The Divergence Constraint in Magnetohydrodynamic Simulations

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Outline

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Magnetohydrodynamics (MHD) and Plasmas

- **Set of equations**
  - Combination of Navier-Stokes equations and Maxwell's equations
  - Derived by taking moments of the kinetic equation and some approximations
    - We do not ask what the plasma ions and electrons are doing separately
    - We describe the plasma as a single conducting fluid
- **Ideal MHD**
  - Perfectly conducting fluid
    - Field line freezing
  - Strongly collisional plasma
  - Length and time scale restrictions

Ideal MHD Equations

- Set of nonlinear hyperbolic equations:

$$\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) &= 0 \quad (1)\\
\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla (p + \tfrac{1}{2} \mathbf{B} \cdot \mathbf{B}) - \nabla \cdot (\mathbf{B} \otimes \mathbf{B}) &= 0 \quad (2)\\
\partial_t \varepsilon + \nabla \cdot [(\varepsilon + p + \tfrac{1}{2} \mathbf{B} \cdot \mathbf{B})\, \mathbf{u} - (\mathbf{u} \cdot \mathbf{B})\, \mathbf{B}] &= 0 \quad (3)\\
\partial_t \mathbf{B} + \nabla \cdot (\mathbf{B} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{B}) &= 0 \quad (4)
\end{aligned}$$

The hydrodynamic pressure p is given by the equation of state

$$p = (\gamma - 1) \left(\varepsilon - \frac{\rho}{2} \mathbf{u} \cdot \mathbf{u} - \frac{1}{2} \mathbf{B} \cdot \mathbf{B} \right)$$

- Divergence constraint $\quad \nabla \cdot \mathbf{B} = 0$

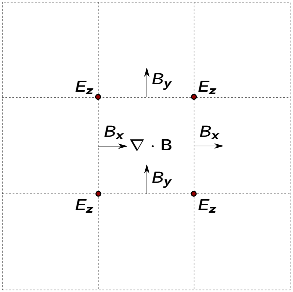
Outline

Divergence Constraint

- Initial data satisfy $\nabla \cdot \mathbf{B} = 0$
  $\Rightarrow$ Exact solution will satisfy this constraint for all times
- Formulated in a strong sense
  - $\partial_t \mathbf{B} + \nabla \cdot (\mathbf{B} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{B}) = 0$ can be written as
    $$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{u}) = 0 \quad (5)$$
  - We have $\nabla \cdot (\nabla \times \cdot) \equiv 0$
- Usually, discrete divergence of discrete curl is not exactly zero
  - $\nabla \cdot \mathbf{B}$ errors arise and may increase
  - Unphysical behavior
- Several methods aim to maintain this constraint numerically
  - Magnetic vector potential
  - **Constrained transport**
  - **Divergence cleaning**

Constrained Transport [1]

- Staggered mesh formulation
  - Inherently divergence-free
  - Interpolation of **B** to the control volume center needed in eq. (3) introduces an error in the conservation of the total energy



2D staggered grid

[1] S. Fromang, P. Hennebelle, and R. Teyssier. A high order Godunov scheme with constrained transport and adaptive mesh refinement for astrophysical MHD. *A&A*, 457:371-384, 2006.

Divergence Cleaning [2]

- Cell-centered grid $\Rightarrow$ divergence cleaning step
- Solve system (1)-(4), with equation (4) replaced by

$$\begin{aligned} \partial_t \mathbf{B} + \nabla \cdot (\mathbf{B} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{B}) + \nabla \psi &= 0 \qquad (6) \\ \mathcal{D}(\psi) + \nabla \cdot \mathbf{B} &= 0 \qquad (7) \end{aligned}$$

- GLM-MHD system consists of equations (1), (2), (6), (3), (7)
  - **Hyperbolic correction**

    $\mathcal{D}(\psi) = \frac{1}{c_h^2} \partial_t \psi$ with $c_h \in (0, \infty)$
  - **Mixed correction**

    $\mathcal{D}(\psi) = \frac{1}{c_h^2} \partial_t \psi + \frac{1}{c_p^2} \psi$ with $c_h, c_p \in (0, \infty)$

    This choice offers both propagation and dissipation of divergence errors

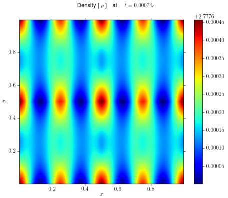
[2] A. Dedner, F. Kemm, D. Kröner, C.-D. Munz, T. Schnitzer, and M. Wesenberg. Hyperbolic divergence cleaning for the MHD equations. *J. Comput. Phys.*, 175(2):645-673, 2002.

Outline

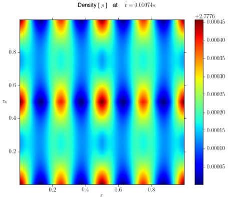
Code

- HERACLES
  - Second-order finite volume method on structured grids
  - 3D parallel code used to simulate astrophysical flows
    - Hydrodynamics
    - MHD
    - Radiative transfer
    - Gravity
- Riemann problems for MHD
  - HLLC
  - **HLLD**
- Enforcing divergence-free magnetic field
  - Constrained transport
  - Divergence cleaning

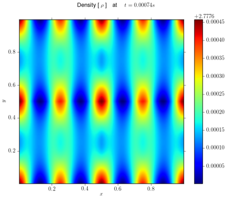
Orszag-Tang



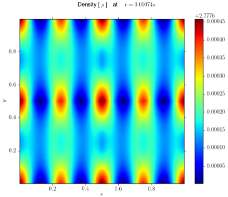
Constrained Transport



No Cleaning



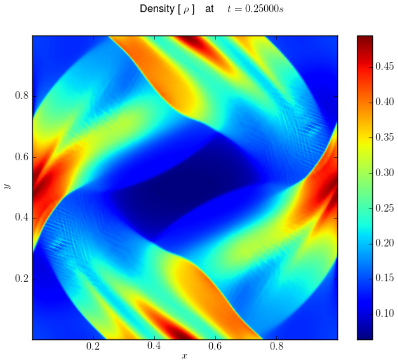
Hyper GLM

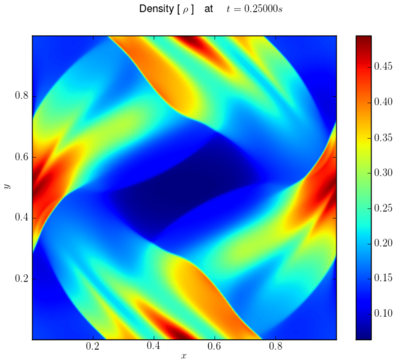


Mixed GLM

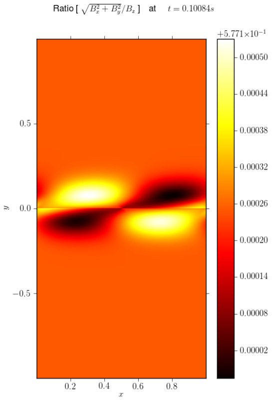
Noise Comparison

No Cleaning vs. Hyper GLM



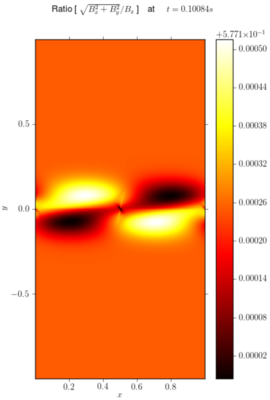


Kelvin-Helmholtz Instability



Constrained Transport

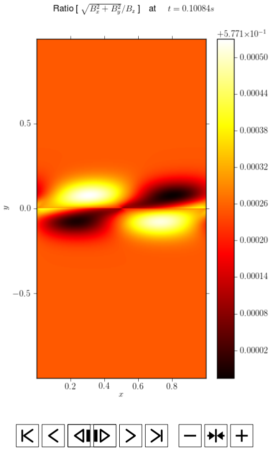
Staggered Grid



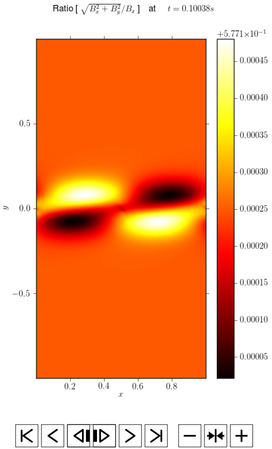
No Cleaning

Centered Grid

Kelvin-Helmholtz Instability

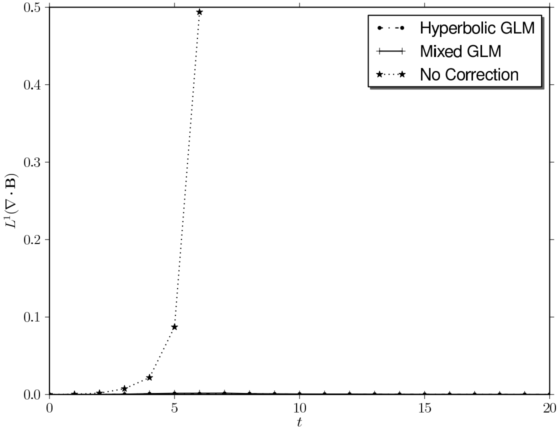


Constrained Transport



Mixed GLM

Total Divergence



Outline

Current[3] and Future Work

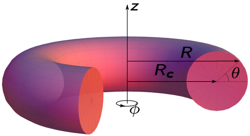
Conclusions

- Showed the importance of maintaining $\nabla \cdot \mathbf{B} = 0$ numerically
- New modules in HERACLES working as expected

Future work

- Three-dimensional numerical scheme adapted to toroidal geometry
  - Curvilinear coordinates
  - Well-balanced
  - Large-scale parallel architectures
  - Cell-centered unstructured grid

$\Rightarrow$ Less diffusive divergence cleaning method



[3] J. Vides, B. Braconnier, E. Audit, C. Berthon, and B. Nkonga. A Godunov-type solver for the numerical approximation of gravitational flows, *submitted*

Thank you for your attention.

The Divergence Constraint in MHD Simulations

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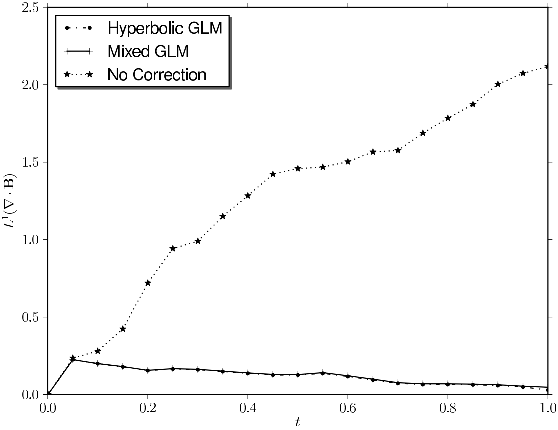
Total Divergence



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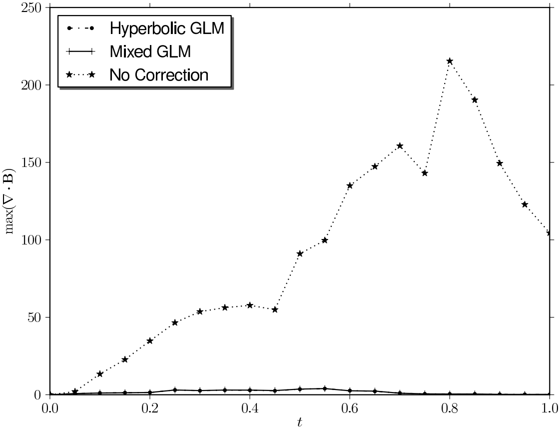
Maximum Divergence



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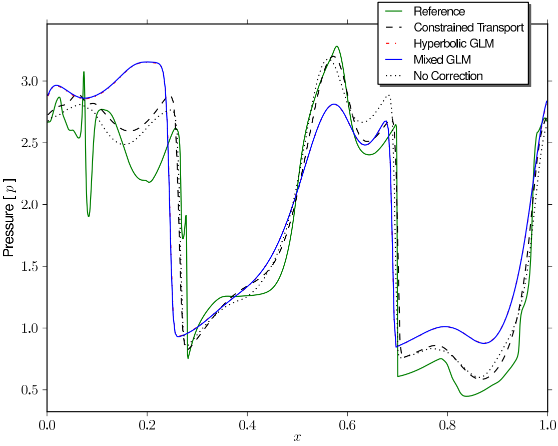
Horizontal Cut (First Order)

- Horizontal cut at $y = 0.3125$ at $t = 0.5$ using the HLLD scheme



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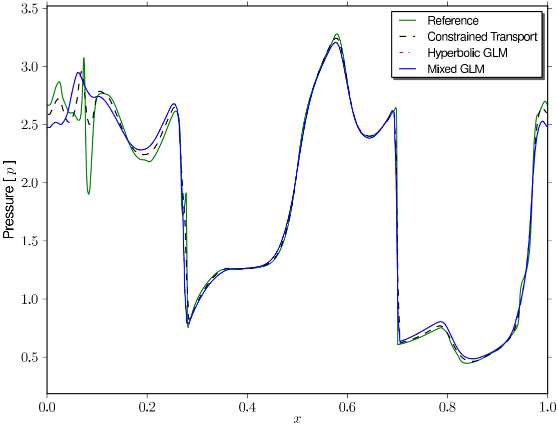
Horizontal Cut (Second Order)

- Horizontal cut at $y = 0.3125$ at $t = 0.5$ using the HLLD scheme



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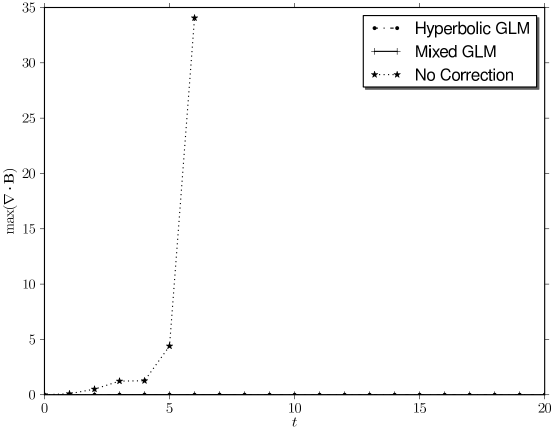
Maximum Divergence



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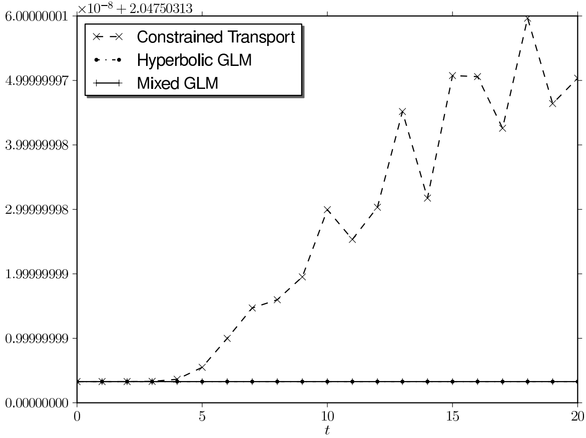
Total Energy



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