ALGEBRAIC VARIETIES ARE HOMEOMORPHIC TO VARIETIES DEFINED OVER NUMBER FIELDS

ADAM PARUSIŃSKI AND GUILLAUME ROND

Abstract. We show that every affine or projective algebraic variety defined over the field of real or complex numbers is homeomorphic to a variety defined over the field of algebraic numbers. We construct such a homeomorphism by choosing a small deformation of the coefficients of the original equations. This method is based on the properties of Zariski equisingular families of varieties.

Moreover we construct an algorithm, that, given a system of equations defining a variety V, produces a system of equations with coefficients in $\overline{\mathbb{Q}}$ of a variety homeomorphic to V.

1. Introduction

In computational algebraic geometry one is interested in computing the solution set of a given system of polynomial equations, or at least in computing various algebraic or geometric invariants of such a solution set. Here by computing we mean producing an algorithm that gives the desired result when it is implemented in an appropriate computer system.

Such an algorithm should be stable with respect to a small perturbation of the coefficients. But in general the shape of the solution set may change drastically under a small perturbation of the coefficients. This difficulty is particularly apparent if one has to deal with polynomial equations whose coefficients are neither rational nor algebraic numbers. The main goal of this paper is to show that, when we are interested in the topology of such a solution set, one can always assume that the polynomial equations have algebraic number coefficients precisely by choosing in an effective way a particular perturbation of its coefficients.

Theorem 1. *Let $V \subset \mathbb{K}^n$ (resp. $V \subset \mathbb{P}^n_{\mathbb{K}}$) be an affine (resp. projective) algebraic set, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then there exist an affine (resp. projective) algebraic set $W \subset \mathbb{K}^n$ (resp. $W \subset \mathbb{P}^n_{\mathbb{K}}$) and a homeomorphism $h : \mathbb{K}^n \longrightarrow \mathbb{K}^n$ (resp. $h : \mathbb{P}^n_{\mathbb{K}} \longrightarrow \mathbb{P}^n_{\mathbb{K}}$) such that:*

- *(i) the homeomorphism maps V onto W,*
- *(ii) W is defined by polynomial equations with coefficients in $\overline{\mathbb{Q}} \cap \mathbb{K}$.*

Remark 2. In fact we prove a more precise and stronger result: see Theorem 10 for a precise statement.

In the proof of Theorem 10 we show that W can obtained by a small deformation of the coefficients of the equations defining V. Let denote these coefficients by $g_{i,\alpha}$. The deformation is denoted by $t \mapsto g_{i,\alpha}(t)$ with $g_{i,\alpha}(0) = g_{i,\alpha}$. This deformation is constructed in such a way that every polynomial relation with rational coefficients satisfied by the $g_{i,\alpha}$ is satisfied by the $g_{i,\alpha}(t)$ for every t. So we can prove that the deformation is equisingular in the sense of Zariski since this equisingularity condition can be encoded in term of the vanishing and the non-vanishing of polynomial relations on the $g_{i,\alpha}(t)$. In particular this deformation is topologically trivial.

The first part of this paper is devoted to the proof of an approximation result similar to the one given in [Ro]. In the next part we recall the notion of Zariski equisingularity. The proof of Theorem 10 is given in the following part. Finally we provide an algorithm that, given the equations defining V, computes the equations defining W. This algorithm is based on the proof of Theorem 10.

Notation For a vector of indeterminates $x = (x_1, \dots, x_n)$, x^i denotes the vector of indeterminates $(x_1, \dots, x_i)$. To avoid confusion, variables will be denoted by normal letters t and elements of $\mathbb{K}$ will be denoted by bold letters $\mathbf{t}$.
For a field $\mathbb{K}$ the ring of algebraic power series is denoted by $\mathbb{K}\langle x\rangle$.

2. An approximation result

Let $\Omega \subset \mathbb{K}^r$ be open and non-empty and let φ be an analytic function on Ω. We say that f is a *Nash function* at $a \in \Omega$ if there exist an open neighborhood U of a in Ω and a nonzero polynomial $P \in \mathbb{K}[t_1, \dots, t_r, z]$ such that $P(\mathbf{t}, \varphi(\mathbf{t})) = 0$ for $\mathbf{t} \in U$ or, equivalently, if the Taylor series of φ at a is an algebraic power series. An analytic function on Ω is a Nash function if it is a Nash function at every point of Ω. An analytic mapping $\varphi : \Omega \to \mathbb{K}^N$ is a *Nash mapping* if each of its components is a Nash function.

We call a Nash function $\varphi : \Omega \to \mathbb{K}$, $\mathbb{Q}$-algebraic (and we note $\varphi \in \mathrm{QA}(\Omega)$) if (locally) it is algebraic over $\mathbb{Q}[t]$, i.e. it satisfies a polynomial relation $P(\mathbf{t}, \varphi(\mathbf{t})) = 0$ for every $\mathbf{t} \in \Omega$ with $P \in \mathbb{Q}[t_1, \dots, t_r, z]$, $P \neq 0$. It is well known that this means that the Taylor expansion of φ at a point with coordinates in $\mathbb{Q}$ (or $\overline{\mathbb{Q}}$) is an algebraic power series whose coefficients lie in a common finite field extension of $\mathbb{Q}$, i.e. this Taylor expansion belongs to $\Bbbk\langle x\rangle$ where $\Bbbk$ is a subfield of $\mathbb{K}$ which is finite over $\mathbb{Q}$ (see [RD84] for example).

Theorem 3. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathbf{y} \in \mathbb{K}^m \setminus \overline{\mathbb{Q}}^m$. Denote by $\Bbbk$ the field extension of $\mathbb{Q}$ generated by the coefficients of the components $\mathbf{y}_i$ of $\mathbf{y}$. The field $\Bbbk$ is a primitive extension of a transcendental finitely generated field extension of $\mathbb{Q}$: $\Bbbk = \mathbb{Q}(\mathbf{t}_1, \dots, \mathbf{t}_r)(\mathbf{z})$ where the $\mathbf{t}_i \in \mathbb{K}$ are algebraically independent over $\mathbb{Q}$ and $\mathbf{z} \in \mathbb{K}$ is finite of degree d over $\mathbb{Q}(\mathbf{t}_1, \dots, \mathbf{t}_r)$.*
Then there exist new variables $t = (t_1, \dots, t_r)$ and z, an open non-empty neighborhood of $\mathbf{t}$, $\mathcal{U} \subset \mathbb{K}^r$, a vector function

$$y(t, z) \in \mathbb{Q}(t)[z]^m$$

and a function $z(t) \in \mathrm{QA}(\mathcal{U})$ such that

$$\mathbf{z} = z(\mathbf{t}),\ \mathbf{y} = y(\mathbf{t}, \mathbf{z}),$$

and for all $f(y) \in \mathbb{Q}[y]^p$, where $y = (y_1, \dots, y_m)$ is a vector of indeterminates, such that $f(\mathbf{y}) = 0$, we have

$$f(y(t, z)) = 0.$$

Moreover, the function $z(t)$ is algebraic of degree d over $\mathbb{Q}(t)$.

Proof of Theorem 3. Let us denote by $\Bbbk$ the field extension of $\mathbb{Q}$ generated by the components of $\mathbf{y}$, so $\Bbbk \subset \mathbb{K}$. Since there are finitely many such coefficients, $\Bbbk$ is a finitely generated field extension of $\mathbb{Q}$. Let $\mathbf{t}_1, \dots, \mathbf{t}_r$ be a transcendence basis

of $\mathbb{k}$ over $\mathbb{Q}$ (with $r \geq 1$ because $\mathbf{y} \notin \overline{\mathbb{Q}}^m$) and set $\mathbb{L} := \mathbb{Q}(\mathbf{t}_1, \ldots, \mathbf{t}_r)$. By the primitive element theorem, there exists an element $\mathbf{z} \in \mathbb{k}$ finite over $\mathbb{L}$ and such that $\mathbb{L}(\mathbf{z}) = \mathbb{k}$.
For every $i = 1, \ldots, m$ we can write

$$\mathbf{y}_i = \sum_{k=0}^{d-1} \frac{p_{i,k}(\mathbf{t}_1, \ldots, \mathbf{t}_r)}{q_{i,k}(\mathbf{t}_1, \ldots, \mathbf{t}_r)} \mathbf{z}^k$$

where d is the degree of $\mathbf{z}$ over $\mathbb{L}$, $p_{i,k}(t)$, $q_{i,k}(t) \in \mathbb{Q}[t]$ and $q_{i,k}(\mathbf{t}_1, \ldots, \mathbf{t}_r) \neq 0$. By multiplying the $p_{i,k}$ by some well-chosen polynomials we may assume that all the $q_{i,k}(t)$ are equal, let us say to $q(t)$.

Let $P(t, z) \in \mathbb{Q}(t)[z]$ be the monic polynomial of minimal degree in z such that

$$P(\mathbf{t}_1, \ldots, \mathbf{t}_r, \mathbf{z}) = 0.$$

Set $\mathbf{t} := (\mathbf{t}_1, \ldots, \mathbf{t}_r) \in \mathbb{K}^r$ and let $D \subset \mathbb{K}^r$ be the discriminant locus of $P(t, z)$ seen as a polynomial in z (i.e. D is the locus of points $a \in \mathbb{K}^r$ such that a is a pole of one of the coefficients of P or such that $P(a, z)$ has at least one multiple root). Since $P(\mathbf{t}_1, \ldots, \mathbf{t}_r, z)$ has no multiple roots in an algebraic closure of $\mathbb{L}$, the point $\mathbf{t}$ is not in D. Then there exist $\mathcal{U} \subset \mathbb{K}^r \backslash D$ a simply connected open neighborhood of $\mathbf{t}$ and analytic functions

$$w_i : \mathcal{U} \longrightarrow \mathbb{K}, \quad i = 1, \ldots, d$$

such that

$$P(t, z) = \prod_{i=1}^{d} (z - w_i(t))$$

and $w_1(\mathbf{t}_1, \ldots, \mathbf{t}_r) = \mathbf{z}$.
Moreover the $t \longmapsto w_i(t)$ are algebraic functions over $\mathbb{Q}[t]$. Let us set $\mathbb{Q}_{\mathbb{K}} = \mathbb{Q}$ when $\mathbb{K} = \mathbb{R}$ and $\mathbb{Q}_{\mathbb{K}} = \mathbb{Q} + i\mathbb{Q}$ when $\mathbb{K} = \mathbb{C}$. Then $w_1 \in \mathrm{QA}(\mathcal{U})$ and the Taylor series of w_1 at a point of $\mathcal{U} \cap \mathbb{Q}_{\mathbb{K}}^r$ is an algebraic power series whose coefficients belong to a finite field extension of $\mathbb{Q}$.
Since the polynomial q is not vanishing at $\mathbf{t}$ the function

$$t \in \mathbb{K}^r \backslash \{q = 0\} \longmapsto \frac{1}{q(t)}$$

is a Nash function whose Taylor series at a point of $\mathbb{Q}_{\mathbb{K}}^r \backslash \{q = 0\}$ is an algebraic power series with rational coefficients.
Let $b := (b_1, \ldots, b_r) \in \mathbb{Q}_{\mathbb{K}}^r \cap \mathcal{U} \backslash \{q = 0\}$. Let us denote by $\varphi_1(t)$ the Taylor series of w_1. Then let $\varphi_2(t) \in \mathbb{Q}\langle t \rangle$ denote the Taylor series of $t \longmapsto \frac{1}{q(t)}$ at b. For simplicity we can make a translation and assume that b is the origin of $\mathbb{K}^r$.
For every $i = 1, \ldots, m$ let us define

$$y_i(t_1, \ldots, t_r, z, v) := v \sum_{k=0}^{d-1} p_{i,k}(t_1, \ldots, t_r) z^k \tag{2.1}$$

and

$$y(t, z, v) := (y_1(t, z, v), \ldots, y_m(t, z, v)) .$$

Let $f(y) \in \mathbb{Q}[y]^p$ such that $f(\mathbf{y}) = 0$. We have that

$$f(y(\mathbf{t}_1, \ldots, \mathbf{t}_r, \mathbf{z}, \frac{1}{q(\mathbf{t})})) = 0$$

or equivalently

$$f(y(\mathbf{t}_1,\ldots,\mathbf{t}_r,\varphi_1(\mathbf{t}_1,\ldots,\mathbf{t}_r),\varphi_2(\mathbf{t}_1,\ldots,\mathbf{t}_r)))=0.$$

But the function

$$t\longmapsto f(y(t_1,\ldots,t_r,\varphi_1(t_1,\ldots,t_r),\varphi_2(t_1,\ldots,t_r)))$$

is an algebraic function over $\mathbb{Q}[t]$ and $\mathbf{t}_1$, ..., $\mathbf{t}_r$ are algebraically independent over $\mathbb{Q}$. Thus we have that

$$f\big(y(t_1,\ldots,t_r,\varphi_1(t_1,\ldots,t_r),\varphi_2(t_1,\ldots,t_r))\big)=0.$$

Indeed, write $f=(f_1,\ldots,f_m)$ and for each $i=1,\ldots,m$ consider the complex valued function

$$\varphi(t)=f_i\big(y(t_1,\ldots,t_r,\varphi_1(t_1,\ldots,t_r),\varphi_2(t_1,\ldots,t_r))\big).$$

Note that $\varphi\in\mathrm{QA}(\mathcal{U})$ because so is w_1. Let $P_1\in\mathbb{Q}[t_1,\ldots,t_r,z]$ be a polynomial of minimal degree such that $P_1(t,\varphi(t))=0$. Note that P_1 is irreducible. Write

$$P_1(t,z)=\sum_{k=0}^{degP_1}a_k(t)z^k.$$

Then $a_0(\mathbf{t})=0$ since $\varphi(\mathbf{t})=0$. Therefore $a_0(t)=0$ because $\mathbf{t}_1$, ..., $\mathbf{t}_r$ are algebraically independent over $\mathbb{Q}$. Hence $P_1=P_2z$ for some polynomial P_2, which means $\varphi(t)$ vanishes identically on $\mathcal{U}$.
This proves the theorem by defining

$$y(t,z)=y(t,z,\varphi_2(t))$$

and $z(t)=w_1(t)$. Indeed the coefficients of the components of $y(t,z,\varphi_2(t))$ seen as polynomials in z are rational power series by (2.1). Moreover $\varphi_1(t)$ belongs to a finite field extension of $\mathbb{Q}(t)$ of degree d since $P(t,z)$ is irreducible. □

The following lemma will be used in the proof of the main theorem:

Lemma 4. *Let $\varphi(t)\in\mathbb{C}\langle t\rangle$ be a power series algebraic over $\mathbb{Q}[t]$ and $P(t,\Phi)\in\mathbb{Q}[t,\Phi]$ be a nonzero polynomial of degree d in Φ such that $P(t,\varphi(t))=0$. Let $\mathbf{t}\in\mathbb{Q}^r$ be such that $\mathbf{t}$ is not in the vanishing locus of the coefficients of $P(t,\Phi)$ seen as a polynomial in Φ, and such that $\mathbf{t}$ belongs to the domain of convergence of $\varphi(t)$. Then $\varphi(\mathbf{t})$ is an algebraic number of degree $\leq d$ over $\mathbb{Q}$.*

Proof. The proof is straightforward: just replace t by $\mathbf{t}$ in the relation $P(t,\varphi(t))=0$. □

3. Zariski equisingularity

Zariski equisingularity of families of singular varieties was introduced by Zariski in [Za71] (originally it was called the algebro-geometric equisingularity). Answering a question of Zariski, Varchenko showed [Va72, Va73, Va75] that Zariski equisingular families are locally topologically trivial. In the papers [Va73, Va75] Varchenko considers the families of local singularities while the paper [Va75] deals with the families of affine or projective algebraic varieties. A new method of proof of topological triviality, giving much stronger statements, was given recently in [PP17]. The case of families of algebraic varieties was considered in sections 5 and 9 of [PP17]. The version presented below follows from the proof of the main theorem, Theorem 3.3, of [PP17], see also Theorems 3.1 and 4.1 of [Va72] in the complex case and Theorems 6.1 and 6.3 [Va72] in the real case, and Proposition 5.2 and Theorem 9.2 of [PP17] where the algebraic global case is treated.

Theorem 5. [Va72],[PP17] *Let $\mathcal{V}$ be an open connected neighborhood of $\mathbf{t}$ in $\mathbb{K}^r$ and let $\mathcal{O}_{\mathcal{V}}$ denote the ring of $\mathbb{K}$-analytic functions on $\mathcal{V}$. Let $t = (t_1, \ldots, t_r)$ denote the variables in $\mathcal{V}$ and let $x = (x_1, \ldots, x_n)$ be a set of variables in $\mathbb{K}^n$. Suppose that for $i = k_0, \ldots, n$, there are given*

$$F_i(t, x^i) = x_i^{d_i} + \sum_{j=1}^{d_i} a_{i-1,j}(t, x^{i-1}) x_i^{d_i - j} \in \mathcal{O}_{\mathcal{V}}[x^i],$$

with $d_i > 0$, such that

- *(i) for every $i > k_0$, the first non identically equal to zero generalized discriminant of $F_i(t, x^{i-1}, x_i)$ equals $F_{i-1}(t, x^{i-1})$ up to a multiplication by a nowhere vanishing function of $\mathcal{O}_{\mathcal{V}}$.*
- *(ii) the first non identically zero generalized discriminant of F_{k_0} is independent of x and does not vanish at $\mathbf{t}$.*

Let us set for every $\mathbf{q} \in \mathcal{V}$, $V_{\mathbf{q}} = \{(\mathbf{q}, \mathbf{x}) \in \mathcal{V} \times \mathbb{K}^n \mid F_n(\mathbf{q}, \mathbf{x}) = 0\}$.

Then for every $\mathbf{q} \in \mathcal{V}$ there is a homeomorphism

$$h_{\mathbf{q}} : \{\mathbf{t}\} \times \mathbb{K}^n \to \{\mathbf{q}\} \times \mathbb{K}^n$$

such that $h_{\mathbf{q}}(V_{\mathbf{t}}) = V_q$.

Moreover if $F_n = G_1 \cdots G_s$ then for every $j = 1, \ldots, s$

$$h_{\mathbf{q}}\left(G_j^{-1}(0) \cap (\{\mathbf{t}\} \times \mathbb{K}^n)\right) = G_j^{-1}(q) \cap (\{0\} \times \mathbb{K}^n).$$

Remark 6. The homeomorphism $h_{\mathbf{q}}$ of Theorem 5 is written as $h_{\mathbf{q}}(\mathbf{t}, x) = (\mathbf{q}, \Psi_{\mathbf{q}}(x))$ for every x. In the case where the F_i are homogeneous polynomials with respect to the indeterminates x, the homeomorphisms $\Psi_{\mathbf{q}}$ satisfy

$$\forall \lambda \in \mathbb{K}^*, \forall x \in \mathbb{K}^n \quad \Psi_{\mathbf{q}}(\lambda x) = \lambda \Psi_{\mathbf{q}}(x)$$

Hence if we define $\mathbb{P}(V_{\mathbf{q}}) = \{(\mathbf{q}, \mathbf{x}) \in \mathcal{V} \times \mathbb{P}^n_{\mathbb{K}} \mid F_n(\mathbf{q}, \mathbf{x}) = 0\}$, the homeomorphism $h_{\mathbf{q}}$ induces an homeomorphism between $\mathbb{P}(V_{\mathbf{t}})$ and $\mathbb{P}(V_{\mathbf{q}})$.

Remark 7. By construction of [Va72, Va73, Va75] and [PP17], the homeomorphisms $h_{\mathbf{q}}$ can be obtained by a local topological trivialization. That is there is a neighborhood $\mathcal{W}$ of $\mathbf{t}$ in $\mathbb{K}^r$ and a homeomorphism

$$\Phi : \mathcal{W} \times \mathbb{K}^n \to \mathcal{W} \times \mathbb{K}^n$$

such that $\Phi(\mathbf{q}, x) = (\mathbf{q}, h_{\mathbf{q}}(x))$. Moreover, as follows from [PP17], we may require that:

1. The homeomorphism Φ is subanalytic. In the algebraic case, that is in the case considered in this paper, we replace in the assumptions $\mathcal{O}_{\mathcal{V}}$ by the ring of $\mathbb{K}$-valued Nash functions on $\mathcal{V}$. Then Φ can be chosen semialgebraic.
2. Φ is arc-wise analytic, see Definition 1.2 of [PP17]. In particular each $h_{\mathbf{q}}$ is arc-analytic. It means that for every real analytic arc $\gamma(s) : (-1, 1) \to \mathbb{K}^n$, $h_{\mathbf{q}} \circ \gamma$ is real analytic and the same property holds for $h_{\mathbf{q}}^{-1}$.

4. Generic linear changes of coordinates

Let $f \in \mathbb{K}[x]$ be a polynomial of degree d and let $\Bbbk$ be a field extension of $\mathbb{Q}$ containing the coefficients of f. We denote by $\overline{f}$ the homogeneous part of degree d of f. The polynomial

$$\overline{f}(x_1,\ldots,x_{n-1},1) \not\equiv 0$$

otherwise $\overline{f}$ would be divisible by x_n-1 which is impossible since $\overline{f}$ is a homogeneous polynomial. So there exists

$$(\mu_1,\ldots,\mu_{n-1}) \in \mathbb{Q}^{n-1}$$

such that $c := \overline{f}(\mu_1,\ldots,\mu_{n-1},1) \neq 0$. Then let us denote by φ_μ the linear change of coordinates defined by

$$\begin{array}{ccc} x_i & \longmapsto & x_i+\mu_i x_n \text{ for } i<n \\ x_n & \longmapsto & x_n. \end{array}$$

We have that

$$\varphi_\mu(\overline{f}) = cx_n^d + h$$

where h is a homogeneous polynomial of degree d belonging to the ideal generated by $x_1,\ldots,x_{n-1}$. So we have that

$$\varphi_\mu(f) = cx_n^d + \sum_{j=1}^{d} a_j(x^{n-1})x_n^{d-j}$$

for some polynomials $a_j(x^{n-1}) \in \Bbbk[x^{n-1}]$.

Remark 8. If $g = g_1\ldots g_s$ is a product of polynomials of $\Bbbk[x]$, then the linear change of coordinates φ_μ defined above also satisfies

$$\varphi_\mu(g_i) = c_i x_n^{d_i} + \sum_{j=1}^{d} a_{i,j}(x^{n-1})x_n^{d-j}$$

for some nonzero constants $c_i \in \Bbbk$.

Remark 9. For every $f \in \Bbbk[x]$ of degree d where $\Bbbk \subset \mathbb{K}$ and $\mu \in \mathbb{Q}^{n-1}$, we have that

$$\varphi_{-\mu}\circ\varphi_\mu(f) = f.$$

Let us define the support of $f = \sum_{\alpha\in\mathbb{N}^n} f_\alpha x^\alpha$ as

$$\operatorname{Supp}(f) := \{\alpha \in \mathbb{N}^n \mid f_\alpha \neq 0\}.$$

Let us assume that

$$\varphi_\mu(f) = \sum f'_\alpha x^\alpha$$

for some polynomials $f'_\alpha \in \Bbbk$. Then the coefficient f_α has the form

$$f_\alpha = P_\alpha(\mu_1,\ldots,\mu_{n-1},f'_\beta)$$

for some polynomial $P_\alpha \in \mathbb{Z}[\mu, f'_\beta]$ depending only on the f'_β with $|\beta| = |\alpha|$.

5. Main theorem

We can state now our main result:

Theorem 10. *Let $V \subset \mathbb{K}^n$ (resp. $V \subset \mathbb{P}^n_{\mathbb{K}}$) be an affine (resp. projective) algebraic set, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then there exist an affine (resp. projective) algebraic set $W \subset \mathbb{K}^n$ (resp. $W \subset \mathbb{P}^n_{\mathbb{K}}$) and a homeomorphism $h : \mathbb{K}^n \longrightarrow \mathbb{K}^n$ (resp. $h : \mathbb{P}^n_{\mathbb{K}} \longrightarrow \mathbb{P}^n_{\mathbb{K}}$) such that:*

- *(i) The homeomorphism maps V onto W,*
- *(ii) W is defined by polynomial equations with coefficients in $\overline{\mathbb{Q}} \cap \mathbb{K}$,*
- *(iii) The variety W is obtained from V by a Zariski equisingular deformation. In particular the homeomorphism h can be chosen semialgebraic and arc-analytic,*
- *(iv) Let $g_1, \ldots, g_s$ be the generators of the ideal defining V. Let us fix $\varepsilon > 0$. Then W can be chosen so that the ideal defining it is generated by polynomials $g'_1, \ldots, g'_s \in \overline{\mathbb{Q}}[x]$ such that if we write*
$$g_i = \sum_{\alpha \in \mathbb{N}^n} g_{i,\alpha} x^\alpha \text{ and } g'_i = \sum_{\alpha \in \mathbb{N}^n} g'_{i,\alpha} x^\alpha$$
then for every i and α we have that:
$$|g_{i,\alpha} - g'_{i,\alpha}| < \varepsilon.$$
- *(v) Every polynomial relationship with rational coefficients between the $g_{i,\alpha}$ will also be satisfied by the $g'_{i,\alpha}$.*

Remark 11. We will see in the proof the following: let $\Bbbk$ denote the field extension of $\mathbb{Q}$ generated by the coefficients of the g_i and assume that $\Bbbk \neq \overline{\mathbb{Q}}$ (otherwise there is nothing to prove). By the primitive element theorem, $\Bbbk$ is a simple extension of a purely transcendental extension $\mathbb{L}$ of $\mathbb{Q}$, i.e. $\mathbb{Q} \longrightarrow \mathbb{L}$ is purely transcendental and $\mathbb{L} \longrightarrow \Bbbk$ is a simple extension of degree d. Then the coefficients of the g'_i belong to a finite extension of $\mathbb{Q}$ of degree $\leq d$.
In particular if $\mathbb{Q} \longrightarrow \Bbbk$ is purely transcendental then W is defined over $\mathbb{Q}$.

Remark 12. In fact (v) implies that
$$\forall i, \alpha \quad g_{i,\alpha} = 0 \Longrightarrow g'_{i,\alpha} = 0.$$
If ε is chosen small enough we may even assume that
$$\forall i, \alpha \quad g_{i,\alpha} = 0 \Longleftrightarrow g'_{i,\alpha} = 0.$$

Proof of Theorem 10. Let us consider a finite number of polynomials $g_1, \ldots, g_s \in \mathbb{K}[x]$ generating the ideal defining V and let us denote by $g_{i,\alpha}$ their coefficients as written in the theorem. After a linear change of coordinates φ_μ with $\mu \in \mathbb{Q}^{n-1}$ as in Section 4 we can assume that
$$\varphi_\mu(g_r) = c_r x_n^{p_r} + \sum_{j=1}^{p_r} b_{n-1,r,j}(x^{n-1}) x_n^{p_r - j} = \sum_{\beta \in \mathbb{N}^n} a_{n,r,\beta} x^\beta \quad \forall r = 1, \ldots, s.$$
Moreover by multiplying each $\varphi_\mu(g_r)$ by $1/c_r$ we can assume that $c_r = 1$ for every r. We denote by f_n the product of the $\varphi_\mu(g_r)$ and by a_n the vectors of coefficients $a_{n,r,\beta}$. The entries of a_n are polynomial functions in the $g_{i,\alpha}$ with rational coefficients, let us say

$$a_n = A_n(g_{i,\alpha}) \tag{5.1}$$

for some $A_n = (A_{n,r,\beta})_{r,\beta} \in \mathbb{Q}(u_{i,\alpha})^{N_n}$ for some integer $N_n > 0$ and some new indeterminates $u_{i,\alpha}$.

Let l_n be the smallest integer such that

$$\Delta_{n,l_n}(a_n) \not\equiv 0$$

where $\Delta_{n,l}$ denotes the l-th generalized discriminant of f_n. In particular we have that

$$\Delta_{n,l}(a_n) \equiv 0 \ \ \forall l < l_n.$$

After a linear change of coordinates x^{n-1} with coefficients in $\mathbb{Q}$ we can write $\Delta_{n,l_n}(a_n) = e_{n-1}f_{n-1}$ with

$$f_{n-1} = \sum_{\beta\in\mathbb{N}^n} a_{n-1,\beta}x^\beta = x_{n-1}^{d_{n-1}} + \sum_{j=1}^{d_{n-1}} b_{n-2,j}(x^{n-2})x_{n-1}^{d_{n-1}-j}$$

for some constants $e_{n-1}, a_{n-1,\beta} \in \mathbb{k}$, with $e_{n-1} \neq 0$, and some polynomials $b_{n-2,j} \in \mathbb{k}[x^{n-2}]$. We denote by a_{n-1} the vector of coefficients $a_{n-1,\beta}$. Let l_{n-1} be the smallest integer such that

$$\Delta_{n-1,l_{n-1}}(a_{n-1}) \not\equiv 0$$

where $\Delta_{n-1,l}$ denotes the l-th generalized discriminant of f_{n-1}.

We repeat this construction and define a sequence of polynomials $f_j(x^j)$, $j = k_0, \ldots, n-1$ for some k_0, such that

$$\Delta_{j+1,l_{j+1}}(a_{j+1}) = e_j\left(x_j^{d_j} + \sum_{k=1}^{d_j} b_{j-1,k}(x^{j-1})x_j^{d_j-k}\right) = e_j\left(\sum_{\beta\in\mathbb{N}^n} a_{j,\beta}x^\beta\right) = e_jf_j$$

is the first non identically zero generalized discriminant of f_{j+1}, where a_j denotes the vectors of coordinates $a_{j,\beta}$ in $\mathbb{k}^{N_j}$, i.e. we have that :

$$\Delta_{j+1,l_{j+1}}(a_{j+1}) = e_j\left(\sum_{\beta\in\mathbb{N}^n} a_{j,\beta}x^\beta\right) = e_jf_j \tag{5.2}$$

and

$$\Delta_{j+1,l}(a_{j+1}) \equiv 0 \ \ \forall l < l_{j+1}, \tag{5.3}$$

until we get $f_{k_0} = 1$ for some $k_0 \geq 1$.

By (5.2) we see that the entries of a_j and e_j are rational functions in the entries of a_{j+1} for every $j < n$. Thus by (5.1) we see that the entries of the a_k and the e_j are rational functions in the $g_{i,\alpha}$ with rational coefficients, let us say

$$a_k = A_k(g_{i,\alpha}), \quad e_j = E_j(g_{i,\alpha})$$

for some $A_k \in \mathbb{Q}(u_{i,\alpha})^{N_k}$ and $E_j \in \mathbb{Q}(u_{i,\alpha})$ for every k and j, where the $u_{i,\alpha}$ are new indeterminates for every i and α.

By Theorem 3, there exist a new set of indeterminates $t = (t_1, \ldots, t_r)$, an open domain $\mathcal{U} \subset \mathbb{K}^r$, $(\mathbf{t}_1, \ldots, \mathbf{t}_r) \in \mathcal{U}$, polynomials $g_{i,\alpha}(t, z) \in \mathbb{Q}(t)[z]$, for every i and α, and $z(t) \in \mathrm{QA}(\mathcal{U})$ such that $g_{i,\alpha}(\mathbf{t}, z(\mathbf{t})) = g_{i,\alpha}$ for every i and α.

Since the a_l and the e_j are rational functions with rational coefficients in the $g_{i,\alpha}$, the system of equations (5.3) is equivalent to a system of polynomial equations with rational coefficients

$$f(g_{i,\alpha}) = 0. \tag{5.4}$$

By Theorem 3 the functions $g_{i,\alpha}(t, z(t))$ are solutions of this system of equations.

Because $e_j(g_{i,\alpha}) = e_j(g_{i,\alpha}(\mathbf{t}, z(\mathbf{t}))) \neq 0$ for every j, we have that none of the $e_j(g_{i,\alpha}(t, z(t)))$ vanishes in a small open ball $\mathcal{V} \subset \mathcal{U}$ centered at $\mathbf{t}$.

In particular we have that

$$e_j(g_{i,\alpha}(t, z(t))) \neq 0 \ \ \forall j, \ \forall t \in \mathcal{V}. \tag{5.5}$$

For $t \in \mathcal{V}$ and $r = 1, \ldots, s$, we define

$$\begin{aligned} g'_r(t, x) &:= \sum_{\alpha \in \mathbb{N}^n} g_{i,\alpha}(t, z(t))x^\alpha, \\ F_n(t, x) &= \prod_{r=1}^{s} G_r(t, x), \text{ where} \\ G_r(t, x) &= \sum_{\alpha \in \mathbb{N}^n} A_{n,r,\alpha}(g_{i,\alpha}(t, z(t)))x^\alpha, r = 1, \ldots, s, \end{aligned}$$

and for $j = k_0, \ldots, n-1$

$$F_j(t, x) = \sum_{\alpha \in \mathbb{N}^n} A_{j,\alpha}(g_{i,\alpha}(t, z(t)))x^\alpha.$$

In particular we have that $G_r(\mathbf{t}, x) = \varphi_\mu(g_r(x))$ for every r.

Let us denote by $\mathcal{O}_\mathcal{V}$ the ring of $\mathbb{K}$-analytic functions on $\mathcal{V}$. In particular we have that $F_j(t, x)$ is a polynomial of $\mathcal{O}_\mathcal{V}[x^i]$, of degree d_j in x_j, such that the coefficient of $x_j^{d_j}$ is $e_j(g_{i,\alpha}(t, z(t)))$ and whose discriminant (seen as a polynomial in x_j) is equal to $e_{j-1}(g_{i,\alpha}(t, z(t)))F_{j-1}(t, x) \in \mathcal{O}_\mathcal{V}[x^{j-1}]$ for every j by (5.4).

Thus the family $(F_j)_j$ satisfies the hypothesis of Theorem 5 by (5.5). Hence the algebraic hypersurfaces $X_0 := \{F_n(0, x) = 0\}$ and $X_1 := \{F_n(1, x)\}$ are homeomorphic. Moreover the homeomorphism between them maps every component of X_0 defined by $G_r(0, x) = 0$ onto the component of X_1 defined by $G_r(1, x) = 0$. This proves that the algebraic variety defined by $\{\varphi_\mu(g_1) = \cdots = \varphi_\mu(g_s) = 0\}$ is homeomorphic to the algebraic variety $\{G_1(0, x) = \cdots = G_s(0, x) = 0\}$ which is defined by polynomial equations over $\overline{\mathbb{Q}}$. Thus V is homeomorphic to $W := \{g'_1(x) = \cdots = g'_s(x) = 0\}$.

Moreover since $\overline{\mathbb{Q}} \cap \mathbb{K}$ is dense in $\mathbb{K}$, we can choose $\mathbf{q} \in \overline{\mathbb{Q}}$ as close as we want to $\mathbf{t}$. In particular by choosing $\mathbf{q}$ close enough to $\mathbf{t}$ we may assume that (iv) in Theorem 10 is satisfied, since γ, $t \mapsto z(t)$ and the $g_{i,\alpha}$ are continuous functions.

Finally we have that $z(\mathbf{q})$ is algebraic of degree $\leq d$ over $\mathbb{Q}$ by Lemma 4 (see Remark 11).
Thus Theorem 10 is proven in the affine case. The projective is proven is the same way by Remark 6. □

We can remark that Theorem 10 (v) implies that several algebraic invariants are preserved by the homeomorphism h. For instance we have the following corollary:

Corollary 13. *For $\varepsilon > 0$ small enough the Hilbert-Samuel function of W is the same as that of V.*

Proof of Corollary 13. Let $h_1, \ldots, h_m$ be a Gröbner basis of the ideal I of $\mathbb{K}[x]$ generated by the g_i with respect to a given monomial order $\preceq$. Let us recall that for $f = \sum_{\alpha\in\mathbb{N}^n} f_\alpha x^\alpha \in \mathbb{K}[x]$, $f \neq 0$, we denote the leading term of f by $\mathrm{LT}(f) = f_{\alpha_0} x^{\alpha_0}$ where α_0 is the largest nonzero exponent of the support of f with respect to the monomial order:

$$\mathrm{Supp}(f) = \{\alpha \in \mathbb{N}^n \mid f_\alpha \neq 0\}.$$

A Gröbner basis of I is computed by considering S-polynomials and divisions (see [CLO07] for more details):
- the S-polynomial of two nonzero polynomials f and g is defined as follows: set $\mathrm{LT}(f) = ax^\alpha$ and $\mathrm{LT}(g) = bx^\beta$ with a, $b \in \mathbb{K}^*$, and let x^δ be the least common multiple of x^α and x^β. Then the S-polynomial of f and g is

$$S(f, g) := \frac{x^{\delta-\alpha}}{a} f - \frac{x^{\delta-\beta}}{b} g.$$

- the division of a polynomial f by polynomials $f_1, \ldots, f_l$ is defined inductively as follows: firstly after some renumbering one assume that $\mathrm{LT}(f_1) \preceq \mathrm{LT}(f_2) \preceq \cdots \preceq \mathrm{LT}(f_l)$. Then we consider the smallest integer i such that $\mathrm{LT}(f)$ is divisible by $\mathrm{LT}(f_i)$. If such a i exists one sets $q_i^{(1)} = \dfrac{\mathrm{LT}(f)}{\mathrm{LT}(f_i)}$ and $q_j^{(1)} = 0$ for $j \neq i$ and $r^{(1)} = 0$. Otherwise one sets $q_j^{(1)} = 0$ for every j and $r^{(1)} = \mathrm{LT}(f)$. We repeat this process by replacing f by $f - \sum_{j=1}^{l} q_j^{(1)} f_j - r^{(1)}$. After a finite number of steps we obtain a decomposition

$$f = \sum_{j=1}^{l} q_j f_j + r$$

where none of the elements of $\mathrm{Supp}(r)$ is divisible by any $\mathrm{LT}(f_i)$. The polynomial r is called the remainder of the division of f by the f_i.

Thus we can make the following remark: *every remainder of the division of some S-polynomial $S(g_i, g_j)$ by $g_1, \ldots, g_s$ is a polynomial whose coefficients are rational functions on the coefficients of the g_k.*

The Buchberger's Algorithm is as follows: we begin with $g_1, \ldots, g_s$ the generators of I and we compute all the S-polynomials of every pair of polynomials among the g_i. Then we consider the remainders of the divisions of these S-polynomials by the g_i. If some remainders are nonzero we add them to the family $\{g_1, \ldots, g_s\}$. Then we repeat the same process that stops after a finite number of setps.

By the previous remark if h denotes the remainder of the division of a S-polynomial $S(g_i, g_j)$ by the g_k then we have a relation of the form

$$S(g_i, g_j) = \sum_{l=1}^{s} b_l g_l + h \tag{5.6}$$

and the b_l belong to the field extension of $\mathbb{Q}$ generated by the coefficients of the g_l. Let h_β denote the coefficient of x^β in h. Then for those h_β that are nonzero, Equation (5.6) provides an expression of them as rational functions in the $g_{i,\alpha}$, let us say

$$\forall \beta \in \mathrm{Supp}(h) \quad h_\beta = H_\beta(g_{i,\alpha}). \tag{5.7}$$

For those h_β equal to zero Equation (5.6) provides a polynomial relation with rational coefficients between the $g_{i,\alpha}$:

$$\forall \beta \notin \mathrm{Supp}(h) \qquad H_\beta(g_{i,\alpha}) = 0. \tag{5.8}$$

And Equation (5.6) is equivalent to the systems of equations (5.7) and (5.8). This remains true if we replace the g_i by polynomials whose coefficients are rational functions in the $g_{i,\alpha}$ with rational coefficients. Thus the fact that $h_1,\ldots,\ h_m$ is a Gröbner basis obtained from the g_i by Buchberger's Algorithm is equivalent to a system of equations

$$Q_k(g_{i,\alpha}) = 0 \text{ for } k \in E \tag{5.9}$$

where E is a (not necessarily finite) set. Thus by Theorem 10 (v) we see that these equations are satisfied by the coefficients of the g'_i, hence the ideal defining W has a Gröbner basis $h'_1,\ \ldots,\ h'_m$ obtained from the g'_i by doing exactly the same steps in Buchberger's Algorithm, and $\mathrm{Supp}(h'_i) \subset \mathrm{Supp}(h_i)$ for every i.
Moreover the initial terms of the h_i are rational functions in the $g_{i,\alpha}$ with rational coefficients, let us say $H_i(g_{i,\alpha})$ for some rational functions H_i. By choosing ε small enough we insure that

$$H_i(g'_{i,\alpha}) \neq 0 \quad \forall i.$$

Thus the leading exponents of the h'_i are equal to those of the h_i. In particular the Hilbert-Samuel function of W is the same as that of V, and (iv) in Theorem 10 is proven. □

Remark 14. In fact we have proven that the ideal of leading terms of I is the same as the ideal of leading terms of the ideal defining W. So we could have concluded by [GP08, Theorem 5.2.6] for instance.

6. Complete algorithm

6.1. Settings.

Input:

1. polynomials $g_1,\ \ldots,\ g_s \in \mathbb{K}[x]$ whose coefficients $g_{i,\alpha}$ belong to a finitely generated field extension $\Bbbk$ over $\mathbb{Q}$.
2. a presentation
$$\Bbbk = \mathbb{Q}(\mathbf{t}_1, \ldots, \mathbf{t}_r, \mathbf{z})$$
where the $\mathbf{t}_i$ are algebraically independent over $\mathbb{Q}$ and $\mathbf{z}$ is finite over $\mathbb{Q}(\mathbf{t}_1, \ldots, \mathbf{t}_r)$.
3. the minimal polynomial $P(z)$ of $\mathbf{z}$ over $\mathbb{Q}(\mathbf{t}_1, \ldots, \mathbf{t}_r)$.
4. the $g_{i,\alpha}$ are given as rational functions in the $\mathbf{t}_i$ and $\mathbf{z}$, and the coefficients of $P(z)$ as rational functions in the $\mathbf{t}_i$.
5. a positive real number ε.

Moreover we assume that the $\mathbf{t}_i$ and $\mathbf{z}$ are computable numbers [Tu36]. Let us recall that a real computable number is a number $\mathbf{x} \in \mathbb{R}$ for which there is a Turing machine that computes a sequence $(\mathbf{q}_n)$ of rational numbers such that $|\mathbf{x} - \mathbf{q}_n| < \frac{1}{n}$ for every $n \geq 1$. A computable number is a complex number whose real and imaginary parts are real computable numbers. The set of computable numbers is an algebraically closed field [Ri54]. More precisely if $\mathbf{t}_1,\ldots,\ \mathbf{t}_r$ are computable numbers such that for every $i \in \{1, \ldots, r\}$ $(\mathbf{q}_{i,n})_n$ is a sequence of $\mathbb{Q} + i\mathbb{Q}$ computed by a Turing machine with $|\mathbf{t}_i - \mathbf{q}_{i,n}| < \frac{1}{n}$ for every n, and if $\mathbf{z} \in \mathbb{C}$ satisfies $P(\mathbf{t}, \mathbf{z}) = 0$

for some reduced polynomial $P(t,z)\in\mathbb{Q}[t,z]$, then one can effectively find a Turing machine that computes a sequence $(\mathbf{q}_n)_n$ of $\mathbb{Q}+i\mathbb{Q}$ such that $|\mathbf{z}-\mathbf{q}_n|<\frac{1}{n}$ for every integer n.

Output: polynomials $g'_1,\ldots,g'_s\in(\overline{\mathbb{Q}}\cap\mathbb{K})[x]$ with the properties:

(1) the pairs $(V(g_i),\mathbb{K}^n)$ and $(V(g'_i),\mathbb{K}^n)$ are homeomorphic, and the homeomorphism is subanalytic and arc-analytic.

(2) the coefficients $g'_{i,\alpha}$ of g'_i satisfy the properties:
$$g'_{i,\alpha}\neq 0 \Longleftrightarrow g_{i,\alpha}\neq 0$$
$$|g'_{i,\alpha}-g_{i,\alpha}|<\varepsilon \text{ for every } i \text{ and } \alpha$$
and every polynomial relation with coefficients in $\mathbb{Q}$ satisfied by the $g_{i,\alpha}$ is also satisfied by the $g'_{i,\alpha}$.

6.2. **Algorithm:** We present here the successive steps of the algorithm.

(1) We make a linear change of coordinates with coefficients in $\mathbb{Q}$, denoted by φ_μ with $\mu\in\mathbb{Q}^{n-1}$, such that each of the g_i is a monic polynomial in x_n of degree $\deg(g_i)$.
We denote by f_n the product of the g_i after this change of coordinates, and by a_n the vector of the coefficients of the g_i after this change of coordinates (seen as a polynomial in x_n).

(2) For every j from n to 1 we do the following: let us assume that f_j is a polynomial in $x_1,\ldots,x_j$ having a nonzero monomial $e_jx_j^{\deg(f_j)}$, $e_j\in\mathbb{k}^*$. We denote by a_j the vector of the coefficients of f_j seen as a polynomial in x_j. We consider the generalized discriminants $\Delta_{j,l}$ of f_j with respect to x_j, and we denote by l_j the smallest integer such that
$$\Delta_{j,l_j}(a_j)\neq 0.$$
These polynomials $\Delta_{j,l_j}(a_j)$ can be effectively computed (see 6.3).
We perform a linear change of coordinates (with coefficients in $\mathbb{Q}$) in $x_1,\ldots,x_{i-1}$ such that $\Delta_{j,l_j}(a_j)$ becomes a unit times a monic polynomial of degree $\deg(\Delta_{j,l_j}(a_j))$ in x_{j-1}, and we denote by f_{j-1} this new monic polynomial.

(3) We stop the process once we have that f_{j-1} is a nonzero constant.

(4) We consider (5.2) as a system of polynomial equations that will allow us to compute the expression of the e_j as rational functions in the $g_{i,\alpha}$.

(5) We denote by $P(t,z)$ the monic polynomial of $\mathbb{Q}(t)[z]$ such that $P(\mathbf{t},z)=P(z)$ is the minimal polynomial of $\mathbf{z}$ over $\mathbb{Q}(\mathbf{t})$.
By replacing ε by a smaller positive number we may assume that
$$d(\mathbf{t},\Delta_P)>2\varepsilon$$
where Δ_P denotes the discriminant locus of $P(t,z)$ seen as a polynomial in z. This discriminant is computed as $\mathrm{Res}_z(P,\partial P/\partial z)$. See [DGY96] or [BM95, Theorem C] for a practical way of choosing such a ε.

(6) Let $\mathbf{z}_1:=\mathbf{z}$, $\mathbf{z}_2,\ldots,\mathbf{z}_d$ be the distincts roots of $P(\mathbf{t},z)$. These are computable

numbers and so we can compute $\eta \in \mathbb{Q}_{>0}$ such that all the differences between two $\mathbf{z}_i$ are strictly greater than η.
Let $z(t)$ be the root of $P(t, z)$ such that $z(\mathbf{t}) = \mathbf{z}$. We can write

$$w(t) := z(t + \mathbf{t}) = \sum_{\alpha \in \mathbb{N}^r} \mathbf{z}_\alpha t^\alpha, \quad \mathbf{z}_0 = \mathbf{z}.$$

We write

$$P(t, z) = p_0(t) + p_1(t)z + \cdots + p_{d-1}(t)z^{d-1} + z^d$$

where the $p_i(t) \in \mathbb{K}(t)$. Set

$$M := 1 + \max_{0 \leq i \leq d-1} \max_{\mathbf{t}' \in B(0, 2\varepsilon)} |p_i(\mathbf{t}')|.$$

Then $|\mathbf{z}_\alpha| \leq \dfrac{M}{(2\varepsilon)^{|\alpha|}}$ for every α (by the Cauchy bounds for the roots of a monic polynomial since $w(t)$ is convergent on $B(0, 2\varepsilon)$ by (5)). Let $W_k(t)$ be the homogeneous term of degree k in the Taylor expansion of $w(t)$: $W_k(t) = \sum_{|\alpha|=k} \mathbf{z}_\alpha t^\alpha$. Then

$$\forall \mathbf{t}' \in B(0, \varepsilon) \quad |W_k(\mathbf{t}')| \leq \frac{M}{2^k} \binom{k + r - 1}{r - 1}.$$

We have that

$$\forall k \geq r \quad \frac{M}{2^k} \binom{k + r - 1}{r - 1} \leq \frac{M 2^r}{(r - 1)!} \frac{k^r}{2^k}.$$

Then choose $k_0 \geq r$ such that $\dfrac{k^r}{\sqrt{2}^k} \leq 1$ forall $k \geq k_0$. Therefore

$$\forall \mathbf{t}' \in B(0, \varepsilon), \forall k \geq k_0 \quad |W_k(\mathbf{t}')| \leq \frac{M 2^r}{(r - 1)!} \frac{1}{\sqrt{2}^k}$$

and

$$\forall \mathbf{t}' \in B(0, \varepsilon), \forall k \geq k_0 \quad \left| \sum_{l \geq k} W_l(\mathbf{t}') \right| \leq \frac{M 2^r}{(r - 1)!} \frac{\sqrt{2}}{\sqrt{2} - 1} \frac{1}{\sqrt{2}^k} = C \frac{1}{\sqrt{2}^k}.$$

(7) Now we want to determine the computable number $z(\mathbf{q}) = w(\mathbf{q} - \mathbf{t})$ for a given choice of $\mathbf{q} \in B(\mathbf{t}, \varepsilon) \cap (\mathbb{Q} + i\mathbb{Q})^r$. This number is one of the roots of $P(\mathbf{q}, z)$. These roots are computable numbers and so we can bound from below all the differences between each two of them: let $\delta \in \mathbb{Q}_{>0}$ be such a bound.
By the previous step one can compute an integer k such that

$$\forall \mathbf{t}' \in B(0, \varepsilon), \quad \left| \sum_{l \geq k} W_l(\mathbf{t}') \right| \leq C \frac{1}{\sqrt{2}^k} \leq \frac{\delta}{2}.$$

So we can distinguish $z(\mathbf{q})$ from the other roots of $P(\mathbf{q}, z)$.

(8) Choose $\mathbf{q} \in (\mathbb{Q} + i\mathbb{Q})^r$ such that

(i) $\mathbf{q}$ is not in the discriminant locus of $P(t, z)$ seen as a polynomial in z,
(ii) $\|\mathbf{q} - \mathbf{t}\| < \varepsilon$,
(iii) $e_j(g_{i,\alpha}(\mathbf{q}, z(\mathbf{q}))) \neq 0$.

The first condition is insured by the choice of ε in (5).
In order to check that $e_j(g_{i,\alpha}(\mathbf{q}, z(\mathbf{q}))) \neq 0$, one only has to choose $\mathbf{q}$ such that $\|\mathbf{t}-\mathbf{q}\|$ is small enough and this can be done effectively. Indeed the $e_j(g_{i,,\alpha})$ are rational functions in the t_i and z, thus we can effectively bound the variations of e_j locally around $\mathbf{t}$.

(9) Then we evaluate the $g_{i,a}(t, z(t))$ at $(\mathbf{q}, z(\mathbf{q}))$. We denote these values by $g'_{i,\alpha}$ and we define the polynomials

$$g'_i := \sum_{\alpha\in\mathbb{N}^n} g'_{i,\alpha}x^\alpha.$$

6.3. **Generalized discriminants.** We follow Appendix IV of [Wh72] or [Roy06]. The generalized discriminant Δ_l of a polynomial

$$f = x^d + \sum_{j=1}^d b_j x^{d-j}$$

can be defined as follows:
Let $\xi_1,\ldots, \xi_d$ be the roots of f (with multiplicities), and set $s_i = \sum_{k=1}^d \xi_k^i$ for every $i \in \mathbb{N}$. Then

$$\Delta_{d+1-l} = \begin{vmatrix} s_0 & s_1 & \cdots & s_{l-1} \\ s_1 & s_2 & \cdots & s_l \\ \cdots & \cdots & \cdots & \cdots \\ s_{l-1} & s_l & \cdots & s_{2l-2} \end{vmatrix}.$$

Thus Δ_1 is the classical discriminant of f. The polynomials Δ_l may be effectively computed in term of the s_i, and those can be effectively computed in terms of the b_i. The polynomial f admits exactly k distinct complex roots if and only if $\Delta_1 = \cdots = \Delta_{d-k} = 0$ and $\Delta_{d-k+1} \neq 0$.

REFERENCES

[BPR17] M. Bilski, A. Parusiński, G. Rond, Local topological algebraicity of analytic function germs, *J. Algebraic Geom.*, **26**, (2017), 177-197.

[BM95] L. P. Bos, P. D. Milman, Sobolev-Gagliardo-Nirenberg and Markov type inequalities on subanalytic domains, *Geom. Funct. Anal.*, **5**, (1995), no. 6, 853-923.

[CLO07] D. Cox, J. Little, D. O'Shea, Ideals, Varieties, and Algorithms, An introduction to computational algebraic geometry and commutative algebra, Springer, New York, 2007.

[DGY96] J.-P. Dedieu, X. Gourdon, J.-C. Yakoubsohn, Computing the distance from a point to an algebraic hypersurface, The mathematics of numerical analysis (Park City, UT, 1995), 285-293, Lectures in Appl. Math., 32, Amer. Math. Soc., Providence, RI, 1996.

[GP08] M. Greuel, G. Pfister, A SINGULAR introduction to commutative algebra, *Springer, Berlin*, 2008.

[Mo84] T. Mostowski, Topological equivalence between analytic and algebraic sets, *Bull. Polish Acad. Sci. Math.*, **32**, (1984), no. 7-8, 393-400.

[PP17] L. Păunescu, A. Parusiński, Arc-wise analytic stratification, Whitney fibering conjecture and Zariski equisingularity, *Adv. Math.*, **309**, (2017), 254-305.

[RD84] P. Ribenboim, L. Van den Dries,, The absolute Galois group of a rational function field in characteristic zero is a semi-direct product, *Can. Math. Bull.*, **27**, (1984), 313-315.

[Ri54] H. G. Rice, Recursive real numbers, *Proc. Amer. Math. Soc.*, **5**, (1954), 784-791.

[Ro] G. Rond, Local topological algebraicity with algebraic coefficients of analytic sets or functions, *Algebra Number Theory*, **12**, (2018), no. 5, 1215-1231.

[Roy06] M.-F. Roy, Subdiscriminant of symmetric matrices are sums of squares, *Mathematics, Algorithms, Proofs*, volume 05021 of Dagstuhl Seminar Proceedings, Internationales Begegnungs und Forschungszentrum für Informatik (IBFI), Schloss Dagstuhl, Germany, (2005).

[Te90] B. Teissier, Un exemple de classe d'équisingularité irrationnelle, *C. R. Acad. Sci. Paris Sér. I Math.*, **311**, (1990), no. 2, 111-113.

[Tu36] A. M. Turing, On Computable Numbers, with an Application to the Entscheidungsproblem, *Proc. London Math. Soc. (2)*, **42**, (1936), no. 3, 230-265.

[Va72] A. N. Varchenko, Theorems on the topological equisingularity of families of algebraic varieties and families of polynomial mappings, *Math. USSR Izviestija*, **6**, (1972), 949-1008.

[Va73] A. N. Varchenko, The relation between topological and algebro-geometric equisingularities according to Zariski. *Funkcional. Anal. Appl.* **7** (1973), 87–90.

[Va75] A. N. Varchenko, Algebro-geometrical equisingularity and local topological classification of smooth mappings. Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974), Vol. 1, pp. 427–431. Canad. Math. Congress, Montreal, Que., 1975.

[Wh72] H. Whitney, Complex Analytic Varieties, Addison-Wesley Publ. Co., Reading, Massachusetts 1972.

[Za71] O. Zariski, Some open questions in the theory of singularities, *Bull. Amer. Math. Soc.* **77** (1971) 481–491; Oscar Zariski: Collected Papers, Volume IV, MIT Press, 238–248.

E-mail address: `adam.parusinski@unice.fr`

Université Nice Sophia Antipolis, CNRS, LJAD, UMR 7351, 06108 Nice, France

E-mail address: `guillaume.rond@univ-amu.fr`

Aix Marseille Univ, CNRS, Centrale Marseille, I2M, Marseille, France