THE WEIGHT FILTRATION FOR REAL ALGEBRAIC VARIETIES II: CLASSICAL HOMOLOGY

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To Heisuke Hironaka on the occasion of his 80th birthday

Abstract. We associate to each real algebraic variety a filtered chain complex, the weight complex, which is well-defined up to filtered quasi-isomorphism, and which induces on classical (compactly supported) homology with $\mathbb{Z}_2$ coefficients an analog of the weight filtration for complex algebraic varieties. This complements our previous definition of the weight filtration of Borel-Moore homology.

We define the weight filtration of the homology of a real algebraic variety by first addressing the case of smooth noncompact varieties. As in Deligne's definition [5] of the weight filtration for complex varieties, given a smooth variety X we consider a *good compactification*, a smooth compactification $\overline{X}$ of X such that $D = \overline{X} \setminus X$ is a divisor with normal crossings. Whereas Deligne's construction can be interpreted in terms of the action of a torus $(S^1)^N$, we use the action of a discrete torus $(S^0)^N$ to define a filtration of the chains of a semialgebraic compactification of X associated to the divisor D. The resulting filtered chain complex is functorial for pairs $(\overline{X}, X)$ as above, and it behaves nicely for a blowup with a smooth center that has normal crossings with D.

We apply a result of Guillén and Navarro Aznar [6, Theorem (2.3.6)] to show that our filtered complex is independent of the good compactification of X (up to quasi-isomorphism) and to extend our definition to a functorial filtered complex, the *weight complex*, that is defined for all varieties and enjoys a generalized blowup property (Theorem 7.1). For compact varieties the weight complex agrees with our previous definition [9] for Borel-Moore homology.

We work with homology rather than cohomology to take advantage of the topology of semialgebraic chains [9, Appendix]. We denote by $H_k(X)$ the kth classical homology group of X, with compact supports and coefficients in $\mathbb{Z}_2$, the integers modulo 2. The vector space $H_k(X)$ is dual to $H^k(X)$, the classical kth cohomology group with closed supports. On the other hand, let $H_k^{BM}(X)$ denote the kth Borel-Moore homology group of X (*i.e.* homology with closed supports) with coefficients in $\mathbb{Z}_2$. Then $H_k^{BM}(X)$ is dual to $H_c^k(X)$, the kth cohomology group with compact supports.

Our work owes much to the foundational paper [6] of Guillén and Navarro Aznar. In particular we have been influenced by the viewpoint of section 5 of that paper, on the theory of motives. Using Guillén and Navarro Aznar's extension theorems, Totaro [13] observed that there is a functorial weight filtration for the cohomology with compact supports of a real analytic variety with a given compactification. In [9] we developed this theory in detail for real algebraic varieties, working with Borel-Moore homology. Our task was simplified by the strong additivity property of Borel-Moore homology (or compactly supported cohomology) [9, Theorem 1.1]. For classical homology or cohomology

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one does not have such an additivity property, and so the present construction of the weight filtration is more involved.

In section 1 below we define the weight filtration of a smooth, possibly noncompact, variety X, in terms of a good compactification $\overline{X}$ with divisor D at infinity. First we define a semialgebraic compactification X', the *corner compactification* of X, with X' contained in a principal bundle over $\overline{X}$ with group a discrete torus $\{1,-1\}^N$. We use the action of this group to define the *corner filtration* of the semialgebraic chain group of X'. The filtered *weight complex* is obtained from the corner filtration by an algebraic construction, the *Deligne shift*. In section 2 we analyze the relation of the weight complex to the homological *Gysin complex* of the divisor D. Section 3 contains the proof of the crucial fact that the weight complex is functorial for pairs $(\overline{X}, X)$, together with an analysis of the functoriality of the Gysin complex.

Sections 4, 5, and 6 treat the blowup properties of the weight complex of a smooth variety. A key role is played by the Gysin complex. For example, in section 6 we use the fact that a homomorphism of weight complexes is a filtered quasi-isomorphism if and only if it induces an isomorphism of the homology of the corresponding Gysin complexes.

In section 7 we use the theorems of Guillén and Navarro Aznar to extend the definition of the weight complex to singular varieties, and we describe some elementary examples. The appendix (section 8) is devoted to a canonical filtration of the $\mathbb{Z}_2$ group algebra of a discrete torus group. This is in effect a local version of the weight filtration.

By a *real algebraic variety* we mean an affine real algebraic variety in the sense of Bochnak-Coste-Roy [3]: a topological space with a sheaf of real-valued functions isomorphic to a real algebraic set $X \subset \mathbb{R}^N$ with the Zariski topology and the structure sheaf of regular functions. A *regular function on* X is the restriction of a rational function on $\mathbb{R}^N$ that is everywhere defined on X. By a *regular mapping* we mean a regular mapping in the sense of Bochnak-Coste-Roy [3].

For instance, the set of real points of a reduced projective scheme over $\mathbb{R}$, with the sheaf of regular functions, is an affine real algebraic variety in this sense. This follows from the fact that real projective space is isomorphic, as a real algebraic variety, to a subvariety of an affine space [3, Theorem 3.4.4]. We also adopt from [3] the notion of an algebraic vector bundle. We recall that such a bundle is, by definition, a subbundle of a trivial vector bundle, and hence it is the pullback of the universal vector bundle on the Grassmannian, and its fibers are generated by global regular sections [3, Chapter 12].

By a *smooth real algebraic variety* we mean a nonsingular affine real algebraic variety.

1. The weight filtration of a smooth variety

In this section we define the weight filtration of the classical homology of a smooth variety X. We use a smooth compactification $\overline{X}$ with a normal crossing divisor at infinity to define a semialgebraic compactification X' of X and a surjective map $\pi : X' \to \overline{X}$ with finite fibers. This map is used to define the weight filtration of the semialgebraic chain complex of X' with $\mathbb{Z}_2$ coefficients. Thus we obtain the weight filtration of the homology of X', which is canonically isomorphic to the homology of X. We will prove in Section 7 that this filtration of $H_*(X)$ does not depend on the choice of compactification $\overline{X}$.

1.1. **The corner compactification.** Let M be a compact smooth real algebraic variety and let $D \subset M$ be a smooth divisor. Associated to D there is an algebraic line bundle L over M that has a section s such that D is the variety of zeroes of s. Let $S(L)$ be the space of oriented directions in the fibers of L. It can be given the structure of a real algebraic variety as follows. By [3, Remark 12.2.5], L is isomorphic to an algebraic subbundle of the trivial bundle $M \times \mathbb{R}^N$. Denote by $\Psi : L \to \mathbb{R}^N$ the regular map defined by this isomorphism. The scalar product on $\mathbb{R}^N$ defines a regular metric on L. We identify $S(L)$ with the unit zero-sphere bundle of L; that is, with the real algebraic variety $\Psi^{-1}(S^{N-1})$. This structure is uniquely defined. Indeed, the standard projection $L \setminus M \to \Psi^{-1}(S^{N-1})$ is a regular map, and therefore two such unit sphere bundles are biregularly isomorphic. Finally, L is the pullback of the universal line bundle on $\mathbb{P}^{N-1}$ under the regular map $M \to \mathbb{P}^{N-1}$ induced by Ψ.

Thus $S(L)$ is a smooth real algebraic variety, and the projection $\pi_L : S(L) \to M$ is an algebraic double covering. Now the subvariety $\pi_L^{-1} D$ of $S(L)$ is the zero set of the regular function $\varphi : S(L) \to \mathbb{R}$ defined by $\varphi(x, \ell) \cdot \ell = s(x)$, where $x \in M$ and ℓ is a unit vector in the fiber $L_x = \pi_L^{-1}(x)$. Note that the generator τ of the group of covering transformations of $S(L)$ changes the sign of φ, for $\varphi(\tau(x, \ell)) = \varphi(x, -\ell) = -\varphi(x, \ell)$.

Let X be a smooth n-dimensional variety, and let $\overline{X}$ be a *good compactification* of X [12, p. 89]: $\overline{X}$ is a compact smooth variety containing X, and $D = \overline{X} \setminus X$ is a divisor with simple normal crossings. Thus D is a finite union of smooth codimension one subvarieties D_i of $\overline{X}$,

$$D = \bigcup_{i \in I} D_i \ , \tag{1.1}$$

and the divisors D_i meet transversely. Note that we do not assume that the divisors D_i are irreducible.

For $i \in I$, let L_i be the line bundle on $\overline{X}$ associated to D_i and let s_i be a section of L_i that defines the divisor D_i. Let $\widetilde{\pi} : \widetilde{X} \to \overline{X}$ be the covering of degree $2^{|I|}$ defined as the fiber product of the double covers $\pi_{L_i} : S(L_i) \to \overline{X}$, and let $\widetilde{\varphi}_i : \widetilde{X} \to \mathbb{R}$ be the pullback of the function $\varphi_i : S(L_i) \to \mathbb{R}$ corresponding to the section s_i, so that the variety $\widetilde{\pi}^{-1} D_i$ is the zero space of $\widetilde{\varphi}_i$. The *corner compactification* of X associated to the good compactification $(\overline{X}, D)$ is the semialgebraic set $X' \subset \widetilde{X}$ defined by

$$X' = \text{Closure}\{\widetilde{x} \in \widetilde{X} \mid \widetilde{\varphi}_i(\widetilde{x}) > 0, \ i \in I\}. \tag{1.2}$$

In the terminology of [11, §3.2], X' is the variety $\overline{X}$ *cut along the divisor* D. Let $\pi : X' \to \overline{X}$ be the restriction of the covering map $\widetilde{\pi} : \widetilde{X} \to \overline{X}$.

Let T be the group of covering transformations of the covering space $\widetilde{\pi} : \widetilde{X} \to \overline{X}$, with $\widetilde{\tau}_i \in T$ the pullback of the nontrivial covering transformation τ_i of the double cover $\pi_{L_i} : S(L_i) \to \overline{X}$. There is a canonical isomorphism $\theta : T \to G$, where G is the multiplicative group of functions $g : I \to \{1, -1\}$, given by $\theta(\widetilde{\tau}_i) = g_i$, with $g_i(i) = -1$ and $g_i(j) = 1$ for $i \neq j$. To emphasize the role of the group G we prefer to consider

$$\xi = (\widetilde{\pi} : \widetilde{X} \to \overline{X}) \tag{1.3}$$

as a *principal* G*-bundle* for the group $G = \{1, -1\}^I$ and then

$$\begin{aligned} \widetilde{\varphi}_i(g_i \cdot \widetilde{x}) &= -\widetilde{\varphi}_i(\widetilde{x}), \\ \widetilde{\varphi}_j(g_i \cdot \widetilde{x}) &= \widetilde{\varphi}_j(\widetilde{x}), \ i \neq j. \end{aligned} \tag{1.4}$$

If $U \subset \overline{X}$ is contractible then $\xi | U$ is trivial, i.e.

$$\widetilde{\pi}^{-1}(U) \simeq U \times G \tag{1.5}$$

as a principal G-bundle. This isomorphism is uniquely defined by a choice of $x \in U$ and a point $\widetilde{x} \in \widetilde{\pi}^{-1}(x)$, which we identify via (1.5) with $(x, 1) \in U \times G$.

Proposition 1.1. *The semialgebraic map $\pi : X' \to \overline{X}$ is surjective. If $x \in \overline{X}$ let $J(x) = \{i \in I \mid x \in D_i\}$ and $G(x) = \{g \in G \mid g(i) = 1,\ i \notin J(x)\}$. The fiber $\pi^{-1}(x) = \{\widetilde{x} \in \widetilde{\pi}^{-1}(x) \mid \widetilde{\varphi}_i(\widetilde{x}) > 0$ for $i \in J(x)\}$. Thus $\pi^{-1}(x)$ is a regular orbit of the action of $G(x)$ on $\widetilde{X}$; i.e., a $G(x)$-torsor. Hence the number of points in $\pi^{-1}(x)$ is $2^{|J(x)|}$.*

Proof. If $x \in X = \overline{X} \setminus D$, then $J(x) = \emptyset$ and $G(x)$ is trivial. For each $i \in I$ let $\ell_i(x) \in (L_i)_x$ be the unit vector such that $s_i(x)$ is a positive multiple of $\ell_i(x)$; that is, $\varphi_i(x, \ell_i(x)) > 0$. Let $\widetilde{x} = (x, \ell_i(x))_{i \in I} \in \widetilde{X}$. Then by definition $\widetilde{x} \in X'$ and $\pi^{-1}(x) = \{\widetilde{x}\}$. If U is a contractible, open neighborhood of x in X then the principal bundle $\xi|U$ is trivial, and $\pi^{-1}(U)$ is a connected component of $\widetilde{\pi}^{-1}(U)$. Denote

$$X'_+ = \{\widetilde{x} \in \widetilde{X} \mid \widetilde{\varphi}_i(\widetilde{x}) > 0,\ i \in I\}. \tag{1.6}$$

Thus $\pi^{-1}X = X'_+$, and π maps X'_+ homeomorphically onto X.

Since the divisor D has simple normal crossings (1.1), it follows that for every $x \in \overline{X}$ there is a regular system of parameters $u_1, \ldots, u_n$ for $\overline{X}$ at x, and a semialgebraic open neighborhood U of x such that $(u_1, \ldots, u_n)$ is a real analytic, semialgebraic coordinate system on U with $(u_1(x), \ldots, u_n(x)) = 0$, and for each $i \in J(x)$ there is an index $k(i) \in \{1, \ldots, n\}$ such that $D_i \cap U$ is the coordinate hyperplane $u_{k(i)} = 0$ and

$$X \cap U = \{y \in U \mid u_{k(i)}(y) \neq 0 \text{ for all } i \in J(x)\}.$$

Then

$$\begin{aligned} X \cap U &= \bigcup_{g \in G(x)} X_g(U), \\ X_g(U) &= \{y \in U \mid g(i) u_{k(i)}(y) > 0 \text{ for all } i \in J(x)\}, \end{aligned} \tag{1.7}$$

the set of points of U such that each of the coordinates $u_{k(i)}$ has the sign $g(i)$.

We say that $(U, (u_1, \ldots, u_n))$ is a *good local coordinate system* on $(\overline{X}, D)$ at $x \in \overline{X}$ if, moreover, U and all $X_g(U)$ for $g \in G(x)$ are contractible. Thus $X \cap U$ has exactly $2^{|J(x)|}$ connected components.

Let $y \in X_1(U)$, *i.e.* $u_{k(i)}(y) > 0$ for all $i \in J(x)$, and let $\widetilde{U}_1$ be the component of $\widetilde{\pi}^{-1}U$ containing $\pi^{-1}X_1(U)$. We choose the isomorphism in (1.5) so that $\widetilde{U}_1$ corresponds to $U \times \{1\}$. Then, by (1.4) ,

$$\pi^{-1} X_g(U) = \widetilde{\pi}^{-1} X_g(U) \cap g\widetilde{U}_1.$$

In other words $\pi^{-1}X_g(U)$ corresponds to $X_g(U) \times \{g\}$ via the isomorphism (1.5). In particular, $\pi^{-1}(x) = \{x\} \times G(x)$ as claimed. □

As a corollary of the proof we have that every $x' \in X'$ has a neighborhood in X' semialgebraically homeomorphic to a quadrant $\{(u_1, \ldots, u_n) \mid u_i \geq 0,\ i = 1, \ldots, m\} \subset \mathbb{R}^n$, where $m = |J(\pi(x'))|$. Thus X' is a semialgebraic manifold with boundary $\partial X'$, and $X' \setminus \partial X' = X'_+$. The inclusion $X \hookrightarrow \overline{X}$ factors through π,

$$\begin{array}{ccc} & & X' \\ & \nearrow & \downarrow \pi \\ X & \hookrightarrow & \overline{X} \end{array}$$

and the restriction of π to $X' \setminus \partial X' = X'_+$ is a semialgebraic homeomorphism onto X. Thus the inclusion $\lambda : X \to X'$ is a homotopy equivalence, and so $\lambda_* : H_k(X) \to H_k(X')$ is an isomorphism for all $k \geq 0$, where $H_k(X)$ denotes classical homology (with compact supports) with coefficients in $\mathbb{Z}_2$.

Proposition 1.2. *The corner compactification X' of X does not depend on the choice of sections s_i.*

Proof. Suppose that for all $i \in I$ we have sections s_i and $\hat{s}_i$ of L_i defining D_i, and these sets of sections define corner compactifications X' and $\widehat{X}'$, respectively. Suppose there is an index $j \in I$ such that $s_i = \hat{s}_i$ for all $i \neq j$. If $s_j(x)$ and $\hat{s}_j(x)$ lie in the same component of the fiber $L_x \setminus \{0\}$ for $x \notin D_j$, then the corresponding functions φ_j and $\hat{\varphi}_j$ have the same sign, so $X' = \widehat{X}'$. If $s_j(x)$ and $\hat{s}_j(x)$ lie in different components of the fiber $L_x \setminus \{0\}$ for $x \notin D_j$, then the corresponding functions φ_j and $\hat{\varphi}_j$ have opposite signs. Thus $g_j(X') = \widehat{X}'$. □

Proposition 1.3. *The corner compactification X' does not depend on the choice of decomposition* (1.1) *of the divisor D into smooth subvarieties; that is, two such compactifications are canonically semialgebraically homeomorphic.*

Proof. Suppose that the divisor D_j is the union of two nonempty smooth divisors D_a and D_b, $D_a \cap D_b = \emptyset$, and we replace D_j with $D_a \cup D_b$ in the decomposition (1.1). Then the line bundle L_j equals $L_a \otimes L_b$, and we can take $s_j = s_a \otimes s_b$. If we choose the metric on L_j to be the product of the metrics on L_a and L_b, then $\varphi_j = \varphi_a \cdot \varphi_b$, *i.e.* $\varphi_j(x, \ell_a \otimes \ell_b) = \varphi_a(x, \ell_a)\varphi_b(x, \ell_b)$, and we have a double cover $S(L_a) \times_{\overline{X}} S(L_b) \to S(L_j)$ given by $((x, \ell_a), (x, \ell_b)) \mapsto (x, \ell_a \otimes \ell_b)$. Let $\widetilde{X}(j)$ be the fiber product of the double covers $S(L_i)$ with L_j replaced by L_a and L_b, and let $X'(j)$ be the resulting corner compactification. Then the double cover $p : \widetilde{X}(j) \to \widetilde{X}$ restricts to a semialgebraic homeomorphism $X'(j) \to X'$. To prove this it suffices to show that p restricts to a bijection $X'(j) \setminus \partial X'(j) \to X' \setminus \partial X'$. Suppose that $\widetilde{x} = (x, \ell_i)_{i \in I} \in X' \setminus \partial X'$. Then $\widetilde{\varphi}_i(x, \ell_i) > 0$ for all $i \in I$, and in particular $\varphi(x, \ell_j) > 0$. Let $\ell_j = \ell_a \otimes \ell_b$, so that $\varphi_j(x, \ell_j) = \varphi_a(x, \ell_a)\varphi_b(x, \ell_b) > 0$. Now $p^{-1}(\widetilde{x}) = \{\widetilde{y}, \widetilde{z}\}$, where $\widetilde{y}$ is obtained from $\widetilde{x}$ by replacing (x, ℓ_j) with $((x, \ell_a), (x, \ell_b))$ and $\widetilde{z}$ is obtained from $\widetilde{x}$ by replacing (x, ℓ_j) with $((x, -\ell_a), (x, -\ell_b))$. Thus if $\varphi_a(x, \ell_a) > 0$ we have $\widetilde{y} \in X'(j)$ and $\widetilde{z} \notin X'(j)$, and if $\varphi_a(x, \ell_a) < 0$ we have $\widetilde{y} \notin X'(j)$ and $\widetilde{z} \in X'(j)$. □

1.2. **The corner filtration.** We will use the map $\pi : X' \to \overline{X}$ and the action of the group G on $\widetilde{X}$ to define a filtration of the semialgebraic chain complex $C_*(X')$ of the corner compactification X'. Given the decomposition (1.1) of D, for $J \subset I$ we define

$$D_J = \bigcap_{i \in J} D_i \ , \ D_\emptyset = \overline{X},$$

$$\mathring{D}_J = D_J \setminus \bigcup_{i \notin J} D_i \ , \ \mathring{D}_\emptyset = X. \tag{1.8}$$

Then $\{\mathring{D}_J\}_{J \subset I}$ is a stratification of $\overline{X}$. This is a local condition that follows from the fact that every $x \in \overline{X}$ is contained in a good coordinate system $(U, (u_1, \ldots, u_n))$, with

$$D \cap U = \{y \in U \mid u_{k(i)} = 0 \text{ for some } i \in J(x)\}.$$

In these coordinates, for $J \subset J(x)$ we have

$$D_J \cap U = \{y \in U \mid u_{k(i)}(y) = 0,\ i \in J\},$$
$$\mathring{D}_J \cap U = (D_J \cap U) \cap \{y \in U \mid u_{k(i)}(y) \neq 0,\ i \in J(x) \setminus J\}.$$

Similarly we can stratify X' by taking $\pi^{-1}\mathring{D}_J$ as strata, and $\pi^{-1}D_J$ is the closure in X' of the stratum $\pi^{-1}\mathring{D}_J$. To prove these assertions, let $(U, (u_1, \dots, u_n))$ be a good coordinate system as above, and let $\widetilde{U}_1$ be the component of $\widetilde{\pi}^{-1}U$ containing $\pi^{-1}X_1(U)$. For $g \in G(x)$ let

$$\overline{X}_g(U) = \{y \in U \mid g(i)u_{k(i)}(y) \geq 0,\ i \in J(x)\}, \tag{1.9}$$

the closure in U of $X_g(U)$. We have that $\widetilde{\pi}$ maps $(g\widetilde{U}_1, \pi^{-1}\overline{X}_g(U))$ homeomorphically onto $(U, \overline{X}_g(U))$, and $\pi^{-1}U$ is the disjoint union of the sets $\pi^{-1}\overline{X}_g(U)$. Clearly $\{\mathring{D}_J \cap \overline{X}_g(U)\}_{J \subset J_0}$ is a stratification of $\overline{X}_g(U)$, and the closure of $\mathring{D}_J \cap \overline{X}_g(U)$ in $\overline{X}_g(U)$ is $D_J \cap \overline{X}_g(U)$.

Now for each $J \subset I$ such that $D_J \neq \emptyset$, let $G(J) = \{g \in G \mid g(i) = 1,\ i \notin J\}$. Then $G(J)$ is isomorphic to $\{1, -1\}^{|J|}$ and for each $x \in \mathring{D}_J$ we have $G(x) = G(J)$. Thus the action of $G(J)$ on $\widetilde{X}$ preserves $\pi^{-1}D_J$. Consider the inclusion $C_*(\pi^{-1}D_J) \to C_*(X')$. We denote by $F^J C_*(X')$ the image in $C_*(X')$ of the subcomplex of $C_*(\pi^{-1}D_J)$ of $G(J)$-invariant chains. Then $F^J C_*(X')$ is a subcomplex of $C_*(X')$. For $p \geq 0$ let $F^p C_*(X')$ be the subcomplex of $C_*(X')$ generated by the $F^J C_*(X')$ with $|J| = p$.

If $J \subset K$ then $D_J \supset D_K$ and $G(J) \subset G(K)$. Therefore $F^J C_*(X') \supset F^K C_*(X')$. So for all $p \geq 0$ we have $F^{p+1}C_*(X') \subset F^p C_*(X')$. We obtain a filtration

$$C_*(X') = F^0 C_*(X') \supset F^1 C_*(X') \supset F^2 C_*(X') \supset \cdots, \tag{1.10}$$

with $F^{n-k+1}C_k(X') = 0$ for all $k \geq 0$, where $n = \dim X$. We call this filtered complex the *corner complex* of the good compactification $(\overline{X}, D)$ of X.

The *corner spectral sequence* $\widehat{E}^r_{p,q}$ is the spectral sequence associated to the increasing filtration $\widehat{F}_*$ obtained by setting $\widehat{F}_{-p} = F^p$,

$$\cdots \subset \widehat{F}_{-2}C_*(X') \subset \widehat{F}_{-1}C_*(X') \subset \widehat{F}_0 C_*(X') = C_*(X'). \tag{1.11}$$

This is a second quadrant spectral sequence: If $\widehat{E}^r_{p,q} \neq 0$ then (p, q) lies in the closed triangle with vertices $(0,0)$, $(0,n)$, $(-n,n)$, $n = \dim X$. The corner spectral sequence converges to the homology of the corner compactification X',

$$\widehat{E}^r_{p,q} \implies H_{p+q}(X').$$

It will be useful to describe the corner filtration on the level of semialgebraic sets, using the definition of semialgebraic chains given in the appendix of [9]. If Γ is a closed k-dimensional semialgebraic subset of a semialgebraic set X, then $c = [\Gamma] \in C_k(X)$ is the semialgebraic chain represented by Γ.

The vector subspace $F^p C_k(X')$ is generated by the subspaces $F^J C_k(X')$ for $J \subset I$, and our definition implies that $c \in F^J C^k(X')$ if and only if $c = [\Gamma]$, where $\Gamma \subset X'$ and $G(J)\Gamma = \Gamma$.

Next we give an alternative description of the corner filtration. For each $J \subset I$ such that $D_J \neq \emptyset$ consider the corner compactification D'_J of $\mathring{D}_J$ associated to the good compactification D_J of $\mathring{D}_J$ with divisor $\bigcup_{i \notin J}(D_i \cap D_J)$, and let $\pi_J : D'_J \to D_J$ be the projection.

Proposition 1.4. *The projection $\pi^{-1}D_J \to D_J$ factors through D'_J. The induced map $\rho_J : \pi^{-1}D_J \to D'_J$ is a principal $G(J)$-bundle, and hence it is a covering space of degree $2^{|J|}$.*

Proof. Let $\xi = (\widetilde{\pi} : \widetilde{X} \to \overline{X})$ be the principal G-bundle associated to the good compactification $\overline{X}$ of X (1.3). The restriction $\xi|D_J = (\widetilde{\pi} : \widetilde{\pi}^{-1}D_J \to D_J)$ is a principal G-bundle. Let $\xi_J = (\widetilde{\pi}_J : \widetilde{D}_J \to D_J)$ be the principal $G(I \setminus J)$-bundle associated to the good compactification D_J of $\mathring{D}_J$. Let $\zeta_J = (\widetilde{\rho}_J : \widetilde{\pi}^{-1}D_J \to \widetilde{D}_J)$ be the principal $G(J)$-bundle such that $\widetilde{\rho}_J$ is the quotient map of the action of $G(J)$ on $\widetilde{\pi}^{-1}D_J$. On $\widetilde{\pi}^{-1}D_J$ we have $\widetilde{\pi} = \widetilde{\pi}_J \circ \widetilde{\rho}_J$.

Now $(\widetilde{\rho}_J)^{-1}D'_J = \pi^{-1}D_J$. If $\rho_J = \widetilde{\rho}_J|\pi^{-1}D_J$ then on $\pi^{-1}D_J$ we have $\pi = \pi_J \circ \rho_J$, and $\zeta_J|D'_J = (\rho_J : \pi^{-1}D_J \to D'_J)$ is a principal $G(J)$-bundle. □

Corollary 1.5. *There is a finite semialgebraic open cover $\mathcal{U}_J$ of $\mathring{D}_J$ such that over each $U \in \mathcal{U}_J$ the projection $\pi^{-1}U \to U$ is a trivial $G(J)$-bundle, i.e. $\pi^{-1}U = U \times G(J)$.*

Proof. This is true for $\rho_J : \pi^{-1}D_J \to D'_J$ because D'_J is compact. Now $\pi_J : \pi_J^{-1}(\mathring{D}_J) \to \mathring{D}_J$ is an isomorphism and hence $\mathring{D}_J$ can be identified with $\pi_J^{-1}(\mathring{D}_J) \subset D'_J$. Thus it suffices to restrict to $\mathring{D}_J$ the corresponding open cover of D'_J. □

Associated to the principal bundle $\rho_J : \pi^{-1}D_J \to D'_J$ of Proposition 1.4, we have the *inverse image map* $\rho_J^* : C_*(D'_J) \to C_*(\pi^{-1}D_J)$ defined by $\rho_J^*([\Gamma]) = [\rho_J^{-1}\Gamma]$. The function ρ_J^* commutes with the boundary map, and so ρ_J^* is an injective morphism of complexes. (The map ρ_J^* is the chain-level *transfer homomorphism* of the covering map ρ_J.) Let $i_J : C_*(\pi^{-1}D_J) \to C_*(X')$ be the inclusion. Then $F^J C_*(X')$ is the image in $C_*(X')$ of the composition $\eta_J = i_J \circ \rho_J^*$,

$$\eta_J : C_*(D'_J) \xrightarrow{\rho_J^*} C_*(\pi^{-1}D_J) \xrightarrow{i_J} C_*(X'), \tag{1.12}$$

and η_J is an isomorphism of the complexes $C_*(D'_J)$ and $F^J C_*(X')$. Thus $c \in F^J C_*(X')$ if and only if $c = [\Gamma]$ for $\Gamma \subset \pi^{-1}D_J$ with $\Gamma = \rho_J^{-1}B$, where $B \subset D'_J$.

From Corollary 1.5 we obtain the following useful local characterization of the corner filtration. The vector space $F^J C_*(X')$ is generated by the chains $c \in C_*(X')$ such that $c = [\Gamma]$ with $\Gamma \subset \pi^{-1}D_{J'}$ for $J' \supset J$ (so $D_{J'} \subset D_J$), and $\Gamma = \operatorname{Closure}\mathring{\Gamma}$, where $\mathring{\Gamma} \subset \pi^{-1}\mathring{B}$, with $\mathring{B} \subset \mathring{D}_{J'}$, $\pi^{-1}\mathring{B} = \mathring{B} \times G(J')$ (*i.e.* $\pi^{-1}\mathring{B} \to \mathring{B}$ is a trivial $G(J')$-bundle) and $\mathring{\Gamma} = \mathring{B} \times gG(J)$ for some $g \in G(J')$. In other words, $\mathring{\Gamma}$ is an orbit of the action of $G(J)$ on $\pi^{-1}\mathring{B}$.

Let $\mathring{B} \subset \mathring{D}_{J'}$ be a semialgebraic set such that $\pi^{-1}\mathring{B} = \mathring{B} \times G(J')$, let $\dim \mathring{B} = k$, and let B be the closure of $\mathring{B}$. Then

$$C_k(\pi^{-1}B) = C_k(B) \otimes C_0(G(J')), \tag{1.13}$$

where we consider $G(J')$ as a discrete topological space. In particular, $C_0(G(J')) = \mathbb{Z}_2[G(J')]$ the $\mathbb{Z}_2$ group algebra of $G(J')$. Using this algebra structure we define in the Appendix (8.2) a filtration $\mathcal{I}^*$ on $\mathbb{Z}_2[G(J')]$ and hence on $C_0(G(J'))$.

Lemma 1.6. *$C_k(\pi^{-1}B) \cap F^p C_k(X') = C_k(B) \otimes \mathcal{I}^p C_0(G(J'))$.*

Proof. This follows from Proposition 8.5. By Proposition 8.6, the right hand side does not depend on the choice of isomorphism $\pi^{-1}\mathring{B} = \mathring{B} \times G(J')$. □

Proposition 1.7. *The homomorphism* $\pi_* : C_*(X') \to C_*(\overline{X})$ *induces an exact sequence*

$$0 \to F^1C_*(X') \to C_*(X') \to C_*(\overline{X}) \to 0.$$

Proof. Fix $\mathring{B} \subset \mathring{D}_{J'}$, $\dim \mathring{B} = k$, as above. It suffices to check the exactness for k-chains over B; that is, the exactness of the sequence

$$0 \to C_k(\pi^{-1}B) \cap F^1C_k(X') \to C_k(\pi^{-1}B) \to C_k(B) \to 0.$$

This follows from Lemma 1.6 and the definition of $\mathcal{I}^1$ as the kernel of the augmentation map $\epsilon : C_0(G(J')) \to \mathbb{Z}_2$; see the Appendix (8.1). □

Now we compute the successive quotients of the corner filtration. In Section 2 below we will use the following result to show that the $(\widehat{E}^1, \hat{d}^1)$ term of the corner spectral sequence is isomorphic to the Gysin complex of the divisor D (Corollary 2.3).

Proposition 1.8. *For each* $p \geq 0$ *there is an isomorphism of chain complexes*

$$\psi_p : \bigoplus_{|J|=p} C_*(D_J) \xrightarrow{\approx} \frac{F^pC_*(X')}{F^{p+1}C_*(X')}\ .$$

Proof. First we consider the case $p = 0$. By Proposition 1.7, $\pi_* : C_*(X') \to C_*(\overline{X})$ induces an isomorphism $\psi : C_*(\overline{X}) \to C_*(X')/F^1C_*(X')$ that can be described geometrically as follows. Given a chain $b \in C_k(\overline{X})$ represented by the set B, then $c = \psi(b)$ is represented modulo $F^1C_*(X')$ by the closure Γ of the image of any semialgebraic (not necessarily continuous) section of π over B.

Similarly we construct ψ_p for any $p \geq 0$. Let $b = [B] \in C_k(D_J)$, $p = |J|$, and let $\Gamma \subset X'$ be the closure of the image of any semialgebraic section of π over B. Then we define $\psi_p(b) = c \in F^pC_k(X')$ (mod $F^{p+1}C_k(X')$), where $c = [G(J)\Gamma]$. We have to show that ψ_p is well-defined, injective, surjective, and that it commutes with the boundary. For this we use the characterization of the corner filtration F^* given in Lemma 1.6 and the Appendix.

Let $b = [B]$, with $B \subset D_J$, and let Γ', $\Gamma'' \subset X'$ be the closures of the images of semialgebraic sections of π over B. By Corollary 1.5, after a subdivision of B we may suppose that B is the closure of $\mathring{B}$, where $\mathring{B} \subset \mathring{D}_{J'}$, $J \subset J'$, and that $\pi^{-1}\mathring{B}$ is isomorhpic to $\mathring{B} \times G(J')$ as a principal $G(J')$-bundle. Moreover, by a choice of this isomorphism, and another subdivision of B if necessary, we may also suppose that Γ' is the closure of $\mathring{\Gamma}'$, and Γ'' is the closure of $\mathring{\Gamma}''$, where $\mathring{\Gamma}' = \mathring{B} \times \{1\}$ and $\mathring{\Gamma}'' = \mathring{B} \times \{g\}$. If $g \in G(J)$ then $G(J)\mathring{\Gamma}' = G(J)\mathring{\Gamma}''$ so suppose $g \notin G(J)$. Let G' be the subgroup of $G(J')$ generated by $G(J)$ and g. Then $[G(J)\Gamma'] - [G(J)\Gamma''] = [G'\Gamma'] \in F^{p+1}C_k(X')$, by Lemma 1.6 and Lemma 8.1. This shows that ψ_p is well-defined.

We now show the injectivity of ψ_p. By a reduction as in the previous argument it suffices to show the following claim. Let $b = [B] \in C_k(D_{J'})$, $|J'| \geq p$, where B is the closure of $\mathring{B}$, with $\mathring{B} \subset \mathring{D}_{J'}$ and $\pi^{-1}\mathring{B} = \mathring{B} \times G(J')$. Let Γ be the closure of the image of a semialgebraic section of π over B. We claim that if

$$\sum_{J \subset J', |J|=p} a_J[G(J)\Gamma] \in F^{p+1}C_k(X'), \quad a_J \in \mathbb{Z}_2,$$

then $a_J = 0$ for all J. Now, in terms of the isomorphism (1.13),

$$\sum_{J \subset J', |J|=p} a_J[G(J)\Gamma] = [\Gamma] \otimes \left(\sum_{J \subset J', |J|=p} a_J[G(J)] \right),$$

so the claim follows from Corollary 8.3.

We now reinterpret the restriction of ψ_p to $C_*(D_J)$. We denote this restriction by ψ_J. Denote by $\psi_{J,0}$ the isomorphism of complexes from $C_*(D'_J)/F^1C_*(D'_J)$ to $C_*(D_J)$. Then $\psi_J = \eta'_J \circ (\psi_{J,0})^{-1}$, where η'_J equals η_J modulo $F^{p+1}C_*(X')$ (see (1.12)). It follows that ψ_J commutes with the boundary. Since the images of all η_J, $|J| = p$, generate $F^pC_*(X')$, the images of ψ_J, $|J| = p$, generate $F^pC_*(X')/F^{p+1}C_*(X')$. This shows ψ_p is surjective. $\square$

1.3. **The weight filtration.** The *weight filtration* $\mathcal{W}_*$ of $C_*(X')$ is defined by

$$\begin{aligned}\mathcal{W}_pC_k(X') &= \mathrm{Ker}[\partial : \widehat{F}_{p+k}C_k(X') \to C_{k-1}(X')/\widehat{F}_{p+k-1}C_{k-1}(X')] \\ &= (\mathrm{Dec}\,\widehat{F})_pC_k(X'),\end{aligned} \tag{1.14}$$

where $\mathrm{Dec}\,\widehat{F}_*$ is the *Deligne shift* of the filtration $\widehat{F}_*$ (1.11) [5, (1.3.3], [12, A.50]. Thus the weight filtration $\mathcal{W}_*$ runs

$$0 = \mathcal{W}_{-n-1}C_k(X') \subset \cdots \subset \mathcal{W}_{-k-1}C_k(X') \subset \mathcal{W}_{-k}C_k(X') = C_k(X'). \tag{1.15}$$

We denote this filtered complex by $\mathcal{W}C_*(X')$. It is the *weight complex* of the good compactification $\overline{X}$ of X.

The *weight spectral sequence* $E^r_{p,q}$ is the spectral sequence associated to the weight complex. It is a second quadrant spectral sequence: If $E^r_{p,q} \neq 0$ then (p, q) lies in the closed triangle with vertices $(0, 0)$, $(-n, 2n)$, $(-n, n)$, $n = \dim X$. We have

$$E^r_{p,q} = \widehat{E}^{r+1}_{2p+q,-p}$$

for all $r \geq 1$ and all p, q. In particular the E^1 term of the weight spectral sequence equals the reindexed $\widehat{E}^2$ term of the corner spectral sequence. The weight spectral sequence converges to the homology of the corner compactification X',

$$E^r_{p,q} \implies H_{p+q}(X').$$

In Section 7 below we will prove that, up to filtered quasi-isomorphism, the weight complex is independent of the good compactification of X. Thus the induced filtration on $H_*(X)$ and all the terms of the weight spectral sequence $(E^r_{p,q}, d^r)$ for $r \geq 1$ are algebraic invariants of X.

2. The Čech and Gysin complexes

We show that the corner complex of a good compactification $(\overline{X}, D)$ of X is filtered quasi-isomorphic to the cohomology Čech complex of the divisor D. It follows that the term $(\widehat{E}^1, \hat{d}^1)$ of the corner spectral sequence is isomorphic to the Gysin complex of the divisor D.

The semialgebraic cohomology groups of a variety Y are dual to the semialgebraic homology groups of Y (coefficients in $\mathbb{Z}_2$). The cohomology groups $H^k(Y)$, $k \geq 0$, are the homology groups of the semialgebraic cochain complex $(C^*(Y), \delta)$, where $C^k(Y) = \hom(C_k(Y), \mathbb{Z}_2)$ and the coboundary map $\delta_k : C^k(Y) \to C^{k+1}(Y)$ is the adjoint of the boundary map $\partial_{k+1} : C_{k+1}(Y) \to C_k(Y)$.

2.1. **The Čech complex.** Consider the double complex

$$C^{p,q} = \bigoplus_{|J|=p} C^q(D_J) \tag{2.1}$$

with first differential $\delta' : C^{p,q} \to C^{p+1,q}$ the sum of the restriction maps $C^q(D_J) \to C^q(D_{J'})$ ($|J| = p$, $|J'| = p+1$, and $J \subset J'$) (1.8), and second differential $\delta'' : C^{p,q} \to C^{p,q+1}$ the sum of the coboundary maps $C^q(D_J) \to C^{q+1}(D_J)$. The cohomology *Čech complex* of $(\overline{X}, D)$ is the complex $\check{C}^l(\overline{X}, D) = \bigoplus_{p+q=l} C^{p,q}$ with differential $\delta = \delta' + \delta''$ and decreasing filtration $F^p\check{C}^l(\overline{X}, D) = \bigoplus_{j\geq p}\bigoplus_{j+q=l} C^{j,q}$. The cohomology spectral sequence $\check{E}_r^{p,q}$ associated to this filtration [7, chapter XI, section 8] satisfies

$$\check{E}_1^{p,q} = \bigoplus_{|J|=p} H^q(D_J),$$

where the differential $\check{d}_1^{p,q} : \check{E}_1^{p,q} \to \check{E}_1^{p+1,q}$ is equal to the sum of the restriction maps $H^q(D_J) \to H^q(D_{J'})$. This spectral sequence converges to the relative cohomology of the pair $(\overline{X}, D)$,

$$\check{E}_1^{p,q} \implies H^{p+q}(\overline{X}, D).$$

2.2. **The Gysin complex.** If $f : M \to N$ is a continuous map of compact manifolds without boundary, the *Gysin homomorphism* $f^* : H_*(N) \to H_*(M)$ is defined as follows. Let $m = \dim M$ and $n = \dim N$, and for all $l \geq 0$ let $\mathcal{D}_M : H^l(M) \to H_{m-l}(M)$ and $\mathcal{D}_N : H^l(N) \to H_{n-l}(N)$ be the Poincaré duality isomorphisms. For all $k \geq 0$ we let

$$f^* = \mathcal{D}_M \circ H^{n-k}(f) \circ \mathcal{D}_N^{-1} : H_k(N) \to H_{k+m-n}(M),$$

where $H^{n-k}(f) : H^{n-k}(N) \to H^{n-k}(M)$ is the homomorphism induced by f on cohomology.

The *Gysin complex* $G(\overline{X}, X)$ of the good compactification $(\overline{X}, D)$ of X is the chain complex

$$G_p(\overline{X}, X) = \bigoplus_{|J|=p}\bigoplus_{k} H_k(D_J) \tag{2.2}$$

with differential d, where $d_p : G_p(\overline{X}, X) \to G_{p+1}(\overline{X}, X)$ is the sum of the Gysin maps $i^*_{J,J'} : H_k(D_J) \to H_{k-1}(D_{J'})$ for $J \subset J'$ with $|J| = p$ and $|J'| = p+1$, and $i_{J,J'} : D_{J'} \to D_J$ is the inclusion.

Note that while the Čech complex is the filtered complex associated to a double complex, the Gysin complex is a simple complex without filtration.

Proposition 2.1. *The $\check{E}_1$ term of the Čech spectral sequence of a good compactification of X is canonically isomorphic to the Gysin complex,*

$$(\check{E}_1^{p,*}, \check{d}_1^{p,*}) \cong (G_p(\overline{X}, X), d_p).$$

Proof. The isomorphism $\check{E}_1^{p,*} \to G_p(\overline{X}, X)$ is the sum of the Poincaré duality isomorphisms $H^q(D_J) \to H_{n-p-q}(D_J)$, where $n = \dim X$. □

2.3. **Poincaré-Lefschetz duality.** The following isomorphism corresponds to the classical duality isomorphism $H^{n-k}(\overline{X}, D) \cong H_k(X)$, $n = \dim X$, $k \geq 0$.

Theorem 2.2. *Let $(\overline{X}, D)$ be a good compactification of X, and let X' be the associated corner compactification of X. There is a quasi-isomorphism of filtered complexes*

$$\Psi : (\check{C}^*(\overline{X}, D), F^*) \to (C_*(X'), F^*)$$

from the cohomology Čech complex to the corner complex. More precisely, there is a chain homomorphism $\Psi = (\Psi_l)$, $\Psi_l : \check{C}^l(\overline{X}, D) \to C_{n-l}(X')$, such that for all $p, l \geq 0$ we have

$$\Psi_l(F^p\check{C}^l(\overline{X}, D)) \subset F^pC_{n-l}(X'),$$

and for all $p \geq 0$ the resulting chain homomorphism

$$\Psi_* : \frac{F^p\check{C}^*(\overline{X}, D)}{F^{p+1}\check{C}^*(\overline{X}, D)} \to \frac{F^pC_{n-*}(X')}{F^{p+1}C_{n-*}(X')}$$

induces an isomorphism in homology.

Corollary 2.3. *The $\widehat{E}_1$ term of the corner spectral sequence of a good compactifcation of X is isomorphic to the Gysin complex of the divisor D at infinity. More precisely, for every $p, k \geq 0$ there is an isomorphism*

$$\widehat{E}^1_{-p,k+p} = H_k\left(\frac{F^pC_*(X')}{F^{p+1}C_*(X')}\right) \cong \bigoplus_{|J|=p} H_k(D_J).$$

Under this isomorphism the differential

$$\hat{d}^1_{-p,k+p} : H_k\left(\frac{F^pC_*(X')}{F^{p+1}C_*(X')}\right) \to H_{k-1}\left(\frac{F^{p+1}C_*(X')}{F^{p+2}C_*(X')}\right)$$

corresponds to the sum of the Gysin homomorphisms

$$i^*_{J,J'} : H_k(D_J) \to H_{k-1}(D_{J'}),$$

where $|J| = p$, $|J'| = p+1$, and $J \subset J'$.

Proof. This is an immediate consequence of Theorem 2.2 and Proposition 2.1. □

Now we turn to the proof of Theorem 2.2. We construct Ψ as the composition of the three filtered quasi-isomorphisms described in subsections 2.4, 2.5, 2.6 below.

2.4. **The simplicial Čech complex.** Let K be a semialgebraic triangulation of $\overline{X}$ such that for all $J \subset I$ the subvariety $D_J = \bigcap_{i \in J} D_i$ is a subcomplex of K. There exists a unique triangulation K' of X' such that the map $\pi : X' \to \overline{X}$ is simplicial, and such a triangulation K' is semialgebraic. We say that (K', K) is an *adapted triangulation* of π.

Let (K', K) be an adapted triangulation of $\pi : X' \to \overline{X}$. For each $J \subset I$ let K_J be the subcomplex of K that triangulates D_J. Let $C^q(K_J)$ be the q-th simplicial cochain group of K_J. Consider the double complex

$$C^{p,q}(K) = \bigoplus_{|J|=p} C^q(K_J). \tag{2.3}$$

The *simplicial Čech complex* is the filtered complex defined by $\check{C}^l(K) = \bigoplus_{p+q=l} C^{p,q}(K)$, with filtration $F^p\check{C}^l(K) = \bigoplus_{j \geq p} \bigoplus_{j+q=l} C^{j,q}(K)$. The map of double complexes $C^{p,q} \to C^{p,q}(K)$ that is the sum of the chain maps $C^*(D_J) \to C^*(K_J)$ adjoint to the inclusion $C_*(K_J) \to C_*(D_J)$ defines a filtered quasi-isomorphism $\check{C}^*(\overline{X}, D) \to \check{C}^*(K)$ from the Čech complex (2.1) to the simplicial Čech complex (2.3).

2.5. **The cellular dual complex.** Let K^* be the dual cell complex of the simplicial complex K [10, §64]. For each $J \subset I$, the cells of the dual complex K_J^* of the triangulation K_J of D_J are the intersections with D_J of the cells of K^*. The dimension of the smooth variety D_J is $n-p$, where $p = |J|$. The double complex $C^{p,q}(K)$ is isomorphic to the double complex

$$C_{p,k}(K^*) = \bigoplus_{|J|=p} C_k(K_J^*) \tag{2.4}$$

via the classical cellular Poincaré duality isomorphism $C^q(K_J) \to C_{n-p-q}(K_J^*)$ which assigns to a simplex $\sigma \in K_J$ the dual cell $\sigma_J^* \in K_J^*$. The second differential of this double complex $\partial'' : C_{p,k}(K_J^*) \to C_{p,k-1}(K_J^*)$ is given by the cellular boundary map. The first differential $\partial' : C_{p,k}(K^*) \to C_{p+1,k-1}(K^*)$ is given by the cellular Gysin maps $i(J,J')^* : C_k(K_J^*) \to C_{k-1}(K_{J'}^*)$, where $|J| = p$, $|J'| = p+1$, and $J \subset J'$. By definition the Gysin map on cellular chains is Poincaré dual to the restriction map on simplicial cochains. If $\sigma \in K_{J'} \subset K_J$ then $i(J,J')^*(\sigma_J^*) = \sigma_{J'}^* = \sigma_J^* \cap D_{J'}$. Thus the Gysin map $i(J,J')^*$ applied to a cellular chain in K_J^* is the intersection of the chain with $D_{J'}$.

Thus there is a filtered chain isomorphism from the simplicial Čech complex to the *cellular dual complex*

$$\widetilde{C}_k(K^*) = \bigoplus_p C_{p,k}(K^*)$$

with boundary map $\partial = \partial' + \partial''$ and filtration $F^p\widetilde{C}_k(K^*) = \bigoplus_{j\geq p} C_{j,k}(K^*)$. The term (E^1, d^1) of the spectral sequence of this filtered complex is the Gysin complex (2.2).

2.6. **The cellular pullback.** We define a filtered quasi-isomorphism from the cellular dual complex to the corner complex,

$$\phi : \widetilde{C}_*(K^*) \to C_*(X') \ .$$

For $|J| = p$ we define $\phi : C_k(K_J^*) \to C_k(X')$ as follows. For each k-cell $B \in C_k(K_J^*)$ let $\phi(B) = [\pi^{-1}B] \in C_k(X')$. Since $\pi^{-1}B = \rho_J^{-1}(\pi_J^{-1}B)$ we have $[\pi^{-1}B] \in F^pC_*(X')$.

We claim that $\phi(\partial B) = \partial\phi(B)$ for all k-cells $B \in C_k(K_J^*)$, and so ϕ is a chain map. By definition $\partial B = \partial' B + \partial'' B$.

If $J' \supset J$ with $|J'| = p+1$ then $\dim(B \cap D_{J'}) = k-1$; in fact, $B \cap D_{J'}$ is a $(k-1)$-cell of $K_{J'}^*$. Thus $\partial' B = \sum_{J'} B \cap D_{J'}$, summed over all $J' \supset J$ with $|J'| = p+1$. Therefore

$$\phi(\partial' B) = \sum_{J'} \phi(B \cap D_{J'}) = \sum_{J'} [\pi^{-1}(B \cap D_{J'})] = [(\pi^{-1}B) \cap \partial(\pi^{-1}D_J)],$$

where $\partial(\pi^{-1}D_J) = \bigcup_{J'} \pi^{-1}D_{J'}$ is the boundary of the manifold $\pi^{-1}D_J$.

Let $\mathrm{bd}(B)$ denote the cellular boundary of the k-cell B. In other words, if $h : \mathbb{B}^k \to B$ is a semialgebraic homeomorphism from the unit ball in $\mathbb{R}^k$ onto B, then $\mathrm{bd}(B) = h(\partial\mathbb{B}^k)$. We have $\partial'' B = \sum B'$, summed over all $(k-1)$-cells B' of K_J^* with $B' \subset \mathrm{bd}(B)$. So

$$\phi(\partial'' B) = \sum \phi(B') = [\pi^{-1}\,\mathrm{bd}(B)].$$

On the other hand,

$$\partial\phi(B) = \partial[\pi^{-1}B] = [(\pi^{-1}B) \cap \partial(\pi^{-1}D_J)] + [\pi^{-1}\,\mathrm{bd}(B)],$$

which gives the claim.

Finally, we show that the chain map ϕ is a filtered quasi-isomorphism. For a k-cell B of K^*_J, the set $\pi_J^{-1}B$ is the image of a semialgebraic section of $\pi_J : D'_J \to D_J$ over B. Thus the induced map

$$\phi_p : \frac{F^p\widetilde{C}_*(K^*)}{F^{p+1}\widetilde{C}_*(K^*)} = \bigoplus_{|J|=p} C_*(K^*_J) \to \frac{F^pC_*(X')}{F^{p+1}C_*(X')}$$

factors as $\phi_p = \psi_p \circ \xi_p$, where

$$\xi_p : \bigoplus_{|J|=p} C_*(K^*_J) \to \bigoplus_{|J|=p} C_*(D_J)$$

is the inclusion $\xi_p(B) = [B]$, and

$$\psi_p : \bigoplus_{|J|=p} C_*(D_J) \to \frac{F^pC_*(X')}{F^{p+1}C_*(X')}$$

is given by Proposition 1.8. The chain map ξ_p induces an isomorphism in homology, and ψ_p is a chain isomorphism by Proposition 1.8. Thus ϕ_p induces an isomorphism in homology.

This completes the proof of Theorem 2.2.

2.7. **Duality with Borel-Moore homology.** If X is a smooth n-dimensional real algebraic variety with good compactification $(\overline{X}, D)$ and associated corner compactification X', the weight filtration (1.15) on the complex of semialgebraic chains $C_*(X')$ gives the weight filtration of the classical (compactly supported) homology groups $H_k(X)$, $0 \le k \le n$,

$$0 = \mathcal{W}_{-n-1}H_k(X) \subset \cdots \subset \mathcal{W}_{-k-1}H_k(X) \subset \mathcal{W}_{-k}H_k(X) = H_k(X). \tag{2.5}$$

We will show in section 7 that this filtration does not depend on the choice of good compactification of X. In previous work [9] we defined a weight filtration on the complex of semialgebraic chains with closed supports $C^{BM}_*(X)$ [9], which gives the weight filtration of the Borel-Moore homology groups $H^{BM}_{n-k}(X)$, $0 \le k \le n$,

$$0 = \mathcal{W}_{-n+k-1}H^{BM}_{n-k}(X) \subset \cdots \subset \mathcal{W}_{-1}H^{BM}_{n-k}(X) \subset \mathcal{W}_0H^{BM}_{n-k}(X) = H^{BM}_{n-k}(X). \tag{2.6}$$

For each k, $0 \le k \le n$, Poincaré-Lefschetz duality gives a nonsingular bilinear intersection pairing

$$\langle\ ,\rangle : H_k(X) \times H^{BM}_{n-k}(X) \to \mathbb{Z}_2.$$

We show that the weight filtrations (2.5) and (2.6) on these groups are dual under this pairing.

Theorem 2.4. *Let X be a smooth n-dimensional variety. For all $p \le 0$ and $k \ge 0$,*

$$\mathcal{W}_pH_k(X) = \{\alpha \in H_k(X) \mid \langle\alpha, \beta\rangle = 0 \textit{ for all } \beta \in \mathcal{W}_{-p-n-1}H^{BM}_{n-k}(X)\}.$$

Proof. This is a consequence of a more basic duality of filtered chain complexes. The weight filtration on $H_*(X)$ is induced by the weight filtration on the complex $C_*(X')$, where X' is the corner compactification of X. The weight filtration on $C_*(X')$ is by definition the Deligne shift of the corner filtration on $C_*(X')$ (1.14). The complex $C_*(X')$ with the corner filtration is in turn filtered quasi-isomorphic to the cohomology Čech complex $\check{C}^*(\overline{X}, D)$

with its standard filtration (Theorem 2.2). The cohomology Čech complex is dual to the homology Čech complex $\check{C}_*(\overline{X}, D)$, where

$$\check{C}_l(\overline{X}, D) = \bigoplus_{p+q=l} C_{p,q},$$

$$C_{p,q} = \bigoplus_{|J|=p} C_q(D_J),$$

$$F_p\check{C}_l(\overline{X}, D) = \bigoplus_{j\leq p}\bigoplus_{j+q=l} C_{j,q}.$$

Finally, the complex $C_*^{BM}(X)$ with its weight filtration is quasi-isomorphic to the homology Čech complex $\check{C}_*(\overline{X}, D)$ with the Deligne shift of the standard filtration [9, Theorem (1.1)(2), proof of Proposition (1.9)]. This last quasi-isomorphism corresponds to the isomorphism $H_l(\overline{X}, D) \cong H_l^{BM}(X)$. □

3. Functoriality

In this section we prove that the weight filtration is functorial for maps of pairs $(\overline{X}, X)$, where $\overline{X}$ is a good compactification of X. First we show that a regular map $(\overline{f}, f) : (\overline{X}, X) \to (\overline{Y}, Y)$ induces a semialgebraic map $f' : X' \to Y'$ of corner compactifications. The group actions on the principal bundles containing X' and Y' are used to prove that the chain map induced by f' preserves the corner filtration. Finally, we compute the corresponding homomorphism of Gysin complexes.

Let $f : X \to Y$ be a regular map of smooth varieties that extends to a regular map $\overline{f} : \overline{X} \to \overline{Y}$ of good compactifications. The divisor $D = \overline{X} \setminus X$ is a finite union of smooth codimension one subvarieties D_i, $i \in I_X$ (1.1), and the divisor $E = \overline{Y} \setminus Y$ is a finite union of smooth codimension one subvarieties E_j, $j \in I_Y$. In this section we assume that all the divisors D_i and E_j are *irreducible*.

For $x \in \overline{X}$ let $J(x) = \{i \in I_X \mid x \in D_i\}$, and for $y \in \overline{Y}$ let $J(y) = \{j \in I_Y \mid y \in E_j\}$. For every $x \in \overline{X}$ and $y \in \overline{Y}$, there exist good local coordinates $(U, (u_1, \ldots, u_n))$ on $(\overline{X}, D)$ with $(u_1(x), \ldots, u_n(x)) = 0$, and $(V, (v_1, \ldots, v_m))$ on $(\overline{Y}, E)$ with $(v_1(y), \ldots, v_m(y)) = 0$.

Let G_X be the group of functions $g : I_X \to \{1, -1\}$, and let G_Y be the group of functions $h : I_Y \to \{1, -1\}$. Let $G(x) = \{g \in G_X \mid g(i) = 1,\ i \notin J(x)\}$ and $G(y) = \{h \in G_Y \mid h(j) = 1,\ j \notin J(y)\}$.

Theorem 3.1. *Let X and Y be smooth real algebraic varieties with good compactifications $\overline{X}$ and $\overline{Y}$, and let $f : X \to Y$ be a regular map that extends to a regular map $\overline{f} : \overline{X} \to \overline{Y}$. Let X' and Y' be the corner compactifications associated to $\overline{X}$ and $\overline{Y}$. There exists a unique continuous semialgebraic map $f' : X' \to Y'$ such that $f'|X = f$. Moreover*

$$\overline{f} \circ \pi_X = \pi_Y \circ f',$$

where $\pi_X : X' \to \overline{X}$ and $\pi_Y : Y' \to \overline{Y}$ are the projections.

If Z is a smooth real algebraic variety with good compactification $\overline{Z}$, and $\xi : Y \to Z$ is a regular map that extends to a regular map $\overline{\xi} : \overline{Y} \to \overline{Z}$, then

$$(\xi \circ f)' = \xi' \circ f'.$$

Proof. Given $x' \in X'$, let $x = \pi_X(x')$ and $y = \overline{f}(x)$. Choose good coordinate neighborhoods U of x and V of y as above, with $\overline{f}(U) \subset V$. For $g \in G(x)$ and $h \in G(y)$ consider the sets $X_g(U) \subset X$, $\overline{X}_g(U) \subset \overline{X}$ and $Y_h(V) \subset Y$, $\overline{Y}_h(V) \subset \overline{Y}$ (1.7) and (1.9).

Let $X'(U) = \pi_X^{-1}U$, $X'_g(U) = \pi_X^{-1}\overline{X}_g(U)$, and $Y'(V) = \pi_Y^{-1}V$, $Y'_h(V) = \pi_Y^{-1}\overline{Y}_h(V)$. Then $X'(U) = \bigsqcup_g X'_g(U)$ and $Y'(V) = \bigsqcup_h Y'_h(V)$. Now $x' \in X'_{g_0}(U)$ for a unique $g_0 \in G(x)$. The open sets $X_g(U)$ and $Y_h(V)$ are connected, so there is a unique h_0 with $f(X_{g_0}(U)) \subset Y_{h_0}(V)$. Let y' be the unique element of $Y'_{h_0}(V)$ such that $\pi_Y(y') = y$, and set $f'(x') = y'$.

By construction $\overline{f}(\pi_X(x')) = \pi_Y(f'(x'))$, so $\overline{f}|X = f$, and the graph of f' is the closure in $X' \times Y'$ of the graph of f. Therefore f' is continuous and semialgebraic. The function f' is uniquely determined by f since X is dense in X'. It follows that if $f : X \to Y$ and $\xi : Y \to Z$ are as above, then $(\xi \circ f)' = \xi' \circ f'$. □

If $f : X \to Y$ is a regular map of smooth varieties that extends to a regular map $\overline{f} : \overline{X} \to \overline{Y}$ of good compactifications, with $D = \overline{X} \setminus X$ and $E = \overline{Y} \setminus Y$, then $\overline{f}^{-1}(E) \subset D$. Let $\overline{f}(x) = y$ and let U and V be good coordinate neighborhoods of x and y, respectively, as above, with $\overline{f}(U) \subset V$. Suppose that for $i \in J(x)$ the divisor $D_i \cap U$ of U is given by $u_{k_U(i)} = 0$, and for $j \in J(y)$ the divisor $E_j \cap V$ of V is given by $v_{k_V(j)} = 0$. For every $i \in J(x)$ and $j \in J(y)$ there are non-negative integers a_{ij} and a real analytic function $r_j : U \to \mathbb{R}$ such that on U we have

$$v_{k_V(j)} \circ \overline{f} = r_j \prod_{i \in I(x)} (u_{k_U(i)})^{a_{ij}}. \tag{3.1}$$

Moreover, $r_j^{-1}(0) \subset D$ and $\dim r_j^{-1}(0) \leq n-2$, and therefore r_j has constant sign on $U \setminus D$. Since the divisors D_i and E_j are irreducible, the exponents a_{ij} do not depend on the choice of x and y or on the choice of good local coordinates. Indeed, they are defined by the condition that the divisor

$$\overline{f}^{-1}(E_j) - \sum_{i \in I_X} a_{ij} D_i,$$

described locally by $r_j = 0$, has real part of dimension strictly less than $n-1$. (See Remark 3.7 for an example.) Thus the numbers a_{ij} are well-defined not only for the divisors D_i and E_j such that $D_i \cap \overline{f}^{-1}(E_j) \neq \emptyset$ by (3.1), but also we have that if $D_i \cap \overline{f}^{-1}(E_j) = \emptyset$, then $a_{ij} = 0$.

We define a homomorphism $\varphi : G(x) \to G(y)$ by

$$\varphi(g)(j) = \prod_{i \in J(x)} g(i)^{a_{ij}}. \tag{3.2}$$

Proposition 3.2. *Let $f : X \to Y$ be a regular map of smooth varieties that extends to a regular map $\overline{f} : \overline{X} \to \overline{Y}$ of good compactifications, and let $f' : X' \to Y'$ be the associated map of corner compactifications. If $\overline{f}(x) = y$, then*

$$f'(g \cdot x') = \varphi(g) \cdot f'(x')$$

for all $g \in G(x)$ and $x' \in \pi_X^{-1}(x)$.

Proof. Let $f(X_1(U)) \subset Y_h(V)$. Then by (3.1) we have $f(X_g(U)) \subset Y_{\varphi(g)h}(V)$, and this gives the proposition. □

Theorem 3.3. *Let $f : X \to Y$ be a regular map of smooth varieties that extends to a regular map $\overline{f} : \overline{X} \to \overline{Y}$ of good compactifications. If $f' : X' \to Y'$ is the associated map of corner compactifications, then for all $k, p \geq 0$,*

$$f'_*(F^p C_k(X')) \subset F^p C_k(Y').$$

Proof. By Lemma 1.6 and Corollary 8.3, it suffices to show the claim for $c \in F^pC_k(X')$ of the form

$$c = [B] \otimes [G(I)] \in C_k(B) \otimes C_0(G(I')),$$

where $I \subset I'$, $|I| = p$, and B is the closure of $\mathring{B}$, with $\mathring{B} \subset \mathring{D}_{I'}$. Suppose, moreover, that $\overline{f}(\mathring{B}) \subset \mathring{E}_{J'}$. We define a homomorphism $\varphi_{I'J'} : G(I') \to G(J')$ by

$$\varphi_{I'J'}(g)(j) = \prod_{i \in I'} g(i)^{a_{ij}}.$$

Suppose first that $\varphi_{I'J'}$ restricted to $G(I)$ is injective. Then by Proposition 3.2 we have

$$f'_*([B] \otimes [G(I)]) = \overline{f}_*([B]) \otimes [\varphi_{I'J'}(G(I))], \tag{3.3}$$

which lies in $F^pC_k(Y')$ by Lemma 8.1 and Proposition 8.5.

If $\varphi_{I'J'}$ restricted to $G(I)$ is not injective, then for every $x \in \mathring{B}$ the fibers of $f' : \{x\} \times G(I) \to \{\overline{f}(x)\} \times G(J')$ have even cardinality. Therefore the pushforward $f'_*([B] \otimes [G(I)])$ is equal to 0. □

By Proposition 1.8, f' induces a morphism of complexes

$$f_p : \bigoplus_{|I|=p} C_*(D_I) \to \bigoplus_{|J|=p} C_*(E_J). \tag{3.4}$$

We now show that f_p is a combination of pushforwards with weights.

To a pair (I, J) with $I \subset I_X$, $J \subset I_Y$, and $|J| = |I| = p$, we associate the number $a_{IJ} = \det(a_{ij})_{i \in I, j \in J}$.

Lemma 3.4. *Let D_I^0 be an irreducible component of D_I, and suppose $D_I^0 \cap \overline{f}^{-1}(E_J) \neq \emptyset$, $|I| = |J| = p$. If $D_I^0 \not\subset \overline{f}^{-1}(E_J)$ then $a_{IJ} = 0$.*

Proof. Let $x \in D_I^0$ and $y = \overline{f}(x) \in E_J$. Choose good local coordinates $(U, (u_1, \dots, u_n))$ on $(\overline{X}, D)$ and $(V, (v_1, \dots, v_m))$ on $(\overline{Y}, E)$ as above, with $\overline{f}(U) \subset V$, and such that $D_I \cap U = \{u_1 = \cdots = u_p = 0\}$ and $E_J \cap V = \{v_1 = \cdots = v_p = 0\}$. If $D_I^0 \not\subset \overline{f}^{-1}(E_J)$, then by (3.1) there exists $j \in \{1, \dots, p\}$ such that $a_{ij} = 0$ for all $i \in \{1, \dots, p\}$. Hence $a_{IJ} = 0$. □

Proposition 3.5. *Let $f : X \to Y$ be a regular map of smooth varieties that extends to a regular map $\overline{f} : \overline{X} \to \overline{Y}$ of good compactifications, with the divisors $D = \overline{X} \setminus X = \bigcup_{i \in I_X} D_i$, $E = \overline{Y} \setminus Y = \bigcup_{j \in I_Y} E_j$. Then for every $I \subset I_X$, $|I| = p$, and for every irreducible component D_I^0 of D_I, the morphism f_p of* (3.4) *restricted to D_I^0 is given by*

$$f_p | D_I^0 = \bigoplus_J a_{IJ} (\overline{f}^0_{IJ})_* \ , \tag{3.5}$$

where the sum is taken over all $J \subset I_Y$, $|J| = p$, such that $\overline{f}(D_I^0) \subset E_J$, with $\overline{f}^0_{IJ} : D_I^0 \to E_J$ the restriction of $\overline{f}$, and the other components of $f_p|D_I^0$ are zero.

Proof. Let $c \in C_k(D_I^0)$ and suppose that $c = [B] \otimes [G(I)]$, where B is the closure of $\mathring{B}$ and $\mathring{B} \subset \mathring{D}_{I'}$, $I \subset I'$, and that $\overline{f}(\mathring{B}) \subset \mathring{E}_{J'}$. If $\varphi_{I'J'}$ restricted to $G(I)$ is not injective then $f'_*(c) = 0$ and $a_{IJ} = 0$ for all $J \subset J'$.

Suppose now that $\varphi_{I'J'}$ restricted to $G(I)$ is injective. By formula (3.3) it suffices to decompose the image of $\varphi_{I'J'}(G(I))$ in $\mathcal{I}^p(G(J'))/\mathcal{I}^{p+1}(G(J'))$ with respect to the basis given by Corollary 8.3. Then (3.5) follows from Corollary 8.4 both in the case when $\overline{f}(D_I^0) \subset E_J$ and when $\overline{f}(D_I^0) \not\subset E_J$. Indeed, in the latter case the claim follows from the fact that $a_{IJ} = 0$ by Lemma 3.4. □

Recall that a good compactification gives rise to a Gysin complex defined by (2.2). Thus $\overline{f} : \overline{X} \to \overline{Y}$ induces a morphism of Gysin complexes $G(\overline{X}, X) \to G(\overline{Y}, Y)$ that can be computed using Proposition 3.5. We will encounter in the following sections several examples of morphisms of Gysin complexes that are simply the *homology pushforward*

$$\bigoplus_{|I|=p} H_*(D_I) \to \bigoplus_{|J|=p} H_*(E_J)$$

given by the sum of all the induced maps $D_I \to E_J$.

Corollary 3.6. *Let $f : X \to Y$ be a regular map of smooth varieties that extends to a regular map $\overline{f} : \overline{X} \to \overline{Y}$ of good compactifications. Suppose that for all p, and all $I \subset I_X$, $J \subset I_Y$ such that $|I| = |J| = p$, either*

(1) *$\overline{f}(D_I) \subset E_J$ and then $a_{IJ} = 1$, or*
(2) *$\dim D_I \cap \overline{f}^{-1}(E_J) < \dim D_I$.*

Then the induced morphism of Gysin complexes $G(\overline{X}, X) \to G(\overline{Y}, Y)$ is the homology pushforward.

Proof. This is an immediate consequence of Corollary 2.3 and Proposition 3.5. □

Remark 3.7. We say that $\overline{f}$ is a *monomial map* (with respect to the divisors D and E) if for every $x \in \overline{X}$ and $y = \overline{f}(x)$, the functions r_j of (3.1) are never zero. Then $\overline{f} : \overline{X} \to \overline{Y}$ is a topological *tico map* with respect to the ticos D and E [1, III.2]. ("Tico" stands for "transversely intersecting codimension one.") In this case the coefficients a_{ij} can be simply defined by

$$\overline{f}^{-1}(E_j) = \sum_i a_{ij} D_i \ .$$

In the complex algebraic case a regular map $\overline{f} : (\overline{X}, \overline{X} \setminus D) \to (\overline{Y}, \overline{Y} \setminus E)$, where D and E are divisors with normal crossings, is automatically monomial in this sense [1, p 176]. This is not true in the real algebraic case, as the following example shows.

Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, and $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2/(1 + y^2)$. Let $\overline{X} = \mathbb{P}^2$ with coordinates $[x : y : z]$, so that $D = \{z = 0\}$, and let $\overline{Y} = \mathbb{P}^1$ with coordinates $[s : t]$, so that $E = \{t = 0\}$. Let $\overline{f} : \mathbb{P}^2 \to \mathbb{P}^1$, $\overline{f}[x : y : z] = [x^2 : y^2 + z^2]$. Then $\overline{f}^{-1}(E)$ is a single point of D.

4. Blowup squares

In this section we analyze the homology of a classical blowup square. A key tool is the Leray-Hirsch theorem on the homology of a projectivized vector bundle. In sections 5 and 6 we will apply this special case to understand the behaviour of the weight filtration under a blowup with smooth center contained in a good compactification of a smooth variety.

A *blowup square* (also called an *elementary acyclic square*) is a cartesian diagram of compact irreducible nonsingular real algebraic varieties and regular morphisms

$$\begin{array}{ccc} E & \xrightarrow{s} & \widetilde{M} \\ \downarrow q & & \downarrow p \\ C & \xrightarrow{r} & M \end{array} \tag{4.1}$$

such that C is a subvariety of M with inclusion r, $\widetilde{M}$ is the blowup of M with center C and projection p, and $E = p^{-1}(C)$ is the exceptional divisor.

In what follows we suppose $\dim C < \dim M$.

Lemma 4.1. *The composition $p_* \circ p^*$ is the identity map, so $p_* : H_*(\widetilde{M}) \to H_*(M)$ is surjective and $p^* : H_*(M) \to H_*(\widetilde{M})$ is injective.*

Proof. If $\alpha \in H_*(M)$, by Poincaré duality there exists $\beta \in H^*(M)$ with $\alpha = \beta \frown [M]$. Then $p^*(\alpha) = p^*(\beta) \frown [\widetilde{M}]$, and $p_*[\widetilde{M}] = [M]$ since p has degree 1. Thus

$$p_*p^*(\alpha) = p_*(p^*(\beta) \frown [\widetilde{M}]) = \beta \frown p_*[\widetilde{M}] = \beta \frown [M] = \alpha.$$

□

Proposition 4.2. *Given a blowup square* (4.1)*, for every $k > 0$ there is a short exact sequence*

$$0 \to H_k(E) \xrightarrow{i_*} H_k(C) \oplus H_k(\widetilde{M}) \xrightarrow{j_*} H_k(M) \to 0, \tag{4.2}$$

where $i_*(\alpha) = (q_*(\alpha), s_*(\alpha))$ and $j_*(\beta, \gamma) = r_*(\beta) + p_*(\gamma)$. Moreover, q_* is surjective and s_* induces an isomorphism $\ker q_* \xrightarrow{\simeq} \ker p_*$

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
\longrightarrow & H_{k+1}(\widetilde{M}, E) & \longrightarrow & H_k(E) & \xrightarrow{s_k} & H_k(\widetilde{M}) & \longrightarrow & H_k(\widetilde{M}, E) & \longrightarrow \\
 & \downarrow p'_{k+1} & & \downarrow q_k & & \downarrow p_k & & \downarrow p'_k & \\
\longrightarrow & H_{k+1}(M, C) & \longrightarrow & H_k(C) & \xrightarrow{r_k} & H_k(M) & \longrightarrow & H_k(M, C) & \longrightarrow
\end{array}$$

The rows are exact, the maps p'_k are isomorphisms, and the maps p_k are surjective by Lemma 4.1. The proposition is proved by a diagram chase. □

The exactness of the sequence (4.2) can be paraphrased by saying that the square

$$\begin{array}{ccc}
H_k(E) & \xrightarrow{s_*} & H_k(\widetilde{M}) \\
\downarrow q_* & & \downarrow p_* \\
H_k(C) & \xrightarrow{r_*} & H_k(M)
\end{array} \tag{4.3}$$

is commutative and acyclic.

Corollary 4.3. *Given a blowup square* (4.1)*, for every $k > 0$ there is a short exact sequence*

$$0 \leftarrow H^k(E) \xleftarrow{i^*} H^k(C) \oplus H^k(\widetilde{M}) \xleftarrow{j^*} H^k(M) \leftarrow 0, \tag{4.4}$$

where $i^*(\beta, \gamma) = q^*(\beta) + s^*(\gamma)$ and $j^*(\delta) = (r^*(\delta), p^*(\delta))$. Moreover, q^* is injective and s^* induces an isomorphism $\operatorname{im} q^* \xleftarrow{\simeq} \operatorname{im} p^*$.

The exactness of the sequence (4.4) says that the square

$$\begin{array}{ccc}
H^k(E) & \xleftarrow{s^*} & H^k(\widetilde{M}) \\
\uparrow q^* & & \uparrow p^* \\
H^k(C) & \xleftarrow{r^*} & H^k(M)
\end{array} \tag{4.5}$$

is commutative and acyclic. Equivalently, if $\dim M - \dim C = m > 0$, the square of Gysin homomorphisms

$$\begin{array}{ccc} H_{k-1}(E) & \overset{s^*}{\longleftarrow} & H_k(\widetilde{M}) \\ \uparrow q^* & & \uparrow p^* \\ H_{k-m}(C) & \overset{r^*}{\longleftarrow} & H_k(M) \end{array} \tag{4.6}$$

is commutative and acyclic.

Lemma 4.4.
(1) $H_k(\widetilde{M}) = \ker p_* \oplus \operatorname{im} p^*$.
(2) $H_{k-1}(E) = \operatorname{im} q^* \oplus s^*(\ker p_*)$.

Proof. (1) follows from Lemma 4.1. We prove (2) as follows. Let $\alpha \in H_{k-1}(E)$. By Corollary 4.3 there are $\beta \in H_{k-m}(C)$ and $\gamma \in H_k(\widetilde{M})$ and such that $\alpha = q^*(\beta) + s^*(\gamma)$. Then $\gamma_1 = \gamma - p^*p_*(\gamma) \in \ker p_*$ and $\alpha = q^*(\beta + r^*p_*(\gamma)) + s^*(\gamma_1)$. If $q^*(\beta) + s^*(\gamma) = 0$ then, by Corollary 4.3, $\beta \in \operatorname{im} r^*$ and $\gamma \in \operatorname{im} p^*$. If, moreover, $\gamma \in \ker p_*$, then since $\ker p_* \cap \operatorname{im} p^* = 0$ we have $\gamma = 0$. □

Theorem 4.5. *Let $m = \dim M - \dim C$. For all $k > 0$ there is a unique homomorphism $\widetilde{q}_* : H_{k-1}(E) \to H_{k-m}(C)$ such that $\widetilde{q}_* \circ q^*$ is the identity and the following diagram is commutative and acyclic:*

$$\begin{array}{ccc} H_{k-1}(E) & \overset{s^*}{\longleftarrow} & H_k(\widetilde{M}) \\ \downarrow \widetilde{q}_* & & \downarrow p_* \\ H_{k-m}(C) & \overset{r^*}{\longleftarrow} & H_k(M) \end{array} \tag{4.7}$$

Proof. By Lemma 4.4, $\alpha \in H_{k-1}(E)$ can be written uniquely $\alpha = q^*(\beta) + s^*(\gamma)$, where $\beta \in H_{k-m}(C)$ and $\gamma \in \ker p_*$. We require $\widetilde{q}_* q^*(\beta) = \beta$, and $\widetilde{q}_* s^*(\gamma) = r^*p_*(\gamma) = 0$, so we must have $\widetilde{q}_*(\alpha) = \beta$, and β is unique since q^* is injective. Straightforward computations using Lemma 4.4 and Corollary 4.3 give that (4.7) is commutative and the associated simple complex is exact. □

In the blowup square (4.1), the map $q : E \to C$ is the projectivization of the normal bundle of C in M. To give a geometric description of the homomorphism $\widetilde{q}_*$ we apply the classical *Leray-Hirsch Theorem*:

Theorem 4.6. *Let $A \to B$ be vector bundle of rank m, and let $\pi : \mathbb{P}(A) \to B$ be its projectivization. Let $e \in H^1(\mathbb{P}(A))$ be the Euler class of the tautological line bundle. The cohomology group $H^*(\mathbb{P}(A))$ is a free module over $H^*(B)$ with basis $1, e, e^2, \ldots, e^{m-1}$. In other words, every element $u \in H^*(\mathbb{P}(A))$ can be written uniquely*

$$u = \pi^*(u_0) + \pi^*(u_1) \smile e + \cdots + \pi^*(u_{m-1}) \smile e^{m-1},$$

where $u_0, u_1, \ldots, u_{k-1} \in H^*(B)$.

Proof. The proof uses the Leray-Serre spectral sequence [8, Theorem 5.10, p. 48]. □

If the base B of the vector bundle is a topological manifold of dimension b, then $\mathbb{P}(A)$ is a manifold of dimension $b + m - 1$, and by Poincaré duality the Leray-Hirsch Theorem gives that every element $\alpha \in H_*(\mathbb{P}(A))$ can be written uniquely

$$\alpha = \pi^*(a_0) + e \frown \pi^*(a_1) + \cdots + e^{m-1} \frown \pi^*(a_{m-1}), \tag{4.8}$$

where $a_0, a_1, \dots, a_{m-1} \in H_*(B)$.

Lemma 4.7. *If $\pi : \mathbb{P}(A) \to B$ is the projectivization of an m-plane bundle and $\alpha \in H_*(\mathbb{P}(A))$ is given by* (4.8)*, then $\pi_*(\alpha) = a_{m-1}$.*

Proof. Suppose that $\alpha = e^i \frown \pi^*(a_i)$, $i \in \{0, \dots, m-1\}$, and $a_i = u_i \frown [B]$. Let $\varepsilon^i = e^i \frown [\mathbb{P}(A)]$. Then

$$\alpha = e^i \frown (\pi^*(u_i) \frown [\mathbb{P}(A)]) = \pi^*(u_i) \frown (e^i \frown [\mathbb{P}(A)]) = \pi^*(u_i) \frown \varepsilon^i$$

and so

$$\pi_*(\alpha) = \pi_* \left(\pi^*(u_i) \frown \varepsilon^i\right) = u_i \frown \pi_*(\varepsilon^i).$$

Now $\varepsilon^i \in H_{b+m-1-i}(\mathbb{P}(A))$, and if $i < m-1$ then $b + m - 1 - i > b$, so $\pi_*(\varepsilon^i) = 0$.

On the other hand, we claim that $\pi_*(\varepsilon^{m-1}) = [B]$, and so if $\alpha = e^{m-1} \frown \pi^*(a_{m-1})$ then $\pi_*(\alpha) = u_{m-1} \frown [B] = a_{m-1}$. Now $\pi_*(\varepsilon^{m-1}) \in H_b(B)$, and we have $\pi_*(\varepsilon^{m-1}) = [B]$ if and only if $\rho_x(\pi_*(\varepsilon^{m-1})) \neq 0$ for all $x \in B$, where $\rho_x : H_b(B) \to H_b(B, B \setminus \{x\})$ is the restriction map. There is a commutative square

$$\begin{array}{ccc} H_b(\mathbb{P}(A)) & \xrightarrow{\sigma_x} & H_b(\mathbb{P}(A), \mathbb{P}(A) \setminus \pi^{-1}(x)) \\ \downarrow \pi_* & & \downarrow \pi_* \\ H_b(B) & \xrightarrow{\rho_x} & H_b(B, B \setminus \{x\}) \end{array}$$

with $H_b(\mathbb{P}(A), \mathbb{P}(A) \setminus \pi^{-1}(x)) = H_b(B, B \setminus \{x\}) \otimes H_0(\pi^{-1}(x))$ and $\pi_*(\alpha \otimes \beta) = \phi(\beta)\alpha$, where $\phi : H_0(\pi^{-1}(x)) \to \mathbb{Z}_2$ is the augmentation isomorphism. By the local triviality of the bundle $\pi : \mathbb{P}(A) \to B$, we have

$$\begin{aligned} \sigma_x(\varepsilon^{m-1}) &= \sigma_x(e^{m-1} \frown [\mathbb{P}(A)]) \\ &= \rho_x[B] \otimes \left((e^{m-1}|\pi^{-1}(x)) \frown [\pi^{-1}(x)]\right) \\ &= \rho_x[B] \otimes \left((e|\pi^{-1}(x))^{m-1} \frown [\pi^{-1}(x)]\right). \end{aligned}$$

Now $\pi^{-1}(x) = \mathbb{P}^{m-1}$, and $e|\pi^{-1}(x)$ is the Euler class of the tautological line bundle, so $(e|\pi^{-1}(x))^{m-1} \neq 0$, and hence $\rho_x(\pi_*(\varepsilon^{m-1})) \neq 0$. □

Now we show that the homomorphism $\widetilde{q}_* : H_{k-1}(E) \to H_{k-m}(C)$ of (4.7) can be defined geometrically in terms of the excess bundle, which is defined as follows. Let $\mathcal{N}_C$ be the normal bundle of C in M and denote by $e(C) \in H^m(C)$ its Euler class (*i.e.* the top Stiefel-Whitney class). Similarly we denote by $\mathcal{N}_E$ the normal bundle of E in $\widetilde{M}$ and by $e(E) \in H^1(E)$ its Euler class. Then $q : E \to C$ is the projectivization of $\mathcal{N}_C$, and $\mathcal{N}_E$ is the tautological line bundle. The *excess bundle* is the quotient bundle $\mathcal{E} = q^*\mathcal{N}_C/\mathcal{N}_E$. The Euler class $e(\mathcal{E})$ satisfies $q^*e(C) = e(\mathcal{E})e(E)$.

Proposition 4.8. *Let $\alpha \in H_{k-1}(E)$. Then $\widetilde{q}_*(\alpha) = q_*(e(\mathcal{E}) \frown \alpha)$.*

Proof. Since $\alpha = q^*(\beta) + s^*(\gamma)$, where $\beta \in H_{k-m}(C)$ and $\gamma \in \ker p_*$, it suffices to consider two cases, $\alpha = q^*(\beta)$ or $\alpha = s^*(\gamma)$ with $\gamma \in \ker p_*$.

If $\alpha = q^*(\beta)$ then we have to show that $q_*(e(\mathcal{E}) \frown q^*(\beta)) = \beta$. The Whitney formula for the total Stiefel-Whitney class $w(q^*\mathcal{N}_C) = w(\mathcal{N}_E)w(\mathcal{E}) = (1 + e(E))w(\mathcal{E})$ yields

$$e(\mathcal{E}) = w_{m-1}(\mathcal{E}) = \sum_{i=0}^{m-1} e(E)^{m-1-i} \smile q^*(w_i(\mathcal{N}_C)).$$

Therefore by Lemma 4.7,

$$\begin{aligned} q_*(e(\mathcal{E}) \frown q^*(\beta)) &= q_* \left(\sum_{i=0}^{m-1} (e(E)^{m-1-i} \smile q^*(w_i(\mathcal{N}_C))) \frown q^*(\beta)) \right) \\ &= \sum_{i=0}^{m-1} q_* \left(e(E)^{m-1-i} \frown (q^*(w_i(\mathcal{N}_C)) \frown q^*(\beta)) \right) \\ &= q_* \left(e(E)^{m-1} \frown (q^*(w_0(\mathcal{N}_C) \frown q^*(\beta))) \right) \\ &= \beta. \end{aligned}$$

If $\alpha = s^*(\gamma)$ with $\gamma \in \ker p_*$, then we have to show that $q_*(e(\mathcal{E}) \frown s^*(\gamma)) = 0$. By Proposition 4.2 there is $\widetilde{\alpha} \in H_k(E)$ such that $\gamma = s_*(\widetilde{\alpha})$ and $q_*(\widetilde{\alpha}) = 0$. Therefore $\alpha = s^*(\gamma) = s^*(s_*(\widetilde{\alpha})) = e(E) \frown \widetilde{\alpha}$, and so we have

$$\begin{aligned} q_*(e(\mathcal{E}) \frown s^*(\gamma)) &= q_*((e(\mathcal{E}) \smile e(E)) \frown \widetilde{\alpha}) \\ &= q_*(q^*(e(C)) \frown \widetilde{\alpha}) \\ &= e(C) \frown q_*(\widetilde{\alpha}) \\ &= 0. \end{aligned}$$

This completes the proof of the Proposition. □

5. Blowup with center transverse to the divisor at infinity

In section 7 we will apply the main theorem of Guillén and Navarro Aznar [6] to extend our weight filtration to singular varieties. Their key extension criterion describes the behavior of the weight filtration for the blowup of a good compactification with center transverse to the divisor at infinity. In the present section we verify this extension criterion. The key result is the acyclicity of the Gysin diagram (5.3) of a blowup square.

Let X be a smooth n-dimensional variety and let $W = \overline{X}$ be a good compactification of X, with $W \setminus X = D$ a divisor with normal crossings, so that $D = \bigcup_{i \in I} D_i$, where D_i are smooth hypersurfaces meeting transversely. Let Z be an irreducible smooth m-dimensional subvariety of X and let $Y = \overline{Z}$ be the closure of Z in W. Suppose that Y is a smooth subvariety of W such that Y has normal crossings with D [6, (2.3.1)] and $Y \not\subset D$. Then for every $x \in W$ there is a good local coordinate system $(U, (u_1, \ldots, u_n))$ about x, and for each $i \in J(U)$ there is an index $k(i) \in \{1, \ldots, n\}$, $k(i) \leq m$, such that $D_i \cap U$ is the coordinate hyperplane $u_{k(i)} = 0$, and $Y \cap U$ is given by $u_{m+1} = \cdots = u_n = 0$. Thus Y is transverse to the divisor D [1, III.3].

From this data we obtain the blowup square of pairs $(W_\bullet, X_\bullet)$:

$$(5.1) \qquad \begin{array}{ccc} (\widetilde{Y}, \widetilde{Z}) & \longrightarrow & (\widetilde{W}, \widetilde{X}) \\ \downarrow & & \downarrow b \\ (Y, Z) & \xrightarrow{a} & (W, X) \end{array}$$

Here a is the inclusion, b is the blowup of (W, X) along (Y, Z), and $(\widetilde{Y}, \widetilde{Z}) = b^{-1}(Y, Z)$. Since Y has normal crossings with D, it follows that $\widetilde{W}$, Y, and $\widetilde{Y}$ are good compactifications of $\widetilde{X}$, Z, and $\widetilde{Z}$, respectively.

Theorem 5.1. Blowup with center transverse to the divisor at infinity. *Given a blowup square of pairs* (5.1)*, the corresponding square of corner compactifications induces an acyclic diagram of weight complexes*

$$\begin{array}{ccc} \mathcal{W}C_*(\widetilde{Z}') & \longrightarrow & \mathcal{W}C_*(\widetilde{X}') \\ \downarrow & & \downarrow b'_* \\ \mathcal{W}C_*(Z') & \xrightarrow{a'_*} & \mathcal{W}C_*(X') \end{array}$$

In other words, the simple filtered complex of this diagram is quasi-isomorphic to the zero complex.

For the definition of the simple filtered complex of a diagram of filtered complexes, see [9, p. 125].

Recall that the weight filtration $\mathcal{W}_*$ (1.14) is the Deligne shift of the filtration $\widehat{F}_*$, where $\widehat{F}_{-p} = F^p$, and F^* is the corner filtration (1.10) (1.11) (1.14). Thus to prove the theorem it suffices to show that the spectral sequence of the simple filtered complex associated to the diagram

$$\begin{array}{ccc} (C_*(\widetilde{Z}'), F^*) & \longrightarrow & (C_*(\widetilde{X}'), F^*) \\ \downarrow & & \downarrow b'_* \\ (C_*(Z'), F^*) & \xrightarrow{a'_*} & (C_*(X'), F^*) \end{array} \tag{5.2}$$

has trivial E^2 term. This in turn is equivalent to the statement that the simple complex $\mathbf{s}\, G(W_\bullet, X_\bullet)$ associated to the diagram of Gysin complexes

$$\begin{array}{ccc} G(\widetilde{Y}, \widetilde{Z}) & \longrightarrow & G(\widetilde{W}, \widetilde{X}) \\ \downarrow & & \downarrow b_* \\ G(Y, Z) & \xrightarrow{a_*} & G(W, X) \end{array} \tag{5.3}$$

is acyclic. By Corollary 3.6 the arrows in (5.3) are homology pushforward. We will prove that the complex $\mathbf{s}\, G(W_\bullet, X_\bullet)$ is acyclic by induction on the *complexity* of the divisor $D = W \setminus X$, which is defined as follows.

Let D be a divisor of the compact nonsingular variety W, and suppose that D has simple normal crossings (1.1). A *nonsingular decomposition* of D is a set $\mathcal{D} = \{D_i\}_{i \in I}$ of nonsingular divisors of W such that $D = \bigcup_{i \in I} D_i$. The *complexity* $c(D)$ of the divisor D is the minimum cardinality of a nonsingular decomposition of D.

If the divisor D has simple normal crossings in W, there is a one-to-one correspondence between nonsingular decompositions of D and partitions of the set $\mathcal{C}(D)$ of irreducible components of D such that if C_i and C_j belong to the same member of the partition then $C_i \cap C_j = \emptyset$. The nonsingular decomposition $\mathcal{D} = \{D_i\}_{i \in I}$ corresponds to the partition $\{\mathcal{D}_i\}_{i \in I}$ of $\mathcal{C}(D)$, where $\mathcal{D}_i = \{C_j \mid C_j \subset D_i\}$.

Remark 5.2. If D is a simple normal crossing divisor of W, let $\Gamma(D)$ be the corresponding graph. The vertices of $\Gamma(D)$ are the irreducible components of D, and there is an edge of $\Gamma(D)$ between C_i and C_j if and only if $C_i \cap C_j \neq \emptyset$. Thus nonsingular decompositions of D are in one-to-one correspondence with *graph partitions* of $\Gamma(D)$, and the complexity $c(D)$ is the *chromatic number* of $\Gamma(D)$.

Now the inductive proof of Theorem 5.1 proceeds as follows. In the base case $c(D) = 0$ the divisor D is empty, and the diagram (5.3) reduces to

$$\begin{array}{ccc} H_*(\widetilde{Y}) & \longrightarrow & H_*(\widetilde{W}) \\ \downarrow & & \downarrow \\ H_*(Y) & \longrightarrow & H_*(W) \end{array}$$

which is acyclic by Proposition 4.2.

Now suppose that $c(D) > 0$. Let $\mathcal{D} = \{D_i\}_{i \in I}$ be a nonsingular decomposition of D with $|\mathcal{D}| = c(D)$. Let $D = D'' \cup V$ and $D' = D'' \cap V$, where $V = D_0 \in \mathcal{D}$. The cubical diagram $(D_J)_{J \subset I} \to W$ $(J \neq \emptyset)$ is equal to the diagram

$$\begin{array}{ccc} (D'_J)_{0 \notin J} & \longrightarrow & V \\ \downarrow & & \downarrow \\ (D''_J)_{0 \notin J} & \longrightarrow & W \end{array}$$

where the vertical maps are inclusions. It follows from the definition of the homological Gysin complex that this diagram yields a short exact sequence of chain complexes,

$$0 \to G(V, V \setminus D')[1] \to G(W, W \setminus D) \to G(W, W \setminus D'') \to 0.$$

Blowing up along Y transverse to D we obtain a short exact sequence of chain complexes

$$0 \to \mathbf{s}\, G(V_\bullet, (V \setminus D')_\bullet)[1] \to \mathbf{s}\, G(W_\bullet, (W \setminus D)_\bullet) \to \mathbf{s}\, G(W_\bullet, (W \setminus D'')_\bullet) \to 0.$$

Now $\mathcal{D}'' = \mathcal{D} \setminus \{V\}$ is a nonsingular decomposition of D'' with $|\mathcal{D}''| = |\mathcal{D}| - 1 = c(D) - 1$, so $c(D'') < c(D)$. Also $\mathcal{D}' = \{D_i \cap V \mid i \neq 0\}$ is a nonsingular decomposition of D' with $|\mathcal{D}'| = |\mathcal{D}| - 1$, so $c(D') < c(D)$. Thus by induction on $c(D)$ the complexes $\mathbf{s}\, G(V_\bullet, (V \setminus D')_\bullet)$ and $\mathbf{s}\, G(W_\bullet, (W \setminus D'')_\bullet)$ are acyclic. It follows that $\mathbf{s}\, G(W_\bullet, (W \setminus D)_\bullet)$ is acyclic, as desired. This completes the proof of Theorem 5.1.

6. Blowup with center contained in the divisor at infinity

To see that the weight filtration of a smooth variety X does not depend on the choice of a good compactification $\overline{X}$ we show that the weight filtration is invariant, up to quasi-isomorphism, under a blowup of $\overline{X}$ with center contained in the divisor D at infinity. This follows from the fact that the corresponding homomorphism of Gysin complexes induces an isomorphism in homology.

Again let $W = \overline{X}$ be a good compactification of the smooth variety X, and let $D = W \setminus X$. Let Y be an irreducible smooth m-dimensional subvariety of W such that $Y \subset D$, and suppose that Y has normal crossings with D. Thus for every $x \in W$ there is a good coordinate system $(U, (u_1, \ldots, u_n))$ about x, with $Y \cap U$ given by $u_{m+1} = \cdots = u_n = 0$, such that for each $i \in J(U)$ there is an index $k(i) \in \{1, \ldots, n\}$ with $D_i \cap U$ the coordinate hyperplane $u_{k(i)} = 0$, and there exists $i \in I$ such that $k(i) > m$. Thus Y intersects the divisor D *cleanly* [1, III.3].

From this data we obtain the square

$$\begin{array}{ccc} (\widetilde{Y},\emptyset) & \longrightarrow & (\widetilde{W},\widetilde{X}) \\ \downarrow & & \downarrow b \\ (Y,\emptyset) & \xrightarrow{a} & (W,X) \end{array} \tag{6.1}$$

where a is the inclusion, b is the blowup of W along Y (so b maps $\widetilde{X}$ isomorphically onto X), and $\widetilde{Y} = b^{-1}(Y)$. Since Y has normal crossings with D, it follows that $\widetilde{W}$ is a good compactification of $\widetilde{X}$.

Theorem 6.1. Blowup with center contained in the divisor at infinity, clean intersection. *Given a blowup square of pairs* (6.1)*, the homomorphism*

$$b'_* : \mathcal{W}C_*(\widetilde{X}') \to \mathcal{W}C_*(X')$$

is a quasi-isomorphism of filtered complexes.

By definition of the weight filtration, to prove the theorem it suffices to show that the homomorphism of corner complexes

$$b'_* : (C_*(\widetilde{X}'), F^*) \to (C_*(X'), F^*)$$

induces an isomorphism on the E^2 term of the corner spectral sequence (1.11). This is equivalent to the statement that the corresponding homomorphism of Gysin complexes

$$b_* : G(\widetilde{W}, \widetilde{X}) \to G(W, X)$$

induces an isomorphism in homology.

First we prove the special case when the divisor D is nonsingular. This is the most involved part of the proof.

Let $W = \overline{X}$ be a good compactification of the smooth variety X, and suppose that $D = W \setminus X$ is a non-singular divisor in W. Let Y be an irreducible smooth m-dimensional subvariety of W such that $Y \subset D$. We assume that the codimension of Y in W is bigger than 1. From this data we obtain a blowup square of pairs (6.1), where the divisor $\widetilde{D} = \widetilde{W} \setminus \widetilde{X}$ is the union of the proper transform $\widehat{D}$ of the divisor D and the divisor $\widetilde{Y}$; *i.e.* $\widetilde{D} = \widehat{D} \cup \widetilde{Y}$. We let $\widehat{E} = \widehat{D} \cap \widetilde{Y}$.

Proposition 6.2. *The following diagram is commutative and acyclic,*

$$\begin{array}{ccccc} H_k(\widetilde{W}) & \xrightarrow{(s^*,\widetilde{a}^*)} & H_{k-1}(\widetilde{Y}) \oplus H_{k-1}(\widehat{D}) & \longrightarrow & H_{k-2}(\widehat{E}) \\ \downarrow p_* & & p_{1*} \downarrow p_{2*} & & \downarrow \\ H_k(W) & \xrightarrow{a^*} & H_{k-1}(D) & \longrightarrow & 0 \end{array} \tag{6.2}$$

where the horizontal arrows are Gysin morphisms and the vertical arrows are pushforward maps induced by p.

Proof. The bottom row of diagram (6.2) is the Gysin complex of $(W, W \setminus D)$ and the top row is the Gysin complex of $(\widetilde{W}, \widetilde{W} \setminus \widetilde{D})$. Thus the commutativity of (6.2) follows from Corollary 3.6.

To show the acyclicity of (6.2) we consider the following augmented version of (6.2),

$$
\begin{array}{ccccccc}
H_k(\widetilde{W}) & \longrightarrow & H_{k-1}(\widetilde{Y}) & \oplus & H_{k-1}(\widehat{D}) & \longrightarrow & H_{k-2}(\widehat{E}) \\
\downarrow p_* & & \downarrow \widetilde{q}_* & \searrow & \downarrow & & \downarrow \widetilde{q}'_* \\
H_k(W) & \longrightarrow & H_{k-m}(Y) & \oplus & H_{k-1}(D) & \longrightarrow & H_{k-m}(Y)
\end{array} \tag{6.3}
$$

where the horizontal arrows are Gysin morphisms. (In particular $H_{k-m}(Y) \to H_{k-m}(Y)$ is the identity.) The morphism $\widetilde{q}_*$, resp. $\widetilde{q}'_*$, is given by Theorem 4.5 for the blowup $p : \widetilde{W} \to W$, resp. $p' : \widehat{D} \to D$. Note that the augmented diagram (6.3) includes two acyclic squares of type (4.7). Taking into account the commutativity of (6.2), in order to establish the commutativity of (6.3) it suffices to show that the square

$$
\begin{array}{ccc}
H_{k-1}(\widetilde{Y}) & \overset{i^*_{\widehat{E},\widetilde{Y}}}{\longrightarrow} & H_{k-2}(\widehat{E}) \\
\widetilde{q}_* \downarrow p_{1*} & & \downarrow \widetilde{q}'_* \\
H_{k-m}(Y) \oplus H_{k-1}(D) & \overset{\mathrm{id} + i^*_{Y,D}}{\longrightarrow} & H_{k-m}(Y)
\end{array} \tag{6.4}
$$

is commutative, which we prove by considering two cases.

Case 1. Let $\beta = q^*\alpha \in H_{k-1}(\widetilde{Y})$, $\alpha \in H_{k-m}(Y)$. Then $\widetilde{q}_*\beta = \widetilde{q}_*q^*\alpha = \alpha$, $p_{1*}\beta = (i_{Y,D})_*q_*q^*\alpha = 0$, and $\widetilde{q}'_* i^*_{\widehat{E},\widetilde{Y}}\beta = \widetilde{q}'_*(q')^*\alpha = \alpha$.

Case 2. Let $\beta = s^*(\alpha) \in H_{k-1}(\widetilde{Y})$, $\alpha \in H_k(\widetilde{W})$. By the commutativity of the left hand subdiagram of (6.3) of type (4.7), $\widetilde{q}_*\beta = i^*_{Y,W}p_*\alpha$. Note that both the top and the bottom rows of (6.3) are complexes (*i.e.* the composition of two consecutive morphisms is zero). Indeed, they are the simple complexes of the Gysin diagrams associated to commutative squares. Hence $i^*_{\widehat{E},\widetilde{Y}}\beta + i^*_{\widehat{E},\widehat{D}}\widetilde{a}^*(\alpha) = 0$ and $\widetilde{q}_*\beta + i^*_{Y,D}(p_{1*}\beta + p_{2*}\widetilde{a}^*(\alpha)) = i^*_{Y,W}p_*\alpha + i^*_{Y,D}i^*_{D,W}p_*\alpha = 0$. Therefore we have $i^*_{\widehat{E},\widetilde{Y}}\beta = i^*_{\widehat{E},\widehat{D}}\widetilde{a}^*(\alpha)$ and $\widetilde{q}_*\beta + i^*_{Y,D}p_{1*}\beta = i^*_{Y,D}p_{2*}\widetilde{a}^*(\alpha)$. Now $\widetilde{q}'_* i^*_{\widehat{E},\widehat{D}}\widetilde{a}^*(\alpha) = i^*_{Y,D}p_{2*}\widetilde{a}^*(\alpha)$ by the commutativity of the right hand subdiagram of (6.3) of type (4.7).

By (b) of Lemma 4.4, $H_{k-1}(\widetilde{Y})$ is generated by $\operatorname{im} q^*$ and $\operatorname{im} s^*$. Thus the commutativity of (6.4) follows from cases 1 and 2.

Recall that to say that the diagram (6.3) is *acyclic* means that the associated simple complex is acyclic. Thus the diagram (6.3) is acyclic since it consists of two acyclic squares of type (4.7). More precisely, the simple complex of (6.3) is acyclic since it equals the simple complex of the following diagram with acyclic rows,

$$
\begin{array}{ccccccc}
H_k(\widetilde{W}) & \longrightarrow & H_{k-1}(\widetilde{Y}) & \oplus & H_k(W) & \longrightarrow & H_{k-m}(Y) \\
\downarrow & & \downarrow & \searrow & \downarrow & & \downarrow \\
H_{k-1}(\widehat{D}) & \longrightarrow & H_{k-2}(\widehat{E}) & \oplus & H_{k-1}(D) & \longrightarrow & H_{k-m}(Y)
\end{array}
$$

It follows that (6.2) is acyclic, since the diagrams (6.2) and (6.3) differ by the acyclic diagram $H_{k-m}(Y) \overset{\mathrm{id}}{\longrightarrow} H_{k-m}(Y)$. □

Now we prove Theorem 6.1 by induction on (r, c), where $r = r(D)$ is the number of smooth components D_i of the divisor D such that $Y \subset D_i$ and $c = c(D)$ is the complexity of D. (Since Y is irreducible, $r(D)$ equals the number of irreducible components of D

that contain Y, so $r(D)$ is independent of the nonsingular decomposition $D = \bigcup_i D_i$.) Proposition 6.2 is the base case $(r,c) = (1,1)$.

Suppose $r(D) = 1$ and $c(D) > 1$. Let $\mathcal{D}$ be a nonsingular decomposition of D with $|\mathcal{D}| = c(D)$. Let $D = D'' \cup V$, where $V = D_0 \in \mathcal{D}$ and $Y \not\subset V$, and let $D' = D'' \cap V$. We have a diagram with exact rows which is commutative by Corollary 3.6:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G(\widetilde{V}, \widetilde{V} \setminus \widetilde{D}')[1] & \longrightarrow & G(\widetilde{W}, \widetilde{W} \setminus \widetilde{D}) & \longrightarrow & G(\widetilde{W}, \widetilde{W} \setminus \widetilde{D}'') & \longrightarrow & 0 \\ & & \downarrow b'_* & & \downarrow b_* & & \downarrow b''_* & & \\ 0 & \longrightarrow & G(V, V \setminus D')[1] & \longrightarrow & G(W, W \setminus D) & \longrightarrow & G(W, W \setminus D'') & \longrightarrow & 0 \end{array} \tag{6.5}$$

Now $c(D') < c(D)$ and $c(D'') < c(D)$, so by induction on $c(D)$ the maps b'_* and b''_* are isomorphisms. Therefore b_* is an isomorphism.

Now suppose $r(D) > 1$. Let $D = D'' \cup V$, where $V = D_0 \in \mathcal{D}$ and $Y \subset V$, and let $D' = D'' \cap V$. Then $r(D') < r(D)$ and $r(D'') < r(D)$, so by induction on $r(D)$ the diagram (6.5) shows that b_* is an isomorphism. This completes the proof of Theorem 6.1.

7. Extension of the weight filtration to singular varieties

Following Guillén and Navarro Aznar [6], let $\mathbf{Sch}(\mathbb{R})$ be the category of reduced separated schemes of finite type over $\mathbb{R}$. In this paper we are interested in the topology of the set of real points of $X \in \mathbf{Sch}(\mathbb{R})$. The set $X(\mathbb{R})$ of real points of X with its sheaf of regular functions is a real algebraic variety in the sense of Bochnak-Coste-Roy [3].

By an *acyclic square* in $\mathbf{Sch}(\mathbb{R})$ we mean a cartesian diagram

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow p \\ Y & \xrightarrow{i} & X \end{array} \tag{7.1}$$

such that i is a closed immersion, p is proper, and p induces an isomorphism $\widetilde{X} \setminus \widetilde{Y} \to X \setminus Y$ [6, (2.1.1)].

Let $\mathbf{V}(\mathbb{R})$ be the category of nonsingular projective schemes over $\mathbb{R}$, and let $\mathbf{V}^2(\mathbb{R})$ be the category of pairs (W, X) such that $W \in \mathbf{V}(\mathbb{R})$, X is an open subscheme of W, and $D = W \setminus X$ is a divisor with normal crossings in W.

An *elementary acyclic square* in $\mathbf{V}^2(\mathbb{R})$ [6, (2.3.1)] is a diagram

$$\begin{array}{ccc} (\widetilde{Y}, \widetilde{Y} \cap \widetilde{X}) & \longrightarrow & (\widetilde{W}, \widetilde{X}) \\ \downarrow & & \downarrow p \\ (Y, Y \cap X) & \xrightarrow{i} & (W, X) \end{array} \tag{7.2}$$

such that $p : \widetilde{W} \to W$ is the blowup of W with smooth center Y that has normal crossings with the divisor $D = W \setminus X$, where $\widetilde{X} = p^{-1}X$ and $\widetilde{Y} = p^{-1}Y$. (The condition that Y has normal crossings with D includes both $Y \not\subset D$ and $Y \subset D$.)

Let $\mathcal{C}$ be the category of bounded complexes of $\mathbb{Z}_2$ vector spaces with increasing bounded filtration. Following [6] we denote by $Ho\,\mathcal{C}$ the category $\mathcal{C}$ localized with respect to filtered quasi-isomorphisms. By Proposition $(1.7.5)^{op}$ of [6], the category $\mathcal{C}$ with this notion of quasi-isomorphism and the simple complex operation for cubical diagrams is a category of homological descent [9, §1A].

A $\mathbf{\Phi}$-*rectification* of a functor G with values in a derived category $Ho\,\mathcal{C}$ is an extension of G to a functor of finite orderable diagrams, with values in the derived category of diagrams, satisfying certain naturality properties [6, (1.6.5)], [9, p. 125]. A factorization of G through the category $\mathcal{C}$ determines a canonical rectification of G.

We define a functor

$$\mathbf{F} : \mathbf{V}^2(\mathbb{R}) \to Ho\,\mathcal{C}$$

as follows. If $(W, X) \in \mathbf{V}^2(\mathbb{R})$, then the real algebraic variety $W(\mathbb{R})$ is a good compactification of $X(\mathbb{R})$. Let X' be the associated corner compactification of $X(\mathbb{R})$, and set

$$\mathbf{F}(W, X) = \mathcal{W}C_*(X'),$$

the weight complex of this good compactification (1.15). We have that $\mathbf{F}$ is a functor by the functoriality of the semialgebraic chain complex $C_*(X')$ [9, Appendix] and Theorem 3.3.

Theorem 7.1. *There is a covariant* $\mathbf{\Phi}$*-rectified functor*

$$\mathcal{W}C_* : \mathbf{Sch}(\mathbb{R}) \to Ho\,\mathcal{C}$$

such that

(1) if $(W, X) \in \mathbf{V}^2(\mathbb{R})$ *there is a natural isomorphism* $\mathcal{W}C_*(X) \cong \mathbf{F}(W, X)$*,*

(2) $\mathcal{W}C_*$ *satisfies the following acyclicity property: For an acyclic square (7.1) the simple filtered complex of the diagram*

$$\begin{array}{ccc} \mathcal{W}C_*(\widetilde{Y}) & \longrightarrow & \mathcal{W}C_*(\widetilde{X}) \\ \downarrow & & \downarrow \\ \mathcal{W}C_*(Y) & \longrightarrow & \mathcal{W}C_*(X) \end{array}$$

is acyclic (quasi-isomorphic to the zero complex).

Such a functor $\mathcal{W}C_*$ *is unique up to a unique quasi-isomorphism.*

Proof. This theorem follows from applying [6, Theorem $(2.3.6)^{op}$] to the functor $\mathbf{F}$. Since $\mathbf{F}$ factors through $\mathcal{C}$, it is automatically $\mathbf{\Phi}$-rectified [6, (1.6.5), (1.1.2)]. Clearly $\mathbf{F}$ is additive for disjoint unions (condition (2.1.5) (F1) of [6]). It remains to check condition (2.1.5) (F2) of [6]: Given an elementary acyclic square (7.2), the simple filtered complex associated to the square

$$\begin{array}{ccc} \mathbf{F}(\widetilde{Y}, \widetilde{Y} \cap \widetilde{X}) & \longrightarrow & \mathbf{F}(\widetilde{W}, \widetilde{X}) \\ \downarrow & & \downarrow \\ \mathbf{F}(Y, Y \cap X) & \longrightarrow & \mathbf{F}(W, X) \end{array}$$

is acyclic. This follows from our blowup results Theorem 5.1 and Theorem 6.1. □

Remark 7.2. This theorem shows not only that the weight complex functor extends to singular varieties, but also that, up to quasi-isomorphism, the weight complex (1.15) of a smooth real algebraic variety X does not depend on the choice of a good compactification.

Proposition 7.3. *For all* $X \in \mathbf{Sch}(\mathbb{R})$ *the homology of the weight complex* $\mathcal{W}C_*(X)$ *is the classical compactly supported homology of* X *with* $\mathbb{Z}_2$ *coefficients,*

$$H_*(\mathcal{W}C_*(X)) = H_*(X).$$

If X has dimension n, for each $k \geq 0$ the filtration of $H_k(X)$ given by this identification satisfies

$$0 = \mathcal{W}_{-n-1}H_k(X) \subset \mathcal{W}_{-n}H_k(X) \subset \cdots \subset \mathcal{W}_0 H_k(X) = H_k(X). \tag{7.3}$$

Proof. The proof of the first assertion is parallel to the proof of of [9, Proposition 1.5]. (One considers the forgetful functor from the category $\mathcal{C}$ to the category $\mathcal{D}$ of bounded complexes of $\mathbb{Z}_2$ vector spaces.)

The second assertion follows from the fact that the weight complex $\mathcal{W}C_*(X)$ can be computed as the simple filtered complex associated to the diagram of filtered complexes given by a *cubical hyperresolution* of X. This is the basic construction of Guillén and Navarro Aznar [6]. If $\dim X = n$ there is an n-cubical diagram $X_\bullet$ in $\mathbf{Sch}(\mathbb{R})$, *i.e.* a contravariant functor from the set of subsets of $\{0, \dots, n\}$ to $\mathbf{Sch}(\mathbb{R})$, with $X = X_\bullet(\emptyset)$, and $X_\bullet(S)$ smooth for $S \neq \emptyset$. For $q \geq 0$, if $X^{(q)}$ is the disjoint union of the smooth schemes $X_\bullet(S)$ for $|S| = q+1$, then $\dim X^{(q)} \leq n - q$, and we have

$$\begin{gathered} \mathcal{W}_i C_k(X) = \bigoplus_{l+q=k} \mathcal{W}_i C_l(X^{(q)}), \\ \partial : \mathcal{W}_i C_l(X^{(q)}) \to \mathcal{W}_i C_{l-1}(X^{(q)}) \oplus \mathcal{W}_i C_l(X^{(q-1)}), \end{gathered} \tag{7.4}$$

where $\partial c = \partial' c + \partial'' c$, with ∂' the boundary map of the chain complex $C_l(X^{(q)})$ and ∂'' the chain homomorphism induced by the map $X^{(q)} \to X^{(q-1)}$ given by the cubical diagram.

By (1.15) we have $\mathcal{W}_i C_l(X^{(q)}) = 0$ for $i < -\dim X^{(q)} = -n+q$ and $\mathcal{W}_i C_l(X^{(q)}) = C_l(X^{(q)})$ for $i \geq -l$. Since $q \geq 0$ and $l \geq 0$, we have $\mathcal{W}_i C_k(X) = 0$ for $i < -n$ and $\mathcal{W}_i C_k(X) = C_k(X)$ for $i \geq 0$. □

The filtration (7.3) is the *weight filtration* of the homology of X. It is an interesting problem to describe the relation of this filtration to Deligne's weight filtration [4] for the complex points of X.

If $X \in \mathbf{Sch}(\mathbb{R})$ let $\mathcal{W}C_*^{BM}(X)$ denote the weight complex of Borel-Moore chains (semialgebraic chains with closed supports) of $X(\mathbb{R})$ defined in [9, Theorem 1.1].

Proposition 7.4. *There is a natural transformation of functors $\theta : \mathcal{W}C_* \to \mathcal{W}C_*^{BM}$. If $X \in \mathbf{Sch}(\mathbb{R})$ the canonical homomorphism $\varphi_X : H_*(X) \to H_*^{BM}(X)$ is induced by the morphism $\theta_X : \mathcal{W}C_*(X) \to \mathcal{W}C_*^{BM}(X)$, and so φ_X is compatible with the weight filtrations. If X is compact (*i.e. *proper over $\mathbb{R}$) the morphism θ_X is a quasi-isomorphism.*

Proof. Let $\mathbf{Sch}^{\mathbf{2}}_{\mathbf{Comp}}(\mathbb{R})$ be the category of pairs (W, X), where W is compact and X is an open subscheme of W. Theorem (2.3.6) of [6] is proved in two steps. The first step is [6, Theorem (2.3.3)], the extension property for the inclusion $\mathbf{V}^2(\mathbb{R}) \to \mathbf{Sch}^{\mathbf{2}}_{\mathbf{Comp}}(\mathbb{R})$. By this theorem, our functor $\mathbf{F}$ on $\mathbf{V}^2(\mathbb{R})$ extends to a functor $\mathbf{F}'$ on $\mathbf{Sch}^{\mathbf{2}}_{\mathbf{Comp}}(\mathbb{R})$ satisfying the conditions of [6, Theorem (2.1.5)]. For the second step, the proof of [6, Theorem (2.3.6)] shows that restriction of $\mathbf{F}'$ to the second factor gives a well-defined functor $\mathcal{W}C_*$ on $\mathbf{Sch}(\mathbb{R})$ satisfying [6, (2.1.5)].

Thus we have a sequence of natural morphisms

$$\mathcal{W}C_*(X) \cong \mathbf{F}'(\overline{X}, X) \to \mathbf{F}'(\overline{X}, \overline{X}) \cong \mathcal{W}C_*(\overline{X}) \cong \mathcal{W}C_*^{BM}(\overline{X}) \to \mathcal{W}C_*^{BM}(X),$$

where $\overline{X}$ is any compactification of X, the first and second quasi-isomorphisms are given by the extension results described above, and the third quasi-isomorphism is given by [6, Corollary (2.3.7)]. If X is compact then we can take $\overline{X} = X$, in which case the second and fifth morphisms above are identities. □

Consider the weight filtration (7.3) of a variety X. If X is nonsingular and quasi-projective then, by (1.15), $\mathcal{W}_{-k}H_k(X) = H_k(X)$. If Y is compact then, by Proposition 7.4 and [9, p. 129], $\mathcal{W}_{-k-1}H_k(Y) = 0$. Thus if $f : X \to Y$ is a regular morphism from a nonsingular quasi-projective variety to a compact variety, then $\operatorname{im}[f_k : H_k(X) \to H_k(Y)] \subset \mathcal{W}_{-k}H_k(Y)$ and $\mathcal{W}_{-k-1}H_k(X) \subset \ker[f_k : H_k(X) \to H_k(Y)]$. Thus if $[c] \in \mathcal{W}_{-k-1}H_k(X)$ then c is a boundary in any algebraic compactification of X.

In the special case of a good compactification this result is sharp:

Proposition 7.5. *Let X be a nonsingular quasi-projective variety and let $i : X \to \overline{X}$ be the inclusion in a good compactification of X. Then for all $k \geq 0$, $\ker[i_k : H_k(X) \to H_k(\overline{X})] = \mathcal{W}_{-k-1}H_k(X)$. In particular, $\ker i_k$ does not depend on the choice of a good compactification.*

Proof. This follows from Proposition 1.7. □

Example 7.6. Let $X \subset \mathbb{R}^2$ be given by $xy \neq 0$. The embedding $X \subset \mathbb{P}^2(\mathbb{R})$ is a good compactification of X and $H_0(X) = (\mathbb{Z}_2)^4$, $\mathcal{W}_{-1}H_0(X) = (\mathbb{Z}_2)^3$, and $\mathcal{W}_{-2}H_0(X) = \mathbb{Z}_2$. Let $Y \subset \mathbb{R}^2$ be given by $x(x-1)(x+1) \neq 0$. The embedding $Y \subset \mathbb{P}^2(\mathbb{R})$ is not a good compactification since $\mathbb{P}^2(\mathbb{R}) \setminus Y$ is the union of four lines intersecting at one point. By blowing up this point we obtain a good compactification of Y. Then $H_0(Y) = (\mathbb{Z}_2)^4$, $\mathcal{W}_{-1}H_0(Y) = (\mathbb{Z}_2)^3$, and $\mathcal{W}_{-2}H_0(Y) = 0$. In particular X and Y are not isomorphic.

Example 7.7. Let $X = \mathbb{P}^1(\mathbb{R}) \times \mathbb{R}$. Then $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ is a good compactification of X and $\mathcal{W}_{-2}H_1(X) = 0$. (The generator of $H_1(X)$ is not a boundary in $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$.) Let $Y = \mathbb{R}^2 \setminus 0$. To obtain a good compactification of Y we embed Y in $\mathbb{P}^2(\mathbb{R})$ and blow up the origin. Then $\mathcal{W}_{-2}H_1(X) = \mathbb{Z}_2$. (The generator of $H_1(Y)$ is already a boundary in $\mathbb{P}^2(\mathbb{R})$.) In particular X and Y are not isomorphic.

Example 7.8. (1) The inclusion $\mathbb{R}^* \subset \mathbb{P}^1(\mathbb{R})$ is a good compactification of $\mathbb{R}^*$. Thus $\mathcal{W}_0H_0(\mathbb{R}^*) = H_0(\mathbb{R}^*) = (\mathbb{Z}_2)^2$ and $\mathcal{W}_{-1}H_0(\mathbb{R}^*) = \mathbb{Z}_2$.

(2) Let X be the Bernoulli lemniscate $\{(x^2+y^2)^2 = x^2 - y^2\} \subset \mathbb{R}^2$. The resolution of X is given by blowing up the origin $\pi : \widetilde{X} \to X$, and then $\widetilde{X}$ is diffeomorphic to S^1. The exceptional divisor $E = \pi^{-1}(0)$ is the union of two points. Thus $X_{01} = \{0\}$, $X_{10} = \widetilde{X}$, $X_{11} = E$ is a cubical hyperresolution of X. Hence $\mathcal{W}_{-1}H_1(X) = \mathbb{Z}_2 \subset H_1(X) = (\mathbb{Z}_2)^2$. This also follows from [9, 3.3]. Indeed, $\mathcal{W}_{-k}H_k(W)$ is the lowest filtration that can be non-zero for W compact, and the homology classes of $\mathcal{W}_{-k}H_k(W)$ are precisely those that are represented by the arc-symmetric sets. In our case, the generator of $\mathcal{W}_{-1}H_1(X)$ is the fundamental class of X. The elements of $H_1(X) \setminus \mathcal{W}_{-1}H_1(X)$ are represented by the cycles that are halves of the lemniscate; they are not arc-symmetric.

(3) Let $Y = X \times \mathbb{R}^*$ where X is the Bernoulli lemniscate. Then $Y_{01} = X_{01} \times \mathbb{R}^*$, $Y_{10} = X_{10} \times \mathbb{R}^*$, $Y_{11} = X_{11} \times \mathbb{R}^*$ is a cubical hyperresolution of Y. Hence, using (7.4), we obtain that $\mathbb{Z}_2 = \mathcal{W}_{-2}H_1(Y) \subset \mathcal{W}_{-1}H_1(Y) = (\mathbb{Z}_2)^3$, $H_1(Y) = (\mathbb{Z}_2)^4$. The generator of $\mathcal{W}_{-2}H_1(Y)$ is given by the product of the fundamental class of X with the generator of $\mathcal{W}_{-1}H_0(\mathbb{R}^*)$. The generators of $\mathcal{W}_{-1}H_1(Y)$ are of two types. The first type is the product of the fundamental class of X with a generator of $H_0(\mathbb{R}^*)$. The second type is the product of a generator of $H_1(X)$ (a half of the lemniscate) with the generator of $\mathcal{W}_{-1}H_0(\mathbb{R}^*)$. The sum of these four elements is zero and any three of them generate $\mathcal{W}_{-1}H_1(Y)$ as a $\mathbb{Z}_2$ vector space. Note that the generators of the second type cannot be represented by arc-symmetric cycles.

8. Appendix: Discrete torus groups

The following discussion is adapted from [2]. Let $(G, \cdot)$ be the group of functions $g : \{1, \ldots, n\} \to \{1, -1\}$ with product $(g \cdot h)(k) = g(k)h(k)$. Thus $G = \{1, -1\}^n$, the set of elements of order 2 of the torus $(S^1)^n \subset (\mathbb{C}^*)^n$. We refer to G as a *discrete torus* of rank n.

Let $(V, +)$ be the additive group corresponding to $(G, \cdot)$. If $g \in G$, let g' denote the corresponding element of V, so that $g' + h' = (g \cdot h)'$ and $1' = 0$. Since $g \cdot g = 1$ for all $g \in G$, we have $g' + g' = 0$ for all $g' \in V$, and so V is a vector space over Z_2, with $\dim_{\mathbb{Z}_2} V = n$. If H is a subgroup of G, we say that H has *rank* p if the corresponding subgroup H' of V has dimension p over $\mathbb{Z}_2$.

Let $A = \mathbb{Z}_2[G]$ be the $\mathbb{Z}_2$ group algebra of G. The algebra A is the set of finite formal sums $\sum_i a_i[g_i]$, where $a_i \in \mathbb{Z}_2$ and $g_i \in G$, with addition and multiplication defined by

$$\textstyle\sum_i a_i[g_i] + \sum_i b_i[g_i] = \sum_i (a_i + b_i)[g_i],$$
$$\textstyle(\sum_i a_i[g_i])(\sum_j b_j[g_j]) = \sum_k \sum_{g_i g_j = g_k} (a_i b_j)[g_k].$$

As a vector space over $\mathbb{Z}_2$, the algebra A has dimension $|G| = 2^n$. If S is a subset of G, let $[S] = \sum_{h \in S} [h] \in A$.

Let $\epsilon : A \to \mathbb{Z}_2$ be the *augmentation map*,

$$\textstyle\epsilon(\sum_i a_i[g_i]) = \sum_i a_i, \tag{8.1}$$

and let $\mathcal{I} = \operatorname{Ker} \epsilon$ be the *augmentation ideal.* Consider the filtration of the algebra A by the ideals $\mathcal{I}^p$ for $p \geq 1$,

$$A \supset \mathcal{I}^1 \supset \mathcal{I}^2 \supset \mathcal{I}^3 \supset \cdots . \tag{8.2}$$

Lemma 8.1. *For each $p \geq 1$ the ideal $\mathcal{I}^p$ is spanned as a vector space by the elements $[H]$ such that H is a subgroup of G and* $\operatorname{rank} H = p$.

Proof. We proceed by induction on p. For $p = 1$, we have $\alpha \in \mathcal{I}$ if and only if $\alpha = \sum_{g \in S} [g]$, where $|S|$ is even. Then $\alpha = \sum_{1 \neq g \in S} ([1] + [g])$, and $[1] + [g] = [\{1, g\}]$, with $\operatorname{rank}\{1, g\} = 1$.

Now suppose $\mathcal{I}^p$ is spanned by the elements $[H]$ with $\operatorname{rank} H = p$. Then $\mathcal{I}^{p+1}$ is spanned by elements of the form $([1] + [g])[H]$. If $g \in H$ then $([1] + [g])[H] = [H] + [H] = 0$. If $g \notin H$ then $([1] + [g])[H] = [K]$, where K is the subgroup of rank $p + 1$ generated by H and g. □

Proposition 8.2. *There is a canonical isomorphism $\Phi : \Lambda^* V \xrightarrow{\approx} \operatorname{Gr}_{\mathcal{I}} A$ of graded algebras which induces vector space isomorphisms $\Lambda^p V \cong \mathcal{I}^p / \mathcal{I}^{p+1}$ for each $p \geq 1$. Moreover, Φ is an isomorphism of functors;* i.e. *Φ is functorial with respect to homomorphisms of the group G.*

Proof. We claim that the function $\phi : V \to \mathcal{I}$ given by $\phi(g') = [1] + [g]$ induces a homomorphism of graded algebras

$$\Phi : \Lambda^* V \to \operatorname{Gr}_{\mathcal{I}} A,$$

with $\Phi(\Lambda^p V) = \mathcal{I}^p / \mathcal{I}^{p+1}$. We have

$$\phi(g' + h') = \phi((gh)') = [1] + [gh],$$
$$\phi(g') + \phi(h') = ([1] + [g]) + ([1] + [h]) = [g] + [h].$$

Now

$$([1] + [gh]) + ([g] + [h]) = ([1] + [g])([1] + [h]) \in \mathcal{I}^2,$$

so ϕ defines an additive homomorphism $V \to \mathcal{I}/\mathcal{I}^2$. Thus the function

$$\phi_p : \otimes^p V \to \mathcal{I}^p/\mathcal{I}^{p+1},$$
$$\phi_p(g_1 \otimes \cdots \otimes g_p) = ([1]+[g_1]) \cdots ([1]+[g_p]),$$

is multilinear. Since $([1]+[g])^2 = [1]+2[g]+[g^2] = 0$ for all $g \in G$, the maps ϕ_p define an algebra homomorphism $\Phi : \Lambda^* V \to \mathrm{Gr}_{\mathcal{I}} A$. If $(g'_1, \dots, g'_p)$ is a basis for the subspace $H' \subset V$ corresponding to the subgroup $H \subset G$, then $([1]+[g_1]) \cdots ([1]+[g_p]) = [H]$. Thus Φ is surjective by Lemma 8.1. Since $\dim_{\mathbb{Z}_2} \Lambda^* V = 2^n = \dim_{\mathbb{Z}_2} \mathrm{Gr}_{\mathcal{I}} A$, we conclude that Φ is an isomorphism.

If $\gamma : G \to H$ is a homomorphism of discrete torus groups, the commutativity of the diagram

$$\begin{array}{ccc} V_G & \xrightarrow{\phi_G} & \mathcal{I}_G \\ \downarrow \gamma_* & & \downarrow \gamma_* \\ V_H & \xrightarrow{\phi_H} & \mathcal{I}_H \end{array}$$

implies that Φ is functorial. $\square$

For $J \subset \{1, \dots, n\}$ let $G(J) = \{g \in G \mid g(i) = 1,\ i \notin J\}$.

Corollary 8.3. *For each $p \geq 1$ we have $\dim_{\mathbb{Z}_2} \mathcal{I}^p/\mathcal{I}^{p+1} = \binom{n}{p}$. In particular $\mathcal{I}^{n+1} = 0$. The subset $\{[G(J)] \mid |J| = p\}$ of $\mathcal{I}^p$ maps to a basis of $\mathcal{I}^p/\mathcal{I}^{p+1}$.*

For $n, m \in \mathbb{N}$, let $(G_n, \cdot)$ be the group of functions $g : \{1, \dots, n\} \to \{1, -1\}$, and let $(G_m, \cdot)$ be the group of functions $g : \{1, \dots, m\} \to \{1, -1\}$. Let $\gamma : G_n \to G_m$ be a group homomorphism. Then γ is given by an $m \times n$ matrix (a_{ij}) with coefficients in $\mathbb{Z}_2$,

$$\gamma(g_1, \dots, g_n) = \left(\prod g_i^{a_{i1}}, \dots, \prod g_i^{a_{im}} \right). \tag{8.3}$$

Let $A(G_n)$ and $A(G_m)$ be the group algebras of G_n and G_m, respectively, and let $\mathcal{I}^p(G_n)$ and $\mathcal{I}^p(G_m)$ be the associated filtrations (8.2). The group homomorphism γ induces an algebra homomorphism $\gamma_* : A(G_n) \to A(G_m)$ with $\gamma_*(\mathcal{I}(G_n)) \subset \mathcal{I}(G_m)$, and so for all $p \geq 1$ we have $\gamma_*(\mathcal{I}^p(G_n)) \subset \mathcal{I}^p(G_m)$.

Corollary 8.4. *If $\gamma : G_n \to G_m$ is a group homomorphism, the induced linear map $\gamma_* : \mathcal{I}^p(G_n)/\mathcal{I}^{p+1}(G_n) \to \mathcal{I}^p(G_m)/\mathcal{I}^{p+1}(G_m)$ is given by a matrix (a_{IJ}) with respect to the bases of Corollary 8.3, where for $J \subset \{1, \dots, n\}$, $I \subset \{1, \dots, m\}$, $|I| = |J| = p$,*

$$a_{IJ} = \det(a_{ij})_{i \in I, j \in J}\ .$$

Proposition 8.5. *For all $p \geq 1$ the vector subspace $\mathcal{I}^p \subset A$ is spanned by the set of translates $\{[gG(J)] \mid g \in G,\ |J| = p\}$.*

Proof. Let F^p be the subspace of A spanned by $\{[gG(J)] \mid g \in G,\ |J| = p\}$. Since $\mathcal{I}^p$ is an ideal of A we have $F^p \subset \mathcal{I}^p$.

Let $E^p \subset F^p$ be the subspace spanned by $\{[G(J)] \mid |J| = p\}$. If $|J| = p$ and $|J'| = p+1$, with $J' = J \cup \{i\}$, then $[G(J')] = [G(J)] + [g_i G(J)]$. Thus $E^{p+1} \subset F^p$, and so $F^{p+1} \subset F^p$. Therefore $F^l \subset F^p$ for all $l \geq p$.

It follows that $\mathcal{I}^p \subset F^p$. For if $\alpha \in \mathcal{I}^p$ then by Corollary 8.3 we have

$$\alpha = \alpha_p + \alpha_{p+1} + \cdots + \alpha_n, \quad \alpha_l \in E^l\ .$$

Thus for all l we have $\alpha_l \in F^p$, and so $\alpha \in F^p$. $\square$

Proposition 8.6. *If $g \in G$ let $\psi_g : A \to A$ be the linear isomorphism of A given by translation by g, $\psi_g(\alpha) = [g] \cdot \alpha$. For all $p \geq 1$, $\psi_g(\mathcal{I}^p) = \mathcal{I}^p$, and ψ_g induces the identity map on $\mathcal{I}^p/\mathcal{I}^{p+1}$.*

Proof. We have $\psi_g(\mathcal{I}^p) = \mathcal{I}^p$ by Proposition 8.5. Let $|J| = p$. If $g \in G(J)$ then $\psi_g[G(J)] = [G(J)]$. If $g \notin G(J)$ then

$$[G(J)] - \psi_g[G(J)] = ([1] - [g])[G(J)] \in \mathcal{I}^{p+1}.$$

Thus ψ_g induces the identity map on $\mathcal{I}^p/\mathcal{I}^{p+1}$ by Corollary 8.3. □

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