in $\Omega$
- the model problem is well-posed

- Application to groundwater flows
  - $u$: hydraulic head, $\sigma = -\kappa \nabla u$: Darcy velocity
  - $\kappa$: highly-contrasted hydraulic conductivity
Numerical illustration of high contrasts

- $\sigma = -\kappa \nabla u \in H(\text{div}; \Omega)$
  - the normal component of $\sigma$ is continuous across any interface
  - the normal component of $\nabla u$ is discontinuous if $\kappa$ jumps

- $\Omega = (-1, 1)$ partitioned into $\Omega_1 = (-1, 0)$ and $\Omega_2 = (0, 1)$, $\kappa|_{\Omega_1} = \alpha$ ($\alpha = 0.5$ on left; $\alpha = 0.01$ on right) and $\kappa|_{\Omega_2} = 1$
Discrete setting

- $\kappa$ pcw. constant on a polyhedral partition $P_\Omega = \{\Omega_i\}_{1 \leq i \leq N_\Omega}$ of $\Omega$
- $\mathcal{T}_h$ compatible with $P_\Omega$ ($\kappa$ pcw. constant on $\mathcal{T}_h$)

- Discrete space $V_h := \mathbb{P}_d^k(\mathcal{T}_h)$, $k \geq 1$

- SIP bilinear form

$$ a_h(v_h, w_h) = \int_\Omega \kappa \nabla_h v_h \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h} \int_F \{ \kappa \nabla_h v_h \} \cdot n_F [w_h] $$

$$ - \sum_{F \in \mathcal{F}_h} \int_F [v_h] \{ \kappa \nabla_h w_h \} \cdot n_F + \sum_{F \in \mathcal{F}_h} \eta \frac{\gamma_{\kappa,F}}{h_F} \int_F [v_h] [w_h] $$
Diffusion-dependent penalty

To achieve coercivity, penalty coefficient must depend on $\kappa$

- $\gamma_{\kappa,F} = \{\kappa\}$ [Houston, Schwab & Süli 02]
- for high contrasts, $\gamma_{\kappa,F}$ is controlled by the highest value of $\kappa$ (the most permeable layer)
- ... $\gamma_{\kappa,F}$ should be controlled by the lowest value (the least permeable layer) (as in Mixed FE and FV)

One simple choice is harmonic averaging

$$\gamma_{\kappa,F}^{-1} := \{\kappa^{-1}\}$$

We need to modify the consistency and symmetry terms to maintain coercivity
Symmetric Weighted Interior Penalty (SWIP)

- Weighted average \( \{\{v\}\}_\omega,F := \omega_{T_1,F}v|_{T_1} + \omega_{T_2,F}v|_{T_2} \)
  - \( \omega_{T_1,F} = \omega_{T_2,F} = \frac{1}{2} \) recovers usual arithmetic averages
  - \text{diffusion-dependent weights} \( \omega_{T_1,F} := \frac{k_2}{k_1+k_2}, \omega_{T_2,F} := \frac{k_1}{k_1+k_2} \)
    (homogeneous diffusion yields back arithmetic averages)
  - see [Dryja 03] for idea, [Burman & Zunino 06] for mortaring, [AE, Stephansen & Zunino 09], [Di Pietro, AE & Guermond 08] for advection-diffusion with locally small or zero diffusion

- SWIP bilinear form

\[
a_h(v_h, w_h) = \int_\Omega k_h \nabla_h v_h \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h} \int_F \{\{k_h \nabla_h v_h\}\}_\omega \cdot \mathbf{n}_F [w_h] \\
- \sum_{F \in \mathcal{F}_h} \int_F \{v_h\} \{k_h \nabla_h w_h\}_\omega \cdot \mathbf{n}_F + \sum_{F \in \mathcal{F}_h} \eta \frac{\gamma_{k,F}}{h_F} \int_F [v_h][w_h]
\]

- Strong consistency still holds
Error analysis

- \( C \) denotes a generic constant uniform w.r.t. \( h \) and \( \kappa \)

- **Coercivity:** Assuming \( \eta > C_{tr}^2 N \partial \), \( a_h(v_h, v_h) \geq C_{sta} \| v_h \|_{dG}^2 \) with

\[
\| v_h \|_{dG}^2 := \| \kappa^{1/2} \nabla_h v_h \|^2_{[L^2(\Omega)]^d} + \sum_{F \in \mathcal{F}_h} \frac{\gamma^F}{h_F} \| [v_h] \|_{L^2(F)}^2
\]

- **Boundedness:** \( a_h(v, w_h) \leq C_{bnd} \| v \|_{dG, \#} \| w_h \|_{dG} \) with

\[
\| v \|_{dG, \#}^2 := \| v \|_{dG}^2 + \sum_{T \in \mathcal{T}_h} h_T \| \kappa^{1/2} \nabla v \cdot n_T \|^2_{L^2(\partial T)}
\]

- **Error estimate:** \( \| u - u_h \|_{dG, \#} \leq C \inf_{y_h \in V_h} \| u - y_h \|_{dG, \#} \)

\[
\| u - u_h \|_{dG, \#} \leq C \left( \sum_{T \in \mathcal{T}_h} \kappa_T h_T^{2k} | u |_{H^{k+1}(T)}^2 \right)^{1/2}
\]

- Extension to anisotropic \( \kappa \): use normal component for penalty and averages (error estimate mildly depends on anisotropy ratio \( \sim \rho^{1/2} \))
Outline

- Advection-reaction
- Péclet-robust diffusion-advection-reaction
Model problem

- Let $\Omega$ be a domain in $\mathbb{R}^d$ (open, bounded, connected, strongly Lipschitz set)
- Let $\beta \in [W^{1,\infty}(\Omega)]^d$ and $\mu \in L^\infty(\Omega)$ be s.t.
  \[ \mu - \frac{1}{2} \nabla \cdot \beta \geq \mu_0 > 0 \quad \text{a.e. in } \Omega \]
- Inflow and outflow parts of boundary $\partial \Omega$
  \[ \partial \Omega^\pm = \{ x \in \partial \Omega \mid \pm \beta(x) \cdot n(x) > 0 \} \]
- Let $f \in L^2(\Omega)$; the model problem is
  \[ \begin{cases} 
  \mu u + \beta \cdot \nabla u = f & \text{in } \Omega \\
  u = 0 & \text{on } \partial \Omega^- 
  \end{cases} \]
Functional framework

- Graph space \( W = \{ v \in L^2(\Omega) \mid \beta \cdot \nabla v \in L^2(\Omega) \} \)
  - Hilbert space with norm \( \| v \|_W^2 = \| v \|_{L^2(\Omega)}^2 + \| \beta \cdot \nabla v \|_{L^2(\Omega)}^2 \)

- If \( \partial \Omega^\pm \) are well-separated, there is a bounded trace map \( \gamma : W \rightarrow L^2(|\beta \cdot n|; \partial \Omega) \) s.t. for all \( (v, w) \in W \times W \),
  \[
  \int_{\Omega} (\nabla \cdot \beta)vw + \int_{\Omega} (\beta \cdot \nabla v)w + \int_{\Omega} v(\beta \cdot \nabla w) = \int_{\partial \Omega} (\beta \cdot n)\gamma(v)\gamma(w)
  \]
  - see [AE & Guermond 06]
  - the separation assumption cannot be circumvented for traces in \( L^2(|\beta \cdot n|; \partial \Omega) \)
Weak formulation

- Define on $W \times W$ the bilinear form

$$a(v, w) := \int_{\Omega} \mu vw + (\beta \cdot \nabla v)w + \int_{\partial \Omega} (\beta \cdot n) \otimes vw$$

where for $x \in \mathbb{R}$, $x^{\oplus} = \frac{1}{2}(|x| + x)$ and $x^{\ominus} = \frac{1}{2}(|x| - x)$

- Define on $W$ the linear form $\ell(w) := \int_{\Omega} fw$

- Seek $u \in W$ s.t. $a(u, w) = \ell(w), \forall w \in W$

- BCs are weakly enforced
Well-posedness

- $a$ is $L^2$-coercive on $W$: integrating by parts, we infer that

$$a(v, v) = \int_{\Omega} \left( \mu - \frac{1}{2} \nabla \cdot \beta \right) v^2 + \frac{1}{2} \int_{\partial \Omega} (\beta \cdot n) \gamma(v)^2 + \int_{\partial \Omega} (\beta \cdot n)^\Theta \gamma(v)^2$$

$$\geq \mu_0 \|v\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_{\partial \Omega} |\beta \cdot n| \gamma(v)^2$$

- The weak problem is well-posed
  - $L^2$-coercivity implies uniqueness
  - existence by inf-sup argument [Ern & Guermond 06]
Discrete setting

- Discrete space \( V_h := \mathbb{P}_d^k(T_h), \ k \geq 0 \)
- Discrete problem: Seek \( u_h \in V_h \) s.t. \( a_h(u_h, w_h) = \ell(w_h), \ \forall w_h \in V_h \)
- Main properties of \( a_h \): strong consistency and \( L^2 \)-coercivity on \( V_h \)
- We assume that \( u \in H^s(\Omega), \ s > \frac{1}{2} \); then,

\[
(\beta \cdot n_F)[u] = 0 \quad \forall F \in \mathcal{F}_h^i
\]

(distributional argument)
Centered fluxes

- Use broken gradient in exact bilinear form

- Recover $L^2$-coercivity in a consistent way by setting

\[
a_h^{cf}(v_h, w_h) := \int_{\Omega} \mu v_h w_h + (\beta \cdot \nabla_h v_h)w_h + \sum_{F \in F^b_h} \int_F (\beta \cdot n)^\Theta v_h w_h
- \sum_{F \in F^i_h} \int_F (\beta \cdot n_F) [v_h] [w_h]
\]

- $a_h^{cf}(v_h, v_h) \geq \mu_0 \|v_h\|^2_{L^2(\Omega)} + \sum_{F \in F^b_h} \int_F \frac{1}{2} |\beta \cdot n| v_h^2$

- Error estimate for smooth solution: $\|u - u_h\|_{L^2(\Omega)} \leq Ch^k |u|_{H^{k+1}(\Omega)}$
  - convergence for $k \geq 1$ only, and with suboptimal rate
Local formulation and stencil

- Let $T \in \mathcal{T}_h$, let $\xi \in \mathbb{P}_d^k(T)$ (FV viewpoint)

$$\int_T (\mu - \nabla \cdot \beta) u_h \xi - u_h (\beta \cdot \nabla \xi) + \sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F \phi_F(u_h) \xi = \int_T f \xi$$

with $\epsilon_{T,F} := n_T \cdot n_F = \pm 1$ and numerical fluxes

$$\phi_F(u_h) = \begin{cases} 
(\beta \cdot n_F) \llbracket u_h \rrbracket & \forall F \in \mathcal{F}_h^i \\
(\beta \cdot n) \oplus u_h & \forall F \in \mathcal{F}_h^b
\end{cases}$$

- Standard dG stencil (neighbors in the sense of faces)
Upwind fluxes

- Strengthen discrete stability by penalizing interface jumps in a least-squares sense [Brezzi, Marini & Süli 04]

\[ a_h(v_h, w_h) := a_h^{cf}(v_h, w_h) + s_h(v_h, w_h) \]

with stabilization bilinear form

\[ s_h(v_h, w_h) = \sum_{F \in \mathcal{F}_h} \int_{F} \frac{1}{2} |\beta \cdot n_F | [v_h][w_h] \]

- Strong consistency is preserved
Stability

- Stability norm \((\beta_T := \|\beta\|_{L^\infty(T)^d})\)

\[
\|v_h\|_{dG}^2 := \mu_0\|v_h\|_{L^2(\Omega)}^2 + \sum_{F \in \mathcal{F}_h^b} \int_F \frac{1}{2} |\beta \cdot n| v_h^2 + \sum_{F \in \mathcal{F}_h^i} \int_F \frac{1}{2} |\beta \cdot n_F| \|v_h\|^2 \\
+ \sum_{T \in \mathcal{T}_h} \beta_T^{-1} h_T \|\beta \cdot \nabla v\|_{L^2(T)}^2
\]

- Assume for simplicity \(h_T \mu_0 \leq c_{\mu,\beta} \beta_T, L_{\beta,T} + \|\mu\|_{L^\infty(T)} \leq c_{\mu,\beta} \mu_0\)
  
  - we hide \(c_{\mu,\beta}\) in the generic constants
  
  - general weight on adv. derivative: time-scale \(\tau_T = \min(\mu_0^{-1}, \beta_T^{-1} h_T)\)

- Discrete inf-sup condition [Johnson & Pitkäranta 86]

\[
C_{sta} \|v_h\|_{dG} \leq \sup_{w_h \in V_h \setminus \{0\}} \frac{a_h(v_h, w_h)}{\|w_h\|_{dG}}
\]

- first three terms controlled by coercivity
- bound on advective derivative: test with \(w_h|_{T} = \beta_T^{-1} h_T \langle \beta \rangle_T \cdot \nabla v_h\)
Error analysis

- Boundedness: $a_h(v, w_h) \leq C_{\text{bnd}} \|v\|_{dG, \#} \|w_h\|_{dG}$ with

$$\|v\|^2_{dG, \#} := \|v\|^2_{dG} + \sum_{T \in T_h} \beta_T \left(h_T^{-1} \|v\|^2_{L^2(T)} + \|v\|^2_{L^2(\partial T)}\right)$$

- Error estimate: $\|u - u_h\|_{dG} \leq C \inf_{y_h \in V_h} \|u - y_h\|_{dG, \#}$
  - $\|\cdot\|_{dG, \#}$ and $\|\cdot\|_{dG}$ may not be equivalent on $V_h$, but they lead to the same decay rates of best-approximation errors on smooth functions

$$\|u - u_h\|_{dG} \leq C (\sum_{T \in T_h} \beta_T h_T^{2k+1} |u|_{H^{k+1}(T)^2})^{1/2}$$

- quasi-optimal $L^2$-error estimate $O(h^{k+\frac{1}{2}})$
- optimal error estimate on advective derivative
Local formulation and stencil

- Let $T \in \mathcal{T}_h$, let $\xi \in \mathbb{P}_d^k(T)$

- New numerical fluxes

$$
\phi_F(u_h) = \begin{cases}
(\beta \cdot n_F) \{u_h\} + \frac{1}{2} |\beta \cdot n_F| \|u_h\| & \forall F \in \mathcal{F}_h^i \\
(\beta \cdot n) \oplus u_h & \forall F \in \mathcal{F}_h^b
\end{cases}
$$

- Example: $F = \partial T_1 \cap \partial T_2$, $\beta$ flows from $T_1$ to $T_2$ so that $\beta \cdot n_F \geq 0$

$$
\phi_F(u_h) = (\beta \cdot n_F)(\{u_h\} + \frac{1}{2} \|u_h\|) = (\beta \cdot n_F)\frac{1}{2}(u_h|_{T_1} + u_h|_{T_2} + u_h|_{T_1} - u_h|_{T_2}) = (\beta \cdot n_F)u_h|_{T_1}
$$

- Standard dG stencil (neighbors in the sense of faces)
Further comments

- **$L^2$-coercivity** can be relaxed to $\mu - \frac{1}{2} \nabla \cdot \beta \geq 0$
  - assume that there is $\zeta \in W^{1,\infty}(\Omega)$ s.t. $-\beta \cdot \nabla \zeta \geq \theta_0 > 0$
  - reasonable if $\beta$ has no stationary points or closed curves [Devinatz, Ellis & Friedman 74]

- **Localized error estimate** to avoid global high-order Sobolev norm
  - cut-off functions, exponential decay away from singular layers
  - see [Johnson, Schatz & Wahlbin 87; Guzmán 06]

- **Nonlinear conservation laws**
  - upwinding promotes Gibbs phenomenon [AE & Guermond 13]
  - needs to add nonlinear stabilization mechanism to temper it
Diffusion-advection-reaction

- Model problem
  \[
  \begin{aligned}
  \mu u + \beta \cdot \nabla u &- \nabla \cdot (\kappa \nabla u) = f \quad \text{in } \Omega \\
  u &= 0 \quad \text{on } \partial \Omega
  \end{aligned}
  \]

- Assumptions on the data
  - \(f \in L^2(\Omega)\)
  - \(\beta \in [W^{1,\infty}(\Omega)]^d\), \(\mu \in L^\infty(\Omega)\), \(\mu - \frac{1}{2} \nabla \cdot \beta \geq \mu_0 > 0\)
  - \(\kappa\) scalar-valued, bounded, uniformly positive

- Local Péclet number \(\text{Pe}_T = \frac{\beta_T h_T}{\kappa_T}\) for all \(T \in \mathcal{T}_h\)
  - \(\text{Pe}_T \leq 1\): diffusion-dominated regime
  - \(\text{Pe}_T \geq 1\): advection-dominated regime
  - more generally, \(\text{Pe}_T = \frac{h_T^2}{\tau_T \kappa_T}\) with \(\tau_T = \min(\mu_0^{-1}, \beta_T^{-1} h_T)\)
Discrete setting

- Discrete space $V_h := \mathbb{P}_d^k(T_h)$, $k \geq 1$

- Combine SWIP with upwind fluxes
  - centered fluxes can be used in diffusion-dominated regime
  - Scharfetter–Gummel-type weights can be used as well

- Discrete bilinear form (we drop the symmetry term and integrate by parts the advective derivative)

\[
a_h(v_h, w_h) = \int_{\Omega} (\mu - \nabla \cdot \beta)v_h w_h - v_h(\beta \cdot \nabla h w_h) + \kappa \nabla h v_h \cdot \nabla h w_h
\]

\[- \sum_{F \in \mathcal{F}_h} \int_F (\{\kappa \nabla h v_h\}_\omega + \beta \{v_h\}_\cdot n_F [w_h])
\]

\[+ \sum_{F \in \mathcal{F}_h} \int_F \gamma_{\kappa,\beta,F} [v_h][w_h]
\]

with $\gamma_{\kappa,\beta,F} = \eta h_F + \frac{1}{2} |\beta \cdot n_F|$ if $F \in \mathcal{F}_h^i$ (or $\gamma_{\kappa,\beta,F} = \ldots + (\beta \cdot n_F)$ if $F \in \mathcal{F}_b$)
Error analysis

- **Stability norm**

\[
\|v_h\|_{dG}^2 := \sum_{T \in \mathcal{T}_h} \left( \mu_0 \|v_h\|_{L^2(T)}^2 + \beta_T^{-1} h_T \|\beta \cdot \nabla v_h\|_{L^2(T)}^2 + \kappa_T \|\nabla v_h\|_{[L^2(T)]^d}^2 \right) + \sum_{F \in \mathcal{F}_h} \gamma_{\kappa,\beta,F} \|v_h\|_{L^2(F)}^2
\]

- **Main steps of error analysis**
  - strong consistency
  - discrete inf-sup stability [technical difficulty for anisotropic \(\kappa\)]
  - boundedness in suitable \(\|\cdot\|_{dG,\#}\)-norm

- **Error estimate for smooth solution**

\[
\|u - u_h\|_{dG} \leq C \left( \sum_{T \in \mathcal{T}_h} \left( \mu_0 h_T^2 + \beta_T h_T + \kappa_T \right) h_T^{2k} |u|_{H^{k+1}(T)}^2 \right)^{1/2}
\]

- expected decay in both diffusion- and advection-dominated regimes
Numerical illustrations

- Rotating advective field [AE, Stephansen & Zunino 09]
  - strong $x$- or $y$-diffusion, anisotropy ratio $10^6$
  - SIP+upw enforces zero jumps in under-resolved layers

![Image of SWIP+upw and SIP+upw](image)

- Constant advective field with locally zero anisotropic diffusion [Di Pietro, AE & Guermond 08]

$$\beta = (-1, 0)$$

$$\kappa|_{\Omega_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}$$

$$\kappa|_{\Omega_2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$