VMS stabilization for compressible MHD models

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Taylor Galerkin

System of equations in compact form : $\partial_t \mathbf{w} = -\boldsymbol{\mathcal{L}}(\boldsymbol{\partial}, \mathbf{w})$

Principle of the TG Method (LW, Donea(84))

- Formulate a high-order time-stepping scheme algorithm before the discretization of the spatial variable

$$\mathbf{w}^{n+1} = \mathbf{w}^n + \delta t \left(\partial_t \mathbf{w}\right)^n + \frac{1}{2}\left(\delta t\right)^2 \left(\partial_t^2 \mathbf{w}\right)^n + \frac{1}{6}\left(\delta t\right)^3 \left(\partial_t^3 \mathbf{w}\right)^n + \cdots$$

- Substitute time derivatives by space derivatives:

$$\left(\partial_t \mathbf{w}\right)^n \simeq -\boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}^n\right), \quad \left(\partial_t^{k+1} \mathbf{w}\right)^n \simeq -\left(\partial_t^k \boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}\right)\right)^n$$

- Solve the PDE with $\delta \mathbf{w} = \mathbf{w}^{n+1} - \mathbf{w}^n$

$$\frac{\delta \mathbf{w}}{\delta t} = -\boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}^n\right) - \frac{\delta t}{2}\left(\partial_t \boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}\right)\right)^n - \frac{\left(\delta t\right)^2}{6}\left(\partial_t^2 \boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}\right)\right)^n$$

Stabilization TG2/TG3

Ambrosi & Quartapelle JCP(98)

$$\mathbf{w}^{n+1} = \mathbf{w}^n + \delta t \left(\partial_t \mathbf{w}\right)^n + \frac{1}{2}\left(\delta t\right)^2 \left(\partial_t^2 \mathbf{w}\right)^n + \frac{1}{6}\left(\delta t\right)^3 \left(\partial_t^3 \mathbf{w}\right)^*$$

$$\mathbf{w}^{n+1} - \mathbf{w}^n = \delta t \left(\partial_t \mathbf{w}\right)^{n+1} - \frac{1}{2}\left(\delta t\right)^2 \left(\partial_t^2 \mathbf{w}\right)^{n+1} + \frac{1}{6}\left(\delta t\right)^3 \left(\partial_t^3 \mathbf{w}\right)^*$$

Approximation : $\left(\partial_t^3 \mathbf{w}\right)^* \simeq 3\beta \dfrac{\left(\partial_t^2 \mathbf{w}\right)^{n+1} - \left(\partial_t^2 \mathbf{w}\right)^n}{\delta t}$

General form : $0 \leq \theta \leq 1$ and $0 \leq \beta \leq 1$

$$\frac{\delta \mathbf{w}}{\delta t} + \theta \left(\boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}^{n+1}\right) - \boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}^{n}\right)\right) + \frac{\delta t \xi_I}{2}\left(\left(\partial_t \boldsymbol{\mathcal{L}}\right)^{n+1} - \left(\partial_t \boldsymbol{\mathcal{L}}\right)^{n}\right) = -\boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}^{n}\right) - \frac{\delta t \xi_e}{2}\left(\partial_t \boldsymbol{\mathcal{L}}\right)^{n}$$

where $\xi_I = \theta - \beta$ and $\xi_e = 2\theta - 1$.

Stabilization TG2/TG3

$\beta \leq \theta \leq 1-\beta$ and $\beta \leq 1/2$

1. This is third order accurate only when $\beta = \frac{1}{3}$.
2. Second order accurate for others values.

Linear hyperbolic : $\boldsymbol{\mathcal{L}}(\boldsymbol{\partial}, \mathbf{w}) = (\underline{\mathbf{A}}^* \cdot \boldsymbol{\partial})\,\mathbf{w} = \nabla \cdot (\underline{\mathbf{A}}^* \mathbf{w})$

$$\frac{\delta \mathbf{w}}{\delta t} + \theta \left(\boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}^{n+1}\right) - \boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}^{n}\right) \right) - \frac{\delta t \xi_l}{2} \boldsymbol{\partial}^2_{\underline{\mathbf{A}^*}} \mathbf{w}^{n+1} = -\boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}^{n}\right) + \frac{\delta t (\xi_e - \xi_l)}{2} \boldsymbol{\partial}^2_{\underline{\mathbf{A}^*}} \mathbf{w}^{n}$$

Corrections are dissipative when : $\xi_l \geq 0$ and $\xi_e - \xi_l \geq 0$

Crank-Nicolson scheme : $\theta = 1/2$ and $\beta = 1/2$. In this case $\xi_l = \xi_e = 0$

Stabilization TG2/TG3 :Linearized hyperbolic component.

$$\boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}\right) \simeq \left(\underline{\mathbf{A}}^{*} \cdot \boldsymbol{\partial}\right) \mathbf{w} + \boldsymbol{\mathcal{L}}_{re}\left(\boldsymbol{\partial}, \mathbf{w}\right).$$

$$\begin{cases} \left(\partial_t \boldsymbol{\mathcal{L}}\right)^{n+1} - \left(\partial_t \boldsymbol{\mathcal{L}}\right)^{n} & \simeq & -\boldsymbol{\partial}_{\underline{\mathbf{A}^*}}\left(\mathcal{R}^{n+1}\right) \\ \left(\partial_t \boldsymbol{\mathcal{L}}\right)^{n} & \simeq & -\boldsymbol{\partial}_{\underline{\mathbf{A}^*}}\left(\mathcal{R}^{n}\right) \cancel{+ \boldsymbol{\partial}_{\underline{\mathbf{A}^*}} \partial_t \mathbf{w}^{n-1}} \to 0 \end{cases}$$

where $\mathcal{R}^{k+1} = \left(\partial_t \mathbf{w}\right)^{k} + \boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}^{k+1}\right) \simeq \frac{\mathbf{w}^{k+1} - \mathbf{w}^{k}}{t^{k+1} - t^{k}} + \boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}^{k+1}\right)$

$$\frac{\delta \mathbf{w}}{\delta t} + \theta\left(\boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}^{n+1}\right) - \boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}^{n}\right)\right) - \frac{\delta t \xi_I}{2} \boldsymbol{\partial}_{\underline{\mathbf{A}^*}}\left(\mathcal{R}^{n+1}\right) = -\boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}^{n}\right) + \frac{\delta t \xi_e}{2} \boldsymbol{\partial}_{\underline{\mathbf{A}^*}}\left(\mathcal{R}^{n}\right)$$

Application to Full MHD

$$\partial_t \mathbf{w} + \boldsymbol{\mathcal{L}}(\boldsymbol{\partial}, \mathbf{w}) = 0 \quad \text{with} \quad \mathbf{w} = \begin{pmatrix} \rho \\ \boldsymbol{m} \\ p \\ \mathbf{B} \end{pmatrix}, \quad \boldsymbol{\mathcal{L}}(\boldsymbol{\partial}, \mathbf{w}) = \begin{pmatrix} \boldsymbol{\mathcal{L}}_\rho(\boldsymbol{\partial}, \mathbf{w}) \\ \boldsymbol{\mathcal{L}}_{\boldsymbol{m}}(\boldsymbol{\partial}, \mathbf{w}) \\ \boldsymbol{\mathcal{L}}_p(\boldsymbol{\partial}, \mathbf{w}) \\ \boldsymbol{\mathcal{L}}_{\mathbf{B}}(\boldsymbol{\partial}, \mathbf{w}) \end{pmatrix}$$

where

$$\begin{aligned}
\boldsymbol{\mathcal{L}}_\rho(\boldsymbol{\partial}, \mathbf{w}) &= \nabla \cdot (\rho \boldsymbol{v}) - \nabla \cdot (\underline{\boldsymbol{D}} \nabla \rho) \\
\boldsymbol{\mathcal{L}}_{\boldsymbol{m}}(\boldsymbol{\partial}, \mathbf{w}) &= \nabla \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v} + p\mathbf{I} + \pi \mathbf{I} - \mathbf{B} \otimes \mathbf{B}) + \nabla \cdot \underline{\boldsymbol{\pi}} \\
\boldsymbol{\mathcal{L}}_p(\boldsymbol{\partial}, \mathbf{w}) &= \boldsymbol{v} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{v} - \Gamma \left(\nabla \cdot (\underline{\boldsymbol{\lambda}} \nabla T) + \underline{\boldsymbol{\pi}} : \nabla \boldsymbol{v} + \underline{\boldsymbol{\eta}} \mathbf{J} \cdot \mathbf{J} \right) \\
\boldsymbol{\mathcal{L}}_{\mathbf{B}}(\boldsymbol{\partial}, \mathbf{w}) &= \nabla \times \mathbf{E}
\end{aligned}$$

Application to Full MHD

We are concerned by plasmas dynamically dominated by ideal MHD pattern : $\mathcal{L}(\partial, \mathbf{w}) = \widetilde{\mathcal{L}}(\partial, \mathbf{w}) + \mathcal{L}_{re}(\partial, \mathbf{w})$ with

$$\widetilde{\mathcal{L}}(\partial, \mathbf{w}) = \begin{pmatrix} \nabla \cdot \boldsymbol{m} \\ \nabla \cdot \left(\frac{\boldsymbol{m} \otimes \boldsymbol{m}}{\rho} + p\mathbf{I} + \pi\mathbf{I} - \mathbf{B} \otimes \mathbf{B} \right) \\ \boldsymbol{v} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{v} \\ \nabla \cdot \left(\frac{\boldsymbol{m} \otimes \mathbf{B}}{\rho} - \frac{\mathbf{B} \otimes \boldsymbol{m}}{\rho} \right) \end{pmatrix}$$

Then $\widetilde{\mathcal{L}}(\partial, \mathbf{w}) = \widetilde{\underline{\mathbf{A}}}(\mathbf{w}, \partial)\, \mathbf{w}$ with

$$\widetilde{\underline{\mathbf{A}}}(\mathbf{w}, \partial) = \left(\begin{array}{c|c|c|c} 0 & \boldsymbol{\partial}^{\mathrm{T}} & 0 & \mathbf{0} \\ \hline -(\boldsymbol{v} \otimes \boldsymbol{v})\boldsymbol{\partial} & \boldsymbol{v} \otimes \boldsymbol{\partial} + \boldsymbol{v} \cdot \boldsymbol{\partial} & \boldsymbol{\partial} & (\mathbf{B} \otimes \boldsymbol{\partial})^{\mathrm{T}} - \mathbf{B} \cdot \boldsymbol{\partial} \\ \hline -\frac{\gamma p}{\rho} \boldsymbol{v} \cdot \boldsymbol{\partial} & \frac{\gamma p}{\rho} \boldsymbol{\partial}^{\mathrm{T}} & \boldsymbol{v} \cdot \boldsymbol{\partial} & \mathbf{0} \\ \hline \frac{\boldsymbol{v}\mathbf{B} \cdot \boldsymbol{\partial} - \mathbf{B}\boldsymbol{v} \cdot \boldsymbol{\partial}}{\rho} & \frac{\mathbf{B} \otimes \boldsymbol{\partial}}{\rho} & \mathbf{0} & \boldsymbol{v} \cdot \boldsymbol{\partial} \end{array} \right)$$

Taylor-Galerkin method is defined with $\underline{\mathbf{A}}_e^* = \widetilde{\underline{\mathbf{A}}}(\mathbf{w}_e^n, \partial)$

Simplified semi-implicit and Implicit stabilizations

$$\mathcal{R}^{n+1} \simeq \frac{\delta \mathbf{w}}{\delta t} \quad \text{and} \quad \mathcal{R}^{n} \simeq 0$$

$$\Longrightarrow \left(\mathbf{I} - \frac{\delta t \xi_I}{2} \boldsymbol{\partial}_{\underline{\mathbf{A}^*}}\right) \frac{\delta \mathbf{w}}{\delta t} + \theta \boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}^{n+1}\right) = -\left(1-\theta\right) \boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial}, \mathbf{w}^{n}\right)$$

Simplified semi-implicit and Implicit stabilizations

The Harned and Kerner algorithm : $\theta = 0$ and $\underline{\mathbf{A}}^* = \widetilde{\underline{\mathbf{A}}}(\mathbf{w}^n, \boldsymbol{\partial})$

$$\widetilde{\underline{\mathbf{A}}}(\mathbf{w}, \boldsymbol{\partial}) = \begin{pmatrix} 0 & \boldsymbol{\partial}^{\mathrm{T}} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\partial} & (\mathbf{B} \otimes \boldsymbol{\partial})^{\mathrm{T}} \\ 0 & \frac{\gamma p}{\rho}\boldsymbol{\partial}^{\mathrm{T}} & 0 & \mathbf{0} \\ \mathbf{0} & \frac{\mathbf{B} \otimes \boldsymbol{\partial}}{\rho} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$\begin{aligned} \frac{\delta\rho}{\delta t} - \frac{\delta t \xi_I}{2}\boldsymbol{\partial}^{\mathrm{T}}\frac{\delta \boldsymbol{m}}{\delta t} &= -\boldsymbol{\mathcal{L}}_{\rho} (\\ \frac{\delta \boldsymbol{m}}{\delta t} - \frac{\delta t \xi_I}{2}\left(\boldsymbol{\partial}\frac{\delta p}{\delta t} + (\mathbf{B}^n \otimes \boldsymbol{\partial})^{\mathrm{T}}\frac{\delta \mathbf{B}}{\delta t}\right) &= -\boldsymbol{\mathcal{L}}_{\boldsymbol{m}} \\ \frac{\delta p}{\delta t} - \frac{\delta t \xi_I}{2}\left(\frac{\gamma p^n}{\rho^n}\boldsymbol{\partial}^{\mathrm{T}}\frac{\delta \boldsymbol{m}}{\delta t}\right) &= -\boldsymbol{\mathcal{L}}_{p} (\\ \frac{\delta \mathbf{B}}{\delta t} - \frac{\delta t \xi_I}{2}\left(\frac{\mathbf{B}^n \otimes \boldsymbol{\partial}}{\rho^n}\frac{\delta \boldsymbol{m}}{\delta t}\right) &= -\boldsymbol{\mathcal{L}}_{\mathbf{B}} (\end{aligned}$$

Self consistent implicit scheme for the momentum:

$$\left(\mathbf{I} - \left(\frac{\delta t \xi_I}{2}\right)^2 \mathcal{G}(\mathbf{w}^n, \boldsymbol{\partial})\right)\frac{\delta \boldsymbol{m}}{\delta t} = -\boldsymbol{\mathcal{L}}_{\boldsymbol{m}}(\boldsymbol{\partial}, \mathbf{w}^n) - \frac{\delta t \xi_I}{2}\mathcal{K}(\boldsymbol{\partial}, \mathbf{w}^n) \tag{1}$$

where $\mathcal{G}(\mathbf{w}^n, \boldsymbol{\partial})$ is a self-adjoint (in $(L_2)^3$) linearized operator associated to ideal MHD

$$\mathcal{G}(\mathbf{w}^n, \boldsymbol{\partial}) = \boldsymbol{\partial}\left(\frac{\gamma p^n}{\rho^n}\boldsymbol{\partial}^{\mathrm{T}}\right) + (\mathbf{B}^n \otimes \boldsymbol{\partial})^{\mathrm{T}}\left(\frac{1}{\rho^n}\mathbf{B}^n \otimes \boldsymbol{\partial}\right)$$

Simplified "Physics-based" preconditioning

$$\mathcal{R}^{n+1} \simeq \frac{\delta \mathbf{w}}{\delta t} + \boldsymbol{\partial}_{\underline{\mathbf{A}}*}\mathbf{w}^{n+1} \quad \text{and} \quad \mathcal{R}^{n} \simeq 0 + \boldsymbol{\partial}_{\underline{\mathbf{A}}*}\mathbf{w}^{n}$$

$$\mathbf{I} - \frac{\delta t \xi_I}{2}\boldsymbol{\partial}_{\underline{\mathbf{A}}*} \simeq \underline{\boldsymbol{P}}(\mathbf{w},\boldsymbol{\partial}) = \begin{pmatrix} 1 & -\frac{\delta t \xi_I}{2}\boldsymbol{\partial}^{\mathrm{T}} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} - \left(\frac{\delta t \xi_I}{2}\right)^2 \mathcal{G}(\mathbf{w},\boldsymbol{\partial}) & 0 & \mathbf{0} \\ 0 & -\frac{\delta t \xi_I}{2}\left(\frac{\gamma p}{\rho}\boldsymbol{\partial}^{\mathrm{T}} + \frac{(\nabla p)^{\mathrm{T}}}{\rho}\right) & 1 & \mathbf{0} \\ \mathbf{0} & -\frac{\delta t \xi_I}{2}\frac{\mathbf{B}\otimes\boldsymbol{\partial}}{\rho} & 0 & \mathbf{I} \end{pmatrix}$$

and

$$\underline{\boldsymbol{P}}(\mathbf{w}^n,\boldsymbol{\partial})\frac{\delta \mathbf{w}}{\delta t} + \theta\left(\boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial},\mathbf{w}^{n+1}\right) - \boldsymbol{\mathcal{L}}\left(\boldsymbol{\partial},\mathbf{w}^{n}\right)\right) - \frac{\delta t \xi_I}{2}\boldsymbol{\partial}^2_{\underline{\mathbf{A}}*}\mathbf{w}^{n+1}$$
$$= -\underline{\boldsymbol{\mathcal{K}}}(\mathbf{w}^n,\boldsymbol{\partial})\boldsymbol{\mathcal{L}}(\boldsymbol{\partial},\mathbf{w}^n) + \frac{\delta t \xi_e}{2}\boldsymbol{\partial}^2_{\underline{\mathbf{A}}*}\mathbf{w}^{n}$$

To be done! Thanks