A NEW PROOF OF BRONSHTEIN'S THEOREM

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ABSTRACT. We give a new self-contained proof of Bronshtein's theorem, that any continuous root of a $C^{n-1,1}$-family of monic hyperbolic polynomials of degree n is locally Lipschitz, and obtain explicit bounds for the Lipschitz constant of the root in terms of the coefficients. As a by-product we reprove the recent result of Colombini, Orrú, and Pernazza, that a C^n-curve of hyperbolic polynomials of degree n admits a C^1-system of its roots.

1. Introduction

Choosing regular roots of polynomials whose coefficients depend on parameters is a classical much studied problem with important connections to various fields such as algebraic geometry, partial differential equations, and perturbation theory.

This problem is of special interest for *hyperbolic* polynomials whose roots are all real. Probably the first result in this direction was obtained by Glaeser [9] who studied the square root of a nonnegative smooth function. The most important and most difficult result in this field is Bronshtein's theorem [6]: any continuous root of a $C^{p-1,1}$-curve of monic hyperbolic polynomials, where p is the maximal multiplicity of the roots, is locally Lipschitz with uniform Lipschitz constants; cf. Theorem 2.1. A multiparameter version follows immediately; see Theorem 2.2. A different proof was later given by Wakabayashi [23] who actually proved a more general Hölder version; for a refinement of Bronshtein's method in order to show this generalization see Tarama [22]. Kurdyka and Paunescu [11] used resolution of singularities to show that the roots of a hyperbolic polynomial whose coefficients are real analytic functions in several variables admit a parameterization which is locally Lipschitz; in one variable we have Rellich's classical theorem [20] that the roots may be parameterized by real analytic functions.

A C^p-curve of monic hyperbolic polynomials with at most p-fold roots admits a differentiable system of its roots. Using Bronshtein's theorem, Mandai [12] showed that the roots can be chosen C^1 if the coefficients are C^{2p}, and Kriegl, Losik, and Michor [10] found twice differentiable roots provided that the coefficients are C^{3p}. Recently, Colombini, Orrú and Pernazza [7] proved that C^p (resp. C^{2p}) coefficients suffice for C^1 (resp. twice differentiable) roots and that this statement is best possible.

In this paper we present a new proof of Bronshtein's theorem. Our proof is simple and elementary. The main tool is the splitting principle, a criterion that allows to factorize

Date: May 7, 2014.

2010 *Mathematics Subject Classification.* 26C05, 26C10, 30C15, 26A16.

Key words and phrases. Bronshtein's theorem, Lipschitz and C^1 roots of hyperbolic polynomials.

Supported by the Austrian Science Fund (FWF), Grants P 22218-N13 and P 26735-N25, and by ANR project STAAVF (ANR-2011 BS01 009).

polynomials under elementary assumptions. The coefficients of the factors can be expressed in a simple way in terms of the coefficients of the original polynomials, so that the bounds on the coefficients and their derivatives can be also carried over. Thanks to this we obtain explicit bounds on the Lipschitz constant of the roots. As a by-product we give a new proof of the aforementioned result of Colombini, Orrú, and Pernazza on the existence of C^1-roots; see Theorem 2.4.

Note that the statements of Theorem 2.1, Theorem 2.2, and Theorem 2.4 are best possible in the following sense. If the coefficients are just $C^{p-1,1}$ then the roots need not admit a differentiable parameterization. Moreover, the roots can in general not be parameterized by $C^{1,\alpha}$-functions for any $\alpha > 0$ even if the coefficients are C^∞. Some better conclusions can be obtained if additional assumptions are made; see [1], [3], [4], [5], [15], [17].

Convention. We will denote by $C(n,\ldots)$ *any* constant depending only on $n,\ldots$; it may change from line to line. Specific constants will bear a subscript like $C_1(n)$ or $C_2(n)$.

2. Bronshtein's theorem

Let $I \subseteq \mathbb{R}$ be an open interval and consider a monic polynomial

$$P_a(t)(Z) = P_{a(t)}(Z) = Z^n + \sum_{j=1}^n a_j(t) Z^{n-j}, \quad t \in I.$$

We say that $P_a(t)$, $t \in I$, is a $C^{p-1,1}$-*curve of hyperbolic polynomials* if $(a_j)_{j=1}^n \in C^{p-1,1}(I,\mathbb{R}^n)$ and all roots of $P_a(t)$ are real for each $t \in I$.

Note that ordering the roots of a hyperbolic polynomial $P_a(Z) = Z^n + \sum_{j=1}^n a_j Z^{n-j}$ increasingly, $\lambda_1^\uparrow(a) \le \lambda_2^\uparrow(a) \le \cdots \le \lambda_n^\uparrow(a)$, provides a continuous mapping $\lambda^\uparrow = (\lambda_j^\uparrow)_{j=1}^n : H_n \to \mathbb{R}^n$ on the space of hyperbolic polynomials of degree n, see e.g. [1, 4.1], which can be identified with a closed semialgebraic subset $H_n \subseteq \mathbb{R}^n$, see e.g. [13].

By a *system of the roots* of $P_a(t)$, $t \in I$, we mean any n-tuple $\lambda = (\lambda_j)_{j=1}^n : I \to \mathbb{R}^n$ satisfying

$$P_a(t)(Z) = \prod_{j=1}^n (Z - \lambda_j(t)), \quad t \in I.$$

Note that any *continuous root* μ_1 of $P_a(t)$, $t \in I$, i.e., $\mu_1 \in C^0(I,\mathbb{R})$ and $P_a(t)(\mu_1(t)) = 0$ for all $t \in I$, can be completed to a continuous system of the roots $\mu = (\mu_j)_{j=1}^n$, cf. [16, 6.17].

Theorem 2.1 (Bronshtein's theorem)**.** *Let $P_a(t)$, $t \in I$, be a $C^{p-1,1}$-curve of hyperbolic polynomials of degree n, where p is the maximal multiplicity of the roots of P_a. Then any continuous root of P_a is locally Lipschitz.*

Moreover if $p = n$ then for any pair of intervals $I_0 \Subset I_1 \Subset I$ and for any continuous root $\lambda(t)$ its Lipschitz constant can be bounded as follows

$$\begin{aligned} \text{(2.1)} \qquad \operatorname{Lip}_{I_0}(\lambda) &\le C(n, I_0, I_1) \left(\max_i \|a_i\|_{C^{n-1,1}(\overline{I}_1)}^{\frac{1}{i}} \right) \\ &\le \tilde{C}(n, I_0, I_1) \left(1 + \max_i \|a_i\|_{C^{n-1,1}(\overline{I}_1)} \right), \end{aligned}$$

where the constants $C(n, I_0, I_1)$, $\tilde{C}(n, I_0, I_1)$ depend only on n and the intervals I_0, I_1. (More precise bounds are stated in Subsection 4.6.)

If $p < n$ then there exist uniform bounds on the Lipschitz constant provided the multiplicities of roots are at most p "in a uniform way". These bounds are stated in Subsection 4.7.

For an open subset $U \subseteq \mathbb{R}^m$ and $p \in \mathbb{N}_{\ge 1}$, we denote by $C^{p-1,1}(U)$ the space of all functions $f \in C^{p-1}(U)$ so that each partial derivative $\partial^\alpha f$ of order $|\alpha| = p-1$ is locally Lipschitz. It is a Fréchet space with the following system of seminorms,

$$\|f\|_{C^{p-1,1}(K)} = \|f\|_{C^{p-1}(K)} + \sup_{|\alpha|=p-1} \operatorname{Lip}_K(\partial^\alpha f), \quad \operatorname{Lip}_K(f) = \sup_{\substack{x,y\in K \\ x\neq y}} \frac{|f(x)-f(y)|}{\|x-y\|},$$

where K ranges over (a countable exhaustion of) the compact subsets of U; on $\mathbb{R}^m$ we consider the 2-norm $\| \; \| = \| \; \|_2$.

By Rademacher's theorem, the partial derivatives of order p of a function $f \in C^{p-1,1}(U)$ exist almost everywhere and coincide almost everywhere with the corresponding weak partial derivatives.

Theorem 2.1 readily implies the following multiparameter version.

Theorem 2.2. *Let $U \subseteq \mathbb{R}^m$ be open and let $P_a(x)$, $x \in U$, be a $C^{p-1,1}$-family of hyperbolic polynomials of degree n, where p is the maximal multiplicity of the roots of P_a. Then any continuous root of P_a is locally Lipschitz.*

Moreover, if $p = n$ for any pair of relatively compact subsets $U_0 \Subset U_1 \Subset I$ and for any continuous root $\lambda(t)$ its Lipschitz constant can be bounded as follows

$$\begin{aligned} \operatorname{Lip}_{U_0}(\lambda) &\le C(m,n,U_0,U_1)\,\big(\max_i \|a_i\|^{\frac{1}{i}}_{C^{n-1,1}(\overline{U}_1)}\big) \\ &\le \tilde{C}(m,n,U_0,U_1)\,\big(1+\max_i \|a_i\|_{C^{n-1,1}(\overline{U}_1)}\big), \end{aligned} \tag{2.2}$$

where the constants $C(m,n,U_0,U_1)$, $\tilde{C}(m,n,U_0,U_1)$ depend only on m, n, and the sets U_0, U_1.

Proof. Let λ be a continuous root of P_a. Without loss of generality we may assume that U_0 and U_1 are open boxes parallel to the coordinate axes, $U_i = \prod_{j=1}^m I_{i,j}$, $i = 0,1$, with $I_{0,j} \Subset I_{1,j}$ for all j. Let $x, y \in U_0$ and set $h := y - x$. Let $\{e_j\}_{j=1}^m$ denote the standard unit vectors in $\mathbb{R}^m$. For any z in the orthogonal projection of U_0 on the hyperplane $x_j = 0$ consider the function $\lambda_{z,j} : I_{0,j} \to \mathbb{R}$ defined by $\lambda_{z,j}(t) := \lambda(z + te_j)$. By Theorem 2.1, each $\lambda_{z,j}$ is Lipschitz and $C := \sup_{z,j} \operatorname{Lip}_{I_{0,j}}(\lambda_{z,j}) < \infty$. Thus

$$|\lambda(x) - \lambda(y)| \le \sum_{j=0}^{m-1} \Big|\lambda\big(x + \sum_{k=1}^{j} h_k e_k\big) - \lambda\big(x + \sum_{k=1}^{j+1} h_k e_k\big)\Big| \le C\|h\|_1 \le C\sqrt{m}\|h\|_2.$$

The bounds (2.2) follow from (2.1). □

Corollary 2.3. *Let $U \subseteq \mathbb{R}^m$ be open. The push forward $(\lambda^\uparrow)_* : C^{n-1,1}(U, \mathbb{R}^n) \supseteq C^{n-1,1}(U, H_n) \to C^{0,1}(U, \mathbb{R}^n)$ is continuous.*

Next we suppose that $P_a(t)$, $t \in I$, is a C^p-curve of hyperbolic polynomials of degree n, where p is the maximal multiplicity of the roots of P_a. Then the roots can be chosen C^1. We will give a new proof of this recent result of [7], see Theorem 2.4.

For a function $f(t)$ we denote by $f'^{-}(t_0)$ (resp. $f'^{+}(t_0)$) the left (resp. right) derivative of f at the point t_0.

Theorem 2.4. *Let $P_a(t)$, $t \in I$, be a C^p-curve of hyperbolic polynomials of degree n, where p is the maximal multiplicity of the roots of P_a. Then:*

(1) *Any continuous root $\lambda(t)$ of P_a has both one-sided derivatives at every $t \in I$.*
(2) *These derivatives are continuous: for every $t_0 \in I$ we have*

$$\lim_{t \to t_0^-} \lambda'^{\pm}(t) = \lambda'^{-}(t_0) \qquad \lim_{t \to t_0^+} \lambda'^{\pm}(t) = \lambda'^{+}(t_0).$$

(3) *There exists a differentiable system of the roots.*
(4) *Any differentiable root is C^1.*

3. Preliminaries

3.1. **Tschirnhausen transformation.** A monic polynomial

$$P_a(Z) = Z^n + \sum_{j=1}^{n} a_j Z^{n-j}, \quad a = (a_1, ..., a_n) \in \mathbb{R}^n_a$$

is said to be in *Tschirnhausen form* if $a_1 = 0$. Every P_a can be transformed to such a form by the substitution $Z \mapsto Z - \frac{a_1}{n}$, which we refer to as *Tschirnhausen transformation*,

$$P_{\tilde{a}}(Z) = P_a(Z - \frac{a_1}{n}) = Z^n + \sum_{j=2}^{n} \tilde{a}_j Z^{n-j}, \quad \tilde{a} = (\tilde{a}_2, ..., \tilde{a}_n) \in \mathbb{R}^{n-1}_{\tilde{a}}. \tag{3.1}$$

We identify the set of monic real polynomials P_a of degree n with $\mathbb{R}^n_a$, where $a = (a_1, a_2, \ldots, a_n)$, and those in Tschirnhausen form with $\mathbb{R}^{n-1}_{\tilde{a}}$. In what follows we write the effect of the Tschirnhausen transformation on a polynomial P_a simply by adding tilde, $P_{\tilde{a}}$.

Thus let $P_{\tilde{a}}$ be a monic polynomial in Tschirnhausen form. Then

$$s_2 = -2\tilde{a}_2 = \lambda_1^2 + \cdots + \lambda_n^2,$$

where the s_i denote the Newton polynomials in the roots λ_j of P_a. Thus, for a hyperbolic polynomial $P_{\tilde{a}}$ in Tschirnhausen form,

$$s_2 = -2\tilde{a}_2 \geq 0.$$

Lemma 3.1. *The coefficients of a hyperbolic polynomial $P_{\tilde{a}}$ in Tschirnhausen form satisfy*

$$|\tilde{a}_i|^{\frac{1}{i}} \leq |s_2|^{\frac{1}{2}} = \sqrt{2}\, |\tilde{a}_2|^{\frac{1}{2}}, \quad i = 2, \ldots, n.$$

Proof. Newton's identities give $|\tilde{a}_i| \leq \frac{1}{i} \sum_{j=2}^{i} |s_j| |\tilde{a}_{i-j}|$, where $\tilde{a}_0 = 1$, which together with

$$|s_i|^{\frac{1}{i}} \leq |s_2|^{\frac{1}{2}}, \quad i = 2, \ldots, n, \tag{3.2}$$

will imply the result by induction on i. To show (3.2) we note that it is equivalent to

$$(\lambda_1^i + \cdots + \lambda_n^i)^2 \leq (\lambda_1^2 + \cdots + \lambda_n^2)^i. \tag{3.3}$$

Each mixed term $\lambda_\ell^i \lambda_m^i$ on the left-hand side of (3.3) may be estimated by the sum of all $\lambda_\ell^a \lambda_m^b$ terms with $a, b > 0$ on the right-hand side of (3.3), in fact

$$2\lambda_\ell^i \lambda_m^i = 2\lambda_\ell^2 \lambda_m^2 \lambda_\ell^{i-2} \lambda_m^{i-2} \le \lambda_\ell^2 \lambda_m^2 (\lambda_\ell^{2(i-2)} + \lambda_m^{2(i-2)}) \le \sum_{j=1}^{i-1} \binom{i}{j} \lambda_\ell^{2j} \lambda_m^{2(i-j)}.$$

This implies the statement. □

3.2. **Splitting.** The following well-known lemma (see e.g. [1] or [2]) is an easy consequence of the inverse function theorem.

Lemma 3.2. *Let $P_a = P_b P_c$, where P_b and P_c are monic complex polynomials without common root. Then for P near P_a we have $P = P_{b(P)} P_{c(P)}$ for analytic mappings $\mathbb{R}^n_a \ni P \mapsto b(P) \in \mathbb{R}_b^{\deg P_b}$ and $\mathbb{R}^n_a \ni P \mapsto c(P) \in \mathbb{R}_c^{\deg P_c}$, defined for P near P_a, with the given initial values.*

Proof. The product $P_a = P_b P_c$ defines on the coefficients a polynomial mapping φ such that $a = \varphi(b, c)$, where $a = (a_i)$, $b = (b_i)$, and $c = (c_i)$. The Jacobian determinant $\det d\varphi(b, c)$ equals the resultant of P_b and P_c which is nonzero by assumption. Thus φ can be inverted locally. □

If $P_{\tilde a}$ is in Tschirnhausen form and $\tilde a \ne 0$ then, the sum of its roots being equal to zero, it always splits. The space of hyperbolic polynomials of degree n in Tschirnhausen form can be identified with a closed semialgebraic subset H_n of $\mathbb{R}^{n-1}_{\tilde a}$. By Lemma 3.1, the set $H_n^0 := H_n \cap \{\tilde a_2 = -1\}$ is compact.

Let $p \in H_n \cap \{\tilde a_2 \ne 0\}$. Then the polynomial

$$Q_{\underline{a}}(Z) := |\tilde a_2|^{-\frac{n}{2}} P_{\tilde a}(|\tilde a_2|^{\frac12} Z) = Z^n - Z^{n-2} + |\tilde a_2|^{-\frac32} \tilde a_3 Z^{n-3} + \cdots + |\tilde a_2|^{-\frac{n}{2}} \tilde a_n$$

is hyperbolic and, by Lemma 3.2, it *splits*, i.e., $Q_{\underline{a}} = Q_{\underline{b}} Q_{\underline{c}}$ and $\deg Q_{\underline{b}}, \deg Q_{\underline{c}} < n$, on some open ball $B_p(r)$ centered at p. Thus, there exist real analytic functions ψ_i so that, on $B_p(r)$,

$$\underline{b}_i = \psi_i\big(|\tilde a_2|^{-\frac32} \tilde a_3, \ldots, |\tilde a_2|^{-\frac{n}{2}} \tilde a_n\big), \quad i = 1, \ldots, \deg P_b;$$

likewise for $\underline{c}_j$. The splitting $Q_{\underline{a}} = Q_{\underline{b}} Q_{\underline{c}}$ induces a splitting $P_{\tilde a} = P_b P_c$, where

$$b_i = |\tilde a_2|^{\frac{i}{2}} \psi_i\big(|\tilde a_2|^{-\frac32} \tilde a_3, \ldots, |\tilde a_2|^{-\frac{n}{2}} \tilde a_n\big), \quad i = 1, \ldots, \deg P_b; \tag{3.4}$$

likewise for c_j. Shrinking r slightly, we may assume that all partial derivatives of ψ_i are separately bounded on $B_p(r)$. We denote by $\tilde b_j$ the coefficients of the polynomial $P_{\tilde b}$ resulting from P_b by the Tschirnhausen transformation.

Lemma 3.3. *In this situation we have $|\tilde b_2| \le 2n |\tilde a_2|$.*

Proof. Let $(\lambda_j)_{j=1}^k$ denote the roots of P_b and $(\lambda_j)_{j=1}^n$ those of P_a. Then, as $|b_1| \le \sum_{j=1}^k |\lambda_j| \le (k \sum_{j=1}^k \lambda_j^2)^{1/2}$ and thus $|\lambda_j||b_1| \le k \sum_{j=1}^k \lambda_j^2$,

$$2|\tilde{b}_2| = \sum_{j=1}^k \Big(\lambda_j + \frac{b_1}{k}\Big)^2 \le \frac{1}{k^2} \sum_{j=1}^k (k^2\lambda_j^2 + b_1^2 + 2k|\lambda_j||b_1|)$$
$$\le \frac{1}{k^2} \sum_{j=1}^k \big(k^2\lambda_j^2 + k \sum_{\ell=1}^k \lambda_\ell^2 + 2k^2 \sum_{\ell=1}^k \lambda_\ell^2\big) = 2(k+1) \sum_{j=1}^k \lambda_j^2 \le 2n \sum_{j=1}^n \lambda_j^2 = 4n|\tilde{a}_2|,$$

as required. □

3.3. **Coefficient estimates.** We shall need the following estimates. (Here it is convenient to number the coefficients in reversed order.)

Lemma 3.4. *Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{C}[x]$ satisfy $|P(x)| \le A$ for $x \in [0, B] \subseteq \mathbb{R}$. Then*

$$|a_j| \le (2n)^{n+1} A B^{-j}, \quad j = 0, \ldots, n.$$

Proof. We show the lemma for $A = B = 1$. The general statement follows by applying this special case to the polynomial $A^{-1}P(By)$, $y = B^{-1}x$. Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be equidistant points. By Lagrange's interpolation formula (e.g. [14, (1.2.5)]),

$$P(x) = \sum_{k=0}^n P(x_k) \prod_{\substack{j=0 \\ j \ne k}}^n \frac{x - x_j}{x_k - x_j},$$

and therefore

$$a_j = \sum_{k=0}^n P(x_k) \prod_{\substack{j=0 \\ j \ne k}}^n (x_k - x_j)^{-1} (-1)^{n-j} \sigma_{n-j}^k,$$

where σ_j^k is the jth elementary symmetric polynomial in $(x_\ell)_{\ell \ne k}$. The statement follows. □

A better constant can be obtained using Chebyshev polynomials; cf. [14, Thm. 16.3.1-2].

3.4. **Consequences of Taylor's theorem.** The following two lemmas are classical. We include them for the reader's convenience.

Lemma 3.5. *Let $I \subseteq \mathbb{R}$ be an open interval and let $f \in C^{1,1}(\overline{I})$ be nonnegative or nonpositive. For any $t_0 \in I$ and $M > 0$ such that $I_{t_0}(M^{-1}) := \{t : |t - t_0| < M^{-1}|f(t_0)|^{\frac{1}{2}}\} \subseteq I$ and $M \ge (\mathrm{Lip}_{I_{t_0}(M^{-1})}(f'))^{\frac{1}{2}}$ we have*

$$|f'(t_0)| \le \big(M + M^{-1} \mathrm{Lip}_{I_{t_0}(M^{-1})}(f')\big)|f(t_0)|^{\frac{1}{2}} \le 2M|f(t_0)|^{\frac{1}{2}}.$$

Proof. Suppose that f is nonnegative; otherwise consider $-f$. It follows that the inequality holds true at the zeros of f. Let us assume that $f(t_0) > 0$. The statement follows from

$$0 \le f(t_0 + h) = f(t_0) + f'(t_0)h + \int_0^1 (1 - s) f''(t_0 + hs)\, ds\, h^2$$

with $h = \pm M^{-1}|f(t_0)|^{\frac{1}{2}}$. □

Lemma 3.6. *Let $f \in C^{m-1,1}(\overline{I})$. There is a universal constant $C(m)$ such that for all $t \in I$ and $k = 1, \ldots, m$,*

$$|f^{(k)}(t)| \le C(m)|I|^{-k}\big(\|f\|_{L^\infty(I)} + \mathrm{Lip}_I(f^{(m-1)})|I|^m\big). \tag{3.5}$$

Proof. We may suppose that $I = (-\delta, \delta)$. If $t \in I$ then at least one of the two intervals $[t, t \pm \delta)$, say $[t, t + \delta)$, is included in I. By Taylor's formula, for $t_1 \in [t, t + \delta)$,

$$\Big|\sum_{k=0}^{m-1} \frac{f^{(k)}(t)}{k!}(t_1 - t)^k\Big| \le |f(t_1)| + \Big|\int_0^1 \frac{(1-s)^{m-1}}{(m-1)!} f^{(m)}(t + s(t_1 - t))\, ds\, (t_1 - t)^m\Big|$$
$$\le \|f\|_{L^\infty(I)} + \mathrm{Lip}_I(f^{(m-1)})\delta^m,$$

and for $k \le m - 1$ we may conclude (3.5) by Lemma 3.4. For $k = m$, (3.5) is trivially satisfied. □

4. Proof of Theorem 2.1

4.1. **First reductions.** We assume that the maximal multiplicity p of the roots equals the degree n of P_a. If $p < n$ then we may use Lemma 3.2 to split P_a locally in factors that have this property. We discuss it in more detail at the end of the proof.

So let $P_a(t)$, $t \in I$, be a $C^{n-1,1}$-curve of hyperbolic polynomials of degree n. Without loss of generality we may assume that $n \ge 2$ and that $P_a = P_{\tilde a}$ is in Tschirnhausen form. Let $(\lambda_j(t))_{j=1}^n$, $t \in I$, be any continuous system of the roots of $P_{\tilde a}$. Then

$$\tilde a_2(t) = 0 \quad \Longleftrightarrow \quad \lambda_1(t) = \cdots = \lambda_n(t) = 0.$$

We shall show that, for any relatively compact open subinterval $I_0 \Subset I$ and any $t_0 \in I_0 \setminus \tilde a_2^{-1}(0)$, there exists a neighborhood I_{t_0} of t_0 in $I_0 \setminus \tilde a_2^{-1}(0)$ so that each λ_j is Lipschitz on I_{t_0} and the Lipschitz constant $\mathrm{Lip}_{I_{t_0}}(\lambda_j)$ satisfies

$$\mathrm{Lip}_{I_{t_0}}(\lambda_j) \le C(n, I_0, I_1)\,\big(\max_i \|\tilde a_i\|^{\frac{1}{i}}_{C^{n-1,1}(\overline{I}_1)}\big),$$

where I_1 is any open interval satisfying $I_0 \Subset I_1 \Subset I$. Here, recall, $C(n, I_0, I_1)$ stands for a universal constant depending only on n, I_0, and I_1.

This will imply Theorem 2.1 by the following lemma.

Lemma 4.1. *Let $I \subseteq \mathbb{R}$ be an open interval. If $f \in C^0(I)$ and each $t_0 \in I \setminus f^{-1}(0)$ has a neighborhood $I_{t_0} \subseteq I \setminus f^{-1}(0)$ so that $L := \sup_{t_0 \in I \setminus f^{-1}(0)} \mathrm{Lip}_{I_{t_0}}(f) < \infty$, then f is Lipschitz on I and $\mathrm{Lip}_I(f) = L$.*

Proof. Let $t, s \in I$. It is easy to see that $|f(t) - f(s)| \le L|t - s|$ if t and s belong to the same connected component J of $I \setminus f^{-1}(0)$. By continuity, this estimate also holds on the closed interval $\overline{J}$. If $t \in \overline{J}_1$ and $s \in \overline{J}_2$, $t < s$, and $\overline{J}_1 \cap \overline{J}_2 = \emptyset$, let r_i be the endpoint of $\overline{J}_i$ so that $s \le r_1 < r_2 \le t$. Then

$$|f(t) - f(s))| \le |f(t) - f(r_2)| + |f(r_1) - f(s)| \le L|t - s|.$$

Clearly, $\mathrm{Lip}_I(f) = L$. □

4.2. **Convenient assumption.** The proof of the statement in Subsection 4.1 will be carried out by induction on the degree of P_a. We replace the assumption of Theorem 2.1 by a new assumption that will be more convenient for the inductive step. Before we state it we need a bit of notation.

For open intervals I_0 and I_1 so that $I_0 \Subset I_1 \Subset I$, we set

$$I_i' := I_i \setminus \tilde a_2^{-1}(0), \quad i = 0, 1.$$

For $t_0 \in I_0'$ and $r > 0$ consider the interval

$$I_{t_0}(r) := \big(t_0 - r|\tilde a_2(t_0)|^{\frac12}, t_0 + r|\tilde a_2(t_0)|^{\frac12}\big).$$

Assumption. Let $I_0 \Subset I_1$ be open intervals. Suppose that $(\tilde a_i)_{i=2}^n \in C^{n-1,1}(\overline I_1, \mathbb{R}^{n-1})$ are the coefficients of a hyperbolic polynomial $P_{\tilde a}$ of degree n in Tschirnhausen form. Assume that there is a constant $A > 0$, so that for all $t_0 \in I_0'$, $t \in I_{t_0}(A^{-1})$, $i = 2, \ldots, n$, $k = 0, \ldots, n$,

$$I_{t_0}(A^{-1}) \subseteq I_1, \tag{A.1}$$

$$2^{-1} \le \frac{\tilde a_2(t)}{\tilde a_2(t_0)} \le 2, \tag{A.2}$$

$$|\tilde a_i^{(k)}(t)| \le C(n)\, A^k\, |\tilde a_2(t)|^{\frac{i-k}{2}}, \tag{A.3}$$

where $C(n)$ is a universal constant. For $k = n$, (A.3) is understood to hold almost everywhere, by Rademacher's theorem.

Condition (A.3) implies that

$$\big|\partial_t^k\big(|\tilde a_2|^{-\frac{i}{2}}\tilde a_i\big)(t)\big| \le C(n)\, A^k\, |\tilde a_2(t)|^{-\frac{k}{2}}. \tag{A.4}$$

More generally, if we assign $\tilde a_i$ the weight i and $|\tilde a_2|^{\frac12}$ the weight 1 and let $L(x_2, \ldots, x_n, y) \in \mathbb{R}[x_2, \ldots, x_n, y, y^{-1}]$ be weighted homogeneous of degree d, then

$$\big|\partial_t^k L\big(\tilde a_2, \ldots, \tilde a_n, |\tilde a_2|^{\frac12}\big)(t)\big| \le C(n, L)\, A^k\, |\tilde a_2(t)|^{\frac{d-k}{2}}.$$

4.3. **Inductive step.** Let $P_{\tilde a}$, I_0, I_1, A, t_0 be as in Assumption. We will show by induction on $\deg P_{\tilde a}$ that any continuous system of the roots of $P_{\tilde a}$ is Lipschitz on I_0 with Lipschitz constant bounded from above by $C(n)\, A$. First we establish the following.

- For some constant $C_1(n) > 1$, the polynomial $P_{\tilde a}(t)$ splits on the interval $I_{t_0}(C_1(n)^{-1}A^{-1})$, that is we have $P_{\tilde a}(t) = P_b(t)P_c(t)$, where P_b and P_c are $C^{n-1,1}$-curves of hyperbolic polynomials of degree strictly smaller than n.
- After applying the Tschirnhausen transformation $P_b \rightsquigarrow P_{\tilde b}$, the coefficients $(\tilde b_i)_{i=2}^{\deg P_b}$ satisfy (A.1)–(A.3) for suitable neighborhoods J_0, J_1 of t_0, and a constant $B = C(n)\, A$ in place of A.

We restrict our curve of hyperbolic polynomials $P_{\tilde a}$ to $I_{t_0}(A^{-1})$ and consider

$$\underline{a} := \big(-1, |\tilde a_2|^{-\frac32}\tilde a_3, \ldots, |\tilde a_2|^{-\frac{n}{2}}\tilde a_n\big) : I_{t_0}(A^{-1}) \to \mathbb{R}^{n-1}_{\underline{a}}.$$

Then $\underline{a}$ is continuous, by (A.2), and bounded, by Lemma 3.1. Moreover, by (A.4) and (A.2), there is a universal constant $C_1(n)$ so that, for $t \in I_{t_0}(A^{-1})$,

$$\|\underline{a}'(t)\| \le C_1(n)\, A\, |\tilde a_2(t_0)|^{-\frac12}. \tag{4.1}$$

According to Subsection 3.2, choose a finite cover of H_n^0 by open balls $B_{p_\alpha}(r)$, $\alpha \in \Delta$, on which we have a splitting $P_{\tilde a} = P_b P_c$ with coefficients of P_b given by (3.4). There exists $r_1 > 0$ such that for any $p \in H_n^0$ there is $\alpha \in \Delta$ so that $B_p(r_1) \subseteq B_{p_\alpha}(r)$; $2r_1$ is a Lebesgue number of the covering $\{B_{p_\alpha}(r)\}_{\alpha\in\Delta}$. Then, if $C_1(n)$ is the constant from (4.1),

$$J_1 := I_{t_0}(r_1 C_1(n)^{-1} A^{-1}) \subseteq \underline{a}^{-1}(B_{\underline{a}(t_0)}(r_1)), \tag{4.2}$$

and on J_1 we have a splitting $P_{\tilde a}(t) = P_b(t) P_c(t)$ with b_i given by (3.4). Fix $r_0 < r_1$ and let

$$J_0 := I_{t_0}(r_0 C_1(n)^{-1} A^{-1}). \tag{4.3}$$

(Here we assume without loss of generality that $r_1 \le C_1(n)$.)

Let us show that the coefficients $(\tilde b_i)_{i=2}^{\deg P_b}$ of $P_{\tilde b}$ satisfy (A.1)–(A.3) for the intervals J_1 and J_0 from (4.2) and (4.3). To this end we set

$$J_i' := J_i \setminus \tilde b_2^{-1}(0), \quad i = 0, 1,$$

consider, for $t_1 \in J_0'$ and $r > 0$,

$$J_{t_1}(r) := \big(t_1 - r|\tilde b_2(t_1)|^{\frac12}, t_1 + r|\tilde b_2(t_1)|^{\frac12}\big),$$

and prove the following lemma.

Lemma 4.2. *There exists a constant $\tilde C = \tilde C(n, r_0, r_1) > 1$ such that for $B = \tilde C A$ and for all $t_1 \in J_0'$, $t \in J_{t_1}(B^{-1})$, $i = 2, \ldots, \deg P_b$, $k = 0, \ldots, n$,*

$$J_{t_1}(B^{-1}) \subseteq J_1, \tag{B.1}$$

$$2^{-1} \le \frac{\tilde b_2(t)}{\tilde b_2(t_1)} \le 2, \tag{B.2}$$

$$|\tilde b_i^{(k)}(t)| \le C(n)\, B^k\, |\tilde b_2(t)|^{\frac{i-k}{2}}. \tag{B.3}$$

for some universal constant $C(n)$.

Proof. If

$$B \ge (r_1 - r_0)^{-1}\, 2\sqrt{n}\, C_1(n)\, A,$$

then by Lemma 3.3 and (A.2),

$$B^{-1}|\tilde b_2(t_1)|^{\frac12} \le (r_1 - r_0)\, C_1(n)^{-1}\, A^{-1}\, |\tilde a_2(t_0)|^{\frac12},$$

and hence (B.1) follows from (4.2) and (4.3), since $t_1 \in J_0$.

Next we claim that, on J_1,

$$\big|\partial_t^k \psi_i\big(|\tilde a_2|^{-\frac32}\tilde a_3, \ldots, |\tilde a_2|^{-\frac n2}\tilde a_n\big)\big| \le C(n)\, A^k\, |\tilde a_2|^{-\frac k2}. \tag{4.4}$$

To see this we differentiate the following equation $(k-1)$ times, apply induction on k, and use (A.4),

$$\partial_t \psi_i\big(|\tilde a_2|^{-\frac32}\tilde a_3, \ldots, |\tilde a_2|^{-\frac n2}\tilde a_n\big) = \sum_{j=3}^n (\partial_{j-2}\psi_i)(\underline{a})\partial_t\big(|\tilde a_2|^{-\frac j2}\tilde a_j\big); \tag{4.5}$$

recall that all partial derivatives of the ψ_i's are separately bounded on $\underline{a}(J_1)$ and these bounds are universal. From (3.4) and (4.4) we obtain, on J_1 and for all $i = 1, \ldots, \deg P_b$, $k = 0, \ldots, n$,

$$|b_i^{(k)}| \le C(n)\, A^k\, |\tilde a_2|^{\frac{i-k}{2}}, \tag{4.6}$$

thus, as the Tschirnhausen transformation preserves the weights of the coefficients, cf. (A.4),

$$|\tilde b_i^{(k)}| \le C(n)\, A^k\, |\tilde a_2|^{\frac{i-k}{2}},$$

and so, by Lemma 3.3,

$$|\tilde b_i^{(k)}| \le C(n)\, A^k\, |\tilde b_2|^{\frac{i-k}{2}} \qquad \text{if } i-k \le 0.$$

This shows (B.3) for $i \le k$. (B.3) for $k = 0$ follows from Lemma 3.1. (B.2) and the remaining inequalities of (B.3), i.e., for $0 < k < i$, follow now from Lemma 4.3 below. □

Lemma 4.3. *There exists a constant $C(n) \ge 1$ such that the following holds. If* (A.1) *and* (A.3) *for $k = 0$ and $k = i$, $i = 2, ..., n$, are satisfied, then so are* (A.2) *and* (A.3) *for $k < i$, $i = 2, ..., n$, after replacing A by $C(n)A$.*

Proof. By assumption, $\mathrm{Lip}_{I_{t_0}(A^{-1})}(\tilde a_2')) \le C(n)A^2$. Thus, by Lemma 3.5 for $f = \tilde a_2$ and $M = C(n)^{\frac12} A$, we get

$$|\tilde a_2'(t_0)| \le 2M|\tilde a_2(t_0)|^{\frac12}.$$

It follows that, for $t \in I_{t_0}((6M)^{-1})$,

$$\frac{|\tilde a_2(t) - \tilde a_2(t_0)|}{|\tilde a_2(t_0)|} \le \frac{|\tilde a_2'(t_0)|}{|\tilde a_2(t_0)|}|t-t_0| + \int_0^1 (1-s)|\tilde a_2''(t_0 + s(t-t_0))|ds\, \frac{|t-t_0|^2}{|\tilde a_2(t_0)|} \le \frac12 \tag{4.7}$$

That implies (A.2). The other inequalities follow from Lemma 3.6. □

4.4. **End of inductive step.** In J_1, any continuous root λ_j of $P_{\tilde a}$, where $P_{\tilde a}$ is in Tschirnhausen form, is a root of either P_b or P_c. Say it is a root of P_b. Then it has the form

$$\lambda_j(t) = -\frac{b_1(t)}{\deg P_b} + \mu_j(t), \tag{4.8}$$

where μ_j is a continuous root of $P_{\tilde b}$ defined on a neighborhood of t_0. By the inductive assumption we may assume that μ_j is Lipschitz with Lipschitz constant bounded from above by $C(n)B$. Hence λ_j is Lipschitz with Lipschitz constant bounded from above by $C(n)A$ (the constant $C(n)$ changes), as $B = \tilde C\, A$ and by (4.6) for $i = k = 1$. This ends the inductive step.

4.5. $P_{\tilde a}$ **satisfies Assumption.** Now we show that $P_{\tilde a}$ always satisfies Assumption. The choice of A will provide the upper bound on the Lipschitz constant of the roots.

Proposition 4.4. *Let $P_{\tilde a}(t)$, $t \in I$, be a $C^{n-1,1}$-curve of hyperbolic polynomials of degree n in Tschirnhausen form, and let I_0 and I_1 be open intervals satisfying $I_0 \Subset I_1 \Subset I$. Then its coefficients $(\tilde a_i)_{i=2}^n$ satisfy* (A.1)–(A.3).

Proof. Let δ denote the distance between the endpoints of I_0 and those of I_1. Set

$$A_1 := \max\big\{\delta^{-1}\|\tilde a_2\|^{\frac12}_{L^\infty(I_1)}, (\mathrm{Lip}_{I_1}(\tilde a_2'))^{\frac12}\big\}, \quad A_2 := \max_i \big\{M_i\|\tilde a_2\|^{\frac{n-i}{2}}_{L^\infty(I_1)}\big\}^{\frac1n}, \tag{4.9}$$

where $M_i = \mathrm{Lip}_{I_1}(\tilde a_i^{(n-1)})$. Then we may choose

$$A \geq A_0 = 6\max\{A_1, A_2\}. \tag{4.10}$$

For (A.1)–(A.2) to be satisfied we need only $A \geq 6A_1$. Indeed, clearly, for $t_0 \in I_0'$,

$$I_{t_0}(A_1^{-1}) \subseteq I_1. \tag{4.11}$$

Then Lemma 3.5 implies that

$$|\tilde a_2'(t_0)| \leq 2A_1\,|\tilde a_2(t_0)|^{\frac12}.$$

It follows that, for $t_0 \in I_0'$ and $t \in I_{t_0}((6A_1)^{-1})$, we have (4.7) and hence (A.2). If $t \in I_{t_0}(A^{-1})$ then Lemma 3.6, Lemma 3.1, and (A.2) imply (A.3). This ends the proof of Proposition 4.4. □

4.6. **Bounds for $p = n$.** Let $\lambda(t) \in C^0(I)$ be a root of $P_{\tilde a}$ that is the Tschirnhausen form and let $I_0 \Subset I_1 \Subset I$. By the inductive step 4.3, Proposition 4.4, and Lemma 4.1 we have the following bounds

$$\begin{aligned} \mathrm{Lip}_{I_0}(\lambda) &\leq C(n)\max\Big\{\delta^{-1}\|\tilde a_2\|^{\frac12}_{L^\infty(I_1)}, (\mathrm{Lip}_{I_1}(\tilde a_2'))^{\frac12}, \max_i\big\{M_i\|\tilde a_2\|^{\frac{n-i}{2}}_{L^\infty(I_1)}\big\}^{\frac1n}\Big\} \\ &\leq C(n, I_0, I_1)\,\big(\max_i\|\tilde a_i\|^{\frac1i}_{C^{n-1,1}(\bar I_1)}\big) \\ &\leq C(n, I_0, I_1)\,\big(1 + \max_i\|\tilde a_i\|_{C^{n-1,1}(\bar I_1)}\big), \end{aligned} \tag{4.12}$$

where δ is the distance between the endpoints of I_0 and those of I_1, and $M_i = \mathrm{Lip}_{I_1}(\tilde a_i^{(n-1)})$. Then the bounds stated in Theorem 2.1 follow from

$$\max_i\|\tilde a_i\|^{\frac1i}_{C^{n-1,1}(\bar I_1)} \leq C(n)\,\big(\max_i\|a_i\|^{\frac1i}_{C^{n-1,1}(\bar I_1)}\big) \leq C(n)\,\big(1 + \max_i\|a_i\|_{C^{n-1,1}(\bar I_1)}\big).$$

The first inequality follows from the (weighted) homogeneity of the formulas for $\tilde a_i$ in terms of $(a_1, \ldots, a_n)$. (The opposite inequality does not hold in general. Adding a constant to all the roots of P_a does not change the associated Tschirnhausen form $P_{\tilde a}$ but changes the norm of the coefficients of P_a.)

4.7. **The case $2 \leq p < n$.** To show that the roots are Lipschitz it suffices, using Lemma 3.2, to split $P_{\tilde a}$ locally in factors of degree smaller than or equal to p and apply the case $n = p$.

In order to have a uniform bound we need to know that the multiplicities of roots are at most p "uniformly". For this we order the roots of $P_{\tilde a}$ increasingly, $\lambda_1(t) \leq \lambda_2(t) \leq \cdots \leq \lambda_n(t)$, and consider

$$\alpha(t) := \frac{|\lambda_n(t) - \lambda_1(t)|}{\min_{i=1,\ldots,n-p}|\lambda_{i+p}(t) - \lambda_i(t)|}, \quad \alpha_I := \sup_{t\in I}\alpha(t).$$

We note that the numerator $|\lambda_n(t)-\lambda_1(t)|$ is of the same size as $|\tilde a_2(t)|^{\frac12}$, for $P_{\tilde a}$ in Tschirnhausen form, since then $\lambda_1(t)$ and $\lambda_n(t)$ have opposite signs and

$$n|\lambda_n(t)-\lambda_1(t)| \ge s_2(t)^{\frac12} = \sqrt{2}|\tilde a_2(t)|^{\frac12} \ge \frac12 |\lambda_n(t)-\lambda_1(t)|.$$

There are the following changes in the way we proceed. First in the proof of Proposition 4.4 we have to modify the formula for A_2 as follows

$$A_2 := \max\Big\{\max_{i\le p}\big\{M_i\|\tilde a_2\|_{L^\infty(I_1)}^{\frac{p-i}{2}}\big\}^{\frac1p}, \max_{i>p}\big\{M_i\, m_2^{\frac{p-i}{2}}\big\}^{\frac1p}\Big\}, \tag{4.13}$$

where $M_i = \operatorname{Lip}_{I_1}(\tilde a_i^{(p-1)})$ and $m_2 = \min_{t\in \bar I_0}|\tilde a_2(t)|$.

In (A.3) of Assumption we may consider only the derivatives of order $k\le p$. Therefore the argument of the inductive step (proof of Lemma 4.2) changes as follows. The first part of the proof of Lemma 4.2 does not change. Then we need (B.3) for $i=k$ in order to apply Lemma 4.3. This is not available if $i>p$, which happens if $\deg P_b > p$, and then we have only

$$|\tilde b_i^{(p)}| \le C(n)\, A^p\, |\tilde a_2|^{\frac{i-p}{2}} \le C(n)\, A_b^p\, |\tilde b_2|^{\frac{i-p}{2}},$$

where we may take $A_b = C(n)|\tilde a_2(t_1)/\tilde b_2(t_1)|^{\frac{n-p}{2p}} A$. Then, by Lemma 3.6, we conclude Lemma 4.2 with A replaced by A_b. This modification is no longer necessary when $\deg P_b \le p$. Thus during the induction process, say, $P_{\tilde a}\to P_{\tilde b}\to\cdots\to P_{\tilde d}\to P_{\tilde e}$ with $\deg P_e\le p$, for the intervals $I_{t_0}(A^{-1})\supset I_{t_1}(A_b^{-1})\supset\cdots\supset I_{t_s}(A_d^{-1})$, the constant A is replaced by

$$\tilde A = C(n)\alpha(t_s)^{\frac{n-p}{p}} A \ge C(n)\Big(\Big|\frac{\tilde a_2(t_s)}{\tilde b_2(t_s)}\Big|\cdot\Big|\frac{\tilde b_2(t_s)}{\tilde c_2(t_s)}\Big|\cdots\Big|\frac{\tilde c_2(t_s)}{\tilde d_2(t_s)}\Big|\Big)^{\frac{n-p}{2p}} A.$$

Finally this gives the following bounds on the Lipschitz constant of each of the roots

$$\begin{aligned}
&\operatorname{Lip}_{I_0}(\lambda) \\
&\le C(n)\,\alpha_{I_1}^{\frac{n-p}{p}}\max\Big\{\delta^{-1}\|\tilde a_2\|_{L^\infty(I_1)}^{\frac12}, (\operatorname{Lip}_{I_1}(\tilde a_2'))^{\frac12}, \max_{i\le p}\big\{M_i\|\tilde a_2\|_{L^\infty(I_1)}^{\frac{p-i}{2}}\big\}^{\frac1p}, \max_{i>p}\big\{M_i\, m_2^{\frac{p-i}{2}}\big\}^{\frac1p}\Big\} \\
&\le C(n,I_0,I_1)\,\alpha_{I_1}^{\frac{n-p}{p}}\big(1+m_2^{\frac{p-n}{2p}}\big)\big(1+\max_i\|\tilde a_i\|_{C^{p-1,1}(\bar I_1)}\big).
\end{aligned} \tag{4.14}$$

This completes the proof of Theorem 2.1. □

5. Proof of Theorem 2.4

5.1. **${}^pC^m$-functions.** In the proof of Theorem 2.4 we shall need a result for functions defined near $0\in\mathbb{R}$ that become C^m when multiplied with the monomial t^p.

Definition 5.1. Let $p,m\in\mathbb{N}$ with $p\le m$. A continuous complex valued function f defined near $0\in\mathbb{R}$ is called a *${}^pC^m$-function* if $t\mapsto t^pf(t)$ belongs to C^m.

Let $I\subseteq\mathbb{R}$ be an open interval containing 0. Then $f: I\to\mathbb{C}$ is ${}^pC^m$ if and only if it has the following properties, cf. [21, 4.1], [18, Satz 3], or [19, Thm 4]:

- $f\in C^{m-p}(I)$,
- $f|_{I\setminus\{0\}}\in C^m(I\setminus\{0\})$,

- $\lim_{t\to 0} t^k f^{(m-p+k)}(t)$ exists as a finite number for all $0 \le k \le p$.

Proposition 5.2. *If $g = (g_1, \ldots, g_n)$ is ${}^pC^m$ and F is C^m near $g(0) \in \mathbb{C}^n$, then $F \circ g$ is ${}^pC^m$.*

Proof. Cf. [19, Thm 9] or [17, Prop 3.2]. Clearly g and $F \circ g$ are C^{m-p} near 0 and C^m off 0. By Faà di Bruno's formula [8], for $1 \le k \le p$ and $t \neq 0$,

$$\frac{t^k (F \circ g)^{(m-p+k)}(t)}{(m-p+k)!} = \sum_{\ell \ge 1} \sum_{\alpha \in A} \frac{t^{k-|\beta|}}{\ell!} d^\ell F(g(t)) \Big(\frac{t^{\beta_1} g^{(\alpha_1)}(t)}{\alpha_1!}, \ldots, \frac{t^{\beta_\ell} g^{(\alpha_\ell)}(t)}{\alpha_\ell!} \Big)$$

$$A := \{\alpha \in \mathbb{N}^\ell_{>0} : \alpha_1 + \cdots + \alpha_\ell = m - p + k\}$$

$$\beta_i := \max\{\alpha_i - m + p, 0\}, \quad |\beta| = \beta_1 + \cdots + \beta_\ell \le k,$$

whose limit as $t \to 0$ exists as a finite number by assumption. □

Let us prove Theorem 2.4. We suppose that P_a is in the Tschirnhausen form $P_a = P_{\tilde{a}}$. It suffices to consider the case $n = p$. We show that every $t_0 \in I$ has a neighborhood in I on which (1) and (2) (of Theorem 2.4) hold. If $\tilde{a}_2(t_0) \neq 0$ then $P_{\tilde{a}}$ splits on a neighborhood of t_0 and we may proceed by induction on $\deg P_a$. If $\tilde{a}_2(t_0) = 0$ then $\tilde{a}_2'(t_0) = 0$ and we distinguish two cases

- *Case (i):* $\tilde{a}_2(t_0) = \tilde{a}_2'(t_0) = \tilde{a}_2''(t_0) = 0$.
- *Case (ii):* $\tilde{a}_2(t_0) = \tilde{a}_2'(t_0) = 0$ and $\tilde{a}_2''(t_0) \neq 0$.

To simplify the notation we suppose $t_0 = 0$. Fix a continuous root $\lambda(t)$ defined in a neighborhood of 0.

5.2. **Proof of (1).** In Case (i), $\lambda(t) = o(t)$ and hence λ is differentiable at 0 and $\lambda'(0) = 0$. In Case (ii), $\tilde{a}_2(t) \sim t^2$ and hence $\tilde{a}_i(t) = O(t^i)$. Therefore,

$$\underline{a}(t) := \big(t^{-2}\tilde{a}_2(t), t^{-3}\tilde{a}_3(t), \ldots, t^{-n}\tilde{a}_n(t)\big) : I_1 \to \mathbb{R}^{n-1}_{\underline{a}}$$

defined on a neighborhood I_1 of 0 is continuous. By Lemma 3.2, $P_{\underline{a}}$ splits. The splitting $P_{\underline{a}} = P_{\underline{b}} P_{\underline{c}}$ induces a splitting $P_{\tilde{a}} = P_b P_c$, where the b_i are given by

$$b_i = t^i \psi_i\big(t^{-2}\tilde{a}_2, \ldots, t^{-n}\tilde{a}_n\big), \quad i = 1, \ldots, \deg P_b; \tag{5.1}$$

and similar formulas hold for $\tilde{b}_i$. Then b_i and $\tilde{b}_i$ are of class C^i at 0, by Proposition 5.2, and of class C^n in the complement of 0. Moreover we may choose the splitting such that $\lambda(t)$ for $t \ge 0$ is a root of P_b, and all the roots of $P_{\underline{b}(0)}$ are equal. The latter gives

$$\tilde{b}_2(0) = \tilde{b}_2'(0) = \tilde{b}_2''(0) = 0.$$

Thus, $\lambda(t)$ can be expressed as in (4.8) with b_1 of class C^1 and μ_j differentiable at 0 ($\mu_j'(0) = 0$). This finishes the proof of (1).

5.3. **Proof of (2).** This is the heart of the proof. In Case (i) the continuity of the one-sided derivatives at 0 follows from (4.12) and the following lemma.

Lemma 5.3. *Suppose Case (i) holds. Then for any $\varepsilon > 0$ there is $\delta > 0$ such that for $I_0 = (-\delta, \delta)$ and $I_1 = (-2\delta, 2\delta)$ and A_0 defined by (4.10) we have $A_0 \le \varepsilon$.*

Proof. This follows immediately from the formula (4.10). □

To show the continuity in Case (ii) we need a similar result for $P_{\tilde b}$.

Lemma 5.4. *Suppose Case (ii) holds. Then, under the assumptions of Proof of (1), for any $\varepsilon > 0$ there is a neighborhood I_ε of 0 in I such that for every $t_0 \in I_\varepsilon \setminus \{0\}$ the conditions* (A.1)–(A.3) *are satisfied for $P_{\tilde b}$ with $A \le \varepsilon$.*

Proof. Since $P_{\tilde b}$ is not necessarily of class $C^{\deg P_b}$ we cannot use directly Lemma 5.3 and the induction on $\deg P_a$. But the proof is similar and we sketch it below.

Let $I_1 = I_\delta = (-\delta, \delta)$ and $I_0 = (-\frac{\delta}{2}, \frac{\delta}{2})$. Since $\tilde b_2''(0) = 0$ and $\tilde b_2(t)$ is of class C^2, the constant A_1 of (4.9) for $\tilde b$ can be made arbitrarily small, provided δ is chosen sufficiently small. This is what we need to get (A.1)–(A.2) with arbitrarily small A.

By Lemma 3.1, $\tilde b_i^{(k)}(0) = 0$ for $i = 2, ..., \deg P_b$, $k = 0, ..., i$. Fix $A > 0$. Since every $\tilde b_i$ is of class C^i, there is a neighborhood I_δ in which (A.3) holds for $i = 2, ..., n$, $k = i$, and then, by Lemma 4.3, in a smaller neighborhood, also for $i = 2, ..., n$, $k \le i$.

Finally, given $A > 0$ we show (A.3) for $i < k \le n$ and δ sufficiently small. Let $\hat A$ denote the constant A for which (A.1)–(A.3) holds for P_a. By (4.6),

$$|\tilde b_i^{(k)}(t)| \le C(n) \hat A^k \, |\tilde a_2(t)|^{\frac{i-k}{2}} \le C(n) \hat A^k \, \varphi(t) |\tilde b_2(t)|^{\frac{i-k}{2}},$$

which gives the required result since, for $k > i$, $\varphi(t) = |\tilde b_2(t)/\tilde a_2(t)|^{\frac{k-i}{2}} = o(1)$. □

5.4. **Proof of (3).** We proceed by induction on n. The case $n = 1$ is obvious. So assume $n > 1$. Set $F = \{t \in I : \tilde a_2(t) = \tilde a_2''(t) = 0\}$. Its complement is a countable union of disjoint open intervals, $I \setminus F = \bigcup_k I_k$. At each $t_0 \in I \setminus F$ the polynomial $P_{\tilde a}$ splits, and, by the induction hypothesis, there exists a local differentiable system of the roots of $P_{\tilde a}$ near t_0. We may infer that there exists a differentiable system on each interval I_k. For, if the (say) right endpoint t_1 of the domain I_λ of $\lambda = (\lambda_j)_{j=1}^n$ belongs to I_k, there exists a local system $\mu = (\mu_j)_{j=1}^n$ with $t_1 \in I_\mu$. We may choose $t_2 \in I_\lambda \cap I_\mu$ and extend $(\lambda_j)_j$ by $(\mu_{\sigma(j)})_j$ on the right of t_2 beyond t_1, where σ is a suitable permutation. Extending by 0 on F yields a differentiable system $(\lambda_j)_j$ of the roots on I (the derivatives vanish on F).

5.5. **Proof of (4).** It follows immediately from (2).

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