EQUIVALENCE RELATIONS FOR TWO VARIABLE REAL ANALYTIC FUNCTION GERMS

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ABSTRACT. For two variable real analytic function germs we compare the blow-analytic equivalence in the sense of Kuo to the other natural equivalence relations. Our main theorem states that C^1 equivalent germs are blow-analytically equivalent. This gives a negative answer to a conjecture of Kuo. In the proof we show that the Puiseux pairs of real Newton-Puiseux roots are preserved by the C^1 equivalence of function germs. The proof is achieved, being based on a combinatorial characterisation of blow-analytic equivalence in terms of the real tree model.

We also give several examples of bi-Lipschitz equivalent germs that are not blow-analytically equivalent.

The natural equivalence relations we first think of are the C^r coordinate changes for $r = 1, 2, \cdots, \infty, \omega$, where C^ω stands for real analytic. Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be real analytic function germs. We say that f and g are C^r *(right) equivalent* if there is a local C^r diffeomorphism $\sigma : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ \sigma$. If σ is a local bi-Lipschitz homeomorphism, resp. a local homeomorphism, then we say that f and g are *bi-Lipschitz equivalent*, resp. C^0 *equivalent*. By definition, we have the following implications:

(0.1) C^0-eq. $\Leftarrow$ bi-Lipschitz eq. $\Leftarrow C^1$-eq. $\Leftarrow C^2$-eq. $\Leftarrow \cdots \Leftarrow C^\infty$-eq. $\Leftarrow C^\omega$-eq.

By Artin's Approximation Theorem [2], C^∞ equivalence implies C^ω equivalence. But the other converse implications of (0.1) do not hold. Let $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be polynomial functions defined by

$$f(x, y) = (x^2 + y^2)^2, \quad g(x, y) = (x^2 + y^2)^2 + x^{r+4}$$

for $r = 1, 2, \cdots$. N. Kuiper [14] and F. Takens [21] showed that f and g are C^r equivalent, but not C^{r+1} equivalent.

In the family of germs

$$K_t(x, y) = x^4 + tx^2y^2 + y^4,$$

the phenomenon of continuous C^1 moduli appears: for $t_1, t_2 \in I$, K_{t_1} and K_{t_2} are C^1 equivalent if and only if $t_1 = t_2$, where $I = (-\infty, -6]$, $[-6, -2]$ or $[-2, \infty)$, see example 0.5 below. On the other hand, T.-C. Kuo proved that this family is

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C^0-trivial over any interval not containing -2, by a C^0 trivialisation obtained by the integration of a vector field, c.f. [15]. In the homogeneous case, as that of K_t, the Kuo vector field is Lipschitz and the trivialisation is bi-Lipschitz. Thus Kuo's construction gives examples of bi-Lipschitz equivalent germs that are not C^1 equivalent.

It is easy to construct examples of C^0 equivalent and bi-Lipschitz non-equivalent germs. Let us note that, moreover, the bi-Lipschitz equivalence also has continuous moduli, c.f. [9, 10]. For instance the family

$$A_t(x,y) = x^3 - 3txy^4 + 2y^6, \quad t > 0,$$

is C^0 trivial and if A_{t_1} is bi-Lipschitz equivalent to A_{t_2}, $t_1, t_2 > 0$, then $t_1 = t_2$.

0.1. **Blow-analytic equivalence.** Blow-analytic equivalence was proposed for real analytic function germs by Tzee-Char Kuo [17] as a counterpart of the topological equivalence of complex analytic germs. Kuo showed in [19] the local finiteness (i.e. the absence of continuous moduli) of blow-analytic types for analytic families of isolated singularities.

We say that a homeomorphism germ $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a *blow-analytic homeomorphism* if there exist real modifications $\mu : (M, \mu^{-1}(0)) \to (\mathbb{R}^n, 0)$, $\tilde\mu : (\tilde M, \tilde\mu^{-1}(0)) \to (\mathbb{R}^n, 0)$ and an analytic isomorphism $\Phi : (M, \mu^{-1}(0)) \to (\tilde M, \tilde\mu^{-1}(0))$ so that $\sigma \circ \mu = \tilde\mu \circ \Phi$. The formal definition of real modification is somewhat technical and, since in this paper we consider only the two variable case, we shall use the following criterion of [13]: in two variable case μ is a real modification if and only if it is a finite composition of point blowings-up. Finally, we say that two real analytic function germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are *blow-analytically equivalent* if there exists a blow-analytic homeomorphism $\sigma : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ \sigma$.

For instance, the family K_t, $t \neq -2$, becomes real analytically trivial after the blowing-up of the $t-$axis, cf. Kuo [17]. Thus for $t < -2$, or $t > -2$ respectively, all K_t are blow-analytically equivalent. Similarly, the family A_t becomes real analytically trivial after a toric blowing-up in $x, y-$variables, cf. Fukui - Yoshinaga [4] or Fukui - Paunescu [7], and hence it is blow-analytically trivial. Thus blow-analytic equivalence does not imply neither C^r-equivalence, $r \geq 1$, nor bi-Lipschitz equivalence.

Blow-analytic equivalence is a stronger and more natural notion than C^0 equivalence. For instance, $f(x,y) = x^2 - y^3$, $g(x,y) = x^2 - y^5$ are C^0 equivalent, but not blow-analytically equivalent. The latter fact can be seen using the Fukui invariant [5], that we recall in section 4 below, or it follows directly from the following theorem.

Theorem 0.1. (cf. [13]) *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be real analytic function germs. Then the following conditions are equivalent:*

(1) f and g are blow-analytically equivalent.
(2) f and g have isomorphic minimal resolutions.
(3) The real tree models of f and g are isomorphic.

For more on the blow-analytic equivalence in the general case n-dimensional we refer the reader to recent surveys [6, 8].

What is the relation between blow-analytic equivalence and C^r equivalences, $1 \leq r < \infty$? Kuo states in [17] that his modified analytic homeomorphism is independent of C^r diffeomorphisms, $1 \leq r < \infty$, and confirms his belief at the invited address of the annual convention of the Mathematical Society of Japan, autumn 1984 ([18]), by asserting that blow-analytic equivalence is independent of C^r equivalences. Untill now it was widely believed that this is the case.

0.2. **Main results of this paper.** The main result of this paper is the following.

Theorem 0.2. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be real analytic function germs and suppose that there exists a C^1 diffeomorphism germ $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $f = g \circ \sigma$. Then f and g are blow-analytically equivalent.*

If, moreover, σ preserves orientation, then f and g are blow-analytically equivalent by an orientation preserving blow-analytic homeomorphism.

This gives a negative answer to the above conjecture of Kuo. To give the reader some flavour of this unexpected result we propose the following special case that contains most of the difficulty of the proof and does not refer to blow-analytic equivalence. Recall that a Newton-Puiseux root of $f(x, y) = 0$ is a real analytic arc $\gamma \subset f^{-1}(0)$ parameterised by

$$\gamma : x = \lambda(y) = a_1 y^{n_1/N} + a_2 y^{n_2/N} + \cdots ,$$

where $\lambda(y)$ is a convergent fractional power series $\lambda(y) = a_1 y^{n_1/N} + a_2 y^{n_2/N} + \cdots$. We shall always assume that $n_1/N \geq 1$, that is γ is transverse to the x-axis. We shall call γ real if all a_i are real for $y \geq 0$, and then we understand γ as such real demi-branch of an analytic arc, with the parametrisation restricted to $y \geq 0$.

Proposition 0.3. (cf. proposition 1.11 below)
Let $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a C^1 diffeomorphism and let $f,\ g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be real analytic function germs such that $f = g \circ \sigma$. Suppose that $\gamma \subset f^{-1}(0)$ and $\tilde\gamma \subset g^{-1}(0)$ be Newton-Puiseux roots of f and g respectively such that $\sigma(\gamma) = \tilde\gamma$ as set germs. Then the Puiseux characteristic pairs of γ and $\tilde\gamma$ coincide.

This property has no obvious counterpart in the complex set-up. The Puiseux pairs of plane curve singularities are embedded topological invariants, cf. [24]. We can not dream of any similar statement in the real analytic set-up, all real analytic demi-branches are C^1 equivalent to the positive $y-$axis. In the proof of proposition 1.11 we use two basic assumptions, the arcs are roots and σ conjugates the analytic functions defining the roots: $f = g \circ \sigma$.

There is another major difference to the complex case. The topological type of a complex analytic function germ can be combinatorially characterised in terms of the tree model of [16], that encodes the contact orders between different Newton-Puiseux roots, that give, in particular, the Puiseux pairs of those roots. This is no longer true in real words, the Puiseux pairs cannot be read from these contact orders, see [13].

It was speculated for a long time that there is a relation between blow-analytic and bi-Lipschitz properties. It is not difficult to construct examples showing that

$$\text{blow-analytic-eq.} \not\Rightarrow \text{bi-Lipschitz eq} ,$$

as the example A_t above. In this paper we construct several examples showing that

$$\text{blow-analytic-eq.} \not\Leftarrow \text{bi-Lipschitz eq} .$$

Thus, there is no direct relation between these two notions. Nevertheless, as shown in [13], a blow-analytic homeomorphism that gives blow-analytic equivalence between two 2-variable real analytic function germs, preserves the order of contact between **non**-parameterised real analytic arcs. Note that by the curve selection lemma, a subanalytic homeomorphism is bi-Lipschitz if and only if it preserves the order of contact between parameterised real analytic arcs.

For more than two variables we have another phenomenon. Let $f_t : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$, $t \in \mathbb{R}$, be the Briancoņ-Speder family defined by $f_t(x, y, z) = z^5 + tzy^6 + y^7x + x^{15}$. Although f_0 and f_{-1} are blow-analytically equivalent, any blow-analytic homeomorphism that gives the blow-analytic equivalence between them does not preserve the order of contact between some analytic arcs contained in $f_0^{-1}(0)$, cf. [11].

0.3. **Organisation of this paper.** In section 1 we construct new invariants of bi-Lipschitz and C^1 equivalences. These invariants can be nicely described in terms of the Newton polygon relative to a curve, the notion introduced in [20], though the reader can follow an alternative way that uses equivalent notions: the order function and associated polynomials. Shortly speaking, if $f = g \circ \sigma$ with σ bi-Lipschitz then the Newton boundaries of f relative to an arc γ coincides with the Newton boundary of g relative to $\sigma(\gamma)$. If σ is C^1 and $D\sigma(0) = Id$ then, moreover, the corresponding coefficients on the Newton boundaries are identical. As a direct corollary we get the C^1 invariance of Puiseux pairs of the Newton-Puiseux roots, see proposition 1.11.

In section 2 we show theorem 0.2. The proof is based on theorem 0.1 so we recall in this section the construction of real tree model.

In section 3 we extend the construction of section 1 to all C^1 diffeomorphisms (we drop the assumption $D\sigma(0) = Id$). As a corollary we give a complete classification of C^1 equivalent weighted homogeneous germs of two variables.

Section 4 contains the construction of examples of bi-Lipschitz equivalent and blow-analytically non-equivalent germs. This is not simple since such a bi-Lipschitz equivalence cannot be natural. Let us first recall the construction of invariants of bi-Lipschitz equivalence of [9, 10]. Suppose that the generic polar curve of $f(x, y)$ has at least two branches γ_i. Fix reasonable parametrisations of these branches, either by a coordonate as $x = \lambda_i(y)$ or by the distance to the origin, and expand f along each such branch. Suppose that the expansions along different branches $f(\lambda_i(y), y) = a_i y^s + \cdots$ have the same leading exponent s, and that the term y^s is sufficiently big in comparison to the distance between the branches. Then the ratio of the leading coefficients a_i/a_j is a bi-Lipschitz invariant (and a continuous modulus). Our construction of bi-Lipschitz homeomorphism goes along these lines but in the opposite direction. First we choose carefully $f(x,y)$, $g(x,y)$ so that such the expansions of f, resp. g, along polar branches are compatible, so that we write down explicitly bi-Lipschitz equivalences between horn neighbourhoods of polar curves of f and g, respectively. Then we show that, in our examples, these equivalences can be glued together using partition of unity.

0.4. **Observations.** We shall use freely the following widely known facts. In the general n-variable case, the multiplicity of an analytic function germ is a bi-Lipschitz invariant. For the C^1 equivalence the initial homogeneous form, up to linear equivalence, is an invariant. Indeed, for real analytic functions not identically zero, we have

Lemma 0.4. *Let f, $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be analytic function germs of the form*

$$f(x) = f_m(x) + f_{m+1}(x) + \cdots , \quad f_m \not\equiv 0,$$
$$g(x) = g_k(x) + g_{k+1}(x) + \cdots , \quad g_k \not\equiv 0.$$

Suppose that f and g are C^1-equivalent. Then $k = m$ and f_m and g_m are linearly equivalent.

In particular, if homogeneous polynomial functions are C^1-equivalent, then they are linearly equivalent.

Proof. Since C^1-equivalence is a bi-Lipschitz equivalence, $m = k$. Let $\sigma(x) = (\sigma_1(x), \cdots , \sigma_n(x))$ be a local C^1 diffeomorphism such that $f = g \circ \sigma$. Let us write

$$\sigma_i(x) = \sum_{j=1}^{n} a_{ij} x_j + h_i(x)$$

where $j^1 h_i(0) = 0$, $i = 1, \cdots , n$. Set $A(x) = (\sum_{j=1}^n a_{1j} x_j, \cdots , \sum_{j=1}^n a_{nj} x_j)$. Since σ is a local C^1 diffeomorphism, A is a linear transformation of $\mathbb{R}^n$. Now we have

$$f_m(x) + f_{m+1}(x) + \cdots = f(x) = g(\sigma(x)) = g_m(A(x)) + G(x)$$

where $f_{m+1}(x) + \cdots = o(|x|^m)$ and $G(x) = o(|x|^m)$. Therefore we have $f_m(x) = g_m(A(x))$. Namely, f_m and g_m are linearly equivalent. □

Example 0.5. Let $f_t : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, $t \in \mathbb{R}$, be a polynomial function defined by

$$f_t(x, y) = x^4 + tx^2y^2 + y^4.$$

By an elementary calculation, we can see that there are a, b, c, $d \in \mathbb{R}$ with $ad-bc \neq 0$ such that

$$(ax + by)^4 + t_1(ax + by)^2(cx + dy)^2 + (cx + dy)^4 = x^4 + t_2x^2y^2 + y^4$$

if and only if $t_1 = t_2$ or $(t_1+2)(t_2+2) = 16$. Therefore it follows from lemma 0.4 that f_{t_1} and f_{t_2}, t_1, $t_2 \in \mathbb{R}$, are C^1-equivalent if and only if $t_1 = t_2$ or $(t_1+2)(t_2+2) = 16$.

1. Construction of bi-Lipschitz and C^1 invariants

Let $f(x,y)$ be a real analytic two variable function germ:

$$f(x,y) = f_m(x,y) + f_{m+1}(x,y) + \cdots , \tag{1.1}$$

where f_j denotes the j-th homogeneous form of f. We say that f is *mini-regular in* x if $f_m(1,0) \neq 0$. Unless otherwise specified we shall always assume that the real analytic function germs are mini-regular in x.

We shall consider the demi-branches of real analytic arcs at $0 \in \mathbb{R}^2$ of the following form

$$\gamma : x = \lambda(y) = a_1 y^{n_1/N} + a_2 y^{n_2/N} + \cdots , \quad y \geq 0,$$

where $\lambda(y)$ is a convergent fractional power series, N and $n_1 < n_2 < \cdots$ are positive integers having no common divisor, $a_i \in \mathbb{R}$, $N, n_i \in \mathbb{N}$. We shall call such a demi-branch *allowable* if $n_1/N \geq 1$, that is γ is transverse to the x-axis.

Given f and γ as above. We define *the order function of* f *relative to* γ, $\operatorname{ord}_\gamma f : [1,\infty) \to \mathbb{R}$ as follows. Fix $\xi \geq 1$ and expand

$$f(\lambda(y) + zy^\xi, y) = P_{f,\gamma,\xi}(z) y^{\operatorname{ord}_\gamma f(\xi)} + \cdots , \tag{1.2}$$

where the dots denote higher order terms in y and $\operatorname{ord}_\gamma f(\xi)$ is the smallest exponent with non-zero coefficient. This coefficient, $P_{f,\gamma,\xi}(z)$, is a polynomial function of z.

By the Newton polygon of f *relative to* γ, denoted by $NP_\gamma f$, we mean the Newton polygon of $f(X + \lambda(Y), Y)$ (cf. [20]). Its boundary, called the *Newton boundary* and denoted by $NB_\gamma f$, is the union of compact faces of $NP_\gamma f$.

Remark 1.1. Both the Newton boundary $NB_\gamma f$ and the order function $\operatorname{ord}_\gamma f : [1,\infty) \to \mathbb{R}$ depend only on f and on the demi-branch γ considered as a set germ at the origin. They are independent of the choice of local coordinate system, as long as f is mini-regular in x and γ is allowable. This follows from corollary 1.6.

As for $P_{f,\gamma,\xi}$, it depends on the choice of coordinate system, but only on its linear part, see corollary 1.7 and proposition 3.1 below.

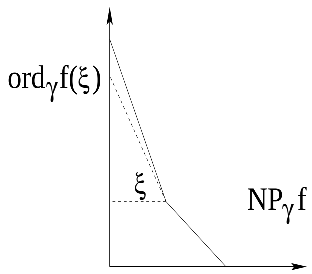
Proposition 1.2. *The Newton boundary* $NB_\gamma f$ *determines the order function* $\operatorname{ord}_\gamma f$ *and vice versa. More precisely, let* $\varphi : (0,m] \to [0,\infty]$ *be the piecewise linear function whose graph* $y = \varphi(x)$ *is* $NB_\gamma f$*, then we have*

$$\begin{aligned} \varphi(x) \quad &= \max_{\xi} \; (\operatorname{ord}_\gamma f(\xi) - \xi x) \quad (\textit{Legendre transform}), \\ \operatorname{ord}_\gamma f(\xi) \quad &= \min_{x} \; (\varphi(x) + \xi x) \quad (\textit{inverse Legendre transform}). \end{aligned}$$

Proof. Let $f(X + \lambda(Y), Y) = \sum_{i,j} c_{ij} X^i Y^j$. Then, see the picture below,

$$\operatorname{ord}_\gamma f(\xi) = \min_{i,j}\{j + i\xi;\ c_{i,j} \neq 0\} = \min_{x}\{\varphi(x) + x\xi\},$$

that shows the second formula.



The first formula follows from the second one. □

The following example illustrates the meaning of proposition 1.2.

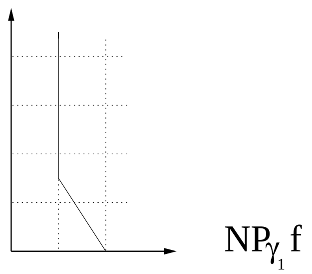
Example 1.3. Let $f(x, y) = x^2 - y^3$, and let $\gamma_1 : x = y^{\frac{3}{2}}$ and $\gamma_2 : x = y^{\frac{3}{2}} + y^{\frac{5}{2}}$. Then

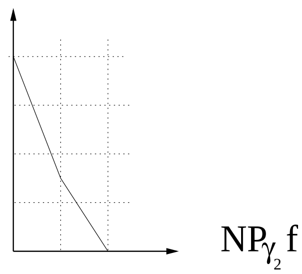
$$f(y^{\frac{3}{2}} + zy^{\xi}, y) = 2zy^{\frac{3}{2}+\xi} + z^2 y^{2\xi},$$
$$f(y^{\frac{3}{2}} + y^{\frac{5}{2}} + zy^{\xi}, y) = 2y^4 + y^5 + 2zy^{\frac{3}{2}+\xi} + 2zy^{\frac{5}{2}+\xi} + z^2 y^{2\xi}.$$

Therefore the order functions of γ_1 and γ_2 are given by

$$\mathrm{ord}_{\gamma_1} f(\xi) = \begin{cases} 2\xi & \text{for } 1 \le \xi \le \frac{3}{2} \\ \frac{3}{2} + \xi & \text{for } \xi \ge \frac{3}{2} \end{cases}$$

$$\mathrm{ord}_{\gamma_2} f(\xi) = \begin{cases} 2\xi & \text{for } 1 \le \xi \le \frac{3}{2} \\ \frac{3}{2} + \xi & \text{for } \frac{3}{2} \le \xi \le \frac{5}{2} \\ 4 & \text{for } \xi \ge \frac{5}{2}. \end{cases}$$





Next we compute the Newton boundaries: $f(X + Y^{\frac{3}{2}}, Y) = X^2 + 2XY^{\frac{3}{2}}$ and $f(X + Y^{\frac{3}{2}} + Y^{\frac{5}{2}}, Y) = X^2 + 2XY^{\frac{3}{2}} + 2XY^{\frac{5}{2}} + 2Y^4 + Y^5$. Therefore

$$\varphi_1(x) = \begin{cases} \infty & \text{for } 0 < x < 1 \\ 3 - \frac{3}{2}x & \text{for } 1 \le x \le 2 \end{cases}$$

$$\varphi_2(x) = \begin{cases} 4 - \frac{5}{2}x & \text{for } 0 < x < 1 \\ 3 - \frac{3}{2}x & \text{for } 1 \le x \le 2. \end{cases}$$

The Newton boundary and the order function give rise to some invariants of bi-Lipschitz and C^1 equivalences of two variable real analytic function germs. We shall introduce them below. For the C^1 equivalence we first consider the equivalence given by C^1-diffeomorphisms $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ with $D\sigma(0) = \mathrm{id}$. The general case will be treated in section 3.

For an allowable real analytic demi-branch $\gamma : x = \lambda(y)$ we define the *horn-neighbourhood of γ with exponent $\xi \geq 1$ and width $N > 0$* by

$$H_\xi(\gamma; N) := \{(x, y);\ |x - \lambda(y)| \leq N|y|^\xi, y > 0\}.$$

Proposition 1.4. *Let $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a bi-Lipschitz homeomorphism, and let f, $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be real analytic function germs such that $f = g \circ \sigma$. Suppose that γ, $\tilde\gamma$ are allowable real analytic demi-branches and that there exist $\xi_0 \geq 1$ and $N > 0$ such that*

$$\sigma(\gamma) \subset H_{\xi_0}(\tilde\gamma; N).$$

Then, for $1 \leq \xi \leq \xi_0$, $\mathrm{ord}_\gamma f(\xi) = \mathrm{ord}_{\tilde\gamma} g(\xi)$ and $\deg P_{f,\gamma,\xi} = \deg P_{g,\tilde\gamma,\xi}$.

Proof. If $\xi = 1$ then $\mathrm{ord}_\gamma f(\xi) = \deg P_{f,\gamma,\xi} = \mathrm{mult}_0 f$ and the claim follows from bi-Lipschitz invariance of multiplicity.

Suppose that $\xi_0 > 1$ and $1 < \xi \leq \xi_0$. Let $\sigma(x, y) = (\sigma_1(x, y), \sigma_2(x, y))$, $\gamma : x = \lambda(y)$, $\tilde\gamma : x = \tilde\lambda(y)$. Then, there exists $C > 0$ such that $\tilde y(y) := \sigma_2(\lambda(y), y)$ satisfies

$$\frac{1}{C} y \leq \tilde y(y) \leq Cy.$$

Lemma 1.5. *For any $1 < \xi \leq \xi_0$ and $M > 0$ there is $\tilde M_\xi$ such that*

$$\sigma(H_\xi(\gamma; M)) \subset H_\xi(\tilde\gamma; \tilde M_\xi).$$

Moreover, $\tilde M_\xi$ can be chosen of the form $\tilde M_\xi = AM$ if $\xi < \xi_0$ and $\tilde M_{\xi_0} = AM + N$ if $\xi = \xi_0$.

Proof. By Lipschitz property, for $(x, y) \in H_\xi(\gamma; M)$ near $0 \in \mathbb{R}^2$,

$$|\sigma_2(x, y) - \tilde y| = |\sigma_2(x, y) - \sigma_2(\lambda(y), y)| \leq LMy^\xi \leq LMC^\xi \tilde y^\xi = o(\tilde y),$$

and

$$\begin{aligned} |\sigma_1(x, y) - \tilde\lambda(\tilde y)| &\leq |\sigma_1(x, y) - \sigma_1(\lambda(y), y)| + |\sigma_1(\lambda(y), y) - \tilde\lambda(\tilde y)| \\ &\leq LMC^\xi \tilde y^\xi + N\tilde y^{\xi_0}. \end{aligned}$$

Finally, for an arbitrary $\varepsilon > 0$, there is a neighbourhood U_ϵ of $0 \in \mathbb{R}^2$ such that for $(x, y) \in H_\xi(\gamma; M) \cap U_\epsilon$,

$$\begin{aligned} |\sigma_1(x, y) - \tilde\lambda(\sigma_2(x, y))| &\leq |\sigma_1(x, y) - \tilde\lambda(\tilde y)| + |\tilde\lambda(\tilde y) - \tilde\lambda(\sigma_2(x, y))| \\ &\leq LMC^\xi \tilde y^\xi + N\tilde y^{\xi_0} + (\tilde\lambda'(0) + \varepsilon)LMC^\xi \tilde y^\xi. \end{aligned}$$

□

Since σ is bi-Lipschitz it can be shown by a similar argument that there exists N' for which $\sigma(H_\xi(\gamma; N')) \supset \tilde\gamma$, that is $\sigma^{-1}(\tilde\gamma) \subset H_{\xi_0}(\gamma; N')$. Thus the assumptions of Proposition 1.4 are symmetric with respect to f and g.

Let $x = \delta(y) = \lambda(y) + cy^\xi$, c arbitrary. On one hand, by lemma 1.5,

$$|g(\sigma(\delta(y), y))| \leq \max\{P_{g,\tilde\gamma,\xi}(z); |z| \leq A|c| + N + 1\}\, \tilde y^{\mathrm{ord}_{\tilde\gamma} g(\xi)}.$$

On the other hand

$$g(\sigma(\delta(y),y)) = f(\lambda(y)+cy^{\xi},y) = P_{f,\gamma,\xi}(c)y^{\operatorname{ord}_{\gamma} f(\xi)} + \cdots .$$

This implies $\operatorname{ord}_{\gamma} f(\xi) \geq \operatorname{ord}_{\tilde{\gamma}} g(\xi)$, then by symmetry $\operatorname{ord}_{\gamma} f(\xi) = \operatorname{ord}_{\tilde{\gamma}} g(\xi)$, and finally $\deg P_{f,\gamma,\xi} = \deg P_{g,\tilde{\gamma},\xi}$. □

Corollary 1.6. *Let $\sigma : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ be a bi-Lipschitz homeomorphism, and let $f,\ g : (\mathbb{R}^2,0) \to (\mathbb{R},0)$ be real analytic function germs such that $f = g\circ\sigma$. Suppose that γ, $\tilde{\gamma}$ are allowable real analytic demi-branches such that $\sigma(\gamma) = \tilde{\gamma}$ as set-germs at $(0,0)$. Then, for all $\xi \geq 1$, $\operatorname{ord}_{\gamma} f(\xi) = \operatorname{ord}_{\tilde{\gamma}} g(\xi)$ and $\deg P_{f,\gamma,\xi} = \deg P_{g,\tilde{\gamma},\xi}$. In particular, $\mathrm{NB}_{\gamma} f = \mathrm{NB}_{\tilde{\gamma}} g$.*

Proposition 1.7. *Let $\sigma : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ be a C^1-diffeomorphism with $D\sigma(0) = \mathrm{id}$, and let $f,\ g : (\mathbb{R}^2,0) \to (\mathbb{R},0)$ be real analytic function germs such that $f = g\circ\sigma$. Suppose that γ, $\tilde{\gamma}$ are allowable real analytic demi-branches such that $\sigma(\gamma) = \tilde{\gamma}$ as set-germs at $(0,0)$. Then for all $\xi \geq 1$*

$$P_{f,\gamma,\xi} = P_{g,\tilde{\gamma},\xi}.$$

Proof. It is more convenient to work in a wider category and assume that f and g are convergent fractional power series of the form

$$\sum_{(i,j)\in\mathbb{N}\times\frac{1}{q}\mathbb{N}} c_{ij}x^i y^j, \tag{1.3}$$

where $c_{ij} \in \mathbb{R}$, $q \in \mathbb{N}$. Such series give rise to function germs well-defined on $y \geq 0$. Let $\gamma : x = \lambda(y)$, $\tilde{\gamma} : x = \tilde{\lambda}(y)$. Then

$$f\circ H_1 = g\circ H_2 \ \circ (H_2^{-1}\circ\sigma\circ H_1),$$

where $H_1(x,y) = (x+\lambda(y),y)$ and $H_2(X,Y) = (X+\tilde{\lambda}(Y),Y)$. Then $\tilde{f} = f\circ H_1$, $\tilde{g} = g\circ H_2$ are fractional power series. The map $\tilde{\sigma} = H_2^{-1}\circ\sigma\circ H_1$ is C^1 and $D\tilde{\sigma}(0) = \mathrm{id}$. Thus by replacing f,g,σ by $\tilde{f},\tilde{g},\tilde{\sigma}$ we may suppose that $\lambda \equiv \tilde{\lambda} \equiv 0$, that is the image of the $y-$axis is the $y-$axis.

If σ preserves the $y-$axis then it is of the form

$$\sigma(x,y) = (x\varphi(x,y), y+\psi(x,y)),$$

with $\varphi(x,y),\psi(x,y)$ continuous and $\varphi(0,0)=1, \psi(0,0)=0$.

Let $g(x,y)$ be a fractional power series as in (1.3). The expansion (1.2) still holds for g and any allowable demi-branch. We use this property for the (positive) $y-$axis as a demi-branch that we denote below by $\underline{0}$ (since it is given by $\lambda \equiv 0$).

Lemma 1.8. *Let $g(x,y)$ be a fractional power series as in (1.3). Then for all $\alpha(y),\beta(y)$ such that $\alpha(y) = o(y), \beta(y) = o(y)$, $\xi \geq 1$, and $z\in\mathbb{R}$ bounded*

$$g((z+\alpha(y))y^{\xi}, y+\beta(y)) = P_{g,\underline{0},\xi}(z)y^{\operatorname{ord}_{\underline{0}} g(\xi)} + o(y^{\operatorname{ord}_{\underline{0}} g(\xi)}).$$

Proof. We have
$$g(zy^\xi, y) = P_{g,\underline{0},\xi}(z) y^{\operatorname{ord}_{\underline{0}} g(\xi)} + o(y^{\operatorname{ord}_{\underline{0}} g(\xi)}).$$
More precisely $g(zy^\xi, y) - P_{g,\underline{0},\xi}(z) y^{\operatorname{ord}_{\underline{0}} g(\xi)} \to 0$ as $y \to 0$ and z is bounded. Then
$$\begin{aligned} g((z+\alpha(y))y^\xi, y+\beta(y)) &= g((z+\alpha(y))(y+\beta(y)-\beta(y))^\xi, y+\beta(y)) \\ &= g((z+\tilde\alpha(y))(y+\beta(y))^\xi, y+\beta(y)) \\ &= P_{g,\underline{0},\xi}(z+\tilde\alpha(y))(y+\beta(y))^{\operatorname{ord}_{\underline{0}} g(\xi)} + o((y)^{\operatorname{ord}_{\underline{0}} g(\xi)}) \\ &= P_{g,\underline{0},\xi}(z) y^{\operatorname{ord}_{\underline{0}} g(\xi)} + o(y^{\operatorname{ord}_{\underline{0}} g(\xi)}), \end{aligned}$$
where $\tilde\alpha(y) = o(y)$. □

To complete the proof of proposition 1.7 we apply lemma 1.8 to g, $\alpha(y) = \varphi(cy^\xi, y) - 1$, and $\beta(y) = \psi(cy^\xi, y)$, where $c \in \mathbb{R}$ is a constant. Then
$$\begin{aligned} f(cy^\xi, y) &= g\circ\sigma(cy^\xi, y) = g(cy^\xi\varphi(cy^\xi, y), y+\psi(cy^\xi, y)) \\ &= P_{g,\underline{0},\xi}(c) y^{\operatorname{ord}_{\underline{0}} g(\xi)} + o(y^{\operatorname{ord}_{\underline{0}} g(\xi)}). \end{aligned}$$
Therefore, by expanding $f(cy^\xi, y) = P_{f,\underline{0},\xi}(c) y^{\operatorname{ord}_{\underline{0}} f(\xi)} + o(y^{\operatorname{ord}_{\underline{0}} f(\xi)})$, we obtain
$$P_{f,\underline{0},\xi}(c) y^{\operatorname{ord}_{\underline{0}} f(\xi)} + o(y^{\operatorname{ord}_{\underline{0}} f(\xi)}) = P_{g,\underline{0},\xi}(c) y^{\operatorname{ord}_{\underline{0}} g(\xi)} + o(y^{\operatorname{ord}_{\underline{0}} g(\xi)}),$$
that shows $P_{f,\underline{0},\xi} = P_{g,\underline{0},\xi}$. This ends the proof of proposition 1.7. □

Proposition 1.9. *Let $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a C^1-diffeomorphism with $D\sigma(0) =$ id, and let f, $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be real analytic function germs such that $f = g\circ\sigma$. Suppose that γ, $\tilde\gamma$ are allowable real analytic demi-branches and that there exist $\xi_0 \geq 1, N > 0$ such that*
$$\sigma(\gamma) \subset H_{\xi_0}(\tilde\gamma; N).$$
Then for all $1 \leq \xi < \xi_0$,
$$P_{f,\gamma,\xi} = P_{g,\tilde\gamma,\xi}.$$
Moreover, P_{f,γ,ξ_0} and $P_{g,\tilde\gamma,\xi_0}$ have the same degrees and their leading coefficients coincide.

Proof. Let $\xi_0 > 1$. Then the tangent directions at the origin to γ and $\tilde\gamma$ coincide. We assume as above that γ, and $\tilde\gamma$ resp., is the (positive) $y-$axis. Write
$$\sigma(x, y) = (\sigma_1(0, y) + x\varphi(x, y), y + \psi(x, y)),$$
with $\varphi(x, y), \psi(x, y)$ continuous and $\varphi(0, 0) = 1, \psi(0, 0) = 0$. The assumption on the image of γ gives
$$|\sigma_1(0, y)| \leq N_1|y|^{\xi_0},$$
with $N_1 = N + \varepsilon$. Then, for $\xi < \xi_0$, $\sigma_1(0, y) = o(y^\xi)$ and
$$\begin{aligned} f(cy^\xi, y) = g\circ\sigma(cy^\xi, y) &= g(cy^\xi\varphi(cy^\xi, y) + \sigma_1(0, y), y + \psi(cy^\xi, y)) \\ &= g((c+\alpha(y))y^\xi, y+\beta(y)) \end{aligned}$$
with $\alpha(y) = o(y), \beta(y) = o(y)$. Thus the first claim follows again from lemma 1.8.

If $\xi = \xi_0 > 1$ then the same computation shows that

$$P_{f,\underline{0},\xi}(c) \in \{P_{g,\underline{0},\xi}(z);\ |z-c| \le N_1\},$$

for all c. That shows that the degrees of $P_{f,\underline{0},\xi}$ and $P_{g,\underline{0},\xi}$ and their leading coefficients coincide.

If $\xi_0 = 1$ then $P_{f,\gamma,1}$ depends only on the initial homogeneous form of f, denoted by f_m in (1.1), and the tangent direction to γ at the origin. Then $m = \deg P_{f,\gamma,1}$ and the leading coefficient of $P_{f,\gamma,1}$ is independent of the choice of γ. But the initial homogeneous forms of f and g coincide by lemma 0.4. This completes the proof of proposition 1.9. □

Let $f(X+\lambda(Y),Y) = \sum_{i,j} c_{ij}X^iY^j$. Then by *the initial Newton polynomial of f relative to* γ, we mean

$$in_\gamma f = \sum_{(i,j)\in NB_\gamma(f)} c_{ij}X^iY^j. \tag{1.4}$$

Note that $in_\gamma f$ is a fractional polynomial, $(i,j) \in \mathbb{Z}\times\mathbb{Q}, i \ge 0, j \ge 0$.

Corollary 1.10. *Let $\sigma : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ be a C^1-homeomorphism with $D\sigma(0) = \mathrm{id}$, and let f, $g : (\mathbb{R}^2,0) \to (\mathbb{R},0)$ be real analytic function germs such that $f = g\circ\sigma$. Suppose that γ, $\tilde\gamma$ are allowable real analytic demi-branches such that $\sigma(\gamma) = \tilde\gamma$ as set-germs at $(0,0)$. Then $in_\gamma f = in_{\tilde\gamma} g$.*

Proof. Let Γ_ξ be a compact face of $NB_\gamma(f)$ of slope $-\xi$. Then

$$P_{f,\gamma,\xi}(z) = \sum_{(i,j)\in\Gamma_\xi} c_{ij}z^i.$$

□

1.1. **$\mathbf{C}^1$ invariance of Puiseux pairs of roots.** Let $\gamma : x = \lambda(y)$ be an allowable real analytic demi-branch. The Puiseux pairs of γ are pairs of relatively prime positive integers $(n_1,d_1),\dots,(n_q,d_q)$, $d_i > 1$ for $i = 1,\dots,q$, $\frac{n_1}{d_1} < \frac{n_2}{d_1d_2} < \cdots < \frac{n_q}{d_1\dots d_q}$, such that

$$\lambda(y) = \sum_\alpha a_\alpha y^\alpha = \tag{1.5}$$

$$= \sum_{j=1}^{[n_1/d_1]} a_j y^j + \sum_{j=n_1}^{[n_2/d_2]} a_{j/d_1} y^{j/d_1} + \sum_{j=n_2}^{[n_3/d_3]} a_{j/d_1d_2} y^{\frac{j}{d_1d_2}} + \cdots + \sum_{j=n_q}^{\infty} a_{j/d_1d_2\cdots d_q} y^{\frac{j}{d_1d_2\cdots d_q}}$$

and $a_{n_i/d_1\cdots d_i} \ne 0$ for $i = 1,\dots,q$, cf. e.g. [23]. The exponents $n_i/d_1\cdots d_i$ will be called *the (Puiseux) characteristic exponents of* γ. The corresponding coefficients $A_i(\gamma) := a_{n_i/d_1\cdots d_i}$ for $i = 1,\dots,q$ will be called *the characteristic coefficients of* γ.

Proposition 1.11. *Let $\sigma : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ be a C^1 diffeomorphism and let f, $g : (\mathbb{R}^2,0) \to (\mathbb{R},0)$ be real analytic function germs such that $f = g\circ\sigma$. Suppose that $\gamma \subset f^{-1}(0)$, $\tilde\gamma \subset g^{-1}(0)$ are allowable real analytic demi-branches such that $\sigma(\gamma) = \tilde\gamma$ as set germs. Then the Puiseux characteristic pairs of γ and $\tilde\gamma$ coincide.*

Moreover, if $D\sigma(0)$ preserves orientation then the signs of characteristic coefficients of γ and $\tilde\gamma$ coincide.

Proof. Let us write f and g as in Lemma 0.4, and let A be the linear transformation of $\mathbb{R}^2$ given in the proof. Define $\tilde g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ by $\tilde g(x) = g(A(x))$. Then $\tilde g$ has the form

$$\tilde g(x) = f_m(x) + \tilde g_{m+1}(x) + \tilde g_{m+2}(x) + \cdots$$

where $\tilde g_j(x) = g_j(A(x))$, $j = m+1,\ m+2, \cdots$. Then $\tilde g$ is linearly equivalent to g and C^1-equivalent to f. Therefore we may assume that $D\sigma(0) = Id$.

Let $(n_1, d_1), \ldots, (n_i, d_i)$ be the first i Puiseux pairs of γ and fix $\xi = n/d_1 \cdots d_i d$, $\gcd(n, d) = 1$, $d > 1$, such that $\xi > \frac{n_i}{d_1 \ldots d_i}$. We may suppose by the inductive assumption that $(n_1, d_1), \ldots, (n_i, d_i)$ are also the first i Puiseux pairs of $\tilde\gamma$.

Consider the truncation γ_ξ of γ at ξ: if $\gamma : x = \lambda(y) = \sum a_\alpha y^\alpha$ then

$$\gamma_\xi : x = \lambda_\xi(y) = \sum_{\alpha<\xi} a_\alpha y^\alpha.$$

Denote by $P(z) = P_{f,\gamma,\xi}$, $P_0(z) = P_{f,\gamma_\xi,\xi}$ the polynomials defined for γ and γ_ξ by the expansion (1.2). Then $P(z) = P_0(z + a_\xi)$. Denote ord $= \operatorname{ord}_{\gamma_\xi} f(\xi) = \operatorname{ord}_\gamma f(\xi)$.

Lemma 1.12. *There exist an integer $k \geq 0$ and a polynomial $\tilde P_0$ such that $P_0(z) = z^k \tilde P_0(z^d)$.*

Proof. Write

$$f(\lambda_\xi(y) + x, y) = \sum_{\xi i + j \geq \mathrm{ord}} c_{ij} x^i y^j = P_0(x/y^\xi) y^{\mathrm{ord}} + \sum_{\xi i + j > \mathrm{ord}} c_{ij} x^i y^j.$$

Since the Puiseux pairs of γ_ξ determine the possible denominators of the exponents of y in $f(\lambda_\xi(y) + x, y)$

$$c_{ij} \neq 0 \Rightarrow j(d_1 \cdots d_i) \in \mathbb{N}. \tag{1.6}$$

Let $i_0 = \deg P_\xi$, $j_0 = \mathrm{ord} - \xi i_0$. Then $P_0(z) = c_{i_0 j_0} z^{i_0} +$ lower degree terms.

We show that for (i, j) such that $\xi i + j = \mathrm{ord}$, the condition $j(d_1 \cdots d_i) \in \mathbb{N}$ of (1.6) is equivalent to $(i - i_o) \in d\mathbb{N}$. Indeed, since $\xi(i - i_0) + (j - j_0) = 0$ and $j_0(d_1 \cdots d_i) \in \mathbb{N}$.,

$$j(d_1 \cdots d_i) \in \mathbb{N} \Leftrightarrow \xi(i - i_0) d_1 \cdots d_i \in \mathbb{N} \Leftrightarrow (i - i_0) n/d \in \mathbb{N} \Leftrightarrow (i - i_0) \in d\mathbb{N}.$$

Thus $P_0 = a_{i_0 j_0} z^k \hat P_0(z^d)$ with $\hat P_0$ unitary. □

By lemma 1.12

$$P(z) = P_0(z + a_\xi) = (z + a_\xi)^k \tilde P_0((z + a_\xi)^d) = A_0 z^{i_0} + A_1 z^{i_0 - 1} + \cdots.$$

If $\gamma \subset f^{-1}(0)$ then $P(0) = 0$ and therefore P is not identically equal to a constant. If P is not a constant then we may compute $a_\xi = \frac{A_1}{i_0 A_0}$. Consequently, by proposition 1.7, ξ is a Puiseux characteristic exponent of γ iff it is a one of $\tilde\gamma$. Moreover, the characteristic coefficients are the same. (Arbitrary linear isomorphisms may change these coefficients but not their signs if the orientation is preserved.) □

2. C^1 EQUIVALENT GERMS ARE BLOW-ANALYTICALLY EQUIVALENT

In this section we show theorem 0.2. The proof is based on the characterisation (3) of theorem 0.1. First we recall briefly the construction of real tree model, for the details see [13].

In [16] Kuo and Lu introduced a tree model $T(f)$ of a complex analytic function germ $f(x,y)$. This model allows one to visualise the numerical data given by the contact orders between the Newton-Puiseux roots of f, in particular their Puiseux pairs, and determines the minimal resolution of f. The real tree model of [13] is an adaptation of the Kuo-Lu tree model to the real analytic world. The main differences are the following. The Newton-Puiseux roots of f

$$x = \lambda(y) = a_1 y^{n_1/N} + a_2 y^{n_2/N} + \cdots \tag{2.1}$$

are replaced by real analytic demi-branches or their horn neighbourhoods obtained by restricting (2.1) to $y \geq 0$, and then truncating it the first non-real coefficient a_i that is replaced by a symbol c signifying a generic $c \in \mathbb{R}$ (in this way we can still keep track of the exponent n_i/N). The later construction can be reinterpreted geometrically in the real world by taking "root horns". The Puiseux pairs of the roots, or of the root horns, are added to the numerical data of the tree. Unlike in the complex case they can not be computed from the contact orders. Finally, the signs of coefficients at the Puiseux characteristic exponents are marked on the tree.

2.1. **Real tree model of f relative to a tangent direction.** Let $f(x,y)$ be a real analytic function germ. Fix v a unit vector of $\mathbb{R}^2$. *The tree model* $\mathbb{R}T_v(f)$ *of* f *relative to* v is defined as follows. Fix any local system of coordinates x,y such that:
- $f(x,y)$ *is mini-regular in* x;
- v *is of the form* (v_1,v_2) *with* $v_2>0$.

Let $x=\lambda(y)$ be a Newton-Puiseux root of f of the form (2.1). If λ is not real and a_i is the first non-real coefficient we replace this root by

$$x = \ a_1 y^{n_1/N} + a_2 y^{n_2/N} + \cdots + c y^{n_i/N}, \quad c \in \mathbb{R} \quad \text{generic}, \tag{2.2}$$

where c is a symbol signifying a generic $c \in \mathbb{R}$. We call (2.2) *a truncated root*. Let Λ_v denote the set of real roots and truncated roots, restricted to $y \geq 0$, that are tangent to v at the origin.

Suppose Λ_v non-empty. We apply the Kuo-Lu construction to Λ_v. We define *the contact order of* λ_i and λ_j of Λ_v as

$$O(\lambda_i,\lambda_j) := \operatorname{ord}_0(\lambda_i - \lambda_j)(y).$$

Let $h \in \mathbb{Q}$. We say that λ_i, λ_j are *congruent modulo* h^+ if $O(\lambda_i,\lambda_j) > h$.

Draw a vertical line as the *main trunk* of the tree. Mark the number m_v of roots in Λ_v counted with multiplicites alongside the trunk. Let $h_1 := \min\{O(\lambda_i,\lambda_j) | 1 \leq i,j \leq m_v\}$. Then draw a bar, B_1, on top of the main trunk. Call $h(B_1) := h_1$ the *height* of B_1.

The roots of Λ_v with the original coefficient a_{h_1} real are divided into equivalence classes, called *bunches*, modulo h_1^+. We then represent each equivalence class by a vertical line segment drawn on top of B_1 in the order corresponding to the order of a_{h_1} coefficients. Each is called a *trunk*. If a trunk consists of s roots we say it has *multiplicity* s, and mark s alongside (if $s = 1$ it is usually not marked). The other roots of Λ_v, that is those with the symbol c as the coefficient at y^{h_1}, do not produce a trunk over B_1 and disappear at B_1.

Now, the same construction is repeated recursively on each trunk, getting more bars, then more trunks, etc.. The height of each bar and the multiplicity of a trunk, are defined likewise. Each trunk has a unique bar on top of it. The construction terminates at the stage where the bar have infinite height, that is is on top of a trunk that contain a single, maybe multiple, real root of f.

To each bar B corresponds a unique trunk supporting it and a unique bunch of roots $A(B)$ bounded by B. In this way there is a one-to-one correspondence between trunks, bars, and bunches. We denote by m_B the multiplicity of the trunk supporting B.

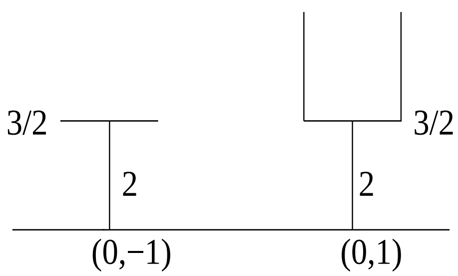
Whenever a bar B gives a new Puiseux pair to a root of $A(B)$ we mark 0 on B. If a trunk T' growing on B corresponds to the roots of A with coefficient $a_{h(B)} = 0$, resp. $a_{h(B)} < 0$, $a_{h(B)} > 0$, then we mark T' as growing at $0 \in B$, resp. to the left of 0, to the right of 0. Graphically, we mark $0 \in B$ by identifying it with the point of B that belongs to the trunk supporting B.

2.2. **Real tree model of f.** The *real tree model* $\mathbb{R}T(f)$ *of* f is defined as follow.

- Draw a bar B_0 that is identified with S^1. We define $h(B_0) = 1$ and call B_0 *the ground bar*.
- Grow on B_0 all non-trivial $\mathbb{R}T_v(f)$ for $v \in S^1$, keeping the clockwise order.
- Let v_1, v_2 be any two subsequent unit vectors for which $\mathbb{R}T_v(f)$ is nontrivial. Mark the sign of f in the sector between v_1 and v_2. Note that one such sign determines all the other signs between two subsequent unit vectors for which $\mathbb{R}T_v(f)$ is nontrivial (passing v changes this sign if and only if Λ_v contains an odd number of roots.)

If the leading homogeneous part f_m of f satisfies $f_m^{-1}(0) = 0$ then B_0 is the only bar of $\mathbb{R}T(f)$.

For instance we give below the real tree model of $f(x, y) = x^2 - y^3$. More examples are presented in section 4 below.



2.3. **Horns.** Recall that for an allowable real analytic demi-branch $\gamma : x = \lambda(y)$ the horn-neighbourhood of γ with exponent $\xi \geq 1$ and width $N > 0$ is given by

$$H_\xi(\gamma; N) := \{(x, y);\ |x - \lambda(y)| \leq N|y|^\xi, y > 0\}.$$

We define *the horn-neighbourhood of* γ *of exponent* ξ as $H_\xi(\gamma; C)$ for C large and we denote it by $H_\xi(\gamma)$. *A horn* is a horn-neighbourhood with exponent $\xi > 1$.

If $\gamma_1 : x = \lambda_1(y)$, $\gamma_2 : x = \lambda_2(y)$, and $O(\lambda_1, \lambda_2) \geq \xi$ then we identify $H_\xi(\gamma_1) = H_\xi(\gamma_2)$ by meaning that for any $C_1 > 0$ there is $C_2 > 0$ such that

$$H_\xi(\gamma_1; C_1) \subset H_\xi(\gamma_2; C_2), \qquad H_\xi(\gamma_2; C_1) \subset H_\xi(\gamma_1; C_2). \tag{2.3}$$

Example 2.1. Let B be a bar of $\mathbb{R}T_v(f)$, $h(B) > 1$. Then B defines a horn

$$H_B := \{(x, y);\ |x - \lambda(y)| \leq C|y|^{h(B)}\},$$

where C is a large constant and $x = \lambda(y)$ is any root of bunch $A(B)$.

Definition 2.2. A horn that equals H_B for a bar B is called *a root horn.*

Let $H = H_\xi(\gamma), \gamma : x = \lambda(y)$, be a horn of exponent ξ. Let $\lambda_H(y)$ denote the truncation of λ at ξ, that is $\lambda_H(y)$ is the sum of all terms of $\lambda(y)$ of exponent $< \xi$. We define the truncated demi-branch by $\gamma_H : x = \lambda_H(y)$ and the generic demi-branch $\gamma_{H,gen} : x = \lambda_{H,gen}(y)$ by

$$\lambda_{H,gen}(y) = \lambda_H(y) + cy^\xi + \cdots, \qquad y \geq 0, \tag{2.4}$$

where $c \in \mathbb{R}$ is a generic constant. The *characteristic exponents of* H are those of $\gamma_{H,gen}$ that are $\leq \xi$. *The signs of characteristic coefficients of* H are those of $\gamma_{H,gen}$ (or of γ_H) corresponding to the exponents $< \xi$. Let $\gamma' : x = \lambda'(y)$ be any allowable real analytic demi-branch contained in H. Then the order function $\operatorname{ord}_{\gamma'} f$, defined by (1.2), restricted to $[1, \xi]$ is independent of the choice of γ' and so is the polynomial $P_{f,\gamma',\xi'}(z)$ for $\xi' < \xi$. The polynomial $P_{f,\gamma',\xi}(z)$ is independent up to a shift of variable z: if the coefficient of $\lambda'(y)$ at y^ξ is a then

$$P_{f,\gamma',\xi}(z) = P_{f,\gamma_H,\xi}(z + a).$$

Proposition 2.3. [compare [13], Proposition 7.5]

Let H be a horn of exponent ξ. Then H is a root horn for $f(x, y)$ if and only if $P_{f,\gamma_H,\xi}(z)$ has at least two distinct complex roots.

If this is the case, $H = H_B$, then $h(B) = \xi$ and $m_B = \deg P_{f,\gamma_H,\xi_H}$.

Proof. Suppose that $H = H_B$ and let $A(B) = \{\gamma_1, \ldots, \gamma_{m_B}\}$ be the corresponding bunch of roots. These roots are truncations of complex Newton-Puiseux roots of f:

$$\gamma_{\mathbb{C},k} : x = \lambda_k(y) = \lambda_H + a_{\xi,k} y^\xi + \cdots, \quad 1 \leq k \leq m_B \tag{2.5}$$

with λ_H real and $a_{\xi,k} \in \mathbb{C}$. Denote by $\gamma_{\mathbb{C},j} : x = \lambda_j(y)$, $j = m_B + 1, \ldots, m$, the remaining complex Newton-Puiseux roots of f. Then

$$f(\lambda_H(y) + zy^\xi, y) = u(x, y) \prod_{i=1}^{m} (\lambda_H(y) - \lambda_i(y) + zy^\xi) = P_{f,\gamma_H,\xi}(z) y^{\operatorname{ord}_{\gamma_H} f(\xi)} + \cdots,$$

where $u(0,0) \neq 0$. Note that $O(\lambda_H, \lambda_j) < \xi$ for $j > m_B$. Therefore

$$\begin{aligned} P_{f,\gamma_H,\xi_H}(z) &= u(0,0) \prod_{i=1}^{m(B)} (z - a_{\xi,i}), \\ \operatorname{ord}_{\gamma_H} f(\xi_H) &= m_B \xi_H + \sum_{j=m(B)+1}^{m} O(\lambda_H, \lambda_j). \end{aligned}$$

By construction of the tree there are at least two roots γ_i and γ_j of (2.5) such that $O(\lambda_i, \lambda_j) = \xi$. Thus $P_{f,\gamma_H,\xi_H}(z)$ has at least two distinct complex roots.

Let $H = H_\xi(\gamma)$ be a horn, where

$$\gamma : x = \lambda_H(y) + a_\xi y^\xi + \cdots .$$

By the Newton algorithm for computing the complex Newton-Puiseux roots of f to each root z_0 of $P_{f,\gamma,\xi}$ of multiplicity s correspond exactly s Newton-Puiseux roots of f, counted with multiplicities, of the form

$$\gamma_0 : x = \lambda_H(y) + (a_\xi + z_0) y^\xi + \cdots .$$

(This is essentially the way the Newton-Puiseux theorem is proved as in [22].) Thus, if $P_{f,\gamma,\xi}$ has at least two distinct roots, then there exist at least two such Newton-Puiseux roots with contact order equal to ξ. This shows that H is of the form H_B, as claimed. $\square$

Proposition 2.3 shows that for a root horn H of width ξ_0, $\deg P_{f,\gamma_H,\xi} > 1$. Moreover, for any (Puiseux) characteristic exponent of γ_H, $\xi < \xi_0$, the horn $H_\xi(\gamma_H)$ is a root horn. Indeed, if $\xi = n_i/d_1 \cdots d_i$, then $\deg P_{f,\gamma_H,\xi} = d_i$ Therefore we may extend the argument of the proof of Proposition 1.11 to the root horn case.

Proposition 2.4. *Let $H = H_\xi(\gamma)$ be a horn root. Let $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a C^1-diffeomorphism and let f, $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be real analytic function germs such that $f = g \circ \sigma$. Suppose that γ, $\tilde\gamma$ are allowable real analytic demi-branches such that*

$$\sigma(\gamma) \subset H_\xi(\tilde\gamma; N).$$

Then the Puiseux characteristic exponents H and $\tilde H$ coincide.

Moreover, if $D\sigma(0)$ preserves orientation then the signs of characteristic coefficients of H and $\tilde H$ coincide.

2.4. **Characterisation of real tree model in terms of root horns.** The real tree model $\mathbb{R}T(f)$ is determined by the root horns and their numerical invariants, cf. [13] subsection 7.3. The root horns are ordered by inclusion and by clockwise order around the origin. Thus H_B is contained in $H_{B'}$ if and only if the bar B grows over B'. The multiplicity m_B and the height $h(B)$ are expressed in terms of invariants of the horn H_B by the formulae of proposition 2.3.

Let $\gamma : x = \lambda(y)$ be a root of $A = A(B)$. Then the Puiseux characteristic exponents of γ that are $< h(B)$ and the corresponding signs of characteristic coefficients are those of $\gamma_{H_B,gen}$ (or, equivalently, of γ_{H_B}). If $\tilde A = A(\tilde B)$ be a sub-bunch of A

containing γ then the invariants of $H_{\tilde{B}}$ determine whether γ takes a new Puiseux pair at $h(B)$ and, if this is the case, the sign of the characteristic coefficient at $h(B)$.

2.5. **End of proof of theorem 0.2.** By propositions 1.4 and 2.3 the image of a root horn H_B is a root horn $H_{\tilde{B}}$. Thus obtained one-to-one correspondence $B \leftrightarrow \tilde{B}$ gives an isomorphism of trees preserving the multiplicities and the heights of bars. The Puiseux characteristic exponents and the corresponding signs of Puiseux coefficients are also preserved as follows from 2.4. If σ preserves the orientation then it preserves the clockwise order of root horns and hence the clockwise order on the trees. Thus the theorem follows from theorem 0.1. □

3. Arbitrary C^1 equivalence.

If f and g are C^1-equivalent by a C^1 diffeomorphism σ, $f = g \circ \sigma$, then usualy we compose f or g with a linear isomorphism and assume that $D\sigma(0) = Id$. Nevertheless, sometimes, it is necessary to construct invariants of the arbitrary C^1-equivalence. This is the case when f and g are weighted homogeneous, a property that is usually destroyed by an arbitrary linear change of coordinates. In this section we construct invariants of the arbitrary C^1-equivalence and apply them to weighted homogeneous polynomials.

Proposition 3.1. *Let $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a C^1-diffeomorphism such that $D\sigma(0)(x, y) = (ax + by, cx + dy)$ and let $f(x, y)$, $g(x, y)$ be real analytic function germs, mini-regular in x, such that $f = g \circ \sigma$. Suppose that γ, $\tilde{\gamma}$ are allowable real analytic demi-branches and that there exist $\xi_0 > 1, N > 0$ such that*

$$\sigma(\gamma) \subset H_{\xi_0}(\tilde{\gamma}; N).$$

Then, for $\xi \in (1, \xi_0)$, $P_{f,\gamma,\xi}$ and $P_{g,\tilde{\gamma},\xi}$ are related by

$$P_{f,\gamma,\xi}(z) = (c\lambda'(0) + d)^{\operatorname{ord}_\gamma f(\xi)} P_{g,\tilde{\gamma},\xi}\Big(\frac{ad - bc}{(c\lambda'(0) + d)^{\xi+1}} z\Big). \tag{3.1}$$

If $\xi = 1$ then

$$P_{f,\gamma,1}(z) = (c\lambda'(0) + d + cz)^m P_{g,\tilde{\gamma},1}\Big(\frac{ad - bc}{c\lambda'(0) + d} \cdot \frac{z}{c\lambda'(0) + d + cz}\Big).$$

Example 3.2. Consider the family

$$A_t(x, y) = x^3 - 3txy^4 + 2y^6.$$

This family is equivalent to the family J_{10} of [1]. For each t, A_t is mini-regular in x, and $\frac{\partial A_t}{\partial x} = 3(x^2 - ty^4)$. For $t > 0$ let us consider the Newton polygon of A_t relative to a polar curve $\gamma_t : x = \sqrt{t}\, y^2$. Then we have

$$A_t(X + \sqrt{t}\, Y^2, Y) = X^3 + 3\sqrt{t}\, X^2 Y^2 + 2(1 - t\sqrt{t}\,)Y^6,$$

and

$$P_{\Gamma_t}(z) = z^3 + 3\sqrt{t}\, z^2 + 2(1 - t\sqrt{t}\,).$$

Suppose that for t, $t' \in (0,\infty)$, there are α, $\beta \neq 0$ such that $P_{\Gamma_{t'}}(z) = \beta^6 P_{\Gamma_t}(\frac{\alpha}{\beta^3} z)$. By an easy computation, we obtain $\alpha^2 = \beta^2 = 1$ and that $P_{\Gamma_t} \equiv P_{\Gamma_{t'}}$ up to a multiplication if and only if $t = t'$ in this case.

Proof. By Proposition 1.9 it suffices to consider only the case of σ linear

$$(\tilde{x},\tilde{y}) = \sigma(x,y) = (ax+by, cx+dy), \quad \det\sigma = ad-bc \neq 0.$$

and $\tilde{\gamma} = \sigma(\gamma)$. Let $\gamma : x = \lambda(y)$, $\tilde{\gamma} : x = \tilde{\lambda}(y)$. Then

$$\lambda(y)a + by = \tilde{\lambda}(c\lambda(y) + dy), \tag{3.2}$$

$c\lambda(y) + dy$ parametrises the positive y-axis, and

$$\tilde{\lambda}'(0) = \frac{a\lambda'(0)+b}{c\lambda'(0)+d}, \qquad c\lambda'(0)+d > 0.$$

Fix $\xi > 1$. Clearly $\operatorname{ord}_\gamma f(\xi) = \operatorname{ord}_{\tilde{\gamma}} g(\xi)$. Put $Y = c(\lambda(y) + zy^\xi) + dy$. Then $Y = (c\lambda'(0)+d)y + o(y)$ and $y = (c\lambda'(0)+d)^{-1}Y + o(Y)$, and consequently

$$\begin{aligned} f(\lambda(y)+zy^\xi, y) &= g(a(\lambda(y)+zy^\xi)+by, c(\lambda(y)+zy^\xi)+dy) \\ &= g(\tilde{\lambda}(c\lambda(y)+dy) + azy^\xi, Y) \\ &= g(\tilde{\lambda}(Y) - \tilde{\lambda}'(0)czy^\xi + azy^\xi + o(y^\xi), Y) \\ &= g(\tilde{\lambda}(Y) + \frac{ad-bc}{(c\lambda'(0)+d)^{\xi+1}} zY^\xi + o(Y^\xi), Y). \end{aligned}$$

Hence, comparing this formula with (1.2), we get

$$P_{f,\gamma,\xi}(z) y^{\operatorname{ord}_\gamma f(\xi)} = P_{g,\tilde{\gamma},\xi}(\frac{ad-bc}{(c\lambda'(0)+d)^{\xi+1}} z) Y^{\operatorname{ord}_{\tilde{\gamma}} g(\xi)},$$

that gives (3.1). The case $\xi = 1$ is left to the reader. □

Corollary 3.3. *Given an analytic function germ $f : (\mathbb{R}^2,0) \to (\mathbb{R},0)$ and a real analytic demi-branch γ. We say (x,y) is an* admissible system of local analytic coordinates for f and γ *if γ is allowable and $f(x,y)$ is mini-regular in x. Then $NB_\gamma f$ is independent of the choice of admissible coordinate systems. Moreover, for each edge $\Gamma \subset NB_\gamma(f)$ with slope smaller than -1, the polynomial $P_\Gamma(z) = \sum_{(i,j)\in\Gamma} c_{ij} z^i$ is well-defined up to left and right multiplications as in* (3.1).

3.1. **C^1−equivalent weighted homogeneous functions.** Using Propositions 3.1 and 1.4 we give below complete bi-Lipschitz and C^1 classifications of weighted homogeneous two variable function germs.

Let $f(x,y)$ be a weighted homogeneous polynomial with weights q,p, $1 \leq p \leq q$, $(p,q) = 1$, and weighted degree d. We may write

$$f(x,y) = y^l(x^{d'/q} + \sum_{qi+pj=d'} a_{ij}x^iy^j) = y^{d/p}P(x/y^\xi), \tag{3.3}$$

where $d' = d - pl$, and $\xi = q/p$. $P(z) := f(z,1)$ is the associated one variable polynomial. We distinguish the following three cases:

(A) homogeneous : $p = q = 1$;
(B) $1 = p < q$;

(C) $1 < p < q$.

In each of these cases we call the following polynomials *monomial-like*:

(Am) $A(ax+by)^k(cx+dy)^l$, $ad-bc \neq 0$;
(Bm) $A(x+by^q)^k y^l$;
(Cm) $Ax^k y^l$.

Proposition 3.4. *Let $f(x,y)$ and $g(x,y)$ be weighted homogeneous polynomials and*

(1) suppose that f and g are bi-lipschitz equivalent. Then
  (a) If f is monomial-like then so is g. Then f and g are analytically equivalent.
  (b) if f is not monomial-like then f and g have the same weights and weighted degree.

(2) suppose that f and g are C^1 equivalent and not monomial-like. Fix the weights q, p, $1 \le p \le q$, $(p,q)=1$. Then
  (a) In case (A), f and g are linearly equivalent.
  (b) In case (B), there exist $c_1 \neq 0, c_2 \neq 0$, and b such that

$$f(x,y) = g(c_1 x - by^q, c_2 y).$$

  (c) In case (C), there exist $c_1 \neq 0, c_2 \neq 0$ such that

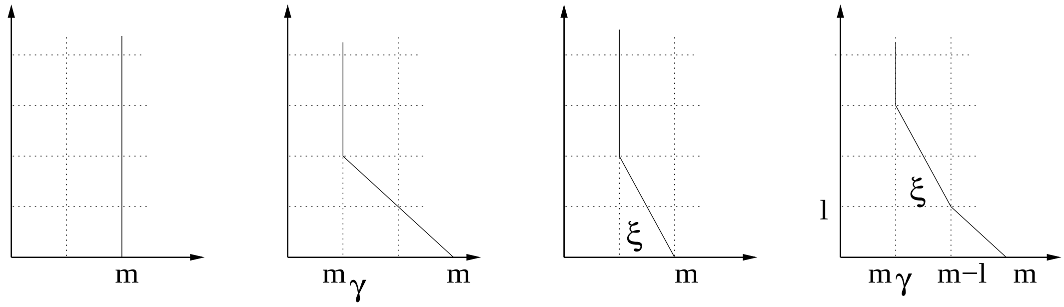
$$f(x,y) = g(c_1 x, c_2 y).$$

Proof. Let f be weighted homogeneous and let γ be a demi-branch of a root of f. In this proof we shall call such γ simply *a root of f* for short. First we list all possiblities for the Newton boundary $NB_\gamma f$ in an admissible system of coordinates, cf. corollary 3.3. Note that in such a system of coordinates f may not be weighted homogeneous. We denote $m = \operatorname{mult}_0 f$ and by m_γ the multiplicity of the root.

(i) If f is monomial-like with $k=0$ or $l=0$ then $m = m_\gamma$ and $NB_\gamma f$ has only one vertex at $(m,0)$.

(ii) If f is monomial-like with $k \neq 0$ and $l \neq 0$, or homogeneous and not monomial-like, then $NB_\gamma f$ has two vertices at $(m,0)$, $(m_\gamma, m-m_\gamma)$, and hence one nontrivial compact edge of slope -1. This is also the Newton boundary for a not monomial-like non-homogeneous f of the form (3.3) and the root $y=0$.

(iii) If f is not homogeneous and not monomial-like, of the form (3.3) and γ is not in $y=0$, then we have two possiblities:
  (a) If $l=0$ then $NB_\gamma f$ has one nontrivial edge of slope $-\xi$ and vertices $(m,0)$, $(m_\gamma, \xi(m-m_\gamma))$.
  (b) If $l \neq 0$ then $NB_\gamma f$ has two nontrivial edges: Γ_1 of slope -1 and vertices $(m,0)$, $(m-l,l)$, and Γ_2 of slope $-\xi$ and vertices $(m-l,l)$, $(m_\gamma, \xi(m-l-m_\gamma))$.



Let $f = g \circ \sigma$, σ bi-Lipschitz. Then $m = \operatorname{mult}_0 f = \operatorname{mult}_0 g$ and $m_\gamma = m_{\tilde\gamma}$ if $\tilde\gamma = \sigma(\gamma)$ for a root γ of f. Moreover, σ preserves the tangency of roots. Therefore f is monomial if and only if it satisfies the following, bi-Lipschitz invariant, property: f has $s = 2$ or 4 roots (demi-branches), mutually not tangent, with the sum of multiplicities equal to $2 \operatorname{mult}_0 f$. This shows (1a).

If f has a root γ such that $NB_\gamma f$ contains an edge of slope $-\xi < -1$ then so does g, and f and g have the same weights. Since the weighted degree can be also read from $NB_\gamma f$, they have the same weighted degree as well. Thus to finish the proof of (1) it suffices to consider the following case.

Special Case. Suppose that f and g are not monomial-like, and that for every root γ and $\tilde\gamma$ of f and g respectively, $NB_\gamma f$ and $NB_{\tilde\gamma} g$ are of the form (ii). (This includes the case where both f and g have isolated zero at the origin.) In this case we shall replace the roots by horn neighbourhoods of polar curves.

Suppose that the weights q, p of f satisfy $\xi = q/p > 1$. Write f as in (3.3). Denote by f_m and g_m the leading homogeneous part of f and g respectively. The real analytic demi-branches δ tangent to a root of f_m are distinguished by the size of f on them, $f(x,y) = o(\|(x,y)\|^m)$ for $(x,y) \in \delta$. The positive (or similarly negative) y-axis is in the zero set of f_m and is not tangent to any root of f. Hence its image by σ is in a horn neighbourhood of a root of g_m that is not tangent to any root of g. Hence $g \neq g_m$, that is g is not homogeneous.

By assumption, $P(z) = f(z,1)$ has no real root, and therefore P' must have one. If $P'(a) = 0$ then the curve $\gamma_a : x = ay^\xi, y \ge 0$, is a polar root of f that is

$$\frac{\partial f}{\partial x}(ay^\xi, y) \equiv 0.$$

Consider the germ at the origin of

$$U_\varepsilon(f) = \{(x,y) \in \mathbb{R}^2;\ r\varepsilon \|\operatorname{grad} f(x,y)\| \le |f(x,y)|\}, \tag{3.4}$$

where $r = \|(x,y)\|$ and $\varepsilon > 0$. If ε is sufficiently small then each polar root γ_a is in $U_\varepsilon(f)$. Indeed, then

$$\|\operatorname{grad} f(ay^\xi, y)\| = |\frac{\partial f}{\partial y}(ay^\xi, y)| = (d/p)|P(a)y^{d-1} + \cdots| \simeq r^{-1}(d/p)|f(ay^\xi, y)|.$$

In general, if a real analytic demi-branch

$$\delta : x = \lambda(y) = a_\xi y^\xi + \sum_{i>N\xi} a_{i/N} y^{i/N}, \quad y \geq 0,$$

is contained in $U_\varepsilon(f)$, then $P'(a_\xi) = 0$ and $a_{i/N} = 0$ for $\xi < i/N < 2\xi - 1$. Hence δ is contained in a horn neighbourhood $H_\mu(\gamma_{a_\xi}, M)$, with $\mu > \xi$. Consequently any local (at the origin) connected component U' of $U_\varepsilon(f) \setminus (0,0)$ satisfies one of the following properties:

- U' is contained in a horn neighbourhood of a polar root $H_\mu(\gamma_a, M)$. Then $f(x,y) \sim r^{d/p}$ on U'. ($d/p > d/q = m$)
- $l > 0$, c.f. (3.3), and U' contains a real analytic demi-branch tangent to $y = 0$ that is a root of f.
- Otherwise $f(x,y) \sim r^m$ on any real analytic demi-branch in U'.

By [9] and [10], $\sigma(U_\varepsilon(f)) \subset U_{\varepsilon'}(g)$ and so the image of a local connected component of the first type has to be contained in a horn neighbourhood of a polar curve of g. Thus the special case follows from the following observation. For any real analytic demi-branch δ in a horn neighbourhood $H_\mu(\gamma_a, M)$ of a polar curve γ_a of f, with $\mu > \xi$, the Newton boundary $NB_\delta f$ is independent of δ (we use $P(a) \neq 0$) and is of the form (iii). This ends the proof of Special Case and completes the proof of (1).

Now we show (2) of the proposition. (a) follows from lemma 0.4. Suppose f is in the form (3.3) with $\xi = q/p > 1$. We assume that P has a root. The proof in Special case is similar, one uses the polar roots instead of the roots. Let $P(a) = 0$. Then $\gamma : x = ay^\xi, y \geq 0$, is a root of f and $\tilde\gamma = \sigma(\gamma)$ is a root of g. Replacing $g(x,y)$ by $g(-x,-y)$, if necessary, we may suppose that $\tilde\gamma : x = \tilde a y^\xi, y \geq 0$.

Let $\tilde P(z) = g(z,1)$. Then $\tilde P(\tilde a) = 0$. Since σ is C^1, by Proposition 3.1, $P_{f,\gamma,\xi}$ and $P_{g,\tilde\gamma,\xi}$ coincide up to the left and right multiplications. Multiplying x by a positive constant, if necessary, we may suppose that

$$P(z-a) = \tilde P(\alpha(z - \tilde a)). \tag{3.5}$$

For $p = 1$ this gives $f(x,y) = g(c_1 x - by^\xi, c_2 y)$ (taking into account of the changes we have made already) and ends the proof of (2b).

If $p > 1$ then

$$P(z) = z^l Q(z^p), \tilde P(z) = z^l \tilde Q(z^p).$$

and therefore the arithmetic mean of complex roots of P, and the one of the roots of $\tilde P$, equals 0. By (3.5), if z is a complex root of P then $\alpha(z + a - \tilde a)$ is a root of $\tilde P$. Thus by comparing both arithmetic means we get $a = \tilde a$. Consequently, $P(z-a) = \tilde P(\alpha(z-a))$ or, by replacing $z - a$ by z, $P(z) = \tilde P(\alpha z)$, and hence we may conclude finally that

$$f(x,y) = g(c_1 x, c_2 y).$$

This ends the proof of proposition 3.4. □

4. Bi-Lipschitz equivalence does not imply blow-analytic equivalence

In this section we present several examples of bi-Lipschitz equivalent real analytic function germs that are not blow-analytically equivalent. In order to distinguish different blow-analytic types we use either the real tree model of [13] or the Fukui invariants. Recall the definition of Fukui invariants of blow-analytic equivalence, c.f. [5]. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function germ. Set

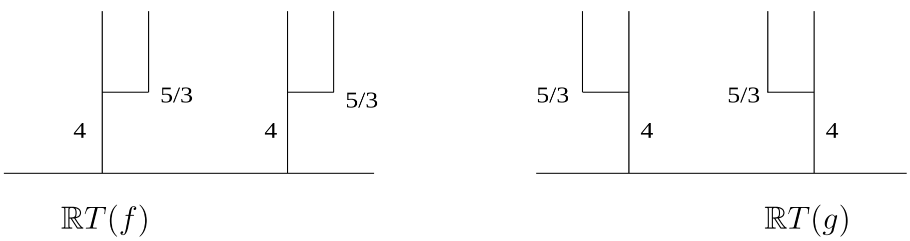
$$A(f) := \{\mathrm{ord}\,(f(\gamma(t))) \in \mathbb{N} \cup \{\infty\}; \gamma : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)\ C^\omega\}.$$

Let $\lambda : U \to \mathbb{R}^n$ be an analytic arc with $\lambda(0) = 0$, where U denotes a neighbourhood of $0 \in \mathbb{R}$. We call λ *nonnegative* (resp. *nonpositive*) *for* f if $(f \circ \lambda)(t) \geq 0$ (resp. ≤ 0) in a positive half neighbourhood $[0, \delta) \subset U$. Then we set

$$\begin{aligned} A_+(f) &:= \{\mathrm{ord}\,(f \circ \lambda); \lambda \text{ is a nonnegative arc through } 0 \text{ for } f\}, \\ A_-(f) &:= \{\mathrm{ord}\,(f \circ \lambda); \lambda \text{ is a nonpositive arc through } 0 \text{ for } f\}. \end{aligned}$$

Fukui proved that $A(f)$, $A_+(f)$ and $A_-(f)$ are blow-analytic invariants. Namely, if analytic functions $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are blow-analytically equivalent, then $A(f) = A(g)$, $A_+(f) = A_+(g)$ and $A_-(f) = A_-(g)$. We call $A(f)$, $A_\pm(f)$ *the Fukui invariant*, *the Fukui invariants with sign*, respectively. Apart from the Fukui invariants, motivic type invariants, *zeta functions*, are also known c.f. [12], [3].

4.1. **Example.** $f(x, y) = x(x^3 - y^5), g(x, y) = x(x^3 + y^5)$.
By [13] f and g are not blow-analytically equivalent by an orientation preserving blow-analytic homeomorphism.



We construct below an orientation preserving bi-Lipschitz homeomorphism $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $f = g \circ \sigma$. The construction uses the fact that f and g are weighted homogeneous with weights 5 and 3. Write

$$f(x, y) = x(x^3 - y^5) = y^{20/3} P(\frac{x}{y^{5/3}}), \qquad P(z) = z^4 - z,$$
$$g(x, y) = x(x^3 + y^5) = y^{20/3} Q(\frac{x}{y^{5/3}}), \qquad Q(z) = z^4 + z.$$

Proposition 4.1. *There exists a unique increasing real analytic diffeomorphism* $\varphi : \mathbb{R} \to \mathbb{R}$ *satisfying* $P = Q \circ \varphi$*. Moreover, for this* φ*,* φ' *and* $\varphi - z\varphi'$ *are globally bounded and* $\varphi(z)/z \to 1$ *as* $z \to \infty$*.*

Proof. P and Q have unique critical points: $z_0 = \sqrt[3]{\frac{1}{4}}, P'(z_0) = 0, \tilde{z}_0 = -z_0, Q'(\tilde{z}_0) = 0$. Therefore $\varphi : (-\infty, z_0] \to (-\infty, \tilde{z}_0]$, defined as $Q^{-1} \circ P$, is continuous and analytic on $(-\infty, z_0)$. Similarly for $\varphi : [z_0, \infty) \to [\tilde{z}_0, \infty)$. Thus $\varphi : \mathbb{R} \to \mathbb{R}$ is well-defined and continuous. In a neighbourhood of z_0, that is a non-degenerate critical point, P is analytically equivalent to $-z^2 + P(z_0)$. Similarly Q near $\tilde{z}_0$ is analytically equivalent to $-z^2 + Q(\tilde{z}_0)$. Finally, since $P(z_0) = Q(\tilde{z}_0)$, P near z_0 is analytically equivalent to Q near $\tilde{z}_0$.

Let $w = \frac{1}{z}$. Consider real analytic function germs

$$p(w) := (P(w^{-1}))^{-1} : (\mathbb{R}, 0) \to (\mathbb{R}, 0), \quad p(w) = w^4 + \cdots,$$
$$q(w) := (Q(w^{-1}))^{-1} : (\mathbb{R}, 0) \to (\mathbb{R}, 0), \quad q(w) = w^4 + \cdots.$$

Then $p = q \circ \psi$ with $\psi(w) = w + \cdots$. Since $\varphi(z) = (\psi(z^{-1}))^{-1}$, the last claim of proposition can be verified easily. □

Corollary 4.2. $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, *defined by*

$$\sigma(x, y) = \begin{cases} (y^{5/3}\varphi(\frac{x}{y^{5/3}}), y) & \text{if } y \neq 0, \\ (x, 0) & \text{if } y = 0, \end{cases}$$

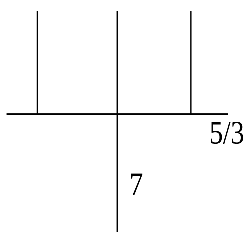
is bi-Lipschitz and $f = g \circ \sigma$.

Proof. We only check that σ is Lipschitz. This follows from the fact that the partial derivatives of σ are bounded

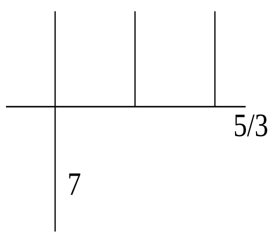
$$\partial\sigma/\partial x = (\varphi'(z), 0), \quad \partial\sigma/\partial y = (5/3y^{2/3}(\varphi(z) - z\varphi'(z)), 1).$$

where $z = \frac{x}{y^{5/3}}$. □

4.2. **Example.** $f(x, y) = x(x^3 - y^5)(x^3 + y^5), g(x, y) = x(x^3 - ay^5)(x^3 - by^5)$, where $0 < a < b$ are constants. The real trees of f and g are not equivalent, see below, hence by [13], f and g are not blow-analytically equivalent.



$\mathbb{R}T_{(0,1)}(f)$



$\mathbb{R}T_{(0,1)}(g)$

Note that the Fukui invariants and the zeta functions of f and g coincide (cf. Example 1.4 in [13]). We show below that for a choice of a and b, f and g are bi-Lipschitz equivalent. Write

$$f(x, y) = y^{35/3}P(\frac{x}{y^{5/3}}), \qquad P(z) = z(z^3 - 1)(z^3 + 1),$$
$$g(x, y) = y^{35/3}Q(\frac{x}{y^{5/3}}), \qquad Q(z) = z(z^3 - a)(z^3 - b).$$

The polynomial P has two non-degenerate critical points $-1 < z_1 < 0$, $z_2 = -z_1$ and $P(z_1) > 0, P(z_2) = -P(z_1) < 0$. The polynomial Q has also two non-degenerate critical points $0 < \tilde z_1 < \sqrt[3]{a} < \tilde z_2 < \sqrt[3]{b}$ and $Q(\tilde z_1) > 0, Q(\tilde z_2) < 0$. Indeed, the discriminant of $Q'(z) = 7z^6 - 4(a+b)z^3 + ab$ with respect to z^3 equals $\Delta = 4(4a^2 + 4b^2 + ab) > 0$. This also shows that these critical points $\tilde z_1(a,b), \tilde z_2(a,b)$ depend smoothly on a, b.

Lemma 4.3. *There exist* a, b, $0 < a < b$, *such that* $Q(\tilde z_1(a,b)) = P(z_1)$, $Q(\tilde z_2(a,b)) = P(z_2)$.

Proof. Fix $b > 0$. If $a \to 0$ then $Q(\tilde z_1) \to 0$ and $Q(\tilde z_2) \to \mathit{const} < 0$. If $a \to b$ then $Q(\tilde z_1) \to \mathit{const} > 0$ and $Q(\tilde z_2) \to 0$. Therefore there is an $a(b)$ such that $Q(\tilde z_1(a(b),b)) = -Q(\tilde z_2(a(b),b))$.

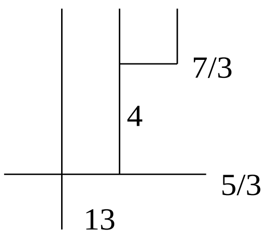
Write $Q_{a,b}$ instead of Q to emphasise that Q depends on a and b. If $\alpha > 0$ then $Q_{a,b}(\alpha z) = \alpha^7 Q_{a/\alpha^3, b/\alpha^3}(z)$. Thus, there is $\alpha > 0$ such that the critical values of $Q_{a(b)/\alpha^3, b/\alpha^3}$ are precisely $P(z_1), P(z_2)$. This shows the lemma. □

Then, for a and b satisfying lemma 4.3, the construction of bi-Lipschitz homeomorphism σ such that $f = g \circ \sigma$ is similar to that of example 4.1.

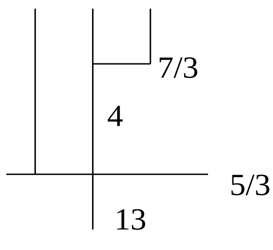
4.3. **Example.**

$$f(x,y) = x(x^3 - y^5)((x^3 - y^5)^3 - y^{17}) \tag{4.1}$$
$$g(x,y) = x(x^3 + ay^5)(x^3 - y^7)(x^6 + by^{10}),$$

where $a > 0, b > 0$ are real constants. As we show below, for a choice of a and b, f and g are bi-Lipschitz equivalent. They have different real tree models, see below, so they are not blow-analytically equivalent. Moreover, in contrast to the previous two examples, f and g have different Fukui invariants.



$\mathbb{R}T_{(0,1)}(f)$



$\mathbb{R}T_{(0,1)}(g)$

Proposition 4.4. *Let* $f,\ g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ *be polynomial functions defined by* (4.1). *Then*

$$A(f) = \{13, 22, 23, 24, \cdots\} \cup \{\infty\}, \quad A(g) = \{13, 23, 25, 26, \cdots\} \cup \{\infty\}.$$

Thus f *and* g *are not blow-analytically equivalent.*

Proof. Let us express an analytic arc at $(0,0) \in \mathbb{R}^2$, $\lambda(t) = (\lambda_1(t), \lambda_2(t))$, as follows:

$$\lambda_1(t) = c_1 t + c_2 t^2 + \cdots, \quad \lambda_2(t) = d_1 t + d_2 t^2 + \cdots.$$

To compute $A(f)$, we consider $f(\lambda(t))$;

$$f(\lambda(t)) = (c_1 t + c_2 t^2 + \cdots)(c_1^3 t^3 + \cdots - d_1^5 t^5 - \cdots)((c_1^3 t^3 + \cdots - d_1^5 t^5 - \cdots)^3 - d_1^{17} t^{17} - \cdots).$$

In case $c_1 \neq 0$, we have $\mathrm{ord}\,(f \circ \lambda) = 13$. In case $c_1 = 0$, we have $\mathrm{ord}\,(f \circ \lambda) \geq 22$. For any $s = 2, 3, \cdots$, $20 + s$ is attained by the arc $\lambda(t) = (t^s, t)$. Therefore we have

$$A(f) = \{13, 22, 23, 24, \cdots\} \cup \{\infty\}.$$

We next compute $A(g)$. Then

$$g(\lambda(t)) = (c_1 t + c_2 t^2 + \cdots)(c_1^3 t^3 + \cdots + a d_1^5 t^5 + \cdots)(c_1 t^3 + \cdots - d_1^7 t^7 - \cdots)$$
$$\times (c_1^6 t^6 + \cdots + b d_1^{10} t^{10} + \cdots).$$

In case $c_1 \neq 0$, we have $\mathrm{ord}\,(f \circ \lambda) = 13$. In case $c_1 = 0$, $c_2 \neq 0$ and $d_1 \neq 0$, we have $\mathrm{ord}\,(f \circ \lambda) = 23$. In case $c_1 = c_2 = 0$ or $c_1 = d_1 = 0$, we have $\mathrm{ord}\,(g \circ \lambda) \geq 25$. For any $s = 3, 4, \cdots$, $22 + s$ is attained by the arc $\lambda(t) = (t^s, t)$. Therefore we have

$$A(g) = \{13, 23, 25, 26, \cdots\} \cup \{\infty\}.$$

□

Next we compute the polar roots of f and g. The one variable polynomial associated to the leading weighted homogeneous part of f with respect to the weigths 5 and 3 equals $P_1(z) = z(z^3 - 1)^4$. Besides a multiple root $z = 1$, it has a unique non-denegenerate critical point a_1, $0 < a_1 < 1$, which gives rise to a polar curve

$$\gamma_1 : x = \lambda_1(y) = a_1 y^{5/3} + \cdots, \qquad f(\lambda_1(y), y) = A_1 y^{21\frac{2}{3}} + O(y^{23\frac{2}{3}})$$

where $A_1 = P_1(a_1)$. The Newton polygon of f relative to $\gamma : x = y^{5/3}$ has two edges: one of slope $-5/3$ and one of slope $-7/3$. The one variable polynomial associated to the latter is $P_2(z) := P_{f,\gamma,7/3}(z) = 3^4 z^4 - 3z$. The unique non-degenerate critical point a_2 of P_2 gives rise to a polar curve

$$\gamma_2 : x = \lambda_2(y) = y^{5/3} + a_2 y^{7/3} + \cdots, \qquad f(\lambda_1(y), y) = A_2 y^{24\frac{1}{3}} + O(y^{25})$$

where $A_2 = P_2(a_2)$.

The one variable polynomial associated to the leading weighted homogeneous part of g equals $Q_1(z) = z^4(z^3 + a)(z^6 + b)$. If $10^2 a^2 - (7 \cdot 39)b < 0$ then $Q_1'(z) = 13z^{12} + 10az^9 + 7bz^6 + 4abz^3$ has a single simple non-zero real root Indeed, let $S(t) = 13t^3 + 10at^2 + 7bt + 4ab$. Then $S'(t) = 39t^2 + 20at + 7b$ and the discriminant of $S'(t)$ is $\Delta/4 = 10^2 a^2 - (7 \cdot 39)b$. Therefore, if we suppose that

$$a > 0, b > 0, 10^2 a^2 - (7 \cdot 39)b < 0, \tag{4.2}$$

then $S(t)$ has a single simple root, that shows our claim on Q_1'. Let $\tilde{a}_1$ denote this critical point of Q_1, $\tilde{a}_1 < 0$. Then there exists a polar curve of g

$$\tilde{\gamma}_1 : x = \tilde{\lambda}_1(y) = \tilde{a}_1 y^{5/3} + \cdots, \qquad g(\tilde{\lambda}_1(y), y) = \tilde{A}_1 y^{21\frac{2}{3}} + O(y^{23\frac{2}{3}}).$$

Finally, the one variable polynomial associated to the face of the Newton polygon of g of slope $-7/3$ is $Q_2(z) = z^4 - z$. It has a single non-degenarate critical point $\tilde a_2$ that gives a polar curve

$$\tilde\gamma_2 : x = \tilde\lambda_2(y) = \tilde a_2 y^{7/3} + \cdots , \qquad g(\tilde\lambda_1(y), y) = \tilde A_2 y^{24\frac{1}{3}} + O(y^{26\frac{1}{3}})$$

where $\tilde A_2 = Q_2(\tilde a_2)$. One checks easily that $\tilde A_2 = A_2$.

Lemma 4.5. *There are constants a, b satisfying* (4.2) *for which $\tilde A_1 = A_1$.*

Proof. Denote by $\tilde a_1(a,b)$ the unique non-zero critical point of Q_1 thus emphasising that it depends on a, b. Note that $\tilde a_1(a,b)$ is between the two roots of Q_1, $-\sqrt[3]{a} < \tilde a_1(a,b) < 0$. For b fixed $Q_1(\tilde a_1(a,b)) \to 0$ as $a \to 0$. Fix a and let $b \to \infty$. Then $Q_1(-\frac{1}{2}\sqrt[3]{a}) \to \infty$ and hence $Q_1(\tilde a_1(a,b)) \to \infty$. Thus there exist a, b for which $Q_1(\tilde a_1(a,b)) = A_1$. □

Next for f, and then for g, we introduce a new system of local coordinates $(\tilde x, \tilde y) = H(x,y)$ in which f has particularly simple form near the polar curves. Firstly, for each polar curve γ_i, $i = 1,2$, separately, we reparametrise λ_i by replacing y by an invertible fractional power series $\tilde y_i(y)$ so that

$$f(\lambda_1(y(\tilde y_1)), y(\tilde y_1)) = A_1 \tilde y_1^{21\frac{2}{3}}, \quad \tilde y_1 = y + O(y^3), \tag{4.3}$$

$$f(\lambda_2(y(\tilde y_2)), y(\tilde y_2)) = A_2 \tilde y_2^{24\frac{1}{3}}, \quad \tilde y_2 = y + O(y^{5/3}). \tag{4.4}$$

Denote $\xi = 5/3$ for short. Let $\varphi_0, \varphi_1, \varphi_2$ be a (C^∞ or $C^k, k \ge 2$, semialgebraic) partition of unity on $\mathbb{R}$ such that

(i) supp φ_1 is a small neighbourhood of a_1 and $\varphi_1 \equiv 1$ in a neighbourhood of a_1.
(ii) supp φ_2 is a small neighbourhood of 1 and $\varphi_2 \equiv 1$ in a neighbourhood of 1.

Then $\varphi_0 = 1 - \varphi_1 - \varphi_2$. We set $\tilde y_0(y) = y$ and define

$$\Phi(x,y) = (x, \tilde y(x,y)) = (x, \sum_{i=0}^{2} \tilde y_i(y)\varphi_i(x/y^\xi)), \quad \Phi(x,0) = (x,0).$$

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a (C^∞ or $C^k, k \ge 2$, semialgebraic) diffeomorphism such that

(i) $\psi(a_1) = 0$ and $\psi(z) = z - a_1$ for z near a_1.
(ii) $\psi(1) = 1$ and $\psi(z) = z$ for z near 1.
(iii) $\psi(z) = z$ for $|z|$ large.

We set

$$\Psi_1(x,y) = (\psi(x/y^\xi)y^\xi, y), \quad \Psi_1(x,0) = (x,0).$$

Let ψ_0, ψ_1, ψ_2 be a (C^∞ or $C^k, k \ge 2$, semialgebraic) partition of unity on $\mathbb{R}$ such that

(i) supp ψ_1 is a small neighbourhood of 0 and $\psi_1 \equiv 1$ in a neighbourhood of 0.
(ii) supp ψ_2 is a small neighbourhood of 1 and $\psi_2 \equiv 1$ in a neighbourhood of 1.

Let $x = \delta_1(y)$ be an equation of $\Psi_1 \circ \Phi(\gamma_1)$ and let $x = \delta_2(y) + y^\xi$ be an equation of $\Psi_1 \circ \Phi(\gamma_2)$. Note that $\delta_i(y) = o(y^\xi)$, $i = 1, 2$. We set $\delta_0 \equiv 0$ and define

$$\Psi_2(x, y) = (\sum_{i=0}^{2}(x - \delta_i(y))\psi_i(x/y^\xi), y), \quad \Psi_2(x, 0) = (x, 0).$$

Proposition 4.6. $H = \Psi_2 \circ \Psi_1 \circ \Phi$ *and* $\tilde{f}(\tilde{x}, \tilde{y}) = f \circ H^{-1}(\tilde{x}, \tilde{y})$ *satisfy the following properties:*

(1) *H is a bi-Lipschitz local homeomorphism. Moreover,* $D^2H = O(y^{-\xi})$.
(2) $\{\partial \tilde{f}/\partial \tilde{x} = 0\} = H(\{\partial f/\partial x = 0\}) = H(\gamma_1) \cup H(\gamma_2)$ *and* $H(\gamma_1) = \{\tilde{x} = 0\}$, $H(\gamma_2) = \{\tilde{x} = \tilde{y}^\xi\}$.
(3) *In a horn neighbourhood of* γ_1, γ_2 *resp., with exponent* ξ, *H is given by*
$$H(x, y) = (x - \lambda_1(y), \tilde{y}_1(y)), \qquad H(x, y) = (x - \lambda_2(y) + \tilde{y}_2^\xi(y), \tilde{y}_2(y))$$
(4) *For C large and* $|x| \geq C|y|^\xi$, $H(x, y) = (x, y)$.

Proof. (3) and (4) are given by construction.

We show that the partial derivatives of Φ, Ψ_1, and Ψ_2 are bounded. For Φ it is convenient to write $\Phi(x, y) = (x, y + \sum_i(\tilde{y}_i - y)\varphi_i(x/y^\xi))$. Then

$$\partial\Phi/\partial x = (1, \sum_i(\tilde{y}_i - y)y^{-\xi}\varphi_i') \qquad \text{bounded},$$
$$\partial\Phi/\partial y = (0, 1 - \sum_i \xi\frac{x}{y^\xi}(\tilde{y}_i - y)y^{-1}\varphi_i' + \sum_i(\tilde{y}_i' - 1)\varphi_i) = (0, 1 + o(y)),$$
$$\partial\Psi_1/\partial x = (\psi', 0),$$
$$\partial\Psi_1/\partial y = (\xi y^{\xi-1}(\psi - \frac{x}{y^\xi}\psi'), 1) = (o(y), 1).$$

For Ψ_2 it is convenient to write $\Psi_2(x, y) = (x - \sum_i \delta_i(y)\psi_i(x/y^\xi), y)$

$$\partial\Psi_2/\partial x = (1 - \sum_i \delta_i(y)y^{-\xi}\psi', 0) = (1 - o(y), 0),$$
$$\partial\Psi_2/\partial y = (-\sum_i \delta_i'(y)\psi_i + \sum_i \xi\frac{x}{y^\xi}y^{-1}\delta_i(y)\psi', 1) = (o(y), 1).$$

Thus H is Lipschitz, $H^{-1}(0) = 0$, and H is a covering over the complement of the origin. Hence it is invertible. The formulae for the partial derivatives also show that the the inverse of Jacobian matrix of H has bounded entries. Thus H^{-1} is also Lipschitz. The last formula of (1) can be verified directly.

To show (2) we note that

$$\frac{\partial\tilde{f}}{\partial\tilde{x}} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial\tilde{x}} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial\tilde{x}}.$$

Note that there is a constant $c > 0$ such that $c \leq \frac{\partial x}{\partial\tilde{x}} \leq c^{-1}$. (2) can be verified easily in the horn neighbourhoods considered in (3) and for $|x/y^\xi|$ large by (4). In the complement of these sets $\partial f/\partial x \sim y^{20}$ and $\partial f/\partial y = O(y^{21\frac{2}{3}})$ and hence

$$\frac{\partial\tilde{f}}{\partial\tilde{x}} \sim y^{20} \sim \tilde{y}^{20},$$

and does not vanish. This shows $\{\partial(f\circ\Phi^{-1})/\partial x=0\}=\Phi(\{\partial f/\partial x=0\})$. Similar results for Ψ_1 and Ψ_2 are obvious. □

We apply the same procedure to $g(x,y)$ and obtain a bi-Lipschitz homeomorphism $\tilde H$ so that $\tilde H$ and $\tilde g(\tilde x,\tilde y)$ satisfy the statement of Proposition 4.6. In what follows we shall drop the "tilda" notation for variables and consider $\tilde f$ and $\tilde g$ as functions of (x,y). We show that the homotopy

$$F(x,y,t)=t\tilde g(x,y)+(1-t)\tilde f(x,y)$$

is bi-Lipschitz trivial and can be trivialised by the vector field

$$v(x,y,t)=\frac{\partial}{\partial t}-\frac{\partial F/\partial t}{\partial F/\partial x}\frac{\partial}{\partial x},\qquad v(x,y,t)=\frac{\partial}{\partial t}\text{ if }\partial F/\partial x=0. \tag{4.5}$$

Thus to complete the proof of bi-Lipschitz equivalence of f and g it suffices to show:

Lemma 4.7. *The vector field $v(x,y,t)$ of* (4.5) *is Lipschitz.*

Proof. The polar curves of $\tilde f$ and $\tilde g$ coincide :

$$\{\partial\tilde f/\partial\tilde x=0\}=\{\partial\tilde g/\partial x=0\}=\{x=0\}\cup\{x=y^\xi\}. \tag{4.6}$$

As we shall show also $\{\partial\tilde F/\partial\tilde x=0\}=\{x=0\}\cup\{x=y^\xi\}$.

We proceed separately in each of the horn neighbourhood with exponent ξ of the polar curves (4.6), for $|x|\ge C|y|^\xi$, C large, and in the complement of these three sets.

Suppose $|x|\le\varepsilon|y|^\xi$, $\varepsilon>0$ and small. By (iii) of Proposition 4.6, $\tilde f$ and $\tilde g$ are fractional convergent power series in x and y. If we pass to new variables $z=x/y^\xi, y$ then, thanks to (4.3),

$$\partial F/\partial x=zy^{20}u(z,y,t)\qquad \frac{\partial F}{\partial t}=\tilde g-\tilde f=z^2y^{21\frac{2}{3}}\eta(z,y),$$

where u and η are fractional power series and $u(0)\neq 0$. Hence

$$\frac{\partial F/\partial t}{\partial F/\partial x}=zy^\xi h(z,y,t)=xh(z,y,t).$$

Thus $\frac{\partial F/\partial t}{\partial F/\partial x}$ is Lipschitz because the partial derivatives of $xh(z,y,t)$ are bounded:

$$\frac{\partial}{\partial x}(xh)=h+\frac{x}{y^\xi}\frac{\partial h}{\partial x},\ \frac{\partial}{\partial y}(xh)=x\frac{\partial h}{\partial z}\frac{\partial z}{\partial y}+x\frac{\partial h}{\partial y}=-\xi\frac{x}{y^\xi}\frac{x}{y}\frac{\partial h}{\partial z}+x\frac{\partial h}{\partial y},\ \frac{\partial}{\partial t}(xh)=x\frac{\partial h}{\partial t}.$$

A similar argument works for a horn neighbourhood of $x=y^\xi$.

If $|x|\ge C|y|^\xi$, C large, then by (iv) of Proposition 4.6, $\tilde f=f$ and $\tilde g=g$. Then in variables $x, w=y^\xi/x$

$$\partial F/\partial x=x^{m-1}u(x,w,t)\qquad \frac{\partial F}{\partial t}=x^m\eta(x,w),$$

where u and η are fractional power series and $u(0)\neq 0$. Hence

$$\frac{\partial F/\partial t}{\partial F/\partial x}=xh(x,w,t).$$

Then, an elementary computation shows that the partial derivatives of $xh(x, w, t)$ are bounded.

Suppose now that x/y^ξ is bounded and that we are not in horn neighbourhoods of the polar curves. By Proposition 4.6 one can verify easily that on this set

$$\begin{aligned}
&\tilde g - \tilde f = O(y^{20+\xi}), \quad D(\tilde g - \tilde f) = O(y^{20+\xi}),\\
&D^2 H^{-1} = O(y^{-\xi})\\
&\partial F/\partial x \sim y^{20}, \quad D(\partial F/\partial x) = O(y^{20-\xi}).
\end{aligned}$$

Now a direct computation shows that the partial derivatives of $\frac{\partial F/\partial t}{\partial F/\partial x}$ are bounded. □

References

[1] V.I. Arnold, A.N. Varchenko, S. Gusein-Zade, *Singularities of differentiable mappings* Vol. 1, Birkhäuser, Moscow, 1982.

[2] M. Artin : *On the solution of analytic functions*, Invent. math. **5** (1968), 277–291.

[3] G. Fichou, *Motivic invariants of Arc-Symmetric sets and blow-Nash equivalence,* Compositio Math. **141** (2005), 655–688.

[4] T. Fukui, E. Yoshinaga : *The modified analytic trivialization of family of real analytic functions*, Invent. math. **82** (1985), 467–477.

[5] T. Fukui : *Seeking invariants for blow-analytic equivalence,* Compositio Math. **105** (1997), 95–107.

[6] T. Fukui, S. Koike, T.-C. Kuo : *Blow-analytic equisingularities, properties, problems and progress*, Real Analytic and Algebraic Singularities (T. Fukuda, T. Fukui, S. Izumiya and S. Koike, ed), Pitman Research Notes in Mathematics Series, **381** (1998), pp. 8–29.

[7] T. Fukui, L. Paunescu : *Modified analytic trivialization for weighted homogeneous function-germs*, J. Math. Soc. Japan **52** (2000), 433–446.

[8] T. Fukui, L. Paunescu : *On blow-analytic equivalence*, in "Arc Spaces and Additive Invariants in Real Algebraic Geometry", Proceedings of Winter School "Real algebraic and Analytic Geometry and Motivic Integration", Aussois 2003, Panoramas et Synthèses **24**, Société Mathématique de France, to appear

[9] J.P. Henry, A. Parusiński : *Existence of moduli for bi-Lipschitz equivalence of analytic functions*, Compositio Math. **136** (2003), 217–235.

[10] J.P. Henry, A. Parusiński : *Invariants of bi-Lipschitz equivalence of real analytic functions*, Banach Center Publications **65** (2004), 67–75.

[11] S. Koike : *On strong C^0-equivalence of real analytic functions*, J. Math. Soc. Japan **45** (1993), 313-320.

[12] S. Koike, A. Parusiński, *Motivic-type invariants of blow-analytic equivalence*, Ann. Inst. Fourier **53** (2003), 2061–2104.

[13] S. Koike, A. Parusiński, *Blow-analytic equivalence of two variable real analytic function germs*, University of Angers preprint no. 234, June 2007, arXiv:0710.1046

[14] N. Kuiper : *C^1-equivalence of functions near isolated critical points*, Symp. Infinite Dimensional Topology, Princeton Univ. Press, Baton Rouge, 1967, R.D. Anderson ed., Annales of Math. Studies **69** (1972), pp. 199–218.

[15] T.-C. Kuo, *On C^0-sufficiency of jets of potential functions,* Topology **8** (1969), 167–171.

[16] T.-C. Kuo, Y.C. Lu, *On analytic function germs of complex variables*, Topology **16** (1977), 299–310.

[17] T.-C. Kuo, *The modified analytic trivialization of singularities,* J. Math. Soc. Japan **32** (1980), 605–614.

[18] T.-C. Kuo : *Blow-analytic equisingularities,* Invited Address at Annual Convention of the Mathematical Society of Japan, Autumn 1984, Tokyo University.

[19] T.-C. Kuo, *On classification of real singularities,* Invent. math. **82** (1985), 257–262.

[20] T.-C. Kuo, A. Parusiński : *Newton polygon relative to an arc*, Real and Complex Singularities (São Carlos, 1998), Chapman & Hall Res. Notes Math. **412** (2000), pp. 76–93.

[21] F. Takens : *A note on sufficiency of jets*, Invent. math. **13** (1971), 225–231.

[22] R. J. Walker: *Algebraic Curves*, Springer-Verlag, New York, Heidelberg, Berlin, 1950.

[23] C.T.C. Wall : *Singular Points of Plane Curves*, London Mathematical Society, Student Texts **63**, Cambridge University Press 2004.

[24] O. Zariski, *On the topology of algebroid singularities*, Amer. Jour. Math. **54** (1932), 453–465.

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