LIFTING DIFFERENTIABLE CURVES FROM ORBIT SPACES

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Dedicated to the memory of Mark Losik

Abstract. Let $\rho : G \to \mathrm{O}(V)$ be a real finite dimensional orthogonal representation of a compact Lie group, let $\sigma = (\sigma_1, \ldots, \sigma_n) : V \to \mathbb{R}^n$, where $\sigma_1, \ldots, \sigma_n$ form a minimal system of homogeneous generators of the G-invariant polynomials on V, and set $d = \max_i \deg \sigma_i$. We prove that for each $C^{d-1,1}$-curve c in $\sigma(V) \subseteq \mathbb{R}^n$ there exits a locally Lipschitz lift over σ, i.e., a locally Lipschitz curve $\overline{c}$ in V so that $c = \sigma \circ \overline{c}$, and we obtain explicit bounds for the Lipschitz constant of $\overline{c}$ in terms of c. Moreover, we show that each C^d-curve in $\sigma(V)$ admits a C^1-lift. For finite groups G we deduce a multivariable version and some further results.

1. Introduction and main results

1.1. Differentiable roots of hyperbolic polynomials

Let us begin by describing the most important special case of our main theorem.

Example 1. (Choosing differentiable roots of hyperbolic polynomials) Let the symmetric group S_n act on $\mathbb{R}^n$ by permuting the coordinates. The algebra of invariant polynomials $\mathbb{R}[\mathbb{R}^n]^{\mathrm{S}_n}$ is generated by the elementary symmetric functions $\sigma_i = \sum_{j_1 < \ldots < j_i} x_{j_1} \cdots x_{j_i}$. Considering the mapping $\sigma = (\sigma_1, \ldots, \sigma_n) : \mathbb{R}^n \to \mathbb{R}^n$, we may identify, in view of Vieta's formulas, each point p of the image $\sigma(\mathbb{R}^n)$ uniquely with the monic polynomial $P_a = z^n + \sum_{j=1}^n a_j z^{n-j}$ whose unordered n-tuple of roots constitutes the fiber of σ over p; two points in the fiber differ by a permutation. So the semialgebraic subset $\sigma(\mathbb{R}^n) \subseteq \mathbb{R}^n$ can be identified with the space of *hyperbolic* polynomials of degree n, i.e., monic polynomials with all roots real.

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Suppose that the coefficients $a = (a_j)_{j=1}^n$ are functions depending in a smooth way on a real parameter t, i.e., $a : \mathbb{R} \to \mathbb{R}^n$ is a smooth curve with $a(\mathbb{R}) \subseteq \sigma(\mathbb{R}^n)$. Then we may ask how regular the roots of P_a can be parameterized. This is a classical much-studied problem with important applications in partial differential equations. We shall just mention three results which will be of interest in this paper.

(1) If a is $C^{n-1,1}$ then any continuous parameterization of the roots of P_a is locally Lipschitz with uniform Lipschitz constant.
(2) If a is C^n then there exists a C^1-parameterization of the roots; actually any differentiable parameterization is C^1.
(3) If a is C^{2n} then there exists a twice differentiable parameterization of the roots.

The first result is a version of Bronshtein's theorem due to [6]; a different proof was given by Wakabayashi [38]. In our recent note [26] we presented another independent proof of (1), the method of which works in the general situation considered in the present paper; see below. For the second and third result we refer to [9]; see also [26] for a different proof, and [22] and [17] for the same conclusions under stronger assumptions. The results (1), (2), and (3) are optimal. Most notably, there are C^∞-curves a so that the roots of P_a do not admit a $C^{1,\omega}$-parameterization for any modulus of continuity ω.

Let V be any finite-dimensional Euclidean vector space. For an open subset $U \subseteq \mathbb{R}^m$ and $p \in \mathbb{N}_{\ge 1}$, we denote by $C^{p-1,1}(U,V)$ the space of all mappings $f \in C^{p-1}(U,V)$ so that each partial derivative $\partial^\alpha f$ of order $|\alpha| = p-1$ is locally Lipschitz. It is a Fréchet space with the following system of seminorms,

$$\|f\|_{C^{p-1,1}(K,V)} = \|f\|_{C^{p-1}(K,V)} + \sup_{|\alpha|=p-1} \operatorname{Lip}_K(\partial^\alpha f),$$
$$\operatorname{Lip}_K(f) = \sup_{\substack{x,y\in K\\ x\ne y}} \frac{\|f(x)-f(y)\|}{\|x-y\|},$$

where K ranges over (a countable exhaustion of) the compact subsets of U; on $\mathbb{R}^m$ we consider the 2-norm $\|\ \| = \|\ \|_2$. By Rademacher's theorem, the partial derivatives of order p of a function $f \in C^{p-1,1}(U,V)$ exist almost everywhere.

1.2. The general setup

Let G be a compact Lie group and let $\rho : G \to \mathrm{O}(V)$ be an orthogonal representation in a real finite-dimensional Euclidean vector space V with inner product $\langle\ |\ \rangle$. For short we shall write $G \circlearrowleft V$. By a classical theorem of Hilbert and Nagata, the algebra $\mathbb{R}[V]^G$ of invariant polynomials on V is finitely generated. So let $\{\sigma_i\}_{i=1}^n$ be a system of homogeneous generators of $\mathbb{R}[V]^G$ which we shall also call a system of *basic invariants*.

A system of basic invariants $\{\sigma_i\}_{i=1}^n$ is called *minimal* if there is no polynomial relation of the form $\sigma_i = P(\sigma_1,\ldots,\widehat{\sigma_i},\ldots,\sigma_n)$, or equivalently, $\{\sigma_i\}_{i=1}^n$ induces a basis of the real vector space $\mathbb{R}[V]_+^G/(\mathbb{R}[V]_+^G)^2$, where $\mathbb{R}[V]_+^G = \{f \in \mathbb{R}[V]^G : f(0) = 0\}$; cf. [12, Sect. 3.6]. The elements in a minimal system of basic invariants may not be unique but its number and its degrees $d_i := \deg \sigma_i$ are unique. Let us set

$$d := \max_{i=1,\ldots,n} d_i.$$

Given a system of basic invariants $\{\sigma_i\}_{i=1}^n$, we consider the *orbit mapping* $\sigma = (\sigma_1, \ldots, \sigma_n) : V \to \mathbb{R}^n$. The image $\sigma(V)$ is a semialgebraic set in the categorical quotient $V/\!\!/G := \{y \in \mathbb{R}^n : P(y) = 0 \text{ for all } P \in \mathscr{I}\}$, where $\mathscr{I}$ is the ideal of relations between $\sigma_1, \ldots, \sigma_n$. Since G is compact, σ is proper and separates orbits of G, and it thus induces a homeomorphism $\widetilde{\sigma}$ between the orbit space V/G and $\sigma(V)$.

Let $H = G_v = \{g \in G : gv = v\}$ be the isotropy group of $v \in V$ and (H) its conjugacy class in G; (H) is called the *type* of the orbit $Gv = \{gv : g \in G\}$. Let $V_{(H)}$ be the union of all orbits of type (H). Then $V_{(H)}/G$ is a smooth manifold and the collection of connected components of the manifolds $V_{(H)}/G$ forms a stratification of V/G by orbit type; cf. [33]. Due to [2], $\widetilde{\sigma}$ is an isomorphism between the orbit type stratification of V/G and the natural stratification of $\sigma(V)$ as a semialgebraic set; it is analytically locally trivial and thus satisfies Whitney's conditions (A) and (B). The inclusion relation on the set of subgroups of G induces a partial ordering on the family of orbit types. There is a unique minimal orbit type, the principal orbit type, corresponding to the open and dense submanifold V_{reg} consisting of points v, where the slice representation $G_v \circlearrowleft N_v$ is trivial; see Subsection 2.3 below. The projection $V_{\text{reg}} \to V_{\text{reg}}/G$ is a locally trivial fiber bundle. There are only finitely many isomorphism classes of slice representations.

A representation $G \circlearrowleft V$ is called *polar*, if there exists a linear subspace $\Sigma \subseteq V$, called a *section*, which meets each orbit orthogonally; cf. [10], [11]. The trace of the G-action on Σ is the action of the *generalized Weyl group* $W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$ on Σ, where $N_G(\Sigma) := \{g \in G : g\Sigma = \Sigma\}$ and $Z_G(\Sigma) := \{g \in G : gs = s \text{ for all } s \in \Sigma\}$. This group is finite, and it is a reflection group if G is connected. The algebras $\mathbb{R}[V]^G$ and $\mathbb{R}[\Sigma]^{W(\Sigma)}$ are isomorphic via restriction, by a generalization of Chevalley's restriction theorem due to [11] and independently [36], and thus the orbit spaces V/G and $\Sigma/W(\Sigma)$ are isomorphic.

We shall fix a minimal system of basic invariants $\{\sigma_i\}_{i=1}^n$ and the corresponding orbit mapping σ. The given data will be abbreviated by the tuple $(G \circlearrowleft V, d, \sigma)$.

1.3. Smooth structures on orbit spaces

We review some ways to endow the orbit space V/G with a smooth structure and stress the connection to the lifting problem studied in this paper. The results and constructions mentioned in this subsection will not be used later in the paper.

A smooth structure on a non-empty set X can be introduced by specifying any of the following families of mappings together with some compatibility conditions:

- the smooth functions on X (differential space)
- the smooth mappings into X (diffeological space)
- the smooth curves in X and the smooth functions on X (Frölicher space).

More precisely: A *differential structure* on X is a family $\mathcal{F}_X$ of functions $X \to \mathbb{R}$, along with the associated initial topology on X, so that

- if $f_1, \ldots, f_n \in \mathcal{F}_X$ and $g \in C^\infty(\mathbb{R}^n)$, then $g \circ (f_1, \ldots, f_n) \in \mathcal{F}_X$

- if $f : X \to \mathbb{R}$ is locally the restriction of a function in $\mathcal{F}_X$, then $f \in \mathcal{F}_X$.

The pair $(X, \mathcal{F}_X)$ is called a *differential space.*

A *diffeology* on X is a family $\mathcal{D}_X$ of mappings $U \to X$, where U is any *domain*, i.e., open in some $\mathbb{R}^n$, so that

- $\mathcal{D}_X$ contains all constant mappings $\mathbb{R}^n \to X$ (for all n)
- for each $p : U \to X \in \mathcal{D}_X$, each domain V, and each $q \in C^\infty(V, U)$, also $p \circ q \in \mathcal{D}_X$
- if $p : U \to X$ is locally in $\mathcal{D}_X$, then $p \in \mathcal{D}_X$.

The pair $(X, \mathcal{D}_X)$ is called a *diffeological space.*

A *Frölicher structure* on X is a pair $(\mathcal{C}_X, \mathcal{F}_X)$ consisting of a subset $\mathcal{C}_X \subseteq X^{\mathbb{R}}$ and a subset $\mathcal{F}_X \subseteq \mathbb{R}^X$ so that

- $f \in \mathcal{F}_X$ if and only if $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $c \in \mathcal{C}_X$
- $c \in \mathcal{C}_X$ if and only if $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $f \in \mathcal{F}_X$.

The triple $(X, \mathcal{C}_X, \mathcal{F}_X)$ is called a *Frölicher space.* The Frölicher structure on X generated by a subset $\mathcal{C} \subseteq X^{\mathbb{R}}$ (respectively $\mathcal{F} \subseteq \mathbb{R}^X$) is the finest (respectively coarsest) Frölicher structure $(\mathcal{C}_X, \mathcal{F}_X)$ on X with $\mathcal{C} \subseteq \mathcal{C}_X$ (respectively $\mathcal{F} \subseteq \mathcal{F}_X$).

A mapping $\phi : X \to Y$ between two spaces of the same kind is called *smooth* if

- $\phi^* \mathcal{F}_Y \subseteq \mathcal{F}_X$ in the case of differential spaces
- $\phi_* \mathcal{D}_X \subseteq \mathcal{D}_Y$ in the case of diffeological spaces
- $\phi_* \mathcal{C}_X \subseteq \mathcal{C}_Y$, equivalently $\phi^* \mathcal{F}_Y \subseteq \mathcal{F}_X$, equivalently $\mathcal{F}_Y \circ \phi \circ \mathcal{C}_X \in C^\infty$ in the case of Frölicher spaces.

Any of the above forms a category, and the category of smooth finite-dimensional manifolds with smooth mappings in the usual sense forms a full subcategory in each of them.

The orbit space V/G can be given a differential structure by defining a function on V/G to be smooth if its composite with the projection $V \to V/G$ is smooth, i.e., $\mathcal{F}_{V/G} = C^\infty(V/G) \cong C^\infty(V)^G$. On the other hand, $\sigma(V)$ has a differential structure defined by restriction of the smooth functions on $\mathbb{R}^n$, i.e., $\mathcal{F}_{\sigma(V)} = \{f|_{\sigma(V)} : f \in C^\infty(\mathbb{R}^n)\}$. By Schwarz' theorem [32], $\sigma^* C^\infty(\mathbb{R}^n) = C^\infty(V)^G$ and so $\widetilde{\sigma}$ is an isomorphism of V/G and $\sigma(V)$ together with their differential structures. In other words, quotient and subspace differential structure coincide. We have

$$\begin{aligned} C^\infty(\mathbb{R}, \sigma(V)) &:= \{c \in C^\infty(\mathbb{R}, \mathbb{R}^n) : c(\mathbb{R}) \subseteq \sigma(V)\} \\ &= \{c \in \sigma(V)^{\mathbb{R}} : f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } f \in C^\infty(V)^G\}. \end{aligned}$$

We may also consider the curves in $\sigma(V)$ that admit a smooth lift over σ,

$$\sigma_* C^\infty(\mathbb{R}, V) = \{\sigma \circ c : c \in C^\infty(\mathbb{R}, V)\}.$$

In general, the inclusion $\sigma_* C^\infty(\mathbb{R}, V) \subseteq C^\infty(\mathbb{R}, \sigma(V))$ is strict (cf. Example 1). The set of functions $C^\infty(V)^G$ on the one hand and the set of curves $\sigma_* C^\infty(\mathbb{R}, V)$ on the other hand give rise to Frölicher space structures on the orbit space $V/G = \sigma(V)$ that turn out to coincide: The Frölicher structure on $\sigma(V)$ generated by $C^\infty(V)^G$

as well as that generated by $\sigma_* C^\infty(\mathbb{R}, V)$ is $(C^\infty(\mathbb{R}, \sigma(V)), C^\infty(V)^G)$. Indeed, we have

$$C^\infty(V)^G \cong \{f \in \mathbb{R}^{\sigma(V)} : f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } c \in \sigma_* C^\infty(\mathbb{R}, V)\},$$

for if $f \circ c \in C^\infty$ for all $c \in \sigma_* C^\infty(\mathbb{R}, V)$ then $f \circ \sigma$ is C^∞, by Boman's theorem [3]. It follows that the quotient and the subspace Frölicher structure coincide on $\sigma(V)$.

However, the quotient diffeology $\mathcal{D}_q$ and the subspace diffeology $\mathcal{D}_s$ on $\sigma(V)$ fall apart. The quotient diffeology $\mathcal{D}_q$ with respect to the orbit mapping $\sigma : V \to \sigma(V)$ is the finest diffeology of $\sigma(V)$ such that $\sigma : V \to \sigma(V)$ is smooth. A mapping $f : U \to \sigma(V)$ belongs to $\mathcal{D}_q$ if and only if it lifts locally over σ, i.e., for each $x \in U$ there is a neighborhood U_0 and a C^∞-mapping $\overline{f} : U_0 \to V$ so that $f = \sigma \circ \overline{f}$ on U_0. The subspace diffeology $\mathcal{D}_s$ on $\sigma(V)$ is the coarsest diffeology of $\sigma(V)$ such that the inclusion $\sigma(V) \hookrightarrow \mathbb{R}^n$ is smooth. A mapping $U \to \sigma(V)$ belongs to $\mathcal{D}_s$ if and only if the composite $U \to \sigma(V) \hookrightarrow \mathbb{R}^n$ is smooth. Evidently, $\mathcal{D}_q \subseteq \mathcal{D}_s$, and the inclusion is strict (cf. Example 1).

The orbit space as a differentiable space

Let us finally consider V/G as a differentiable space in the sense of Spallek [34]. We follow the presentation in [25].

An $\mathbb{R}$-algebra A is called a *differentiable algebra* if it is isomorphic to $C^\infty(\mathbb{R}^n)/\mathfrak{a}$ for some positive integer n and some closed ideal $\mathfrak{a}$ in $C^\infty(\mathbb{R}^n)$. Any differentiable algebra A has a unique Fréchet topology such that the algebra isomorphism $A \cong C^\infty(\mathbb{R}^n)/\mathfrak{a}$ is a homeomorphism; cf. [25, Thm. 2.23]. The real spectrum $\operatorname{Spec}_r A$ of $A = C^\infty(\mathbb{R}^n)/\mathfrak{a}$ is homeomorphic to $\{x \in \mathbb{R}^n : f(x) = 0, \ \forall f \in \mathfrak{a}\}$; cf. [25, Prop. 2.13].

A locally ringed space $(X, \mathcal{O}_X)$ is said to be an *affine differentiable space* if it is isomorphic to the real spectrum $(\operatorname{Spec}_r A, \widetilde{A})$ of some differential algebra A. Here $\widetilde{A}$ is the sheaf associated to the presheaf $U \rightsquigarrow A_U$, where $A_U = \{a/b : a, b \in A, \ b(x) \neq 0, \ \forall x \in U\}$ denotes the localization. A locally ringed space $(X, \mathcal{O}_X)$ is said to be a *differentiable space* if each point $x \in X$ has an open neighborhood U in X such that $(U, \mathcal{O}_X|_U)$ is an affine differentiable space. Sections of $\mathcal{O}_X$ on an open set $U \subseteq X$ are called *differentiable functions* on U. A differentiable space $(X, \mathcal{O}_X)$ is said to be *reduced* if for each open set $U \subseteq X$ and every differentiable function $f \in \mathcal{O}_X(U)$, we have $f = 0$ if and only if $f(x) = 0$ for all $x \in U$.

The space $\mathbb{R}^n$ is a reduced affine differentiable space: let $C^\infty_{\mathbb{R}^n}$ denote the sheaf of C^∞-functions on $\mathbb{R}^n$, then $(\operatorname{Spec}_r C^\infty(\mathbb{R}^n), C^\infty_{\mathbb{R}^n}) \cong (\mathbb{R}^n, C^\infty_{\mathbb{R}^n})$; cf. [25, Example 3.15].

Let Z be a topological subspace of $\mathbb{R}^n$. A continuous function $f : Z \to \mathbb{R}$ is said to be of class C^∞ if each point $z \in Z$ has an open neighborhood U_z in $\mathbb{R}^n$ and there exists $F \in C^\infty(U_z)$ such that $f|_{Z \cap U_z} = F|_Z$. Thus we obtain a sheaf C^∞_Z of continuous functions on Z, and (Z, C^∞_Z) is a reduced affine differentiable space; cf. [25, Cor. 5.8]. The category of reduced differentiable spaces is equivalent to the category of reduced ringed spaces $(X, \mathcal{O}_X)$ with the property that each $x \in X$ has an open neighborhood U such that $(U, \mathcal{O}_X|_U)$ is isomorphic to (Z, C^∞_Z) for some closed subset Z of an affine space $\mathbb{R}^n$; cf. [25, Thm. 3.23].

Let us turn to our situation. We equip the orbit space V/G (with the quotient topology and) with the structural sheaf $\mathcal{O}_{V/G}$, where $\mathcal{O}_{V/G}(U) := \{f \in C^0(U,\mathbb{R}) : f \circ \pi \in C^\infty(\pi^{-1}(U))\} \cong C^\infty(\pi^{-1}(U))^G$ and $\pi : V \to V/G$ denotes the quotient mapping. On the closed subset $\sigma(V)$ of $\mathbb{R}^n$ we consider the structure of reduced affine differentiable space induced by $\mathbb{R}^n$, i.e., $(\sigma(V), C^\infty_{\sigma(V)})$. It follows from Schwarz's theorem and the localization theorem for smooth functions (see [25, p. 28]) that σ induces an isomorphism of the differentiable spaces $(V/G, \mathcal{O}_{V/G})$ and $(\sigma(V), C^\infty_{\sigma(V)})$; see [25, Thm. 11.14]. Note that the reduced affine differentiable space $(V/G, \mathcal{O}_{V/G})$ is the differential space $(V/G, \mathcal{F}_{V/G})$ considered above.

1.4. The main results

In this paper we shall be concerned with the lifting properties of arbitrary elements in $C^\infty(\mathbb{R}, \sigma(V))$ (or in $\mathcal{D}_s$).

Let $I \subseteq \mathbb{R}$ be an open interval and let $c : I \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a curve in the orbit space V/G of $(G \circlearrowleft V, d, \sigma)$. A curve $\overline{c} : I \to V$ is called a *lift of c over σ*, if $c = \sigma \circ \overline{c}$ holds. We will consider curves c in $V/G = \sigma(V)$ that are in some Hölder class $C^{k,\alpha}$; this means that c is $C^{k,\alpha}$ as a curve in $\mathbb{R}^n$ with the image contained in $\sigma(V)$, and it will be denoted by $c \in C^{k,\alpha}(I, \sigma(V))$. Note that any $c \in C^0(I, \sigma(V))$ admits a lift $\overline{c} \in C^0(I, V)$, by [24] or [18, Prop. 3.1]. The problem of lifting curves over invariants is independent of the choice of a system of basic invariants, as any two such choices differ by a polynomial diffeomorphism.

This problem was considered in this generality for the first time in [1]; it was shown that $\sigma_* C^\infty(\mathbb{R}, V)$ contains all elements in $C^\infty(\mathbb{R}, \sigma(V))$ that do not meet lower-dimensional strata of $\sigma(V)$ with infinite order of flatness. A C^d-curve in $\sigma(V)$ admits a differentiable lift, due to [18]. In [19] and [20] the following generalization of Example 1 was obtained: Let G be *finite*, write $V = V_1 \oplus \cdots \oplus V_l$ as an orthogonal direct sum of irreducible subspaces V_i, and set

$$k = \max\{d, k_1, \ldots, k_l\},$$

where k_i is the minimal cardinality of non-zero orbits in V_i. Then C^k (resp. C^{k+d}) curves in V/G admit C^1 (resp. twice differentiable) lifts. This result was achieved by reducing the general case $G \circlearrowleft V$ to the case of the standard action of the symmetric group $\mathrm{S}_n \circlearrowleft \mathbb{R}^n$ and then applying Bronshtein's theorem. This technique works only for finite groups and it yields a corresponding result for polar representations (since the associated Weyl group is finite).

The ideas of our new proof of Bronshtein's theorem in [26] led us to the main results of this paper:

- We show that $C^{d-1,1}$-curves in the orbit space of *any* representation $(G \circlearrowleft V, d, \sigma)$ admit $C^{0,1}$-lifts and we obtain explicit bounds for the Lipschitz constants (Theorem 1).
- We prove that C^d-curves in the orbit space of *any* representation $(G \circlearrowleft V, d, \sigma)$ admit C^1-lifts (Theorem 2).
- If G is a finite group we find that
  — each continuous lift of a $C^{d-1,1}$-curve is $C^{0,1}$ (Corollary 1),
  — each differentiable lift of a C^d-curve is C^1 (Corollary 3),
  — each C^{2d}-curve admits a twice differentiable lift (Corollary 3).

- If G is a finite group we also obtain that each continuous lift of a $C^{d-1,1}$- mapping of *several variables* into the orbit space is $C^{0,1}$ with uniform Lipschitz constants (Corollary 2).
- As a by-product of the problem of gluing together local lifts (see Section 5) we show that real analytic curves in the orbit space of *any* representation $(G \circlearrowleft V, d, \sigma)$ can be lifted *globally* (Theorem 4). This extends a result of [1], who proved the existence of local real analytic lifts, and global ones if $G \circlearrowleft V$ is polar.

Our proofs do not rely on Bronshtein's result, but we reprove it.

Theorem 1. *Let $(G \circlearrowleft V, d, \sigma)$ be a real finite-dimensional orthogonal representation of a compact Lie group. Then any $c \in C^{d-1,1}(I, \sigma(V))$ admits a lift $\overline{c} \in C^{0,1}(I, V)$. More precisely, for any relatively compact subset $I_0 \Subset I$, there is a neighborhood I_1 with $I_0 \Subset I_1 \Subset I$ so that*

$$\begin{aligned} \operatorname{Lip}_{I_0}(\overline{c}) &\le C \left(\max_i \|c_i\|^{1/d_i}_{C^{d-1,1}(\overline{I}_1)} \right) \\ &\le \widetilde{C} \left(1 + \max_i \|c_i\|_{C^{d-1,1}(\overline{I}_1)} \right) \end{aligned} \tag{1.1}$$

for constants C and $\widetilde{C}$ depending only on the intervals I_0, I_1 and on the isomorphism classes of the slice representations of $G \circlearrowleft V$ and respective minimal systems of basic invariants. (*More precise bounds are stated in Subsection* 4.5.)

Remark 1. The statement of Theorem 1 reads "there is a $C^{0,1}$-lift $\overline{c}$ on the whole interval I so that for all $I_0 \Subset I$ there is a neighborhood I_1 such that (1.1) holds". Our proof also yields "for all intervals I_0 and I_1 with $I_0 \Subset I_1 \Subset I$ there is a Lipschitz lift $\overline{c}$ on I_0 satisfying (1.1)".

Convention. We will denote by $C = C(G \circlearrowleft V, \ldots)$ *any* constant depending only on $G \circlearrowleft V, \ldots$; its value may vary from line to line. Specific constants will bear a subscript like $C_0 = C_0(\ldots)$ or $C_1 = C_1(\ldots)$. The dependence on $G \circlearrowleft V$ is to be understood in the following way. For every isomorphism class $H \circlearrowleft W$ of slice representations of $G \circlearrowleft V$ fix a minimal system of basic invariants; note that there are only finitely many slice representations up to isomorphism and that $G \circlearrowleft V$ coincides with its slice representation at 0. Writing $C = C(G \circlearrowleft V)$, we mean that the constant C only depends on the isomorphism classes of the slice representations of $G \circlearrowleft V$ and on the respective fixed minimal systems of basic invariants.

Our second main result is the following.

Theorem 2. *Let $(G \circlearrowleft V, d, \sigma)$ be a real finite-dimensional orthogonal representation of a compact Lie group. Then any $c \in C^d(I, \sigma(V))$ admits a lift $\overline{c} \in C^1(I, V)$.*

Theorem 1 and Theorem 2 will be proved in Section 4 and Section 5, respectively.

For finite groups G we can show more:

Corollary 1. *Let $(G \circlearrowleft V, d, \sigma)$ be a real finite-dimensional orthogonal representation of a finite group. Then any continuous lift $\overline{c}$ of $c \in C^{d-1,1}(I, \sigma(V))$ is locally Lipschitz and satisfies* (1.1) *for all intervals $I_0 \Subset I_1 \Subset I$.*

Proof. Let $\widetilde{c}$ be any continuous lift of c, and let $I_0 \Subset I_1 \Subset I$. Let $\overline{c}$ be the Lipschitz lift on I_0 provided by Remark 1. Let $s,t \in I_0$, $s<t$. For each $g \in G$ consider the closed subset $J_g := \{r \in [s,t] : \widetilde{c}(r) = g\overline{c}(r)\}$ of $[s,t]$. As $[s,t] = \bigcup_{g\in G} J_g$, there exists a subset $\{g_1,\ldots,g_\ell\} \subseteq G$ and finite sequence $s = t_0 < t_1 < \cdots < t_\ell = t$ so that $t_{i-1}, t_i \in J_{g_i}$ for all $i = 1,\ldots,\ell$. Then

$$\|\widetilde{c}(s) - \widetilde{c}(t)\| \le \sum_{i=1}^{\ell} \|g_i\overline{c}(t_{i-1}) - g_i\overline{c}(t_i)\| \le \operatorname{Lip}_{I_0}(\overline{c})\,(t-s),$$

which implies the assertion. □

Corollary 1 readily implies the following result on lifting of mappings in several variables.

Corollary 2. *Let $(G \circlearrowleft V, d, \sigma)$ be a real finite-dimensional orthogonal representation of a finite group. Let $U \subseteq \mathbb{R}^m$ be open and let $f \in C^{d-1,1}(U,\sigma(V))$. Then any continuous lift $\overline{f} : U \supseteq \Omega \to V$ of f, on an open subset Ω of U, is locally Lipschitz. More precisely, for any pair of relatively compact open subsets $\Omega_0 \Subset \Omega_1 \Subset \Omega$ we have*

$$\begin{aligned}\operatorname{Lip}_{\Omega_0}(\overline{f}) &\le C\,\big(\max_i \|f_i\|^{1/d_i}_{C^{d-1,1}(\overline{\Omega}_1)}\big)\\ &\le \widetilde{C}\,\big(1 + \max_i \|f_i\|_{C^{d-1,1}(\overline{\Omega}_1)}\big),\end{aligned} \tag{1.2}$$

for constants $C = C(G \circlearrowleft V, \Omega_0, \Omega_1, m)$ and $\widetilde{C} = \widetilde{C}(G \circlearrowleft V, \Omega_0, \Omega_1, m)$.

Remark.

(1) If G has positive dimension and $\overline{f}$ is a $C^{0,1}$-lift of f, we may obtain a continuous lift of f that is not locally Lipschitz by simply multiplying $\overline{f}$ by a suitable continuous mapping $g : U \to G$.

(2) In general there are representations and smooth mappings into the orbit space of such which do not admit continuous lifts. For instance, the orbit space of a finite rotation group of $\mathbb{R}^2$ is homeomorphic to the set C obtained from the sector $\{re^{i\varphi} \in \mathbb{C} : r \in [0,\infty), 0 \le \varphi \le \varphi_0\}$ by identifying the rays that constitute its boundary. A loop on C cannot be lifted to a loop in $\mathbb{R}^2$ unless it is homotopically trivial in $C \setminus \{0\}$.

Proof. Let $\overline{f} : U \supseteq \Omega \to V$ be a continuous lift of f on Ω. Without loss of generality we may assume that Ω_0 and Ω_1 are open boxes parallel to the coordinate axes, $\Omega_i = \prod_{j=1}^m I_{i,j}$, $i = 0,1$, with $I_{0,j} \Subset I_{1,j}$ for all j. Let $x,y \in \Omega_0$ and set $h := y - x$. Let $\{e_i\}_{i=1}^m$ denote the standard unit vectors in $\mathbb{R}^m$. For any z in the orthogonal projection of Ω_0 on the hyperplane $x_j = 0$ consider the curve $\overline{f}_{z,j} : I_{0,j} \to V$ defined by $\overline{f}_{z,j}(t) := \overline{f}(z + te_j)$. By Corollary 1, each $\overline{f}_{z,j}$ is Lipschitz and $C := \sup_{z,j} \operatorname{Lip}_{I_{0,j}}(\overline{f}_{z,j}) < \infty$. Thus

$$\|\overline{f}(x) - \overline{f}(y)\| \le \sum_{j=0}^{m-1} \Big\|\overline{f}\big(x + \sum_{k=1}^{j} h_k e_k\big) - \overline{f}\big(x + \sum_{k=1}^{j+1} h_k e_k\big)\Big\| \le C\|h\|_1 \le C\sqrt{m}\|h\|_2.$$

The bounds (1.2) follow from (1.1). □

Corollary 3. *Let $(G \circlearrowleft V, d, \sigma)$ be a real finite-dimensional orthogonal representation of a finite group. Then:*

(1) *Any differentiable lift of $c \in C^d(I, \sigma(V))$ is C^1.*
(2) *Any $c \in C^{2d}(I, \sigma(V))$ admits a twice differentiable lift.*

Proof. This follows from Corollary 1. It can be proved as in [19]; see also [20]. □

1.5. Further examples

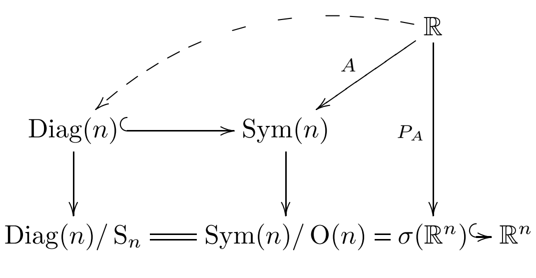
Example 2. (Choosing differentiable eigenvalues of real symmetric matrices) Let the orthogonal group $\mathrm{O}(n) = \mathrm{O}(\mathbb{R}^n)$ act by conjugation on the real vector space $\mathrm{Sym}(n)$ of real symmetric $n \times n$ matrices, $\mathrm{O}(n) \times \mathrm{Sym}(n) \ni (S, A) \mapsto SAS^{-1} = SAS^t \in \mathrm{Sym}(n)$. The algebra of invariant polynomials $\mathbb{R}[\mathrm{Sym}(n)]^{\mathrm{O}(n)}$ is isomorphic to $\mathbb{R}[\mathrm{Diag(n)}]^{\mathrm{S}_n}$ by restriction, where $\mathrm{Diag}(n)$ is the vector space of real diagonal $n \times n$ matrices upon which S_n acts by permuting the diagonal entries. More precisely, $\mathbb{R}[\mathrm{Sym}(n)]^{\mathrm{O}(n)} = \mathbb{R}[\Sigma_1, \ldots, \Sigma_n]$, where $\Sigma_i(A) = \mathrm{Trace}(\bigwedge^i A : \bigwedge^i \mathbb{R}^n \to \bigwedge^i \mathbb{R}^n)$ is the ith characteristic coefficient of A and $\Sigma_i|_{\mathrm{Diag}(n)} = \sigma_i$, where σ_i is the ith elementary symmetric polynomial and we identify $\mathrm{Diag}(n) \cong \mathbb{R}^n$ (cf. [23, 7.1]). This means that the representation $\mathrm{O}(n) \circlearrowleft \mathrm{Sym}(n)$ is polar and Diag(n) forms a section.

A smooth curve $A : \mathbb{R} \to \mathrm{Sym}(n)$ of symmetric matrices induces a smooth curve of hyperbolic polynomials P_A (the characteristic polynomial of A), i.e., a smooth curve in the semialgebraic set $\sigma(\mathrm{Diag}(n)) \cong \sigma(\mathbb{R}^n)$ from Example 1. Then (1), (2), and (3) in Example 1 imply regularity results for the eigenvalues of $t \mapsto A(t)$, which however turn out to be not optimal. In fact, we have the following optimal results.

(1) If A is $C^{0,1}$ then any continuous parameterization of the eigenvalues of A is locally Lipschitz with uniform Lipschitz constant.
(2) If A is C^1 then there exists a C^1-parameterization of the eigenvalues; actually any differentiable parameterization is C^1.
(3) If A is C^2 then there exists a twice differentiable parameterization of the eigenvalues.

The first result follows from a result due to Weyl [39], the second and third were shown in [28]. Actually, these results are true for normal complex matrices and, in appropriate form, even for normal operators in Hilbert space with common domain of definition and compact resolvents; see [28].

Here the curve P_A in the orbit space is the projection of the curve A under $\mathrm{Sym}(n) \to \mathrm{Sym}(n)/\,\mathrm{O}(n)$ and is then lifted over $\mathrm{Diag}(n) \to \mathrm{Diag}(n)/\,\mathrm{S}_n$.



Example 3. (Decomposing nonnegative functions into differentiable sums of squares) Let the orthogonal group $\mathrm{O}(n)$ act in the standard way on $\mathbb{R}^n$. Then the algebra of invariant polynomials $\mathbb{R}[\mathbb{R}^n]^{\mathrm{O}(n)}$ is generated by $\sigma = \sum_{i=1}^n x_i^2$. The orbit space $\mathbb{R}^n/\mathrm{O}(n)$ can be identified with the half-line $\mathbb{R}_{\geq 0} = [0,\infty) = \sigma(\mathbb{R}^n)$. Each line through the origin of $\mathbb{R}^n$ forms a section of $\mathrm{O}(n) \circlearrowleft \mathbb{R}^n$.

Given a smooth nonnegative function f, decomposing f into sums of squares amounts to lifting f over σ. Applying Example 1(1) (actually its multiparameter analogue which follows easily; see Corollary 2) implies that:

(1) Any nonnegative $C^{1,1}$ function $f : \mathbb{R}^m \to \mathbb{R}$ is the square of a $C^{0,1}$ function.

The image of this lift lies in a section of $\mathrm{O}(n) \circlearrowleft \mathbb{R}^n$. This does not apply to the solutions in the following stronger results which benefit from the additionally available space.

(2) Any nonnegative $C^{3,1}$ function $f : \mathbb{R}^m \to \mathbb{R}$ is a sum of $n = n(m)$ squares of $C^{1,1}$ functions.

(3) Let $p \in \mathbb{N}$. Any nonnegative C^{2p} function $f : \mathbb{R} \to \mathbb{R}$ is the sum of two squares of C^p functions.

Result (2) was stated by Fefferman and Phong while proving their celebrated inequality in [14]; see also [16, Lem. 4]. This is sharp in the sense that there exist C^∞ functions $f : \mathbb{R}^m \to \mathbb{R}$, for $m \geq 4$, that are not sums of squares of C^2 functions; see [5]. Result (3) is due to [4]; the decomposition depends on p.

2. Reduction to slice representations

Let $(G \circlearrowleft V, d, \sigma)$ be fixed. Let $V^G = \{v \in V : Gv = v\}$ be the linear subspace of invariant vectors.

2.1. Dominant invariant

We may assume without loss of generality that

$$\sigma_1(v) = \langle v \mid v \rangle = \|v\|^2 \text{ for all } v \in V. \tag{2.1}$$

Indeed, if the invariant polynomial $v \mapsto \langle v \mid v \rangle$ does not belong to the minimal system of basic invariants, we just add it. This does not change d unless $d = 1$. But in the latter case $V = V^G$ and there is nothing to prove. In fact, if $d = 1$ then the elements in a minimal system of basic invariants form a system of linear coordinates on V.

Under the assumption (2.1) the invariant σ_1 is dominant in the following sense: for all $j = 1, \ldots, n$ and all $v \in V$,

$$|\sigma_j(v)|^{1/d_j} \leq C\, |\sigma_1(v)|^{1/d_1} = C\, \|v\|, \tag{2.2}$$

where $C = C(\sigma)$. Indeed, $|\sigma_j(v)| \leq \max_{\|w\|=1} |\sigma_j(w)|\, \|v\|^{d_j}$, by homogeneity.

2.2. Removing fixed points

Let V' be the orthogonal complement of V^G in V. Then we have $V = V^G \oplus V'$, $\mathbb{R}[V]^G = \mathbb{R}[V^G] \otimes \mathbb{R}[V']^G$ and $V/G = V^G \times V'/G$. The following lemma is obvious.

Lemma 1. *Any lift $\overline{c}$ of a curve $c = (c_0, c_1)$ in $V^G \times V'/G$ has the form $\overline{c} = (c_0, \overline{c}_1)$, where $\overline{c}_1$ is a lift of c_1.*

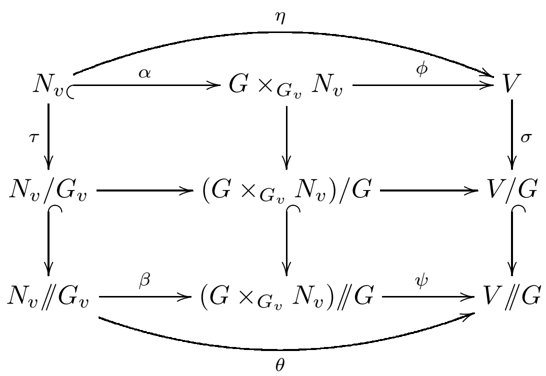
In view of Lemma 1 we may assume that

$$V^G = \{0\}. \tag{2.3}$$

2.3. The slice theorem

For a point $v \in V$ we denote by $N_v = T_v(Gv)^\perp$ the normal subspace of the orbit Gv at v. It carries a natural G_v-action $G_v \circlearrowleft N_v$. The crossed product (or associated bundle) $G \times_{G_v} N_v$ carries the structure of an affine real algebraic variety as the categorical (and geometrical) quotient $(G \times N_v)/\!\!/G_v$ with respect to the action $G_v \circlearrowleft (G \times N_v)$ given by $h(g, x) = (gh^{-1}, hx)$. Denote by $[g, x]$ the element of $G \times_{G_v} N_v$ represented by $(g, x) \in G \times N_v$. The G-equivariant polynomial mapping $\phi : G \times_{G_v} N_v \to V$, $[g, x] \mapsto g(v + x)$, where the action $G \circlearrowleft (G \times_{G_v} N_v)$ is by left multiplication on the first component, induces a polynomial mapping $\psi : (G \times_{G_v} N_v)/\!\!/G \to V/\!\!/G$ sending $(G \times_{G_v} N_v)/G$ into V/G.

The G_v-equivariant embedding $\alpha : N_v \hookrightarrow G \times_{G_v} N_v$ given by $x \mapsto [e, x]$ induces an isomorphism $\beta : N_v/\!\!/G_v \to (G \times_{G_v} N_v)/\!\!/G$ mapping N_v/G_v onto $(G \times_{G_v} N_v)/G$. Set $\eta = \phi \circ \alpha$ and $\theta = \psi \circ \beta$.



Theorem 3 (Cf. [21], [33])**.** *There is an open ball $B_v \subseteq N_v$ centered at the origin such that the restriction of ϕ to $G \times_{G_v} B_v$ is an analytic G-isomorphism onto a G-invariant neighborhood of v in V. The mapping θ is a local analytic isomorphism at 0 which induces a local homeomorphism of N_v/G_v and V/G.*

2.4. Reduction

Let $\{\tau_i\}_{i=1}^m$ be a system of generators of $\mathbb{R}[N_v]^{G_v}$ and let $\tau = (\tau_1, \ldots, \tau_m) : N_v \to \mathbb{R}^m$ be the associated orbit mapping. Consider the slice

$$S_v := v + B_v, \tag{2.4}$$

where B_v is the open ball from Theorem 3. As σ_i is G_v-invariant, there exists $\pi_i \in \mathbb{R}[\mathbb{R}^m]$ so that

$$\sigma_i(x) - \sigma_i(v) = \pi_i(\tau(x - v)), \qquad \text{for } x \in S_v. \tag{2.5}$$

Conversely, every G_v-invariant real analytic function in $x-v$ can be written as a real analytic function in $\sigma(x)-\sigma(v)$ near v, by [32, p. 67], hence there is a real analytic mapping φ defined in a neighborhood of the origin in $\mathbb{R}^n$ with values in $\mathbb{R}^m$ such that

$$\tau(x-v) = \varphi(\sigma(x)-\sigma(v)), \tag{2.6}$$

for x in some neighborhood U_v of v in S_v.

Lemma 2. *Let $c=(c_1,\ldots,c_n)$ be a curve in $\sigma(V)$ with $c_1 \neq 0$ and such that the curve*

$$\underline{c} := \big(1, {c_1}^{-d_2/d_1}c_2, \ldots, {c_1}^{-d_n/d_1}c_n\big)$$

lies in $\sigma(U_v)$. Then $\underline{c}^* := \varphi(\underline{c}-\sigma(v))$ is a curve in $\tau(U_v - v)$ and

$$c^* = (c_1^*,\ldots,c_m^*) := ({c_1}^{e_1/d_1}\underline{c}_1^*, \ldots, {c_1}^{e_m/d_1}\underline{c}_m^*), \quad e_i = \deg \tau_i,$$

is a curve in $\tau(N_v)$. If $\overline{c}^*$ is a lift of c^* over τ then

$$ {c_1}^{1/d_1} v + \overline{c}^* \tag{2.7}$$

is a lift of c over σ.

Proof. Only the last statement is maybe not immediately visible. The curve ${c_1}^{-1/d_1}\overline{c}^*$ is a lift of $\underline{c}^*$ over τ,

$$\tau_i({c_1}^{-1/d_1}\overline{c}^*) = {c_1}^{-e_i/d_1}\tau_i(\overline{c}^*) = {c_1}^{-e_i/d_1}c_i^* = \underline{c}_i^*,$$

and so, by (2.5) and (2.6), ${c_1}^{-1/d_1}\overline{c}^* + v$ is a lift of $\underline{c}$ over σ,

$$\sigma({c_1}^{-1/d_1}\overline{c}^*+v) - \sigma(v) = \pi(\tau({c_1}^{-1/d_1}\overline{c}^* + v - v)) = \pi(\underline{c}^*) = \pi(\varphi(\underline{c}-\sigma(v))) = \underline{c}-\sigma(v).$$

By homogeneity, we find $\sigma_i(\overline{c}^* + {c_1}^{1/d_1}v) = {c_1}^{d_i/d_1}\underline{c}_i = c_i$ as required. $\square$

We can assume that φ and all its partial derivatives are separately bounded. In analogy to (2.1) we may assume that $\tau_1(x) = \|x\|^2$ for all $x \in N_v$, thus $e_1 = 2$. Then the following corollary is evident.

Corollary 4. *We have $|c_1^*| \le C_0\,|c_1|$, where $C_0 = \sup_y |\varphi_1(y)|$.*

The set $\sigma(V)$ is closed in $\mathbb{R}^n_y$. Thus (2.2) implies that the set $\sigma(V) \cap \{y_1 = 1\}$ is compact. It follows that the open cover $\{\sigma(U_v)\}_{v\in V, \|v\|=1}$ of $\sigma(V)\cap\{y_1=1\}$ has a finite subcover

$$\{B_\alpha\}_{\alpha\in\Delta} = \{\sigma(U_{v_\alpha})\}_{\alpha\in\Delta}. \tag{2.8}$$

The following lemma shows that the maximal degree of the basic invariants does not increase by passing to a slice representation. This was shown in [19, Lem. 2.4]; for the convenience of the reader, we include a short proof.

Lemma 3. *Assume that $\{\tau_i\}_{i=1}^m$ is minimal and set $e := \max_i e_i = \max_i \deg \tau_i$. Then $e \le d$.*

Proof. We may assume without loss of generality that the basic invariants τ_i are ordered so that $e_1 \le e_2 \le \cdots \le e_m = e$. Assume that $e_m > d$. We will show that this assumption contradicts minimality of $\{\tau_i\}_{i=1}^m$. It fact, in view of (2.5) it implies that each polynomial π_i is independent of its last entry. Thus, by (2.5) and (2.6), we have for $y \in U_v - v$,

$$\tau_m(y) = \psi_m(\tau'(y)),$$

where $\tau' := (\tau_1, \dots, \tau_{m-1})$ and $\psi_m := \varphi_m \circ \pi$. Expanding into the Taylor series at 0,

$$\tau_m = T_0^\infty \psi_m \circ \tau' = T_0^e \psi_m \circ \tau',$$

we see that τ_m is a polynomial in $\tau_1, \dots, \tau_{m-1}$ (in a neighborhood of 0 and hence everywhere in N_v). This contradicts minimality of $\{\tau_i\}_{i=1}^m$. □

3. Two interpolation inequalities

We recall two classical interpolation inequalities. The first is a version of Glaeser's inequality (cf. [15]).

Lemma 4. *Let $I \subseteq \mathbb{R}$ be an open interval and let $f \in C^{1,1}(\overline{I})$ be nonnegative. For any $t_0 \in I$ and $M > 0$ such that $I_{t_0}(M^{-1}) := \{t : |t - t_0| < M^{-1}|f(t_0)|^{1/2}\} \subseteq I$ and $M^2 \ge \operatorname{Lip}_{I_{t_0}(M^{-1})}(f')$, we have*

$$|f'(t_0)| \le \big(M + M^{-1} \operatorname{Lip}_{I_{t_0}(M^{-1})}(f')\big)|f(t_0)|^{1/2} \le 2M|f(t_0)|^{1/2}.$$

Proof. The inequality holds true at zeros of f. Let us assume that $f(t_0) > 0$. The statement follows from

$$0 \le f(t_0 + h) = f(t_0) + f'(t_0)h + \int_0^1 (1-s) f''(t_0 + hs)\, ds\, h^2$$

with $h = \pm M^{-1}|f(t_0)|^{1/2}$. □

Lemma 5. *Let $f \in C^{m-1,1}(\bar{I})$. There is a universal constant $C = C(m)$ such that for all $t \in I$ and $k = 1, \dots, m$,*

$$|f^{(k)}(t)| \le C|I|^{-k}\big(\|f\|_{L^\infty(I)} + \operatorname{Lip}_I(f^{(m-1)})|I|^m\big). \tag{3.1}$$

Proof. We may suppose $I = (-\delta, \delta)$. If $t \in I$ then at least one of the two intervals $[t, t \pm \delta)$, say $[t, t + \delta)$, is included in I. By Taylor's formula, for $t_1 \in [t, t + \delta)$,

$$\Big|\sum_{k=0}^{m-1} \frac{f^{(k)}(t)}{k!}(t_1 - t)^k\Big| \le |f(t_1)| + \int_0^1 \frac{(1-s)^{m-1}}{(m-1)!}|f^{(m)}(t + s(t_1 - t))|\, ds\, (t_1 - t)^m$$
$$\le \|f\|_{L^\infty(I)} + \operatorname{Lip}_I(f^{(m-1)})\delta^m,$$

and for $k \le m - 1$ we may conclude by Proposition 1 below. For $k = m$, (3.1) is trivially satisfied. □

Proposition 1. *Let* $P(x) = a_0 + a_1 x + \cdots + a_m x^m \in \mathbb{C}[x]$ *satisfy* $|P(x)| \le A$ *for* $x \in [0, B] \subseteq \mathbb{R}$*. Then, for* $j = 0, \ldots, m$,

$$|a_j| \le (2m)^{m+1} A B^{-j}.$$

Proof. We show the lemma for $A = B = 1$. The general statement follows by applying this special case to the polynomial $A^{-1}P(By)$, $y = B^{-1}x$. Let $0 = x_0 < x_1 < \cdots < x_m = 1$ be equidistant points. By Lagrange's interpolation formula (e.g., [27, (1.2.5)]),

$$P(x) = \sum_{k=0}^{m} P(x_k) \prod_{\substack{j=0 \\ j \ne k}}^{m} \frac{x - x_j}{x_k - x_j},$$

and therefore

$$a_j = \sum_{k=0}^{m} P(x_k) \prod_{\substack{j=0 \\ j \ne k}}^{m} (x_k - x_j)^{-1} (-1)^{m-j} \sigma_{m-j}^k,$$

where σ_j^k is the jth elementary symmetric polynomial in $(x_\ell)_{\ell \ne k}$. The statement follows. $\square$

A better constant can be obtained using Chebyshev polynomials; cf. [27, Thms. 16.3.1–2].

4. Proof of Theorem 1

Let $(G \circlearrowleft V, d, \sigma)$ satisfy (2.1) and (2.3), and let $c \in C^{d-1,1}(I, \sigma(V))$.

4.1. Reduction to $G \circlearrowleft (V \setminus \{0\})$

By (2.1) we have $c_1 \ge 0$ and $c_1(t) = 0$ if and only if $c(t) = 0$. We shall show the following statement.

Claim 1. *For any relatively compact open subinterval* $I_0 \Subset I$ *and any* $t_0 \in I_0 \setminus {c_1}^{-1}(0)$*, there exists a Lipschitz lift* $\overline{c}_{t_0}$ *of* c *on a neighborhood* I_{t_0} *of* t_0 *in* $I_0 \setminus {c_1}^{-1}(0)$ *so that*

$$\operatorname{Lip}_{I_{t_0}}(\overline{c}_{t_0}) \le C \left(\max_i \|c_i\|_{C^{d-1,1}(\overline{I}_1)}^{1/d_i} \right),$$

where I_1 *is any open interval satisfying* $I_0 \Subset I_1 \Subset I$ *and* $C = C(G \circlearrowleft V, I_0, I_1)$.

Claim 1 will imply Theorem 1 by the following lemma.

Lemma 6. *Suppose that for each* $t_0 \in I_0 \setminus {c_1}^{-1}(0)$ *there exists a Lipschitz lift* $\overline{c}_{t_0}$ *of* c *on a neighborhood* I_{t_0} *of* t_0 *in* $I_0 \setminus {c_1}^{-1}(0)$ *so that* $L := \sup_{t_0 \in I_0 \setminus {c_1}^{-1}(0)} \operatorname{Lip}_{I_{t_0}}(\overline{c}_{t_0}) < \infty$*. Then there exists a Lipschitz lift* $\overline{c}$ *of* c *on* I_0 *and* $\operatorname{Lip}_{I_0}(\overline{c}) \le L$.

Proof. Let J be any connected component of $I_0 \setminus {c_1}^{-1}(0)$. If $\overline{c}_i$, $i = 1, 2$, are local Lipschitz lifts of c defined on subintervals (a_i, b_i), $i = 1, 2$, of J with $a_1 < a_2 < b_1 < b_2$ and so that $\operatorname{Lip}_{(a_i,b_i)}(\overline{c}_i) \le L$, $i = 1, 2$, then there exists a Lipschitz lift $\overline{c}_{12}$ of c on (a_1, b_2) satisfying $\operatorname{Lip}_{(a_1,b_2)}(\overline{c}_{12}) \le L$. To see this, choose a point $t_{12} \in (a_2, b_1)$. Since $G\overline{c}_1(t_{12}) = G\overline{c}_2(t_{12})$, there exists $g_{12} \in G$ so that $\overline{c}_1(t_{12}) = g_{12}\overline{c}_2(t_{12})$. Define

$\overline{c}_{12}(t) := \overline{c}_1(t)$ for $t \le t_{12}$ and $\overline{c}_{12}(t) := g_{12}\overline{c}_2(t)$ for $t \ge t_{12}$. It is easy to see that c_{12} has the required properties (since G acts orthogonally).

These arguments imply that there exists a Lipschitz lift $\overline{c}_J$ of c with $\mathrm{Lip}_J(\overline{c}_J) \le L$ on each connected component J of $I_0 \setminus {c_1}^{-1}(0)$. Defining $\overline{c}(t) := \overline{c}_J(t)$ if $t \in J$ and $\overline{c}(t) := 0$ if $t \in {c_1}^{-1}(0)$, we obtain a continuous lift of c, since $c_1(t) = \|\overline{c}(t)\|^2$, by (2.1). It is easy to see that $\mathrm{Lip}_{I_0}(\overline{c}) \le L$. □

Let us prove that Claim 1 and Lemma 6 imply Theorem 1. That they imply Remark 1 is obvious. Let $J_1 \subseteq J_2 \subseteq \cdots$ be a countable exhaustion of I by compact intervals so that, for all k, J_k is contained in the interior of J_{k+1}. By Claim 1 and Lemma 6, there exist lifts $\overline{c}_k : J_k \to V$, $k \ge 1$, of c and compact neighborhoods $K_k \supseteq J_k$ in I so that

$$\mathrm{Lip}_{J_k}(\overline{c}_k) \le C\left(\max_i \|c_i\|^{1/d_i}_{C^{d-1,1}(K_k)}\right), \quad k \ge 1,$$

for $C = C(G \circlearrowleft V, J_k, K_k)$. We may construct a $C^{0,1}$-lift $\overline{c} : I \to V$ of c iteratively in the following way. If $\overline{c}$ already exists on J_k we extend it on $J_{k+1} \setminus J_k$ by $g\overline{c}_{k+1}$ for suitable $g \in G$ left and right of J_k (cf. the first paragraph of the proof of Lemma 6). If $I_0 \Subset I$ is relatively compact then $I_0 \subseteq J_N$ for some N. Thus for $t, s \in I_0$, $t < s$, there is a sequence $t =: t_0 < t_1 < \cdots < t_\ell := s$ of endpoints t_i of the intervals J_k (except possibly t_0 and t_ℓ), elements $g_i \in G$, and $k_i \in \{1, \ldots, N\}$ so that

$$\begin{aligned}\|\overline{c}(t) - \overline{c}(s))\| &\le \sum_{i=1}^{\ell} \|g_i \overline{c}_{k_i}(t_i) - g_i \overline{c}_{k_i}(t_{i-1})\| \\ &= \sum_{i=1}^{\ell} \|\overline{c}_{k_i}(t_i) - \overline{c}_{k_i}(t_{i-1})\| \\ &\le \max_{1 \le k \le N} \mathrm{Lip}_{J_k}(\overline{c}_k)\, |t - s|.\end{aligned}$$

Setting $I_1 := \bigcup_{k=1}^N K_k$, we obtain (1.1).

4.2. Convenient assumption

The proof of Claim 1 will be carried out by induction on the *size* of G. If G and H are compact Lie groups we write $H < G$ if and only if $\dim H < \dim G$ or, if $\dim H = \dim G$, H has fewer connected components than G.

We replace the assumption that $c \in C^{d-1,1}(I, \sigma(V))$ by a new (weaker) assumption that will be more convenient for the inductive step. Before stating it we need a bit of notation.

For open intervals I_0 and I_1 so that $I_0 \Subset I_1 \Subset I$, we set

$$I_i' := I_i \setminus {c_1}^{-1}(0), \quad i = 0, 1.$$

For $t_0 \in I_0'$ and $r > 0$ consider the interval

$$I_{t_0}(r) := \big(t_0 - r|c_1(t_0)|^{1/2}, t_0 + r|c_1(t_0)|^{1/2}\big).$$

Assumption. Let $I_0 \Subset I_1$ be open intervals. Suppose that $c \in C^{d-1,1}(\overline{I}_1, \sigma(V))$ and assume that there is a constant $A > 0$ so that for all $t_0 \in I_0'$, $t \in I_{t_0}(A^{-1})$, $i = 1, \ldots, n$, $k = 0, \ldots, d$,

$$I_{t_0}(A^{-1}) \subseteq I_1 \tag{A.1}$$

$$2^{-1} \le \frac{c_1(t)}{c_1(t_0)} \le 2 \tag{A.2}$$

$$|{c_i}^{(k)}(t)| \le C\, A^k\, |c_1(t)|^{(d_i-k)/d_1} \tag{A.3}$$

where $C = C(G \circlearrowleft V) \ge 1$. For $k = d$, (A.3) is understood to hold almost everywhere, by Rademacher's theorem.

Remark. Condition (A.3) implies that

$$\big|\partial_t^k\big({c_1}^{-d_i/d_1} c_i\big)(t)\big| \le C\, A^k\, |c_1(t)|^{-k/d_1}, \tag{A.4}$$

where $C = C(G \circlearrowleft V)$. In fact, if we assign c_i the weight d_i (and ${c_1}^{1/d_1}$ the weight 1) and let $L(x_1, \ldots, x_n, y) \in \mathbb{R}[x_1, \ldots, x_n, y, y^{-1}]$ be weighted homogeneous of degree D, then

$$\big|\partial_t^k L\big(c_1, \ldots, c_n, {c_1}^{1/d_1}\big)(t)\big| \le C\, A^k\, |c_1(t)|^{(D-k)/d_1},$$

for $C = C(G \circlearrowleft V, L)$.

The following two claims clearly imply Claim 1.

Claim 2. *Any curve $c \in C^{d-1,1}(\overline{I}_1, \sigma(V))$ satisfying* (A.1)–(A.3) *has a Lipschitz lift on a neighborhood of any $t_0 \in I_0'$ with Lipschitz constant bounded from above by $C\,A$, where $C = C(G \circlearrowleft V)$.*

Claim 3. *If $c \in C^{d-1,1}(I, \sigma(V))$ then* (A.1)–(A.3) *hold for each pair of open intervals I_0 and I_1 satisfying $I_0 \Subset I_1 \Subset I$ and with $A \le C\,(\max_i \|c_i\|^{1/d_i}_{C^{d-1,1}(\overline{I}_1)})$ for $C = C(I_0, I_1)$.*

4.3. Proof of Claim 2 (inductive step)

Let c, I_0, I_1, A, t_0 be as in the Assumption and hence satisfy (A.1)–(A.3). We will show the following.

- For some constant $C_1 = C_1(G \circlearrowleft V) > 1$, the lifting problem for c reduces on the interval $I_{t_0}({C_1}^{-1} A^{-1})$ to the lifting problem for some associated curve c^* in the orbit space of some slice representation $H \circlearrowleft W$ of $G \circlearrowleft V$ with $H < G$.
- The curve c^* satisfies (A.1)–(A.3) for suitable neighborhoods J_0, J_1 of t_0 and a constant $B = C\,A$ in place of A, where $C = C(G \circlearrowleft V)$.

This will allow us to conclude Claim 2 by induction on the size of G.

Let us restrict c to $I_{t_0}(A^{-1})$ and consider

$$\underline{c} := \big(1, {c_1}^{-d_2/d_1} c_2, \ldots, {c_1}^{-d_n/d_1} c_n\big) : I_{t_0}(A^{-1}) \to \sigma(V) \subseteq \mathbb{R}^n_y.$$

Then $\underline{c}$ is continuous, by (A.2), and bounded, by (2.2). Moreover, by (A.4) and (A.2), for $t \in I_{t_0}(A^{-1})$,

$$\|\underline{c}'(t)\| \le C_1\, A\, |c_1(t_0)|^{-1/d_1}, \tag{4.1}$$

for $C_1 = C_1(G \circlearrowleft V)$. Consider the finite open cover $\{B_\alpha\}_{\alpha\in\Delta} = \{\sigma(U_{v_\alpha})\}_{\alpha\in\Delta}$ of the compact set $\sigma(V) \cap \{y_1 = 1\}$ from (2.8). Let $2r_1 > 0$ be a Lebesgue number of the cover $\{B_\alpha\}_{\alpha\in\Delta}$. Then for any $p \in \sigma(V) \cap \{y_1 = 1\}$ there is $\alpha_p \in \Delta$ so that

$$B_p(r_1) \cap \sigma(V) \cap \{y_1 = 1\} \subseteq B_{\alpha_p},$$

where $B_p(r_1) \subseteq \mathbb{R}^n$ is the open ball centered at p with radius r_1. If C_1 is the constant from (4.1), then

$$J_1 := I_{t_0}(r_1 {C_1}^{-1} A^{-1}) \subseteq \underline{c}^{-1}(B_{\underline{c}(t_0)}(r_1)). \tag{4.2}$$

By Lemma 2 the lifting problem on the interval J_1 reduces to the curve $c^* = (c_i^*)_{i=1}^m$,

$$c_i^* = c_1^{e_i/d_1} \varphi_i\big({c_1}^{-d_2/d_1} c_2, \dots, {c_1}^{-d_n/d_1} c_n\big), \quad e_i = \deg \tau_i, \tag{4.3}$$

in $\tau(N_v)$, where $G_v \circlearrowleft N_v$ is the slice representation at $v = v_{\alpha_{\underline{c}(t_0)}}$ with orbit mapping $\tau = (\tau_1, \dots, \tau_m)$ and where the φ_i are real analytic; the first summand of (2.7) is Lipschitz with Lipschitz constant bounded from above by $C\,A$ with $C = C(G \circlearrowleft V)$ thanks to (A.3). Fix $r_0 < r_1$ and set

$$J_0 := I_{t_0}(r_0 {C_1}^{-1} A^{-1}), \tag{4.4}$$

where C_1 is the constant from (4.1). (Here we assume without loss of generality that $r_1 < C_1$ so that $r_0 {C_1}^{-1} < r_1 {C_1}^{-1} < 1$ and hence $J_0 \subseteq J_1 \subseteq I_{t_0}(A^{-1})$.)

Let us show that the curve c^* satisfies (A.1)–(A.3) for the intervals J_1 and J_0 from (4.2) and (4.4) and a suitable constant $B > 0$ in place of A. To this end we set

$$J_i' := J_i \setminus (c_1^*)^{-1}(0), \quad i = 0, 1,$$

consider, for $t_1 \in J_0'$ and $r > 0$, the interval

$$J_{t_1}(r) := \big(t_1 - r|c_1^*(t_1)|^{1/2}, t_1 + r|c_1^*(t_1)|^{1/2}\big),$$

and prove the following lemma.

Lemma 7. *There is a constant $C = C(G \circlearrowleft V, r_1, r_0) > 1$ such that for $B = C\,A$ and for all $t_1 \in J_0'$, $t \in J_{t_1}(B^{-1})$, $i = 1, \dots, m$, $k = 0, \dots, d$,*

$$J_{t_1}(B^{-1}) \subseteq J_1 \tag{B.1}$$

$$2^{-1} \le \frac{c_1^*(t)}{c_1^*(t_1)} \le 2 \tag{B.2}$$

$$|(c_i^*)^{(k)}(t)| \le \widetilde{C}\, B^k\, |c_1^*(t)|^{(e_i-k)/e_1} \tag{B.3}$$

where $\widetilde{C} = \widetilde{C}(G \circlearrowleft V)$.

Proof. If
$$B \ge (r_1 - r_0)^{-1}\sqrt{2\,C_0}\,C_1\,A,$$
where C_0 and C_1 are the constants from Corollary 4 and (4.1), respectively, then by Corollary 4 and (A.2),
$$B^{-1}|c_1^*(t_1)|^{1/2} \le (r_1 - r_0)\,{C_1}^{-1}\,A^{-1}\,|c_1(t_0)|^{1/2},$$
and so (B.1) follows from (4.2) and (4.4), as $t_1 \in J_0$.

Next we claim that, on J_1,
$$\big|\partial_t^k \varphi_i\big({c_1}^{-d_2/d_1}c_2, \dots, {c_1}^{-d_n/d_1}c_n\big)\big| \le C\,A^k\,|c_1|^{-k/d_1}, \tag{4.5}$$
for $C = C(G \circlearrowleft V)$. To see this we differentiate the following equation $(k-1)$ times, apply induction on k, and use (A.4),
$$\partial_t\varphi_i\big({c_1}^{-d_2/d_1}c_2, \dots, {c_1}^{-d_n/d_1}c_n\big) = \sum_{j=1}^n (\partial_j\varphi_i)(\underline{c})\,\partial_t\big({c_1}^{-d_j/d_1}c_j\big); \tag{4.6}$$
recall that all partial derivatives of the φ_i's are separately bounded on $\underline{c}(J_1)$ and these bounds are universal. From (4.3) and (4.5) we obtain, on J_1 and for all $i = 1, \dots, m$, $k = 0, \dots, d$,
$$|(c_i^*)^{(k)}| \le C\,A^k\,|c_1|^{(e_i-k)/d_1}, \tag{4.7}$$
for $C = C(G \circlearrowleft V)$, and so, by Corollary 4 and as $d_1 = e_1 = 2$,
$$|(c_i^*)^{(k)}| \le C\,A^k\,|c_1^*|^{(e_i-k)/e_1} \quad \text{if } e_i - k \le 0, \tag{4.8}$$
for $C = C(G \circlearrowleft V)$. This shows (B.3) for $k \ge e_i$, and (B.3) for $k = 0$ follows from (2.2). The remaining inequalities, i.e., (B.3) for $0 < k < e_i$ as well as (B.2), follow now from Lemma 8 below (since $d \ge e = \max_i e_i$, by Lemma 3). □

Lemma 8. *There is a constant $C = C(G \circlearrowleft V) \ge 1$ such that the following holds. If* (A.1) *and* (A.3) *for $k = 0$ and $k = d_i$, $i = 1, ..., n$, are satisfied, then so are* (A.2) *and* (A.3) *for $k < d_i$, $i = 1, ..., n$, after replacing A by $C\,A$.*

Proof. By assumption $\operatorname{Lip}_{I_{t_0}(A^{-1})}(c_1') \le C\,A^2$, where C is the constant from (A.3). Thus, by Lemma 4 for $f = c_1$ and $M = C^{1/2}A$, we get
$$|c_1'(t_0)| \le 2M|c_1(t_0)|^{1/2}.$$
It follows that, for $t \in I_{t_0}((6M)^{-1})$,
$$\begin{aligned}\frac{|c_1(t) - c_1(t_0)|}{|c_1(t_0)|} &\le \frac{|c_1'(t_0)|}{|c_1(t_0)|}|t - t_0| + \int_0^1 (1-s)|c_1''(t_0 + s(t-t_0))|ds\,\frac{|t-t_0|^2}{|c_1(t_0)|} \\ &\le 1/2\end{aligned} \tag{4.9}$$
which implies (A.2). The other inequalities follow from Lemma 5. □

We may now finish the proof of Claim 2. By assumption (2.3), $V^G = \{0\}$ and thus $G_v < G$. The inductive hypothesis yields a Lipschitz lift $\overline{c}^*$ of c^* over τ with Lipschitz constant bounded from above by $C\,B$, for $C = C(G_v \circlearrowleft N_v)$. By Lemma 1 and (4.8) for $e_i = k = 1$ (the basic invariants of $G_v \circlearrowleft N_v^{G_v}$ form a system of linear coordinates on $N_v^{G_v}$), we can assume that $N_v^{G_v} = \{0\}$. By Lemma 2,

$$ {c_1}^{1/d_1} v + \overline{c}^* $$

is a lift of c over σ. Thanks to (A.3) for $i = k = 1$ and since there are only finitely many isomorphism types of slice representations, this lift is Lipschitz with Lipschitz constant bounded from above by $C\,A$, for $C = C(G \circlearrowleft V)$. This ends the proof of Claim 2.

4.4. Proof of Claim 3

Let δ denote the distance between the endpoints of I_0 and those of I_1. Set

$$ \begin{aligned} A_1 &:= \max\Big\{\delta^{-1}\|c_1\|^{1/2}_{L^\infty(I_1)}, (\mathrm{Lip}_{I_1}(c_1'))^{1/2}\Big\},\\ A_2 &:= \max_i \Big\{M_i \|c_1\|^{(d-d_i)/2}_{L^\infty(I_1)}\Big\}^{1/d}, \quad M_i := \mathrm{Lip}_{I_1}({c_i}^{(d-1)}), \end{aligned} \tag{4.10} $$

and choose

$$ A \ge A_0 = 6\max\{A_1, A_2\}. \tag{4.11} $$

To have (A.1) and (A.2) it suffices to assume $A \ge 6A_1$. For $t_0 \in I_0'$ obviously $I_{t_0}({A_1}^{-1}) \subseteq I_1$ and thus (A.1). Then Lemma 4 implies

$$ |c_1'(t_0)| \le 2A_1\,|c_1(t_0)|^{1/2}, $$

and so, for $t_0 \in I_0'$ and $t \in I_{t_0}((6A_1)^{-1})$, (4.9) and hence (A.2) holds. Finally, Lemma 5, (2.2), and (A.2) imply (A.3) for $t \in I_{t_0}(A^{-1})$.

4.5. Bounds for the Lipschitz constant

Let $(G \circlearrowleft V, d, \sigma)$ satisfy (2.1) and (2.3), let $c \in C^{d-1,1}(I, \sigma(V))$, and let $I_0 \Subset I$. Then there is a neighborhood I_1 of I_0 with $I_0 \Subset I_1 \Subset I$ such that the lift $\overline{c} \in C^{0,1}(I, V)$ constructed in the above proof satisfies

$$ \begin{aligned} \mathrm{Lip}_{I_0}(\overline{c}) &\le C(G \circlearrowleft V)\max\Big\{\delta^{-1}\|c_1\|^{1/2}_{L^\infty(I_1)}, (\mathrm{Lip}_{I_1}(c_1'))^{1/2}, \max_i\{M_i\|c_1\|^{(d-d_i)/2}_{L^\infty(I_1)}\}^{1/d}\Big\}\\ &\le C(G \circlearrowleft V, I_0, I_1)\,\big(\max_i \|c_i\|^{1/d_i}_{C^{d-1,1}(\overline{I}_1)}\big)\\ &\le C(G \circlearrowleft V, I_0, I_1)\,\big(1 + \max_i \|c_i\|_{C^{d-1,1}(\overline{I}_1)}\big) \end{aligned} \tag{4.12} $$

where δ is the distance between the endpoints of I_0 and those of I_1, and $M_i = \mathrm{Lip}_{I_1}({c_i}^{(d-1)})$. This follows from Claim 2, (4.10), (4.11), and Lemma 6.

5. Proof of Theorem 2

Let $(G \circlearrowleft V, d, \sigma)$ satisfy (2.1) and (2.3), and let $c \in C^d(I, \sigma(V))$. In the proof of Theorem 2, induction on the size of G will provide us with local lifts of class C^1 near points where c is not flat (in the sense that they are not of Case 2 of Subsection 5.5). Moreover, we shall see that the derivatives of these local lifts converge to 0 as t tends to flat points. This faces us with the problem of gluing these local lifts. We tackle this problem first.

5.1. Algorithm for local lifts

We choose a finite cover $\{GU_{v_\alpha}\}_{\alpha \in \Delta}$ of a neighborhood of the sphere $S(V) = {c_1}^{-1}(1)$ in V so that U_v is transverse to all the orbits in GU_{v_α} with the angle very close to $\pi/2$. It induces a cover of $\sigma(V) \cap \{y_1 = 1\}$,

$$\{B_\alpha\}_{\alpha \in \Delta} = \{\sigma(U_{v_\alpha})\}_{\alpha \in \Delta},$$

in analogy to (2.8).

Lemma 2 provides an algorithm for the construction of a lift of c. After removing the fixed points (see Subsection 2.2) we lift c restricted to $I' := \{t \in I : c_1(t) \neq 0\}$ and then extend it trivially to $\{t \in I : c_1(t) = 0\}$. For this we consider

$$\underline{c} := \big(1, {c_1}^{-d_2/d_1} c_2, \ldots, {c_1}^{-d_n/d_1} c_n\big).$$

For each connected component I_1 of the induced cover $\{\underline{c}^{-1}(B_\alpha)\}_{\alpha \in \Delta}$ of I' we lift $c|_{I_1}$ to the slice N_v, $v = v_\alpha$, using Lemma 2 and hence the induction. This reduction ends when $\underline{c}(I) \subseteq B_\alpha$ with B_α in the open stratum (where we keep the notation $\underline{c}$, I, and B_α for the respective reduced objects).

Thus for any $t_0 \in I$ there is a neighborhood I_{t_0} and a lift $\bar{c}$ of c on I_{t_0} that is entirely contained in an affine transverse slice to the orbit over $c(t_0)$ that is close to the normal slice $S_{\overline{c}(t_0)}$ from (2.4). (Note that the orbit over $0 \in \sigma(V)$ is just the origin in V and every slice is a neighborhood of the origin.)

This picture is not complete. One needs to make precise how these local lifts are glued together.

5.2. Change of slice diffeomorphisms

Fix $v \in V$ and let S_v be the normal slice of the orbit Gv at v; see (2.4). Let $H = G_v$ and fix a local analytic section $\varphi_H : G/H \to G$ of the principal bundle $G \to G/H$ such that $\varphi_H([e]) = e$. Then

$$\Phi_v : G/H \times S_v \to V, \quad \Phi_v([g], x) = \varphi_H([g])x \tag{5.1}$$

is a local diffeomorphism and $\Phi_v([e], v) = v$. Indeed, Φ_v equals the following composition

$$G/H \times S_v \xrightarrow{\alpha} G \times_{G_v} S_v \xrightarrow{\phi} V,$$

where $\phi : G \times_{G_v} S_v \to V$, $[g, x] \mapsto gx$, is the slice mapping from Theorem 3, and $\alpha([g], x)$ is the class of $(\varphi_H([g]), x)$. Then α is a diffeomorphism with the inverse

$$\alpha^{-1}([g, x]) = \alpha^{-1}\big([gg^{-1}\varphi_H([g]), (\varphi_H([g]))^{-1}gx]\big) = ([g], (\varphi_H([g]))^{-1}gx).$$

Let M_v be another affine transverse slice at v, and we suppose that the angle between N_v and M_v is small. The second coordinate of the inverse of Φ_v restricted to M_v gives a local diffeomorphism

$$h_{M_v} : M_v \to S_v.$$

The first coordinate of the inverse of Φ_v composed with φ_H gives a mapping

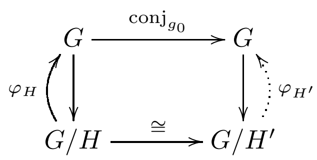
$$s_{M_v} : M_v \to G$$

such that

$$h_{M_v}(x) = (s_{M_v}(x))^{-1}x.$$

By (5.1) the partial derivatives of s_{M_v} and h_{M_v} can be bounded in terms of the partial derivatives of φ_H and the angle between N_v and M_v.

Remark 2. The above construction is uniform in the following sense. If $v' = g_0 v$ then $H = G_v$ and $H' = G_{v'}$ are conjugate, $H' = g_0 H g_0^{-1}$. Conjugation by g_0 on G induces an isomorphism $G/H \to G/H'$, $[g]_H \mapsto [g_0 g g_0^{-1}]_{H'}$. Given φ_H we define $\varphi_{H'}$ by the following diagram:



Thus if we fix φ_H for each conjugacy class and suppose the angle between N_v and M_v is small, we obtain bounds on the derivatives of s_{M_v} and h_{M_v} independent of v (valid in a neighborhood of v whose size depends on the orbit Gv).

5.3. Gluing the local lifts

Suppose that there are local lifts $\overline{c}_1$ and $\overline{c}_2$ of c resulting from the algorithm described in Subsection 5.1 such that the respective domains of definition I_1 and I_2 have nontrivial intersections. Fix $t_0 \in I_1 \cap I_2$. We may assume that $\overline{c}_1(t_0) = \overline{c}_2(t_0)$ and denote this vector by v. Then, by construction, there exist a neighborhood I_{t_0} of t_0 in $I_1 \cap I_2$ and slices M_v^1 and M_v^2 transverse to Gv containing $\overline{c}_1(I_{t_0})$ and $\overline{c}_2(I_{t_0})$, respectively. Then, by Subsection 5.2,

$$I_{t_0} \ni t \mapsto h_{M_v^i}(\overline{c}_i(t)), \quad i = 1, 2,$$

are two lifts of c on I_{t_0} contained in S_v. If we, moreover, assume that $c(I_{t_0})$ belongs to a single stratum, then these two lifts coincide (since all orbits of type (G_v) meet S_v in a single point), and thus, for $t \in I_{t_0}$,

$$s_{M_v^1}(\overline{c}_1(t))^{-1}\ \overline{c}_1(t) = s_{M_v^2}(\overline{c}_2(t))^{-1}\ \overline{c}_2(t). \tag{5.2}$$

Then, there is a universal constant $C > 0$ such that for $i = 1, 2$ and $t \in I_{t_0}$

$$|\partial_t s_{M_v^i}(\overline{c}_i(t))| \le C \max \|\overline{c}_i'(t)\|. \tag{5.3}$$

Lemma 9. *Let $K \Subset J \Subset I$ be intervals and let $s : J \to G$ be of class C^1. Then there is $\widetilde{s} : I \to G$ of class C^1 such that*

(i) $s|_K = \widetilde{s}|_K$.
(ii) $\|s'\|_{L^\infty(K)} = \|\widetilde{s}'\|_{L^\infty(I)}$.
(iii) *$\widetilde{s}$ is constant on each component of $I \setminus J$.*

Proof. We may extend $s|_K$ through the endpoints of $K = (t_-, t_+)$ using the exponential mapping in the direction $s'(t_\pm)$. More precisely, for the right endpoint t_+ set $g = s(t_+) \in G$ and $s'(t_+) = T_e\mu_g.X$ for $X \in \mathfrak{g}$ (where $\mu_g(h) = gh$ denotes left translation on G), and define

$$\widetilde{s}(t) = g \exp(\varphi(t - t_+)X),$$

where $\varphi(t) = \int_0^t \psi(u)\, du$ for

$$\psi(t) = \begin{cases} 1 & t \le 0 \\ 1 - t/\delta & 0 \le t \le \delta \\ 0 & t \ge \delta \end{cases}$$

and where δ denotes the distance of the right endpoints of K and J. □

Fix an open interval $K \Subset I_{t_0}$, $t_0 \in K$. By Lemma 9, we may extend each $s_{M_v^i}(\overline{c}_i(t))$ to a C^1 map $s_i : I_i \to G$ that coincides with $s_{M_v^i}(\overline{c}_i(t))$ on K and is constant in the complement of I_{t_0}. Let us then shrink I_1 and I_2 so that their union $I_1 \cup I_2$ does not change, but $I_1 \cap I_2 = K$. Then we set

$$\overline{c}(t) := s_{M_v^i}(\overline{c}_i(t))^{-1}\ \overline{c}_i(t), \quad \text{if } t \in I_i, \quad i = 1, 2,$$

which is well-defined by (5.2). Moreover,

$$\|\overline{c}'(t)\| \le C \max\{\|\overline{c}_1'(t)\|, \|\overline{c}_2'(t)\|\}, \quad t \in I_1 \cup I_2, \tag{5.4}$$

for a universal constant $C > 0$, where we set $\overline{c}_i'(t) := 0$ if $t \notin I_i$.

5.4. ${}^pC^m$-functions

Later in the proof we shall need a result on functions defined near $0 \in \mathbb{R}$ that become C^m when multiplied with the monomial t^p.

Definition. Let $p, m \in \mathbb{N}$ with $p \le m$. A continuous complex valued function f defined near $0 \in \mathbb{R}$ is called a *${}^pC^m$-function* if $t \mapsto t^p f(t)$ belongs to C^m.

Let $I \subseteq \mathbb{R}$ be an open interval containing 0. Then $f : I \to \mathbb{C}$ is ${}^pC^m$ if and only if it has the following properties; cf. [35, 4.1], [30, Satz 3], or [31, Thm. 4]:

- $f \in C^{m-p}(I)$.
- $f|_{I\setminus\{0\}} \in C^m(I \setminus \{0\})$.
- $\lim_{t\to 0} t^k f^{(m-p+k)}(t)$ exists as a finite number for all $0 \le k \le p$.

Proposition 2. *If $g = (g_1, \dots, g_n)$ is ${}^pC^m$ and F is C^m near $g(0) \in \mathbb{C}^n$, then $F \circ g$ is ${}^pC^m$.*

Proof. Cf. [31, Thm. 9] or [29, Prop. 3.2]. Clearly g and $F \circ g$ are C^{m-p} near 0 and C^m off 0. By Faà di Bruno's formula [13], for $1 \le k \le p$ and $t \ne 0$,

$$\frac{t^k (F\circ g)^{(m-p+k)}(t)}{(m-p+k)!} = \sum_{\ell \ge 1} \sum_{\alpha \in A} \frac{t^{k-|\beta|}}{\ell!} d^\ell F(g(t)) \Big(\frac{t^{\beta_1} g^{(\alpha_1)}(t)}{\alpha_1!}, \dots, \frac{t^{\beta_\ell} g^{(\alpha_\ell)}(t)}{\alpha_\ell!} \Big)$$
$$A := \{\alpha \in \mathbb{N}^\ell_{>0} : \alpha_1 + \cdots + \alpha_\ell = m-p+k\}$$
$$\beta_i := \max\{\alpha_i - m + p, 0\}, \quad |\beta| = \beta_1 + \cdots + \beta_\ell \le k,$$

whose limit as $t \to 0$ exists as a finite number by assumption. □

5.5. End of proof

We distinguish three kinds of points $t_0 \in I$:

Case 0: $c_1(t_0) \ne 0$, or
Case 1: $c_1(t_0) = 0$, thus $c_1'(t_0) = 0$ by (2.1), and $c_1''(t_0) \ne 0$, or
Case 2: $c_1(t_0) = c_1'(t_0) = c_1''(t_0) = 0$.

Near points of Case 0 there are local C^1-lifts, by the algorithm in Subsection 5.1.

Let us prove that we also have local C^1-lifts near points t_0 of Case 1. For simplicity of notation, let $t_0 = 0$. Then $c_1(t) \sim t^2$ and hence $c_i(t) = O(t^{d_i})$. Therefore,

$$\underline{c}(t) := \big(t^{-2} c_1(t), t^{-d_2} c_2(t), \dots, t^{-d_n} c_n(t)\big) : I_1 \to \sigma(V) \subseteq \mathbb{R}^n,$$

defined on a neighborhood I_1 of 0, is continuous. By Lemma 2 the lifting problem reduces to the curve $c^* = (c_i^*)_{i=1}^m$,

$$c_i^*(t) = t^{e_i} \varphi_i\big(t^{-2} c_1(t), t^{-d_2} c_2(t), \dots, t^{-d_n} c_n(t)\big), \quad e_i = \deg \tau_i, \tag{5.5}$$

in the orbit space $\tau(N_v)$ of any slice representation $G_v \circlearrowleft N_v$ so that $v \in \sigma^{-1}(\underline{c}(0))$. Then c_i^* is of class C^{e_i} at 0, by Proposition 2, and of class C^d in the complement of 0. After removing fixed points of $G_v \circlearrowleft N_v$, we may assume that the curve

$$\underline{c}^*(t) := \big(t^{-e_1} c_1^*(t), t^{-e_2} c_2^*(t), \dots, t^{-e_m} c_m^*(t)\big)$$

in $\tau(N_v)$ vanishes at $t = 0$, since $\underline{c}(0) = \sigma(v)$ (cf. (2.6)). Thus $c_i^*(t) = o(t^{e_i})$, for all i.

Lemma 10. *In this situation, for any $\varepsilon > 0$ there is a neighborhood I_ε of 0 in I such that for every $t_0 \in I_\varepsilon \setminus \{0\}$ the assumptions* (A.1)–(A.3) *are satisfied for the reduced curve c^* from* (5.5) *with $A \le \varepsilon$.*

Proof. Here we have to deal with the fact that c^* is not necessarily of class C^e. Let $I_0 = (-\delta, \delta)$ and $I_1 = (-2\delta, 2\delta)$. Since $(c_1^*)''(0) = 0$ and $c_1^*(t)$ is of class C^2, the constant A_1 of (4.10) for c^* can be chosen arbitrarily small. This is what we need to get (A.1)–(A.2) with an arbitrarily small A.

We have $c_i^* \in C^{e_i}$ near 0 (and $c_i^* \in C^d$ off 0) and $(c_i^*)^{(k)}(0) = 0$ for all $k \le e_i$. Therefore for an arbitrary $A > 0$ there is a neighborhood I_1 in which (A.3) holds for all i and $k = e_i$, and then, by Lemma 8, in a smaller neighborhood, for all i and all $k \le e_i$.

Finally, given $A > 0$ we show (A.3) for $k > e_i$ and δ sufficiently small. Let $\hat{A}$ denote the constant A for which (A.1)–(A.3) holds for c. By (4.7), for some constant $C = C(G \circlearrowleft V)$,

$$|(c_i^*)^{(k)}(t)| \le C\hat{A}^k \, |c_1(t)|^{(e_i-k)/2} \le C\hat{A}^k \, \psi(t)|c_1^*(t)|^{(e_i-k)/2},$$

which gives the required result, since $\psi(t) = |c_1^*(t)/c_1(t)|^{(k-e_i)/2} = o(1)$ for $k > e_i$. □

By induction, we may conclude from Lemma 10 that there is a C^1-lift near 0.

We may now glue the local lifts, according to Subsection 5.3. Let J be a connected component of the complement I' of the flat points (i.e., the points in Case 2). Then there exists an open cover $\mathcal{J} = \{J_i\}_{i\in\mathbb{Z}}$ of J, with C^1-lifts $\overline{c}_i$ of $c|_{J_i}$, and such that $J_i \cap J_j \ne \varnothing$ if and only if $|i-j| \le 1$. By Subsection 5.3 we may assume that there are C^1-maps $s_{i,\pm} : J_i \to G$ such that on $J_i \cap J_{i+1}$

$$s_{i,+}(t) \; \overline{c}_i(t) = s_{i+1,-}(t) \; \overline{c}_{i+1}(t). \tag{5.6}$$

Moreover, by Lemma 9, we may assume that there is $t_i \in J_i \setminus (J_{i-1} \cup J_{i+1})$ such that both $s_{i,\pm}$ are constant, say, equal to $g_{i,\pm}$, in a neighborhood J_{t_i} of t_i. Thus we may glue $g_{i,-}^{-1}s_{i,-}$ and $g_{i,+}^{-1}s_{i,+}$ into a single map $s_i : J_i \to G$ that equals $g_{i,-}^{-1}s_{i,-}$ for $t \le t_i$ and $g_{i,+}^{-1}s_{i,+}$ for $t \ge t_i$. Then

$$g_{i,+}s_i(t) \; \overline{c}_i(t) = g_{i+1,-}s_{i+1}(t) \; \overline{c}_{i+1}(t). \tag{5.7}$$

Lemma 11. *There are $h_i \in G$ such that*

$$h_i s_i(t) \; \overline{c}_i(t) = h_{i+1}s_{i+1}(t) \; \overline{c}_{i+1}(t). \tag{5.8}$$

Proof. In view of (5.7) it suffices to find h_i such that $g_{i+1,-}^{-1}g_{i,+} = h_{i+1}^{-1}h_i$. So we may fix $h_0 = e$ and then define them inductively by $h_{i+1} = h_i g_{i,+}^{-1} g_{i+1,-}$.

(Note that the existence of such h_i simply means that the cocycle $g_{i+1,-}^{-1}g_{i,+}$ is a Čech coboundary; that is clear because $\check{H}^1(\mathcal{J}; G) = 0$.) □

In this way we obtain a C^1-lift $\overline{c}$ of c restricted to I' with the property that $\|\overline{c}'(t)\|$ is dominated (up to a universal constant) by A_0 defined by (4.11), thanks to (5.4). The lift $\overline{c}$ extends trivially to flat points t_0 from Case 2. At each such point t_0, $\overline{c}$ is differentiable with $\overline{c}'(t_0) = 0$. It remains to check that $\overline{c}'(t) \to 0$ as $t \to t_0$. This is a consequence of the following lemma, where without loss of generality $t_0 = 0$.

Lemma 12. *If $c_1(0) = c_1'(0) = c_1''(0) = 0$, then for any $\varepsilon > 0$ there is $\delta > 0$ such that for $I_0 = (-\delta, \delta)$, $I_1 = (-2\delta, 2\delta)$, and A_0 defined by (4.11) we have $A_0 \le \varepsilon$.*

Proof. This follows immediately from the formulas (4.11) and (4.10). □

The proof of Theorem 2 is complete.

6. Real analytic lifts

It was shown in [1] that a real analytic curve $c \in C^\omega(I, \sigma(V))$ admits local real analytic lifts near every point $t_0 \in I$, and that the local lifts can be glued to a global real analytic lift if $G \circlearrowleft V$ is polar. We will now show that real analytic gluing is always possible.

Theorem 4. *Let $(G \circlearrowleft V, d, \sigma)$ be a real finite-dimensional orthogonal representation of a compact Lie group. Then any $c \in C^\omega(I, \sigma(V))$ admits a lift $\overline{c} \in C^\omega(I, V)$.*

Proof. The local lifts can be glued thanks to the fact that

$$\check{H}^1(I, G^a) = 0, \tag{6.1}$$

where G^a denotes the sheaf of real analytic maps $I \supseteq U \to G$. This is a deep result, suggested by Cartan in [7], [8], and proven by Tognoli [37].

Indeed, let $\mathcal{I} = \{I_i\}$ be a locally finite cover of I with real analytic lifts $\overline{c}_i$ of $c|_{I_i}$ (which exist by the result of [1]). Then, by Lemma 3.8 of [1], we may assume that if $I_i \cap I_j \neq \varnothing$ then there is real analytic $s_{ij} : I_i \cap I_j \to G$ such that on $I_i \cap I_j$

$$s_{ij}\overline{c}_i = \overline{c}_j.$$

By (6.1), after replacing $\mathcal{I}$ by its refinement if necessary, there are real analytic $h_i : I_i \to G$ such that $s_{ij} = h_j^{-1} h_i$ on $I_i \cap I_j$ and then

$$\overline{c}(t) = h_i(t)\overline{c}_i(t), \quad \text{if} \quad t \in I_i,$$

defines a global lift. $\square$

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