ALGEBRAIC STRATIFIED GENERAL POSITION AND TRANSVERSALITY

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Abstract. The method of Whitney interpolation is used to construct, for any real or complex algebraic stratification, a stratified submersive family of self-maps that yields stratified general position and transversality theorems for semialgebraic subsets.

In classical algebraic topology, general position of chains was used by Lefschetz (1925) to define the intersection pairing on the homology of a manifold. Stratified general position [3] was used by Goresky and MacPherson (1975) to define the intersection pairing on the intersection homology of a complex algebraic variety. Murolo, Trotman, and du Plessis [4] proved a stratified transversality theorem for the most important types of regular stratifications, and they used their result to give geometric meaning to cup and cap product for stratified homology theories.

Here we prove stratified general position and transversality theorems for semialgebraic subsets of algebraic stratifications. Our theorems can be used (as in [4]) to define cup and cap products for semialgebraic homology and the intersection pairing for the intersection homology of semialgebraic chains. (In their initial paper [1], Goresky and MacPherson considered piecewise linear chains with respect to a triangulation.) In a subsequent paper we will apply our transversality theorem to define an intersection pairing for *real intersection homology*, an analog of intersection homology for real algebraic varieties.

1. Submersive families

Let T and M be smooth (that is, C^1) manifolds and let $\Psi : T \times M \to M$ be a smooth mapping. Consider $\Psi_t : M \to M$, $\Psi_t(x) = \Psi(t,x)$, and $\Psi^x : T \to M$, $\Psi^x(t) = \Psi(t,x)$. We say Ψ is a family of diffeomorphisms if for all $t \in T$ the map Ψ_t is a diffeomorphism. The family Ψ is called *submersive* if, for each $(t,x) \in T \times M$, the differential $D\Psi^x$ at t is surjective (*cf.* [2] I.1.3.5). For a stratified set $X = \bigsqcup S_i$ we say that $\Psi : T \times X \to X$ is *a stratified submersive family of diffeomorphisms* if for each stratum S_j, we have $\Psi(T \times S_j) \subset S_j$, and the map $\Psi : T \times S_j \to S_j$ is a submersive family of diffeomorphisms.

By an algebraic, resp. projective, variety we mean an algebraic, resp. projective, variety over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. All sets and functions are supposed to be at least semialgebraic (in the real algebraic sense). Our main goal is to show the following theorem.

Theorem 1.1. *Let $\mathcal{V} = \{V_i\}$ be a finite family of algebraic subsets of projective space $\mathbb{P}^n$. There exists an algebraic stratification $\mathcal{S} = \{S_j\}$ of $\mathbb{P}^n$ compatible with each V_i and a semialgebraic stratified submersive family of diffeomorphisms $\Psi : U \times \mathbb{P}^n \to \mathbb{P}^n$, where*

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U is an open neighborhood of the origin in $\mathbb{K}^{n+1}$, such that $\Psi(0,x) = x$ for all $x \in \mathbb{P}^n$. Moreover, the map $\Phi : U \times \mathbb{P}^n \to U \times \mathbb{P}^n$, $\Phi(t,x) = (t, \Psi(t,x))$, is an arc-wise analytic trivialization of the projection $U \times \mathbb{P}^n \to U$.

Remark 1.2. A similar result holds for affine varieties; see Corollary 3.2.

By a *stratification* of a semialgebraic set X we mean a locally finite partition $\mathcal{S}$ of X into smooth semialgebraic manifolds called the *strata* of $\mathcal{S}$. An *algebraic stratification* of an algebraic variety X is a stratification coming from a filtration of X by algebraic subvarieties

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

such that for each j, either $\dim X_j = j$ or $X_j = X_{j-1}$, and $\operatorname{Sing}(X_j) \subset X_{j-1}$. This filtration induces a decomposition $X = \bigsqcup S_i$, where the S_i are the connected components of all $X_j \setminus X_{j-1}$. The sets S_i are the strata of the algebraic stratification $\mathcal{S} = \{S_i\}$ of X. If $\mathbb{K} = \mathbb{R}$ then each stratum S_j is a nonsingular semialgebraic subset of X. If $\mathbb{K} = \mathbb{C}$ then each stratum S_j is a nonsingular locally closed subvariety of X. (The stratifications we construct in the proof of Theorem 1.1 satisfy the *frontier condition*: If S_1 and S_2 are disjoint strata such that S_1 intersects the closure of S_2, then S_1 is contained in the closure of S_2.)

Arc-wise analytic trivializations were introduced in [5]. In detail the family Ψ of Theorem 1.1 has the following properties. It is continuous and semialgebraic and preserves the strata; that is, for each stratum S, Ψ induces a map $\Psi_S : U \times S \to S$ (we often skip the subscript S if there is no confusion). Moreover, the map $\Phi_S(t,x) = (t, \Psi_S(t,x)) : U \times S \to U \times S$ is a real analytic isomorphism. The arc-wise analyticity means that Ψ is analytic in t and for every real analytic arc $x(s) : (-1,1) \to \mathbb{P}^n$, $(t,s) \mapsto \Psi(t, x(s))$ is analytic ($\mathbb{K}$-analytic in t and real analytic in s). In particular, $\Phi(t,x) = (t, \Psi(t,x)) : U \times \mathbb{P}^n \to U \times \mathbb{P}^n$ is a semialgebraic homeomorphism that is real analytic on real analytic arcs. Finally, we require that Φ^{-1} is also real analytic on real analytic arcs.

We give two applications of Theorem 1.1. Proposition 1.3 is a stratified general position theorem, and Proposition 1.5 is a stratified transversality theorem. Although transversality implies general position, we first present a simple direct proof of general position.

The following result is an algebraic version of the piecewise linear general position theorem of [3] (p. 143).

Proposition 1.3. *Let $\Psi : U \times \mathbb{P}^n \to \mathbb{P}^n$ be a stratified family as in Theorem 1.1, and let $\mathcal{S}$ be the associated algebraic stratification of $\mathbb{P}^n$. Let Z and W be semialgebraic subsets of $\mathbb{P}^n$. There is an open dense semialgebraic subset U' of U such that, for all $t \in U'$ and all strata $S \in \mathcal{S}$,*

$$\dim(Z \cap \Psi_t^{-1}(W) \cap S) \leq \dim(Z \cap S) + \dim(W \cap S) - \dim S.$$

This proposition is a consequence of the following lemma.

Lemma 1.4. *Let T and M be smooth (equidimensional) semialgebraic sets, with $\dim T \geq \dim M$. Let $\Psi : T \times M \to M$ be a semialgebraic submersive family of diffeomorphisms. Let Z and W be semialgebraic subsets of M. There is an open dense semialgebraic subset T' of T such that, for all $t \in T'$,*

$$\dim(Z \cap \Psi_t^{-1}(W)) \leq \dim Z + \dim W - \dim M.$$

Proof. Consider the map $\Theta : T \times M \to M \times M$, $\Theta(t,x) = (x, \Psi(t,x))$. Since Ψ is a submersive family, Θ is a submersion. Therefore for all $(x,y) \in M \times M$, if $\Theta^{-1}(x,y) \neq \emptyset$ then $\dim \Theta^{-1}(x,y) = \dim T - \dim M$. Thus, because $(T \times Z) \cap \Psi^{-1}(W) = \Theta^{-1}(Z \times W)$, we have

$$\dim((T \times Z) \cap \Psi^{-1}(W)) \leq \dim Z + \dim W - \dim M + \dim T.$$

Let $Q = \{t \in T \ ; \ \dim Z \cap \Psi_t^{-1}(W) \leq \dim Z + \dim W - \dim M\}$. If Q is not dense, there exists an open subset Ω of T such that, for all $t \in \Omega$,

$$\dim(Z \cap \Psi_t^{-1}(W)) > \dim Z + \dim W - \dim M.$$

Thus

$$\dim((T \times Z) \cap \Psi^{-1}(W)) > \dim Z + \dim W - \dim M + \dim T,$$

which is a contradiction. The set Q is semialgebraic and dense, so we can take $T' = \operatorname{interior} Q$. □

The following result is an algebraic version of the transversality theorem of [4] (Theorem 3.8, p. 4887). If $\mathcal{S}$ is a stratification of the semialgebraic set X, and $\mathcal{T}$ is a stratification of the semialgebraic subset Y of X, then $(Y, \mathcal{T})$ is a *substratified object* of $(X, \mathcal{S})$ if each stratum of $\mathcal{T}$ is contained in a stratum of $\mathcal{S}$. Two substratified objects $(Z, \mathcal{A})$ and $(W, \mathcal{B})$ of $(X, \mathcal{S})$ are *transverse* in $(X, \mathcal{S})$ if, for every pair of strata $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that A and B are contained in the same stratum $S \in \mathcal{S}$, the manifolds A and B are transverse in S.

Proposition 1.5. *Let $\Psi : U \times \mathbb{P}^n \to \mathbb{P}^n$ be a stratified family as in Theorem 1.1, and let $\mathcal{S}$ be the associated algebraic stratification of $\mathbb{P}^n$. Let Z and W be semialgebraic subsets of $\mathbb{P}^n$, with stratifications $\mathcal{A}$ of Z and $\mathcal{B}$ of W such that $(Z, \mathcal{A})$ and $(W, \mathcal{B})$ are substratified objects of $(\mathbb{P}^n, \mathcal{S})$. There is an open dense semialgebraic subset U' of U such that, for all $t \in U'$, $(Z, \mathcal{A})$ is transverse to $\Psi_t^{-1}(W, \mathcal{B})$ in $(\mathbb{P}^n, \mathcal{S})$,*

This proposition is a consequence of the following lemma.

Lemma 1.6. *Let T and M be smooth (equidimensional) semialgebraic sets, with $\dim T \geq \dim M$. Let $\Psi : T \times M \to M$ be a semialgebraic submersive family of diffeomorphisms. Let A and B be smooth semialgebraic subsets of M. There is an open dense semialgebraic subset T' of T such that, for all $t \in T'$, A is transverse to $\Psi_t^{-1}(B)$.*

Proof. We apply a standard transversality technique (*cf.* [2] Theorem I.1.3.6, p. 39). Since Ψ is a submersive family, the map $\Theta : T \times M \to M \times M$, $\Theta(t,x) = (x, \Psi(t,x))$, is a submersion. So $\Theta^{-1}(A \times B) = (T \times A) \cap \Psi^{-1}(B)$ is a smooth submanifold of $T \times M$. Let $\pi : T \times M \to T$ be the projection, and let

$$P = \{t \in T \ ; \ t \text{ is a critical value of } \pi | (T \times A) \cap \Psi^{-1}(B)\}.$$

Now $t \in P$ if and only if there exists $x \in A$ such that $(t,x) \in (T \times A) \cap \Psi^{-1}(B)$ and

$$\mathbf{T}_{(t,x)}(\{t\} \times A) + \mathbf{T}_{(t,x)}((T \times A) \cap \Psi^{-1}(B)) \neq \mathbf{T}_{(t,x)}(T \times A),$$

where $\mathbf{T}$ denotes the tangent space. This is equivalent to

$$\mathbf{T}_{(t,x)}(\{t\} \times A) + \mathbf{T}_{(t,x)}(\Psi^{-1}(B)) \neq \mathbf{T}_{(t,x)}(T \times M);$$

in other words, $\{t\} \times A$ is not transverse to $\Psi^{-1}(B)$. This is equivalent to the condition that $\Psi_t : A \to M$ is not transverse to B; *i.e.* A is not transverse to $\Psi_t^{-1}(B)$.

By Sard's theorem, P has measure zero in T. (Stratify the semialgebraic map $\pi : (T \times A) \cap \Psi^{-1}(B) \to T$ so that the restriction of π to each stratum is real analytic. Then apply Sard's theorem to these restrictions.) Therefore $Q = T \setminus P$ is dense in T. Since Q is semialgebraic, we can let T' be the interior of Q. □

2. Whitney interpolation

In this section we adapt the Whitney Interpolation of [5] and define the interpolation functions by modifying the function defined in Example 9.14 of [5] by adding a parameter $\eta \in \mathbb{C}$.

Given two subsets $\{a_1, \ldots, a_N\} \subset \mathbb{C}$, $\{b_1, \ldots, b_N\} \subset \mathbb{C}$, such that if $a_i = a_j$ then $b_i = b_j$ and if $a_i = 0$ then $b_i = 0$. We will define a Lipschitz interpolation function $\psi : \mathbb{C} \to \mathbb{C}$, depending continuously on a, b and a parameter $\eta \in \mathbb{C}$, such that $\psi(a_i) = b_i$ and $\psi(0) = 0$. Moreover, if we set

$$\gamma = \max_{a_i \neq a_j} \frac{|D_i - D_j|}{|a_i - a_j|}, \tag{2.1}$$

where $D_i = b_i - a_i$, then we will show that ψ is a bi-Lipschitz homeomorphism provided γ and η are sufficiently small.

Denote by $\sigma_i = \sigma_i(\xi_1, \ldots, \xi_N)$ the elementary symmetric functions in $\xi_1, \ldots, \xi_N \in \mathbb{C}$. Let $P_k = \sigma_k^{\alpha_k}$, where $\alpha_k = (N!)/k$, so that all P_k are homogeneous of the same degree. Then we define

$$f_j(\xi) = \frac{1}{N!} \sum_k \xi_j \frac{\partial P_k}{\partial \xi_j}(\xi) \overline{P_k(\xi)}, \tag{2.2}$$

where the bar denotes complex conjugation, and

$$f(\xi) = \sum f_j(\xi) = \sum_k P_k(\xi) \overline{P_k(\xi)}. \tag{2.3}$$

Note that f is real-valued, $\mathbb{R}$-homogeneous of degree $d = 2N!$, and vanishes only at the origin.

Let

$$\mu_i(z) := f_i((z - a_1)^{-1}, \ldots, (z - a_N)^{-1}), \quad \mu(z) := f((z - a_1)^{-1}, \ldots, (z - a_N)^{-1}).$$

Define the interpolation map $\psi : \mathbb{C} \to \mathbb{C}$ (parametrized by a, b and $\eta \in \mathbb{C}$) by the formula

$$\psi(z) = \psi(z, a, b, \eta) = z + \frac{\sum_{i=1}^N \mu_i(z)(b_i - a_i) + \eta z |z|^{-2N!}}{\mu(z) + |z|^{-2N!}} \tag{2.4}$$

and set formally $\psi(a_i) = b_i$, $\psi(0) = 0$ (as we show later, this is the extension by continuity). Note that

$$\psi(\lambda z, \lambda a, \lambda b, \eta) = \lambda \psi(z, a, b, \eta), \qquad \text{for all } \lambda \in \mathbb{C}. \tag{2.5}$$

We may consider ψ as function of variables z, a, b, η defined on

$$\Xi = \{(z, a, b, \eta) \in \mathbb{C} \times \mathbb{C}^N \times \mathbb{C}^N \times \mathbb{C} \ ; \ \text{ if } a_i = a_j \text{ then } b_i = b_j, \text{ if } a_i = 0 \text{ then } b_i = 0\}.$$

Proposition 2.1. *The function ψ satisfies:*

1. *ψ is continuous on Ξ.*
2. *There is a universal constant $C = C(N)$ such that the map $\psi : \mathbb{C} \to \mathbb{C}$ is Lipschitz with Lipschitz constant $C(\gamma + |\eta|)$. If $C(\gamma + |\eta|) < 1$ then $\psi : \mathbb{C} \to \mathbb{C}$ is a bi-Lipschitz homeomorphism, with $(1 - C(\gamma + |\eta|))^{-1}$ as a Lipschitz constant of ψ^{-1}.*

(3) $\partial\psi/\partial\eta(z,a,b,\eta)=0$ *if and only if* $z=a_i$ *or* $z=0$.

Proof. Sometimes it will be convenient to use the following formula for ψ:

$$\psi(z)=z+\frac{\sum_{i=1}^{N}\mu_i(z)(b_i-a_i)|z|^d+\eta z}{\mu(z)|z|^d+1},\quad d=2N!. \tag{2.6}$$

We begin the proof of (1) by showing that $\psi(z,a,b,\eta)\to 0=\psi(0,a_0,b_0,\eta_0)$ if $(z,a,b,\eta)\to(0,a_0,b_0,\eta_0)$ (we suppress later a,b,η in the notation when it causes no confusion). Since the denominator of the fraction in (2.6) is always real positive ≥ 1, $\frac{\eta z}{\mu(z)|z|^d+1}\to 0$ as $z\to 0$. If $a_{i0}\neq 0$ then $\mu_i(z)(b_i-a_i)|z|^d\to 0$ and hence $\frac{\mu_i(z)(b_i-a_i)|z|^d}{\mu(z)|z|^d+1}\to 0$ as $z\to 0$. Finally consider the case $a_{i0}=0$. Then $b_{i0}=0$ and hence $b_i-a_i\to 0$. Since in general $|\mu_i|\leq C\mu$ for an universal constant C, by conditions (3) and (5) of Appendix 1 of [5], we have $\frac{\mu_i(z)(b_i-a_i)|z|^d}{\mu(z)|z|^d+1}\to 0$ as $z\to 0$ also in this case.

The remaining part of the proof of (1) is similar to the proof of Proposition 9.11 of [5]. Let $(z,a,b,\eta)\to(z_0,a_0,b_0,\eta_0)$, $z_0\neq 0$. We suppose that z_0 equals one of the a_{0i}; otherwise the proof is easy. Thus let $z_0=a_{01}\neq 0$. Then $\psi(z_0,a_0,b_0)=b_{01}$. Denote $J=\{j\in\{1,\dots,n\}\ ;\ a_{0j}=a_{01}\}$. Then

$$\begin{aligned}\psi(z,a,b)-\psi(z_0,a_0,b_0)&=(z-z_0)+\frac{\sum_i\mu_i(z,a)(b_i-a_i)+\eta z|z|^{-d}}{\mu(z,a)+|z|^{-d}}-(b_{01}-a_{01})\\&=(z-z_0)+\frac{\sum_{i\in J}\mu_i(z,a)((b_i-b_{01})-(a_i-a_{01}))}{\mu(z,a)+|z|^{-d}}\\&\quad+\frac{\sum_{i\notin J}\mu_i(z,a)((b_i-b_{01})-(a_i-a_{01}))}{\mu(z,a)+|z|^{-d}}+\frac{\eta z-(b_{01}-a_{01})}{\mu(z,a)|z|^d+1}.\end{aligned}$$

The first three summands converge to 0 as $(z,a,b)\to(z_0,a_0,b_0)$ by the arguments given in [5], and the last one because $\mu(z,a)\to\infty$.

The proof of (2) is similar to that of Proposition 9.10 of [5]. It relies on the formula (9.8) of Appendix 1 of [5], which says that there are universal constants A,B,D such that

$$\begin{aligned}&|\mu_i(z)|\leq A|z-a_i|^{-1}\mu(z)^{1-\frac{1}{d}},\\&|\mu_i'(z)|\leq B|z-a_i|^{-1}\mu(z),\\&|\mu'(z)|\leq D\mu(z)^{\frac{d+1}{d}},\end{aligned}\tag{2.7}$$

where "prime" denotes any directional derivative (in the variable z) $\frac{\partial}{\partial v}$, $v\in\mathbb{C}$, $|v|=1$.

Given $z\in\mathbb{C}$, choose j such that $|z-a_j|=\min_i|z-a_i|$. Then, for all i,

$$|a_i-a_j|\leq 2|z-a_i|. \tag{2.8}$$

By differentiating

$$\begin{aligned}|(\psi(z)-z)'|\leq&\frac{\sum_{i\notin I_j}|\mu_i'(z)(D_i-D_j)||z|^d+\sum_{i\notin I_j}|\mu_i(z)(D_i-D_j)|(|z|^d)'+|\eta|}{|z|^d\mu(z)+1}\\&+\frac{(\sum_{i\notin I_j}|\mu_i(z)(D_i-D_j)||z|^d+|\eta z|)(|z|^d||\mu'(z)|+(|z|^d)'|\mu|)}{(|z|^d\mu(z)+1)^2}.\end{aligned}$$

We have

$$|D_i-D_j|\leq\gamma|a_i-a_j|\leq 2\gamma|z-a_i|. \tag{2.9}$$

Using (2.9) and (2.7) we get

$$|\mu_i'(z)(D_i - D_j)| \le 2B\gamma\mu(z)$$

and

$$|\mu_i(z)(D_i - D_j)||\mu'(z)| \le 2AD\gamma(\mu(z))^2.$$

Moreover $\mu^{1-\frac{1}{d}}|(|z|^d)'| \le (\mu|z|^d + 1)$. All this shows that

$$\frac{\sum_{i\notin I_j} |\mu_i'(z)(D_i - D_j)||z|^d + \sum_{i\notin I_j} |\mu_i(z)(D_i - D_j)|(|z|^d)' + |\eta|}{|z|^d\mu(z)+1} \le (2NB + 2NA)\gamma + |\eta|.$$

Similarly we obtain

$$\frac{(\sum_{i\notin I_j} |\mu_i(z)(D_i - D_j)||z|^d + |\eta z|)(|z|^d|\mu'(z)| + (|z|^d)'\mu)}{(|z|^d\mu(z)+1)^2} \le (2NA + 2NAD)\gamma + |\eta| + D|\eta|.$$

This finally shows that

$$|(\psi(z) - z)'| \le C(\gamma + |\eta|).$$

The claim (3) follows from the fact that

$$\frac{\partial\psi}{\partial\eta} = \frac{z}{|z|^d\mu(z)+1} \tag{2.10}$$

vanishes if $z = 0$ and if $z = a_i$. In the latter case the denominator equals infinity. □

Now we show the arc-wise analyticity of ψ. In the applications (see the next section), the entries of the vectors a and b are the roots of a polynomial. Note that ψ is symmetric in a and b simultaneously; that is, ψ is invariant if we apply the same permutation to the entries of a and of b.

The space $\mathbb{C}^N \ni a$ can be stratified by the type of a, that is by the number of distinct a_i and by the multiplicities, denoted m_s, $s = 0, \dots, k$, that appear in the vector a. By definition $m_s > 0$ for $s > 0$, with only m_0, the number of a_i that are equal to 0, that can be 0. We encode such a type by the multiplicity vector $\mathbf{m} = (m_0, m_1, \dots, m_k)$, $\sum_{s=0}^k m_i = N$, $0 < m_1 \le \cdots \le m_k$. We denote by $S_{\mathbf{m}} \subset \mathbb{C}^N$ the corresponding set of a with the multiplicity vector $\mathbf{m}$. Each stratum, that is each connected component of such $S_{\mathbf{m}}$, is given by a partition $W = \{W_i\}$, $\{1, \dots, N\} = \bigsqcup_s W_s$ with $|W_s| = m_s$, by

$$\{a \in S_{\mathbf{m}} \; ; \; a_i = a_j \text{ if } \exists s \in \{1, \dots, k\}, \text{ s.t. } i, j \in W_s, \text{ and } a_i = 0 \text{ if } i \in W_0\}.$$

Note that the stratification by the type of a can be defined by the signs of symmetric polynomials in the entries of a. More precisely, for a multiplicity vector $\mathbf{m}$, a connected component S of $S_{\mathbf{m}}$ is a connected component of the set described as follows. Let l be a number of non-zero entries in $\mathbf{m}$, $l = k$ or $k+1$. Then S is a connected component of the set given in terms of the generalized discriminants, see [6] Appendix IV or [5] Appendix II, by $D_{l+1} = \cdots = D_N = 0$, $D_l \ne 0$, and $\prod a_i = 0$ if $m_0 > 0$.

Proposition 2.2. *The function ψ satisfies:*

(4) *$\psi(z, a, b, \eta)$ is polynomial of degree 1 in b and η, and for every stratum S of the stratification by the type of a, ψ is real analytic on $\mathbb{C} \times S \times S \times \mathbb{C}$.*
(5) *Let an unordered set of function germs $b(t, s) = (b_1(t, s), \dots, b_N(t, s))$, $(t, s) \in (\mathbb{K}^m \times \mathbb{R}, (0, 0))$, be such that for every symmetric polynomial G in b, $G(b(t, s))$ is analytic in (t, s) (it equals a power series in $(t, s) \in \mathbb{K}^m \times \mathbb{R}$). We also assume that the type of $b(t, s)$ is independent of t and that the last not identically equal*

to zero generalized discriminant $D_l(b_1(t,s),\dots,b_N(t,s))$ *is of the form* $s^k u(t,s)$, $k \ge 0$, $u(0,0) \neq 0$, *and we make a similar assumption on the product of* b_i.

Let $z(s) : (\mathbb{R},0) \to \mathbb{C}$ *be a real analytic germ and set* $a(s) = b(0,s)$. *Then* $\psi(z(s),a(s),b(t,s),\eta)$ *is analytic in* (t,s,η).

Proof. The proof of (4) is similiar to that of Lemma 2.7 of [5]. Let us just sketch it. Let $S = S_W$ for a partition W. We consider only the case $m_0 \neq 0$; the case $m_0 = 0$ is similar. Choose the representatives $i_0,\dots,i_k$ so that $i_s \in W_s$. Write similarly to (2.7) of [5]

$$\psi(z,a,b,\eta) = z + \frac{Q(z,a)\overline{Q}(z,a)\big(\sum_k \sum_j \ Q_{k,j}(z,a)\overline{Q}_k(z,a)(b_j-a_j)|z|^d + \eta z\big)}{N!Q(z,a)\overline{Q}(z,a)(\sum_k Q_k(z,a)\overline{Q}_k(z,a)|z|^d+1)}, \tag{2.11}$$

where

$$Q_k(z,a) = P_k((z-a_1)^{-1},\dots,(z-a_N)^{-1}),$$
$$Q_{k,j}(z,a,b) = (z-a_j)^{-1}\frac{\partial P_k}{\partial \xi_j}((z-a_1)^{-1},\dots,(z-a_N)^{-1})$$
$$Q(z,a) = Q_W(z,a) = \prod_{s=0}^{k}(z-a_{i_s})^{N!}.$$

Then the denominator of the fraction in (2.11) does not vanish on S, and both the numerator and the denominator are real analytic on $\mathbb{C}\times S\times S\times\mathbb{C}$.

The proof of (5) is similar to that of Lemma 2.8 of [5], but because of the presence of the term $|z|^{2N!}$ we cannot apply the reduction to the case $z(s)\equiv 0$ by subtracting $z(s)$ from every component of $b(t,s)$.

We consider $b_i(t,s)$ as the roots of a polynomial $G(z,t,s) = z^N + \sum_{i=1}^N c_i(t,s)$ with coefficients analytic in t and s, that is as power series in $(t,s)\in\mathbb{K}^m\times\mathbb{R}$. The next step is to complexify the variables and consider $c_i(t,s)$ as complex analytic germs of $(t,s)\in(\mathbb{C}^m\times\mathbb{C},(0,0))$. By the assumption on the generalized discriminants and on the product of b_i the type of $b_i(t,s)$ is independent of t also for s complex. Moreover, by the assumption on the generalized discriminants, we may apply to G the Puiseux with parameter theorem, see [5] Section 2. In particular, for s fixed, an ordering of the roots $a_1(s),\dots,a_N(s)$ of $G(z,0,s)$ gives, by continuity in t, an ordering of the roots $b_1(t,s),\dots,b_N(t,s)$ of G. Fix such an ordering and define

$$\varphi(t,s) = \psi(z(s),a(s),b(t,s)),$$

where ψ is given by (2.11). (Q of (2.11) depends on the choice of W. We take the one given by the type of $b(t,s)$, $s\neq 0$.) Thus defined, φ is independent of the choice of an ordering. Since $P_k(\xi)$ is symmetric in ξ, $Q(z(s),a(s))$ and the product $Q(z(s),a(s))Q_k(z(s),a(s))$ are complex analytic in $s\in\mathbb{C}$. Then, as follows from Lemma 2.9 of [5], for each k, the product $Q(z(s),a(s))(\sum_{j=1}^N Q_{k,j}(z(s),a(s))(b_j(t,s)-a_j(s))\in\mathbb{C}\{t,s\}$.

Now we consider again $s\in\mathbb{R}$. It follows from the above that the denominator of the fraction in (2.11) evaluated on $z(s),a(s)$ is real analytic in (one variable) $s\in\mathbb{R}$ and not identically equal to zero. The numerator of this fraction, evaluated on $z(s),b(t,s),a(s)$, is analytic in $t\in\mathbb{K}^m$, $s\in\mathbb{R}$, $\eta\in\mathbb{C}$. Hence $\varphi(t,s)$ is of the form s^{-k} times a power series in (t,s,η). Since, moreover, $\varphi(t,s)$ is bounded in a neighborhood of the origin, it has to be analytic. □

3. Construction of the general position deformation

Consider a family of homogeneous polynomials with coefficients in $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$

$$F_i(x_1, \dots, x_i) = x_i^{d_i} + \sum_{j=1}^{d_i} A_{i,j}(x_1, \dots, x_{i-1}) x_i^{d_i - j}, \quad i = 1, \dots, n, \tag{3.1}$$

satisfying

(1) $A_{i,0} \equiv 0$ for all i such that $d_i > 0$,
(2) For every i, the discriminant of $F_{i,red}$ divides F_{i-1}.

It may happen that $d_i = 0$. Then $F_i \equiv 1$ and we set by convention $F_j \equiv 1$ for $j < i$. We call a family $\mathcal{F} = \{F_i\}$ satisfying the above properties a *system of polynomials.* It is a special case of the systems of pseudopolynomials considered in Section 5 of [5].

We define a non-trivial deformation of the trivial family $F_i(t, x) = F_i(x)$, $t \in \mathbb{K}^n$,

$$\Phi(t, x) = (t, \Psi(t, x)) = (t, \Psi_1(t_1, x_1), \Psi_2(t_1, t_2, x_1, x_2), \dots, \Psi_n(t, x)), \tag{3.2}$$

using the function ψ defined in formula (2.4) of the previous section and the following variant of the formula (3.5) of [5]

$$\Psi_n(t, x) := \psi(x_n, a(0, x'), a(\Phi_{n-1}(t', x')), t_n), \tag{3.3}$$

where we keep the notation of the Section 3 of [5].

Proposition 3.1. *If $U \subset \mathbb{K}^n$ is a sufficiently small neighborhood of the origin, then*

$$\Phi : U \times \mathbb{K}^n \to U \times \mathbb{K}^n$$

is a homeomorphism with the following properties:

(1) *Φ is semialgebraic, satisfies $\Phi(0, x) = (0, x)$ for all $x \in \mathbb{K}^n$, and it is an arc-wise analytic trivialization of the projection $U \times \mathbb{K}^n \to U$.*
(2) *Ψ is $\mathbb{K}^*$-equivariant:*
$$\Psi(t, \lambda x) = \lambda \Psi(t, x)$$
for all $x \in \mathbb{K}^n$ and $\lambda \in \mathbb{K}$.
(3) *Φ preserves the canonical stratification $\mathcal{S} = \{S_i\}$ associated to the system of polynomials $\{F_i\}$. (See Section 5 of* [5] *for the definition.)*
(4) *For each stratum $S \in \mathcal{S}$ the restriction $\Phi : U \times S \to U \times S$ is a real analytic diffeomorphism.*
(5) *For each stratum S and every $x \in S$ the orbit map $\Psi^x : U \to S$ is a $\mathbb{K}$-analytic submersion.*

Proof. Φ is semialgebraic and satisfies $\Phi(0, x) = (0, x)$ for all $x \in \mathbb{K}^n$ by construction. It is an arc-wise analytic trivialization by Proposition 2.2. The proof follows precisely the arguments of Sections 2 and 3 of [5].

Since F_n (and all F_i) is homogeneous its roots $a(t, x')$ satisfy $a(t, \lambda x') = \lambda a(t, x')$. Therefore (2) follows from (2.5).

Condition (3) follows from the construction (see Section 5 of [5]), and (4) follows from (4) of Proposition 2.2.

To show (5) we use the inductive constructions of the canonical stratification and the deformation Ψ. Denote $t' = (t_1, \dots, t_{n-1})$, $x' = (x_1, \dots, x_{n-1})$ and

$$\Psi'(t', x') = (\Psi_1(t_1, x_1), \Psi_2(t_1, t_2, , x_1, x_2), \dots, \Psi_{n-1}(t', x')). \tag{3.4}$$

Let $\mathcal{S}'$ denote the canonical stratification of $\mathbb{K}^{n-1}$ associated to the system of polynomials $\{F_i\}_{i<n}$. By the inductive assumption, there is a neighborhood U' of the origin in $\mathbb{K}^{n-1}$ such that for every stratum $S' \in \mathcal{S}'$, Ψ' induces a real analytic map $\Psi' : U' \times S' \to S'$ for which the associated orbit map ${\Psi'}^{x'} : U' \to S'$ is submersive for every $x' \in S'$. Denote $X = F_n^{-1}(0) \subset \mathbb{K}^n$. By construction the strata of $\mathcal{S}$ are of two types. If $S \in \mathcal{S}$ is of the first type, then $S \subset X$ and there is $S' \in \mathcal{S}'$ such that the projection $\pi : \mathbb{K}^n \to \mathbb{K}^{n-1}$ induces a finite analytic covering $S \to S'$ whose sections are given by the roots of F_n. Since $\psi(a(0, x'), a(0, x'), a(\Phi_{n-1}(t', x')), t_n) = a(\Phi_{n-1}(t', x'))$ independently of t_n we see that $\Psi(t, x)$, $x \in S$, does not depend on t_n, and $\Psi : U \times S \to S$ is the lift of $\Psi' : U' \times S' \to S'$. That is, the following diagram commutes

$$\begin{array}{ccc} U' \times S & \xrightarrow{\ \Psi\ } & S \\ {\scriptstyle id\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ U' \times S' & \xrightarrow{\ \Psi'\ } & S' \end{array}$$

Hence $\pi \circ \Psi^x = {\Psi'}^{\pi(x)}$ is submersive.

If $S \in \mathcal{S}$ is of the second type, then there is $S' \in \mathcal{S}'$ such that S is a connected component of $\pi^{-1}(S') \cap (\mathbb{K}^n \setminus X)$, that is S is open in $S' \times \mathbb{K}$. The jacobian matrix $\frac{\partial \Psi}{\partial t}$ is triangular with non-zero terms on the diagonal. Indeed, the latter claim follows from (3) of Proposition 2.1 for $\frac{\partial \Psi_n}{\partial t_n}$, and by inductive assumption for the other terms. □

Corollary 3.2. *The map $\Psi : U \times \mathbb{K}^n \to \mathbb{K}^n$ is a stratified submersive family of diffeomorphisms.*

Proof of Theorem 1.1. Let $g_{ij} \in \mathbb{K}[x_1, \ldots, x_{n+1}]$ be homogeneous polynomials generating the homogeneous ideals defining $V_i \subset \mathbb{P}^n$. Let $F_{n+1}(x_1, \ldots, x_{n+1})$ be the product of all g_{ij}. We complete F_{n+1} to a system of polynomials and apply the above construction of the deformation Ψ. Thanks to property (2) of Proposition 3.1, Ψ can be projectivised to a map $U \times \mathbb{P}^n \to \mathbb{P}^n$. By Proposition 1.9 of [5], Ψ preserves the zero sets of each g_{ij} and hence each V_i. This shows Theorem 1.1. □

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