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ON THE NON-ANALYTICITY LOCUS OF AN ARC-ANALYTIC FUNCTION

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Abstract

Let X be a real analytic manifold. A function $f : X \to \mathbb{R}$ is called arc-analytic if it is real analytic on each real analytic arc. In real analytic geometry there are many examples of arc-analytic functions that are not real analytic. They appear while studying the arc-symmetric sets and the blow-analytic equivalence.

In this paper we show that the non-analyticity locus of an arc-analytic function is arc-symmetric. We also discuss the behavior of the non-analyticity locus under blowings-up. By a result of Bierstone and Milman, an arc-analytic function $f : X \to \mathbb{R}$ that satisfies a polynomial equation with real analytic coefficients, can be made analytic, over any relatively compact subset of X, by a sequence of blowings-up with smooth centers. We show that these centers can be chosen, at each stage of the resolution, inside the non-analyticity loci.

1. Introduction

Let X be a real analytic manifold. A function $f : X \to \mathbb{R}$ is called *arc-analytic* (cf. [12]), if for every real analytic $\gamma : (-1, 1) \to X$ the composition $f \circ \gamma$ is analytic. The arc-analytic functions are closely related to blow-analytic functions of Kuo (cf. [10]). In particular, we have the following result for the functions with semi-algebraic graphs, conjectured by the first author and proved in [2].

Theorem 1.1. *Let X be a real analytic manifold and let $f : X \to \mathbb{R}$ be an arc-analytic function on X. Suppose that*

$$G(x, f(x)) = 0,$$

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where

$$G(x,y) = \sum_{i=0}^{p} g_i(x) y^{p-i} \tag{1.1}$$

is a non-zero polynomial in y with coefficients $g_i(x)$ which are analytic functions on X. Then there is a mapping $\pi : X' \to X$, which is a composite of a finite sequence of blowings-up with non-singular closed centers over any relatively compact open subset of X, such that $f \circ \pi$ is analytic.

Let $f : X \to \mathbb{R}$ be an arc-analytic subanalytic function. In this paper we study *the set $S(f)$ of non-analyticity of f*. By definition, $S(f)$ is the complement of the set $R(f)$ of points $p \in X$, such that f as a germ is real analytic at p. It is known (cf. [18], [11], and [1]) that $S(f)$ is closed and subanalytic. It follows from [2] or [16], that $\dim S(f) \leq \dim X - 2$. As we show in Theorem 3.1 below, $S(f)$ is arc-symmetric in the sense of [12]. Theorem 3.1 is proved in section 3.

We also study how the set of non-analyticity behaves under blowings-up with smooth centers. This depends on whether the center is entirely contained in $S(f)$ or not. If it is not, then the non-analyticity lifts to the entire fiber; see Proposition 3.11. Note that Theorem 1.1 can also be derived from [16]. Using the method of [16] and Proposition 3.11 we show the following refinement of Theorem 1.1.

Theorem 1.2. *Under the assumptions of Theorem* 1.1*, there is a mapping $\pi : X' \to X$, satisfying the following properties:*

(1) *$f \circ \pi$ is analytic.*
(2) *Over any open relatively compact subset of X, π is a composite of a finite sequence of blowings-up with non-singular closed centers: $\pi = \pi_0 \circ \cdots \circ \pi_k$, and for every $j = 0, \ldots, k$ the center of π_j is contained in the locus of non-analyticity of $f \circ \pi_0 \circ \cdots \circ \pi_{j-1}$.*

In particular, π is an isomorphism over the set of analyticity of f.

1.1. Algebraic case. Theorem 1.1 can be stated in the real algebraic version; see [2]. In this case, if we assume that X is a non-singular real algebraic variety and that the coefficients g_i are regular, then we may require that π is a finite composite of blowings-up with non-singular algebraic centers.

In the algebraic case, we cannot require that the centers of blowings-up are entirely contained in the non-analyticity loci as Example 1.5 shows.

An analytic function on X is called *Nash* if its graph is semialgebraic. It is called *blow-Nash* if it can be made Nash after composing with a finite sequence of blowing-ups with smooth nowhere dense regular centers. Thus the algebraic version of Theorem 1.1 (cf. [2]), says that the function with semialgebraic graph is arc-analytic if and only if it is blow-Nash. Nash morphisms and manifolds form a natural category that contains the algebraic ones (cf. [4]). We note that our refinement of the statement of Theorem 1.1 holds in the Nash category.

Theorem 1.3. *Let X be a Nash manifold and let $f : X \to \mathbb{R}$ be an arc-analytic function on X. Suppose that*

$$G(x, f(x)) = 0,$$

where

$$G(x, y) = \sum_{i=0}^{p} g_i(x) y^{p-i}$$

is a non-zero polynomial in y with coefficients $g_i(x)$ which are Nash functions on X. Then there is a finite composite $\pi = \pi_0 \circ \cdots \circ \pi_k$ of blowings-up of non-singular Nash submanifolds, such that for every k the center of π_{k+1} is contained in the locus of non-analyticity of $f \circ \pi_0 \circ \cdots \circ \pi_k$, and $f \circ \pi$ is Nash.

1.2. Subanalytic case. Less is known for an arc-analytic function with subanalytic graph if it does not satisfy equation (1.1). It is known that an arc-analytic subanalytic function has to be continuous and can be made real analytic by composing with finitely many local blowings-up with smooth centers; see [2] or [16] (we refer the reader to these papers for a precise statement). It is not known whether these blowings-up can be made global; that is, whether the arc-analytic subanalytic functions coincide with the family of blow-analytic functions of T.-C. Kuo (see e.g. [10], [6], and [7]). It is also not known whether the centers of such blowings-up can be chosen in the locus of non-analyticity of the function.

We present below, in Example 1.6, a subanalytic arc-analytic function that cannot be made analytic, even locally, by a blowing-up of a coherent ideal. In particular, it cannot satisfy an equation of type (1.1).

1.3. Examples.

Example 1.4.

(a) The function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \frac{x^3}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$, is arc-analytic, but not differentiable at the origin.

(b) The function $g(x, y) = \sqrt{x^4 + y^4}$ is arc-analytic, but not C^2. This example is due to E. Bierstone and P.D. Milman.

(c) The function $h : \mathbb{R}^2 \to \mathbb{R}$, $h(x, y) = \frac{xy^5}{x^4+y^6}$ for $(x, y) \neq (0, 0)$ and $h(0, 0) = 0$, is arc-analytic, but not Lipschitz. This example is due to L. Paunescu.

(d) We generalize the first example as follows. Fix a real analytic Riemannian metric on X and let Y be a non-singular real analytic subset of X. Then $d_Y^2 : X \to \mathbb{R}$, the square of the distance to Y, is a real analytic function on X. Suppose that Y is of codimension ≥ 2 in X and let $f : X \to \mathbb{R}$ be an analytic function vanishing on Y and not divisible by d_Y^2. Then, $\frac{f^3}{d_Y^2}$ vanishes on Y, is arc-analytic, and is not analytic at the points of Y. Note that $\frac{f^3}{d_Y^2}$ composed with the blowing-up of Y is analytic.

Example 1.5. Let $g(x,y) = y^2 + x(x-1)(x-2)(x-3)$. Then $g^{-1}(0) \subset \mathbb{R}^2$ is irreducible and has two connected compact components, denoted by X_1 and X_2. These connected components can be separated by $h(x,y) = x - 1.5$; that is, $h < 0$ on X_1 and $h > 0$ on X_2. For $\varepsilon > 0$ sufficiently small, $h^2 + \varepsilon g$ is strictly positive on $\mathbb{R}^2$. Define

$$g_1(x,y) = \sqrt{h^2 + \varepsilon g} + h.$$

Then g_1 is analytic, 0 is a regular value of g_1 and $g_1^{-1}(0) = X_1$. Moreover, g_1 is Nash. Then $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x,y,z) = \frac{z^3}{z^2 + g_1^2(x,y)}$$

for $(x,y,z) \neq 0$ and $f(0) = 0$, is arc-analytic and $S(f) = X_1 \times \{0\}$. The function f becomes analytic after the blowing-up of $S(f)$.

Example 1.6. Let $\pi_0 : \widetilde{\mathbb{R}}^3 \to \mathbb{R}^3$ be the blowing-up of the origin and let E be the exceptional divisor of π_0. Let $C \subset E$ be a transcendental (the smallest algebraic subset of E that contains C is E itself) non-singular analytic curve and let $\pi_C : M \to \widetilde{\mathbb{R}}^3$ be the blowing-up of C. Let f be an arc-analytic function on $\mathbb{R}^3$ such that the set of non-analyticity of $f \circ \pi_0$ is C and $f \circ \pi_0 \circ \pi_C$ is analytic. Such a function can be constructed as follows. Using Example 1.4 (d) we may construct an arc-analytic function $g : \widetilde{\mathbb{R}}^3 \to \mathbb{R}$ such that $S(g) = C$. Then we may set $f(x,y,z) = (x^2 + y^2 + z^2)\, g(\pi_0^{-1}(x,y,z))$.

Such f, as a germ at 0, cannot be made analytic by a single blowing-up of an ideal. Indeed, suppose contrary to our claim that there exists an ideal $\mathcal{I}$ of $\mathbb{R}\{x_1, x_2, x_3\}$ such that $f \circ \pi_{\mathcal{I}}$ is analytic, where $\pi_{\mathcal{I}}$ denotes the blowing-up of $\mathcal{I}$. Multiplying $\mathcal{I}$ by the maximal ideal at 0 we may assume that $\pi_{\mathcal{I}}$ factors through π_0, i.e. $\pi_{\mathcal{I}} = \pi_{\mathcal{J}} \circ \pi_0$, where $\mathcal{J}$ is a sheaf of coherent ideals centered on an algebraic subset Y of E. We may assume that $\dim Y \leq 1$. Thus the blowing-up of $\mathcal{J}$, $\pi_{\mathcal{J}} : M_{\mathcal{J}} \to \widetilde{R}^3$ is an isomorphism over the complement of Y that contradicts the construction of f.

2. Arc-meromorphic mappings

In this section, *subanalytic* means subanalytic at infinity. Let us recall (see [18] and [11]), that a subset A of $\mathbb{R}^n$ is called *subanalytic at infinity* if A is subanalytic in some algebraic compactification of $\mathbb{R}^n$. (Then, in fact, it is subanalytic in every algebraic compactification of $\mathbb{R}^n$.) All functions and mappings are supposed to be subanalytic; that is, their graphs are subanalytic at infinity.

Definition 2.1. Let U be an open subanalytic subset of $\mathbb{R}^n$. An everywhere defined subanalytic mapping $f : U \to \mathbb{R}^m$ is called *arc-meromorphic* if for any analytic arc $\gamma : (-1, 1) \to U$ there exist a discrete set $D \subset (-1, 1)$ and φ a meromorphic function on $(-1, 1)$ with poles contained in D and such that $f \circ \gamma = \varphi$ on $(-1, 1) \setminus D$. Note that it may happen that $f \circ \gamma$ does not coincide with φ at some points of D and may be discontinuous at these points.

Example 2.2. The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \frac{xy}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ can be extended to an arc-meromorphic function on $\mathbb{R}^2$ by assigning any value at the origin. Then it becomes discontinuous at $(0, 0)$ even though for every analytic arc $\gamma : (-1, 1) \to \mathbb{R}^2$, $\gamma(0) = (0, 0)$, $f \circ \gamma$ extends to an analytic function.

Remark 2.3. If f is an arc-meromorphic and continuous function on an open set $U \subset \mathbb{R}^n$, then f is arc-analytic.

Remark 2.4. Let f and g be arc-meromorphic functions on an open connected set of U. Assume that $f = g$ on an open non-empty subset $U \subset \mathbb{R}^n$, then $f = g$ except on a nowhere dense subanalytic subset of U.

Lemma 2.5. *Let U be an open bounded subanalytic subset in $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ be an arc-meromorphic mapping. Then there exists $\Gamma \subset \mathbb{R}^n$ a closed nowhere dense subanalytic set, $N \in \mathbb{N}$ and $C > 0$ such that*

$$|f(x)| \leq C \operatorname{dist}(x, \Gamma)^{-N},\ x \in U \setminus \Gamma. \tag{2.1}$$

In particular, we can take as Γ the complement in $\overline{U}$ of the analyticity locus of f.

Proof. It is well known (cf. e.g. [9] and [15]) that there exists a stratification of $\mathbb{R}^n$ which is compatible with $\overline{U}$ and such that f is analytic on each stratum contained in U. We take as Γ the union of all strata contained in $\overline{U}$ of dimension less than n. Let us consider the function defined as follows: $g(x) = |f(x)|$ if $|f(x)| \leq 1$, and $g(x) = |f(x)|^{-1}$ if $|f(x)| \geq 1$. Then $h(x) := dist(x, \Gamma) g(x)$ is a subanalytic and continuous function on $\overline{U}$ which is compact. Moreover, if $\operatorname{dist}(x, \Gamma) = 0$, then $h(x) = 0$. Therefore, by the classical Łojasiewicz's inequality (cf. e.g. [9] and [1]) for subanalytic functions, there exist $N \in \mathbb{N}$ and $c > 0$ such that

$$h(x) \geq c \operatorname{dist}(x, \Gamma)^{N+1},\ x \in U. \tag{2.2}$$

This implies (2.1) with $C = \max\{1/c, M\}$, where $M = \sup_{x \in U} dist(x, \Gamma)^N$. □

We state now an auxiliary lemma on arc-meromorphic functions in two variables.

Lemma 2.6. *Let U be an open subanalytic subset in $\mathbb{R}^2$ and let $f : U \to \mathbb{R}^m$ be an arc-meromorphic mapping. Then, for any $a \in U$, there exists a neighborhood V of a and an analytic function $\varphi : V \to \mathbb{R}$, $\varphi \not\equiv 0$, such that φf is arc-analytic.*

Proof. Let Γ be a subanalytic set associated to f by Lemma 2.5. Clearly we may assume that $a \in \Gamma$, otherwise f is analytic at a and the statement is trivial. Since $\dim \Gamma = 1$, by a result of Łojasiewicz's [14] (see also [13]), the set Γ is actually semianalytic. Then there exists a neighborhood V' of a and an analytic function $\psi : V' \to \mathbb{R}$, $\psi \not\equiv 0$, which vanishes on $V' \cap \Gamma$. Hence, for some compact neighborhood $V \subset V'$ of a, there exists $c > 0$ such that

$$|\psi(x)| \leq c \operatorname{dist}(x, \Gamma),\ x \in V.$$

(This is a consequence of the main value theorem.) Put $\varphi = \psi^{N+1}$, then by Lemma 2.5, the function φf is continuous on V. Clearly φf is arc-meromorphic, so by Remark 2.3, this function is arc-analytic. □

Proposition 2.7. *Let $f : U \to \mathbb{R}$ be an arc-meromorphic function, where U is an open subset in $\mathbb{R}^n$. Assume that f is analytic with respect to the variable x_1. Then the function $\frac{\partial f}{\partial x_1} : U \to \mathbb{R}$ is again arc-meromorphic.*

Proof. First observe that by [11] the function $\frac{\partial f}{\partial x_1}$ is (globally) subanalytic. To prove that $\frac{\partial f}{\partial x_1}$ is arc-meromorphic, let us fix an analytic arc $\gamma : (-1, 1) \to U$. We define an arc-meromorphic function $g : V \to \mathbb{R}$ by $g(s, t) = f(\gamma(t) + s e_1)$, where $e_1 = (1, 0, \dots 0)$ and V is an open neighborhood of $\{0\} \times (-1, 1)$ in $\mathbb{R}^2$. Clearly,

$$\frac{\partial f}{\partial x_1}(\gamma(t)) = \frac{\partial g}{\partial s}(0, t).$$

From the assumption that g is analytic in s it follows that g is continuous at $(0, t)$, for any $|t| > 0$ and small enough. Hence, by Remark 2.3, g is analytic at $(0, t)$. By Lemma 2.6, there exists a neighborhood V of $(0, 0)$ and an analytic function $\varphi : V \to \mathbb{R}$ such that $h := \varphi g$ is arc-analytic on V. Note that $\varphi(0, t) \neq 0$ for $|t| > 0$ and small enough.

By [2] there exists a map $\pi : M \to \mathbb{R}^2$, which is a finite composition of blowings-up of points, such that $h \circ \pi$ is analytic. Consider the arc $\eta(t) := (0, t)$ and let $\tilde{\eta}(t) \in M$ be the unique analytic arc such that $\pi \circ \tilde{\eta} = \eta$. The chain rule gives

$$d_{\tilde{\eta}(t)} h \circ \pi = (d_{\eta(t)} h) \circ (d_{\tilde{\eta}(t)} \pi). \tag{2.3}$$

Note that $d_{\tilde{\eta}(t)} \pi$ is invertible for $t \neq 0$; moreover, the map $t \mapsto (d_{\tilde{\eta}(t)} \pi)^{-1}$ is meromorphic. It follows that $t \mapsto d_{\eta(t)} h$ is meromorphic. In particular,

$t \mapsto \frac{\partial h}{\partial s}(0,t)$ is meromorphic. We have

$$\frac{\partial h}{\partial s}(0,t) = \varphi \frac{\partial g}{\partial s}(0,t) + g \frac{\partial \varphi}{\partial s}(0,t).$$

Since $\varphi(0,t) \neq 0$ for $t \neq 0$, the map $t \mapsto \frac{\partial g}{\partial s}(0,t)$ is meromorphic and Proposition 2.7 follows. □

Remark 2.8. Using Lemma 2.1 in [17], one can give an alternative proof of Proposition 2.7.

Remark 2.9. A repeated application of Proposition 2.7 shows that for every $k \in \mathbb{N}$,

$$\frac{\partial^k f}{\partial x_1^k} : U \to \mathbb{R}$$

is arc-meromorphic. Moreover, there exists a subanalytic stratification $\mathcal{S}$ of U such that for every stratum $S \in \mathcal{S}$ and every $x \in S$ there is $\varepsilon > 0$ and a neighborhood V of x in S such that $f(x + se_1)$ is an analytic function of $(x,s) \in V \times (-\varepsilon, \varepsilon)$. In particular, for every $k \in \mathbb{N}$, $\partial^k f/\partial x_1^k : U \to \mathbb{R}$ is analytic on the strata of $\mathcal{S}$.

3. The non-analyticity locus of an arc-analytic function is arc-symmetric

Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}$ be a bounded function with subanalytic graph. We denote by $S(f)$ the non-analyticity set of f and by $R(f)$ its complement in U. Then $S(f)$ is closed in U and by [18] (see also [11] and [2]) it is a subanalytic set. Moreover, if f is arc-analytic, then it follows from [2] or [16] that $\dim S(f) \leq n-2$.

Theorem 3.1. *Assume that f is arc-analytic with subanalytic graph. Let $\gamma : (-\varepsilon, \varepsilon) \to U$ be an analytic arc such that $\gamma(t) \in R(f)$ for $t < 0$. Then $\gamma(t) \in R(f)$ for $t > 0$ and small. In other words, $S(f)$ is arc-symmetric subanalytic in the sense of* [12].

For the proof we need some basic properties of Gateaux differentials. For each $k \in \mathbb{N}$, we consider

$$h_k(x,v) = \frac{1}{k!}\partial_v^k f(x) = \frac{1}{k!}\frac{d^k}{dt^k} f(x+tv)_{|t=0}. \tag{3.1}$$

Proposition 3.2. *Let $f : U \to \mathbb{R}$ be an arc-analytic function. Then, for any $k \in \mathbb{N}$, the function $h_k(x,v) : U \times \mathbb{R}^n \to \mathbb{R}$ is arc-meromorphic.*

Proof. Let $(x(t), v(t))$ be an analytic arc in $U \times \mathbb{R}^n$. Define an arc-analytic function $g(s,t) = f(x(t) + sv(t))$. Then

$$h_k(x(t), v(t)) = \frac{1}{k!}\frac{\partial^k}{\partial s^k} g(t,s)_{|s=0}$$

is arc-meromorphic by Proposition 2.7. □

For $x \in U$ and $k \in \mathbb{N}$, we denote

$$h_{x,k}(v) = h_k(x,v) = \frac{1}{k!}\partial_v^k f(x).$$

Note that $h_{x,k}$ is a homogeneous function of order k. If f is analytic at x, then $h_{x,k}$ is polynomial. We also have the inverse; see [5].

Theorem 3.3 (Bochnak-Siciak)**.** *Let $f : U \to \mathbb{R}$ be a function, where U is an open subset of $\mathbb{R}^n$. Assume that for some $x \in U$, f satisfies the following conditions:*

(1) *For any affine line L in $\mathbb{R}^n$, such that $x \in L$, the restriction of f to $L \cap U$ is analytic.*
(2) *$h_{x,k}$ is a polynomial for each $k \in \mathbb{N}$.*

Then f is analytic at x.

Traditionally, if $h_{x,k}$ is a polynomial, then it is called the Gateaux differential of f at x of order k. Note that if f is arc-analytic, then the first condition is automatically satisfied.

We call $h_{x,k}$ *generically polynomial* if it is equal to a polynomial, except on a nowhere dense subanalytic (and homogeneous) subset of $\mathbb{R}^n$. Note that, by Remark 2.4, $h_{x,k}$ is generically polynomial if it coincides with a polynomial on an open non-empty set.

Proposition 3.4. *Let $f : U \to \mathbb{R}$ be an arc-analytic function, where U is an open subset in $\mathbb{R}^n$. Let $\gamma : (-\varepsilon, \varepsilon) \to U$ be an analytic arc and $k \in \mathbb{N}$. If $h_{\gamma(t),k}$ is generically polynomial for $t \in (-\varepsilon, 0)$, then there exists a finite set $F_k \subset (0, \varepsilon)$ such that $h_{\gamma(t),k}$ is generically polynomial for each $t \in (0,\varepsilon) \setminus F_k$.*

Proof. Let $\mathbb{R}_k[x_1, \dots, x_n]$ denote the space of homogeneous polynomials of degree k and let $d_k = \binom{n+k-1}{n}$ denote its dimension. We need the classical multivariate interpolation.

Lemma 3.5. *There exists an algebraic nowhere dense subset $\Delta \subset (\mathbb{R}^n)^{d_k}$ such that for $V = (v^1, \dots, v^{d_k}) \in (\mathbb{R}^n)^{d_k} \setminus \Delta$ the map $\Psi_V : \mathbb{R}_k[x_1, \dots, x_n] \to \mathbb{R}^{d(k)}$ given by*

$$\Psi_V(P) = (P(v^1), \dots, P(v^{d_k}))$$

is a linear isomorphism. □

Fix $V = (v^1, \dots, v^{d_k}) \in (\mathbb{R}^n)^{d_k} \setminus \Delta$ generic and denote $\Phi_V = \Psi_V^{-1} : \mathbb{R}^{d(k)} \to \mathbb{R}_k[x_1, \dots, x_n]$. We define an arc-meromorphic map $P_k : (-\varepsilon, \varepsilon) \to$

$\mathbb{R}_k[x_1, \ldots, x_n]$ by

$$P_k(t) := \Phi_V(h_k(\gamma(t), v^1), \ldots, h_k(\gamma(t), v^{d(k)})).$$

The map $p_k : (-\varepsilon, \varepsilon) \times \mathbb{R}^n \to \mathbb{R}$, where $p_k(t, v) = P_k(t)(v)$ is arc-meromorphic. If V is sufficiently generic, then for $t \in (-\varepsilon, 0) \setminus \{\text{finite set}\}$, $p_k(t)$ coincides with $h_{\gamma(t),k}$. Since they both are arc-meromorphic, by Remark 2.4 they coincide on $(-\varepsilon, \varepsilon) \times \mathbb{R}^n \setminus Z_k$, where Z_k is a closed subanalytic set with $\dim Z_k \leq n$. Hence, there exists a finite set $F_k \subset (0, \varepsilon)$ such that for $t \in (0, \varepsilon) \setminus F_k$ the intersection $Z_k \cap (\{t\} \times \mathbb{R}^n)$ is of dimension less than n. Thus, for each $t \in (0, \varepsilon) \setminus F_k$, the function $h_{\gamma(t),k}$ is generically polynomial as claimed. □

The following proposition is a version of the Bochnak-Siciak theorem.

Proposition 3.6. *Assume that f is arc-analytic. If, for every k, there is a non-empty open subset $V_k \subset \mathbb{R}^n$ and a homogeneous polynomial P_k of degree k such that $h_{x,k} \equiv P_k$ on V_k, then f is analytic at x.*

Proof. We first show that $\sum_k P_k(v)$ is convergent in a neighborhood of $0 \in \mathbb{R}^n$.

We may assume that x is the origin. Let π_0 be the blowing up of the origin, $\pi_0(y, s) = (sy, s)$, $s \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$, in a chart. The function $\tilde{f}(y, s) := f(\pi(y, s))$, defined in a neighborhood U' of the exceptional divisor $E : s = 0$, is arc-analytic. The set of non-analyticity of $\tilde{f}$, denoted by $\tilde{S}$, is closed subanalytic and of codimension at least 2. For $y \notin \tilde{S}$, $\tilde{f}$ is analytic in a neighborhood of $(0, y)$ and, moreover, by analytic continuation,

$$h_{x,k}(v) = P_k(v) \qquad \text{for} \quad v = t(y, 1),\ t \in \mathbb{R},\ y \notin \tilde{S}. \tag{3.2}$$

Fix A' an open non-empty subset of E such that the closure of A' does not intersect $\tilde{S}$. Let $A \subset \mathbb{R}^n$ be the cone over A'. Then, by (3.2), $\sum_k P_k(v)$ is convergent in any compact subset of A. The convergence in a neighborhood of 0 in $\mathbb{R}^n$ follows from the following lemma.

Lemma 3.7. *Let $V \subset \mathbb{R}^n$ be starlike with respect to the origin, $a \in V$, and suppose that*

$$|P_k(v)| \leq L \quad \text{on} \quad V' = a + V.$$

Then

$$|P_k(v)| \leq L \quad \text{on} \quad \frac{1}{2e}V.$$

Proof. Since P_k is homogeneous of degree k,

$$P_k(v) = \frac{1}{k!} \sum_{s=0}^{s=k} (-1)^{k-s} \binom{k}{s} P_k(a + sv). \tag{3.3}$$

Indeed, (3.3) can be proved recursively on k using Euler's formula as follows. First note that (3.3) holds for $a = 0$ and that the derivative of the RHS of

(3.3) with respect to a equals

$$0 = \frac{1}{k!}\sum_{s=0}^{s=k}(-1)^{k-s}\binom{k}{s}Q(a+sv), \tag{3.4}$$

where $Q(x) = \sum_{i-1}^{n} a_i \frac{\partial P_k}{\partial x_i}(x)$ is a homogeneous polynomial of degree $k-1$. By the inductive assumption

$$\sum_{s=0}^{s=k}(-1)^{k-s}\binom{k}{s}Q(a+sv) = \sum_{s=0}^{s=k-1}(-1)^{k-1-s}\binom{k-1}{s}Q(a+sv)$$
$$+\sum_{s=1}^{s=k}(-1)^{k-s}\binom{k-1}{s-1}Q(a+sv) = (k-1)!(-Q(v)+Q(v)) = 0.$$

This shows (3.3). Thus, if $v \in \frac{1}{k}V$, $|P_k(v)| \le \frac{1}{k!}L\sum_{s=0}^{k}\binom{k}{s} = L\frac{2^k}{k!}$, that means that for $v \in \frac{1}{2e}V$,

$$|P_k(v)| \le L\frac{(2k)^k}{k!}\frac{1}{(2e)^k} \le L.$$

This ends the proof of Lemma 3.7. □

Then $\sum_k P_k(v)$ is an analytic function in a neighborhood of the origin that coincides with f on a set with non-empty interior. Hence, $f(v) = \sum_k P_k(v)$ in a neighborhood of the origin. This shows Proposition 3.6. □

Proof of Theorem 3.1. We may assume that γ is injective, otherwise the image of $t > 0$ equals the image of $t < 0$ and the statement is obvious. Let $F := \bigcup F_k$, where F_k are finite subsets of $(0,\varepsilon)$ given by Proposition 3.4. Clearly, the complement of F is dense in $(0,\varepsilon)$, so by Proposition 3.6, our function f is analytic at $\gamma(t)$ for $t \in G$, where G is an open dense subset of $(0,\varepsilon)$. Hence, Theorem 3.1 follows. □

Consider the subanalytic sets

$$\tilde{R}_{k_0}(f) = \{x \in U; \forall k \le k_0,\ h_{x,k} \text{ is generically polynomial}\},$$
$$R_{k_0}(f) = \{x \in U; \forall k \le k_0,\ h_{x,k} \text{ is polynomial}\}.$$

Clearly, $\tilde{R}_{k+1}(f) \subset \tilde{R}_k(f)$ and $R_{k+1}(f) \subset R_k(f)$. We recall the following result from [11].

Proposition 3.8 ([11], Proposition 4.4)**.** *Let $f : U \to \mathbb{R}$ be a bounded subanalytic (not necessarily arc-analytic) function on an open bounded $U \subset \mathbb{R}^n$. Then, for any compact $K \subset U$ there is $k \in \mathbb{N}$ such that $R(f) \cap K = R_k(f) \cap K$.*

Proposition 3.9. *Under the assumptions of Proposition* 3.8*, for any compact $K \subset U$ there is $k \in \mathbb{N}$ such that $R(f) \cap K = \tilde{R}_k(f) \cap K$.*

Proof. By Remark 2.9 there exists a stratification $\mathcal{S}$ of $U \times S^{n-1}$ such that for every k, h_k is analytic on the strata. Refining the stratification, if necessary, we may suppose that for every stratum $S \subset U \times S^{n-1}$ its projection to U has all fibers of the same dimension. In the proof, we use only these strata for which all the fibers of projection to U are of maximal dimension $n-1$. We denote their collection by $\mathcal{S}_n$ and their union by Z. Now it is easy to adapt the proof of Lemma 6.1 of [11] (based on multivariate interpolation) and show the following lemma.

Lemma 3.10. *There are analytic subanalytic functions*

$$w_i : U \times S^{n-1} \to \mathbb{R}, \quad i \in \mathbb{N},$$

analytic on each stratum of $\mathcal{S}$ such that $h_{x,i}$ is generically polynomial if and only if $w_i \equiv 0$ generically on $\{x\} \times S^{n-1}$. □

Now Proposition 3.9 follows from Lemma 2.5 of [11] which shows that for every stratum, there exists k such that

$$\bigcap_{i=1}^{\infty} \{w_i = 0\} = \bigcap_{i=1}^{k} \{w_i = 0\}.$$

□

We complete this section with two results, one that controls the change of non-analyticity locus by blowings-up. This result will be crucial in the next section. The last result of this section, Proposition 3.12, though not used in this paper, indicates a possible analogy between our approach and the theory of complex analytic functions.

Proposition 3.11. *Assume that $f : U \to \mathbb{R}$ is subanalytic and arc-analytic. Let T be a closed analytic submanifold of U and let π_T be the blowing-up of T. Suppose that the origin is in the closure of $R(f) \cap T$ and that $f \circ \pi_T$ is analytic at least at one point of $\pi_T^{-1}(0)$ (hence on a neighborhood of this point). Then f is analytic at 0.*

Proof. We choose local coordinates in such way that $T = \{x_k = x_{k+1} = \cdots = x_n = 0\}$. Let $\Pi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be given by $\Pi(x,t,v) = x + tv$ and let $\Pi_T : T \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be the restriction of Π. First, we show that if $f \circ \Pi_T$ is analytic at some points of $\Pi_T^{-1}(0) \cap \{t = 0\}$ and 0 is in the closure of $R(f) \cap T$, then f is analytic at 0. Indeed, suppose that $A' \subset \mathbb{R}^n$ has non-empty interior and suppose that $f \circ \Pi_T$ is analytic in a neighborhood $\{0\} \times \{0\} \times A'$. Let $h_k(x,v)$, $x \in T, v \in \mathbb{R}^n$, be defined by (3.1). Then h_k is arc-meromorphic and analytic on $A = U' \times A'$, where U' is a small neighborhood of 0 in T. For each k, we define by Lemma 3.5,

$$P_k(x,v) = \Psi_V^{-1}(h_k(x,v^1), \ldots, h_k(x,v^{d(k)}))(v), \tag{3.5}$$

where $v^1, \dots, v^{d_k} \in A'$ are generic. Each P_k is analytic on A and equals h_k for $x \in R(f) \cap T$. Therefore, $h_k(0, v) = P_k(0, v)$ for $v \in A'$ and the claim follows from Proposition 3.6.

Thus, it remains to show that $f \circ \Pi_T$ is analytic at some points of $\Pi_T^{-1}(0) \cap \{t = 0\}$. For this we factor Π_T restricted to $\{v_n \neq 0\}$ through π_T and use the assumption on π_T. Write π_T in an affine chart $\pi_T(\tilde{x}, y, s) = (\tilde{x}, sy, s)$, where $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_{k-1})$, $y = (y_k, \dots, y_{n-1})$ and $s \in \mathbb{R}$. Then, on these charts $\Pi_T = \pi_T \circ \varphi$, where

$$(\tilde{x}, y, s) = \varphi(x, t, v) = (x + tv', \frac{1}{v_n} v'', tv_n),$$

where $v' = (v_1, \dots, v_{k-1})$, $v'' = (v_k, \dots, v_{n-1})$. Restricted to $t = 0$, φ is a surjective projection $(x, v) \to (x, \frac{1}{v_n} v'')$ onto $s = 0$. Hence, $R(f \circ \Pi_T) \cap \Pi_T^{-1}(0) \cap \{t = 0\} \supset \varphi^{-1}(R(f \circ \pi_T) \cap \pi^{-1}(0))$ is non-empty. □

Proposition 3.12. *Let $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ and suppose that for every $x_1 > 0$ and small, $f(x_1, x')$ is analytic at $(x_1, 0)$ as a function of x'. Moreover, suppose that for $x_1 > 0$ and small we have a uniform bound*

$$|h_k((x_1, 0), v')| \leq c^k, \qquad \text{for } \|v'\| \leq \varepsilon, k \in \mathbb{N},$$

where $v' = (v_2, \dots, v_n)$. Then f is analytic at the origin.

Proof. The function $h_k((x_1, 0), v')$ is arc-meromorphic as a function of x_1, v'. Moreover, since continuous arc-meromorphic functions of one variable are analytic, using polynomial interpolation Lemma 3.5, we may show that each $h_k((x_1, 0), v')$ extends to an analytic function $\Psi(x_1, v')$ defined in a neighborhood of $(0, 0)$, such that for each x, $v' \to \Psi(x_1, v')$ is a homogeneous polynomial in v'. Moreover, for $x_1 > 0$ and $\|x'\| < \varepsilon/c$,

$$f(x_1, x') = \sum_k h_k((x_1, 0), x')$$

and the series on the right-hand side is convergent.

Fix any $k \in \mathbb{N}$ and $\|v'\| < \varepsilon/c$. Then, for $v = (1, v')$, $0 < t < 1$,

$$f(tv) = \sum_{j=0}^{\infty} h_j((t, 0), tv') = \sum_{j=0}^{\infty} t^j h_j((t, 0), v') = \sum_{j=0}^{k} t^j h_j((t, 0), v') + \varphi(t, v'),$$

where φ is subanalytic and $O(t^{k+1})$. Therefore, for such v,

$$H_k(0, v) := \frac{1}{k!} \frac{d^k}{dt^k} f(tv)_{|t=0} = \frac{1}{k!} \frac{d^k}{dt^k} \sum_{j=0}^{k} h_j((t, 0), tv')_{|t=0}. \tag{3.6}$$

Note that the right-hand side, and hence $H_k(0, v)$ as well, is a polynomial in v. Indeed, this follows from the fact that $x \to \sum_{j=0}^{k} h_j((x_1, 0), x')$ is an analytic function of x and $H_k(0, v)$ coincides with its Gateaux differential. Thus, Proposition 3.12 follows from Proposition 3.6. □

4. Proof of Theorem 1.2

We may suppose that U is connected. We suppose also that the coefficients g_0 and g_p of G and the discriminant $\Delta(x)$ of G are not identically equal to zero. By the resolution of singularities ([8], [3], and [19]), there is a locally finite sequence of blowings-up $\pi : U' \to U$ with non-singular centers such that $(g_0 g_p \Delta) \circ \pi$ is normal crossings. Thus Theorem 1.1 follows from the following.

Proposition 4.1. *Let an arc-analytic function $f(x)$ satisfy the equation* (1.1) *with analytic coefficients g_i. If g_0, g_p and $\Delta(x)$ are simultaneously normal crossings (and hence not identically equal to zero), then f is real analytic.*

Proposition 4.1 was proven in [16] under an additional assumption $g_0 \equiv 1$; see the proof of Theorem 3.1 of [16]. It is easy to reduce the proof to this case by replacing f by $g_p f$. Then, an argument of [16] shows that f can be expanded locally as a fractional power series. Finally, an arc-analytic fractional power series is analytic; see the proof of Theorem 3.1 of [16]. If the discriminant of G vanishes identically, then we replace it by the first non-vanishing higher-order discriminant.

To show Theorem 1.2 we follow, for the product $h(x) = g_0(x) g_p(x) \Delta(x)$, the monomialisation procedure of Bierstone and Milman [3] or Włodarczyk [19]. In this procedure, the center of blowing-up is defined as the locus of points where a local invariant is maximal. Thus, suppose that in a local system of coordinates $x_1, \ldots, x_n$ we have the following. The function $h \circ \pi$, where π denotes the composition of preceding blowings-up, is of the form $h \circ \pi = x^A h_k$, where h_k is the strict transform of h by π and x^A is a monomial in exceptional divisors. Let $m = \operatorname{ord}_x h_k$. We may assume that $H = \{x_n = 0\}$ is a hypersurface of maximal contact, and then

$$h_k(x) = x_n^m + \sum_{j=0}^{m-2} c_j(x') x_n^j, \tag{4.1}$$

where $x' = (x_1, \ldots, x_{n-1})$ and $\operatorname{mult}_0 c_i \geq m - i$.

Let C be the next center given by the procedure and denote by π_C the blowing up of C. We show that $0 \in S(f \circ \pi)$ and $0 \in \overline{R(f \circ \pi) \cap C}$ cannot happen. By Proposition 3.11, it suffices to show that $f \circ \pi \circ \pi_C$ is real analytic at least at one point of $\pi_C^{-1}(0)$. Since C is contained in the equimultiplicity locus of h_k, and hence in $\{x_n = 0\}$, by (4.1), the strict transform of h_k is non-zero at a generic point of $\pi_C^{-1}(0)$. That implies that at a generic point of $\pi_C^{-1}(0)$, the total transform $h \circ \pi \circ \pi_C$ is normal crossing, and hence, by Proposition 4.1, $f \circ \pi \circ \pi_C$ is real analytic at such points.

Let C' denote the connected component of C containing 0. Then either $C' \subset S(f \circ \pi)$ or $C' \cap S(f \circ \pi) = \emptyset$. Theorem 1.2 is proved. □

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