CHARACTERISTIC CLASSES OF SINGULAR VARIETIES VIA CONORMAL GEOMETRY

ADAM PARUSIŃSKI

1. Introduction

The main purpose of this paper is to give a short introduction to the theory of characteristic classes of singular varieties via the conormal geometry approach. Our account is far from being complete and is directed mainly to the students who would like to explore the subject. Thus, instead of giving a long list of statements or an exhaustive historical account our purpose is to present the main constructions and ideas. That's why we found it important to provide some proofs, even if they have to be often sketchy. For more detailed and complete accounts the readers are invited to consult [28], [29], [24], [31].

The paper starts with a construction of the characteristic cycle for a subanalytic subset of an oriented real analytic manifold. A similar construction can be given in a more general set-up, namely for subsets definable in an o-minimal structure, cf. [28]. Then we pass to the complex case. In this case the construction of the characteristic cycle is similar but simpler. In the complex case we work with the algebraic varieties, but the extension to the complex analytic case is not difficult.

2. Motivating example

Let $K \subset \mathbb{R}^n$ be compact and convex. For $r \geq 0$ we denote $K_r = \{x \in \mathbb{R}^n; \operatorname{dist}(x, K) \leq r\}$. Then, by Steiner's formula, the n-dimensional volume of K_r is a polynomial in r,

$$Vol(K_r) = \sum_{i=0}^{n} c_i(K) r^{n-i}, \tag{1}$$

where $c_i(K)$ are constant depending on K. For instance for $n = 2$ we have

$$Area(K_r) = Area(K) + r \cdot perimeter(\partial K) + r^2 \cdot \pi.$$

Weyl Tube Formula [36] states that (1) holds also for K being a smooth compact submanifold of $\mathbb{R}^n$ and $r \geq 0$ small. In the case of a smooth hypersurface, $c_i(K)$'s are the integrals of the elementary symmetric functions of the principal curvatures of K. Similarly, for K smooth or convex, the coefficients $c_i(K)$ can be expressed in terms of the "curvature measures".

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The formula (1) as stated above, does not hold for general singular sets. But the curvature measures were generalized to some singular subsets by Wintgen [37] and Zähle [38], and then in much bigger generality by Fu, [9], [10]. Their approach is based on the notion of *the normal cycle* $N(K)$ of K. The normal cycle of Fu is a Legandrian integral current in the sense of Federer and Fleming [7].

For $K \subset \mathbb{R}^n$ compact and convex its normal cycle, as a set, is defined as follows

$$N(K) = \{(x, v) \in \mathbb{R}^n \times S^{n-1}; x \in \partial K, \forall_{y \in K} < v, y - x > \leq 0\}. \tag{2}$$

Let $\varphi : N(K) \times \mathbb{R} \to \mathbb{R}^n$, be given by $\varphi(x, v, \rho) = x + \rho v$. Then

$$\varphi^*(dx_1 \wedge \ldots \wedge dx_n) = (dx_1 + \rho dv_1 + v_1 d\rho) \wedge \ldots \wedge (dx_n + \rho dv_n + v_n d\rho)$$
$$= \beta + \sum_{i=0}^{n-1} \kappa_i \rho^{n-1-i} d\rho,$$

where the form β does not contain $d\rho$. Hence

$$Vol(K_r) - Vol(K) = \sum_{i=0}^{n-1} (\int_{N(K)} \kappa_i)(\int_0^r \rho^{n-1-i} d\rho) = \sum_{i=0}^{n-1} (\int_{N(K)} \kappa_i) \frac{r^{n-i}}{n-i}.$$

This gives a proof of (1), if φ is injective on $N(K) \times]0, r]$ and provided we know the meaning of integration of $(n-1)$ forms along $N(K)$. That is why it is important to understand $N(K)$ as a current.

For K subanalytic, $N(K)$ is a subanalytic Legandrian integral cycle, that is an object of the following geometric structure. Set theoretically $N(K)$ is a subanalytic subset of $\mathbb{R}^n \times S^{n-1}$ of pure dimension $n-1$, that $N(K)$ can be partitioned into finitely many subanalytic pieces $N(K) = Y \sqcup \bigsqcup_{i=1}^k \Lambda_i$, such that $\dim Y < n-1$ and each Λ_i is an orientable connected real analytic manifold of dimension $n-1$. Fix an orientation of each Λ_i. As an oriented real analytic and subanalytic submanifold, Λ_i defines an element the group of integral subanalytic chains $C_{n-1}(\mathbb{R}^n \times S^{n-1}; \mathbb{Z})$, that we denote by $[\Lambda_i]$. As a integral chain it has the boundary $\partial[\Lambda_i]$, that is again an integral subanalytic chain with support in $\overline{\Lambda_i} \setminus \Lambda_i$. Then there exists integers n_i such that $N(K)$ is a formal integral combination

$$N(K) = \sum n_i [\Lambda_i], \tag{3}$$

that is moreover a cycle, that is $\partial N(K) = \sum n_i \partial [\Lambda_i] = 0$. For instance if K is a convex set, and $N(K)$ is given by (2), then these coefficients are equal ± 1.

Denote the coordinates on $\mathbb{R}^n \times S^{n-1}$ by $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$. The form

$$\alpha = \sum_{i=1}^n y_i \, dx_i$$

is a contact form on $\mathbb{R}^n \times S^{n-1}$: that is $\alpha \wedge (d\alpha)^{n-1} \neq 0$ everywhere. Suppose $SO(n)$ acts on $\mathbb{R}^n \times S^{n-1}$ by the same transformation on x and y. Then α is $SO(n)$ invariant, that is α is independent of the choice of the orthonormal basis of $\mathbb{R}^n$. Then $N(K)$ is Legandrian, that is $\alpha|_{\Lambda_i} \equiv 0$ for all i.

We denote by $T\mathbb{R}^n$ the tangent bundle of $\mathbb{R}^n$ and by $ST\mathbb{R}^n$ its sphere subbundle. We identify $ST\mathbb{R}^n$ with $\mathbb{R}^n \times S^{n-1}$. Let $X \subset \mathbb{R}^n$ be a closed smooth n-dimensional manifold with boundary ∂X and let $\nu : \partial X \to ST\mathbb{R}^n$ be the outer unit normal vector field to ∂X. Then we define $N(X)$ as the graph of ν

$$N(X) = \{(x, v) \in \mathbb{R}^n \times S^{n-1}; x \in \partial X, (x, v) = \nu(x)\}. \tag{4}$$

It is easy to check that such defined $N(X)$ is smooth and Legandrian.

Remark 2.1. The idea of Fu' s definition of the normal cycle of a subanalytic set is the following. Let X be a closed subanalytic subset of $\mathbb{R}^n$. Then, locally, X is the zero set of non-negative Lipschitz function $f : U \to \mathbb{R}$. (By a result of Bierstone, Milman, and Pawłucki quoted in [28], f can be chosen C^1 and subanalytic). Then, locally, $N(X)$ over U as the limit $\lim_{\varepsilon\to 0^+} N(f^{-1}([0, \varepsilon])$, cf. [9].

In order to give to the normal cycle a coordinate free form one should replace it as the *conormal cycle* that lives in the sphere bundle of the conormal bundle $ST^*\mathbb{R}^n$. Then the construction can be done with $\mathbb{R}^n$ replaced by any oriented manifold, see [9].

3. Characteristic cycles in real analytic geometry

Characteristic cycles can be defined for semi-algebraic and subanalytic sets cf. [18], [19], [9], or even for sets defined in any o-minimal structure [28], see also an explicit construction in [29]. More precisely, given an oriented real analytic manifold M, we construct a group isomorphism

$$\mathrm{CC} : F(M) \to \mathcal{L}(M)$$

between the group of subanalytically constructible functions $F(M)$ on M and the group of subanalytic $\mathbb{R}^+$-conical Lagrangian cycles $\mathcal{L}(M)$ in T^*M.

3.1. Lagrangian submanifolds.

Let M be an oriented real analytic manifold of dimension n. The cotangent space T^*M is a symplectic manifold. The Liouville form α on T^*M is defined in an invariant way as

$$\alpha(v) = pr_*(v)(\pi_*(v)),$$

where $v \in T(T^*M)$,$pr : T(T^*M) \to T^*M$, $\pi : T^*M \to M$ and $\pi_* : T(T^*M) \to TM$. If $q_1, \ldots, q_n$ are local coordinates on the base M, the dual coordinates $p_1, \ldots, p_n$ are the coefficients of the decomposition of a covector into linear combination of the differentials dq_i so that

$$\alpha = \sum p_i dq_i.$$

Then $\omega = d\alpha$ is a symplectic form on T^*M: $d\omega = 0$ and ω^n is a volume form. In local coordinates

$$\omega = \sum dp_i \wedge dq_i.$$

We orient T^*M so that $dq_1 \wedge \cdots \wedge dq_n \wedge dp_1 \wedge \cdots \wedge dp_n$ is positive. The Liouville form α is sometimes called the symplectic potential.

A submanifold $N \subset T^*M$ is called *Lagrangian* if $\dim N = n$ and $\omega|_N \equiv 0$.

Example 3.1. If $f : M \to \mathbb{R}$ is C^2 then the graph of its differential

$$Gr(df) = \{(x, df(x)) \in T^*M; x \in M\}$$

is Lagrangian. An orientation of M defines an orientation of $Gr(df)$.

Example 3.2. Let S be a submanifold of M. The *conormal bundle* $T^*_S M \subset T^*M$ to S is defined as the set of all covectors $\xi \in T^*M$ such that $\pi(\xi) \in S$ and ξ vanishes on $T_{\pi(\xi)}S$.

If S is given locally by $q_{k+1} = \cdots = q_n = 0$, then $T^*_S M$ is given by $q_{k+1} = \cdots = q_n = 0 = p_1 = \cdots = p_k$. We define an orientation on $T^*_S M$ as follows. Suppose that $dq_1 \wedge \cdots \wedge dq_n$ is a positive volume form on M. Then we choose the orientation of $T^*_S M$ such that $(-1)^{n-k} dq_1 \wedge \cdots \wedge dq_k \wedge dp_{k+1} \wedge \cdots \wedge dp_n$ is a positive volume form.

Note that $\alpha|_{T^*_S M} \equiv 0$ and that $T^*_S M$ is $\mathbb{R}^*$-conical i.e. for any $\lambda \in \mathbb{R}^*$ and any $(x, \xi) \in T^*_S M$ we have $(x, \lambda\xi) \in T^*_S M$.

It is easy to check that $T^*_S M$ is subanalytic, resp. real analytic, if and only if so is S.

A special case of the above example is the zero section $T^*_M M$ of T^*M.

Theorem 3.3. *Let $N \subset T^*M$ be a connected subanalytic real analytic submanifold of T^*M and suppose that it is Lagrangian and $\mathbb{R}^+$-conical. Then $S = \pi(N)$ is subanalytic and $N \subset \overline{T^*_{Reg(S)} M}$, where $Reg(S)$ denote the set of regular points of S.*

Proof. Since N is real analytic either $N \subset T^*_M M$ or $N \cap T^*_M M$ is nowhere dense in N. The former case is easy so we assume the latter. Let ST^*M denote a sphere bundle of T^*M. Then $\pi : ST^*M \to M$ is proper and hence $\pi(N \cap ST^*M)$ is subanalytic. Since N is $\mathbb{R}^+$ conical., $\pi(N) = (N \cap T^*_M M) \cup \pi(N \cap ST^*M)$ and hence it is subanalytic.

We show that $\alpha|_N \equiv 0$. This suffices to show on any real analytic curve in N. Let $s \to (q(s), p(s))$ be such a curve in local coordinates and define $\varphi(s,t) = (q(s), tp(s))$. Then

$$\varphi^*(\alpha) = \sum tp_i(s) q_i'(s)\, ds$$
$$\varphi^*(\omega) = \sum p_i(s) q_i'(s)\, dt \wedge ds.$$

SInce N is $\mathbb{R}^+$ conical and Lagrangian, $\varphi^*(\omega) \equiv 0$ and hence $\varphi^*(\alpha) \equiv 0$ as claimed. Thus $\alpha|_N \equiv 0$.

Let (q,p) be a generic point in N so that S is nonsingular at q and $\pi_* : T_{(q,p)}N \to T_q S$ is a submersion. If S is given locally by $q_{k+1} = \cdots = q_n = 0$, then dq_i, $i = 1, \ldots, k$, are independent as sections of T^*N, and hence the coefficients of $\sum_{i=1}^k p_i dq_i = \alpha|_N \equiv 0$ must be zero. That is $p_1 = \cdots = p_k = 0$. $\square$

Remark 3.4. Let $N \subset T^*M$ be a connected $\mathbb{R}^+$-conical subanalytic real analytic submanifold of T^*M. Then N is Lagrangian if and only if $N \cap ST^*M$ is Legandrian.

3.2. **Stratifications.** Let A and B be two smooth submanifolds on M. We assume that $A \subset \overline{B} \setminus B$. We say that the pair (A, B) satisfies Whitney conditions (a) and (b) if

(a) For any point $a \in A$ and a sequence of points $B \ni b_i \to a$, assume that there is a limit of $\lim T_{b_i} B \to L \subset T_a M$. Then $L \supset T_a A$.
(b) In a local coordinate system, for any point $a \in A$ and sequences of points $B \ni b_i \to a$, $A \ni a_i \to a$, write $\overline{b_i a_i}$ for the secant line through b_i and a_i. Think of $\overline{b_i a_i}$ as a subspace of $T_{b_i} M$. Assume that there is a limit $\lim T_{b_i} B \to L \subset T_a M$ and $\lim \overline{b_i a_i} = K \subset T_a M$. Then $L \supset K$.

Note that Whitney condition (a) is equivalent to $T^*_A M \subset \overline{T^*_B M}$.

Let X be a subanalytic subset of M. *A stratification of* X is a filtration

$$X_* : \emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n = X,$$

by closed subanalytic subset such that each difference $X_i^o := X_i \setminus X_{i-1}$ is a real analytic submanifold of M of dimension i or empty. A stratum of X_* is a connected component of some X_i^o. Thus two distinct strata are disjoint.

We say that the stratification X_* is *a Whitney stratification* if for every pair of strata S_α, S_β such that $S_\alpha \cap \overline{S_\beta} \neq \emptyset$ we have $S_\alpha \subset \overline{S_\beta}$ and (S_α, S_β) satisfies Whitney conditions.

Theorem 3.5 (Thom-Mather Isotopy Lemma, [13])**.** *Let $X \subset M$ be Whitney stratified and let N be a manifold. Let $f : X \to N$ be a proper stratified submersion (a continuous map, that is a C^2 submersion on each stratum). Then f is a locally trivial fibration.*

If $X \subset M$ is closed and Whitney stratified and if $f : X \to N$ is a stratified submersion, then f is locally topologically trivial along each stratum.

3.3. **Characteristic Cycle of a subset.** Let $V \subset M$ be closed subanalytic and let $\{S_i\}$ be a subanalytic Whitney stratification of V. Define

$$\Lambda^o := \bigsqcup \Lambda^o_{S_i}, \qquad \Lambda^o_{S_i} = T^*_{S_i} M \setminus \bigcup_{j \neq i} \overline{T^*_{S_j} M}.$$

Then the connected components of $\Lambda^o_{S_i}$ are subanalytic and real analytic manifolds of dimension n. After subdividing each of these connected components, if necessary, we obtain a subanalytic partition $\Lambda^o = Y \sqcup \bigsqcup \Lambda^o_j$ such that $\dim Y < n$ and each Λ^o_j is an orientable $\mathbb{R}^+$-conical connected real analytic manifold of dimension n. In what follows we fix an orientation of each Λ^o_j. Thus, each Λ^o_j with orientation defines element of

$[\Lambda_j^o] \in C_n^{cl}(T^*M;\mathbb{Z})$, the group of integral subanalytic chains with closed supports. This element is supported in $\overline{\Lambda_j^o}$. Then, *the characteristic cycle of* V is a formal combination

$$\mathrm{CC}(V) := \sum_j i(\Lambda_j^o)\ [\Lambda_j^o], \tag{5}$$

for some integers $i(\Lambda_j^o)$ that can be determined as follows.

Let S be a stratum such that $\dim S < n$. By *a normal slice of* S *at* $x \in S$ we mean the germ of a submanifod (N,x) which is transverse to S at x and such that $\dim N = \operatorname{codim} S$. Let $(x,\xi) \in \Lambda_j^o \subset T_S^*M$ and let $f:(M,x)\to(\mathbb{R},0)$ be a real analytic function such that $\xi = df(x)$ and $Gr(df)$ intersects Λ_j^o transversally at $(x, df(x))$. Assume $\xi \neq 0$. The *upper/lower half-link* of f (along S) is by definition

$$L_f^{\pm} = N \cap V \cap B \cap \{f(x) = \pm\delta\},$$

where B is the ball of radius ε centered at x and $0 < \delta \ll \varepsilon \ll 1$. (The ball B can be open or closed since the Euler characteristic is the same.) We define *the normal index of* $i(\Lambda_j^o)$ by

$$i(\Lambda_j^o) = 1 - \chi(L_f^-). \tag{6}$$

It can be shown that $i(\Lambda_j^o)$ is well defined, that is independent of the choices. If $\dim S = n$, then we define $i(\Lambda_j^o) = 1$.

Proposition 3.6. *The formal sum* $\sum i(\Lambda_j^o)[\Lambda_j^o]$ *is a homology cycle.*

By this we mean the following. As a integral chain $[\Lambda_j^o]$ has the boundary $\partial\overline{\Lambda_j^o}$ that is again an integral subanalytic chain supported in $\overline{\Lambda_j^o} \setminus \Lambda_j^o$. Proposition 3.6 says that the sum of all these boundaries with coefficients $i(\Lambda_j^o)$ is zero. Thus $\mathrm{CC}(V)$ gives a homology class $[\mathrm{CC}(V)] \in H_n^{BM}(T^*M;\mathbb{Z})$ in the Borel-Moore homology of T^*M.

Proposition 3.7. $\mathrm{CC}(V)$ *as an integral homology cycle is independent of the choice of stratification.*

3.4. **Morse Theory and Intersection Formula.** Let $f: M \to \mathbb{R}$ be a smooth function. Recall that a critical point $x \in M$ of f is called Morse if the Hessian matrix $H_x(f)$ of f at x is nondegenerate. An easy local computation shows that a critical point $x \in M$ of f is Morse if and only if $Gr(df)$ meets T_M^*M at $(x,0)$ transversally. The index $\lambda(x)$ of x is the number of negative eigenvalues of $H_x(f)$. Then

$$\begin{aligned}(T_M^*M.Gr(df)))_{(x,0)} \\ &= \chi(B \cap \{f(x) \le +\delta\}) - \chi(B \cap \{f(x) \le -\delta\}) \\ &= 1 - \chi(B \cap \{f(x) = -\delta\}) = (-1)^{\lambda(x)}.\end{aligned} \tag{7}$$

Similarly, let S be a smooth submanifold of M, $\dim S < \dim M$. We say that $f: M \to \mathbb{R}$ is a Morse function for S at $x \in S$ if $Gr(df)$ meets T_S^*M at $(x, df(x))$

transversally. Then $f|_S$ is Morse at x and an analog of (7) holds

$$\begin{aligned}&(T^*_S M.Gr(df)))_{(x,df(x))}\\&=\chi(B\cap S\cap\{f(x)\le+\delta\})-\chi(B\cap S\cap\{f(x)\le-\delta\})\\&=1-\chi(B\cap S\cap\{f(x)=-\delta\}).\end{aligned}$$

This leads to the following general formula. For a proof see [19] Thm. 9.5.6, [18] or [30].

Theorem 3.8. (Intersection Formula)
Let $V\subset M$ be subanalytic closed, $x\in V$, and let $\{S_i\}$ be a subanalytic Whitney stratification of V. Let $f:(M,x)\to(\mathbb{R},0)$ be smooth.

(1) *If $(x,df(x))\notin\bigcup T^*_{S_i}M$ then $f|_V$ is locally topologically trivial along the stratum containing x. In particular, the homeomorphism type of $B\cap\{x\in V,f(x)\le\delta\}$, where B is the ball of radius ε centered at p and $\varepsilon\ll 1$, is independent of $|\delta|\ll\varepsilon$. Similarly the homeomorphism type of $B\cap\{x\in V,f(x)=\delta\}$ is independent of $|\delta|\ll\varepsilon$.*

(2) *If $(x,df(x))$ is an isolated point of $\bigcup T^*_{S_i}M\cap Gr(df)$ then*

$$\begin{aligned}&(\mathrm{CC}(V).Gr(df))_{(x,df(x))}\\&=\chi(B\cap\{x\in V,f(x)\le+\delta\})-\chi(B\cap\{x\in V,f(x)\le-\delta\}),\\&=1-\chi(B\cap\{x\in V,f(x)=-\delta\}).\end{aligned}\tag{8}$$

where B denotes the ball of radius ε centered at p and $0<\delta\ll\varepsilon\ll 1$.

In particular, Intersection Formula allows us to recover the set V from its characteristic cycle. Indeed, given $p\in M$, it suffices to intersect $\mathrm{CC}(V)$ with $Gr(df)$, where $f:(M,p)\to(\mathbb{R},0)$ is a Morse function of index 0 (for instance $f(x)=x_1^2+\cdots+x_n^2$ in local coordinates). By Whitney condition (b) the assumptions of (2) of Theorem 3.8 are satisfied and hence

$$(\mathrm{CC}(V).Gr(df))_{(p,df(p))}=\begin{cases}1 & \text{if } p\in V\\ 0 & \text{if } p\notin V\end{cases}\tag{9}$$

Finally, the additivity of the Euler characteristic $\chi(X\cup Y)=\chi(X)+\chi(Y)-\chi(X\cap Y)$ for X,Y subanalytic and compact applied to the right-hand side of (6) or (8) gives the following.

Corollary 3.9. *Let $X,Y\in M$ be subanalytic and closed. Then*

$$\mathrm{CC}(X\cup Y)=\mathrm{CC}(X)+\mathrm{CC}(Y)-\mathrm{CC}(X\cap Y).\tag{10}$$

Remark 3.10. It may happen that the normal index (6) of a stratum is zero. This is always the case if we substratify a Whitney stratification, be adding "not necessary" strata. But it may also happen for a "minimal" Whitney stratificatication. For instance, let $V = \{(x,y,z) \in \mathbb{R}^3; x^2+y^2=z^3\} \cup \{x=y=0\}$. Then $Sing(V) = \{0\}$. A generic normal link of V at $\{0\}$, that is the intersection of V with a generic hyperplane not passing thorough the origin, is either a point or the union of a circle and a point. Thus the normal index (6) of this stratum is 0.

Remark 3.11. There are many possible sign conventions depending for instance on the choices of orientations. Thus for instance the characteristic cycle constructed by Fu [9] corresponds to that of Kashiwara-Schapira [19] after the application of the antipodal map (multiplication by -1 in the fibers of T^*M).

3.5. **Subanalytically constructible functions.** We suppose for simplicity that M is compact. We denote by $F(M)$ the group of integer-valued *constructible functions on* M i.e. finite sums

$$\alpha = \sum_i n_i \mathbb{1}_{V_i} \tag{11}$$

where V_i are subanalytic subsets of M, and by $\mathbb{1}_V : M \to \mathbb{Z}$ the function that equals 1 on V and 0 elsewhere. Note that by taking the closures and differences one may always assume that all V_i of (11) are closed in M.

The *Euler integral* of α is defined as follows, see [35], [27], [19]. If all V_i of (11) are compact, then we put

$$\int \alpha \, d\chi := \sum_i n_i \chi(V_i). \tag{12}$$

(If the V_i's are assumed only locally compact then $\int \alpha \, d\chi := \sum_i n_i \chi_c(V_i)$, where χ_c denotes the Euler characteristic with compact supports). It follows from the identity

$$\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y).$$

for X and Y compact, that the right-hand side of (12) depends only on α and not on its presentation as the sum.

Thus, similarly, we may define the charcteristic cycle of a constructible function by setting

$$\mathrm{CC}(\alpha) := \sum_i n_i \, \mathrm{CC}(V_i). \tag{13}$$

Theorem 3.12. ([18], [19], [9], [28])
The defined above charateristic cycle gives a group isomorphism

$$\mathrm{CC} : F(M) \to \mathcal{L}(M)$$

between the group of subanalytically constructible functions $F(M)$ on M and the group of subanalytic $\mathbb{R}^+$-conical Lagrangian cycles $\mathcal{L}(M)$ in T^*M.

The main point of the proof is to show that actually $\mathrm{CC}(V)$ is a cycle. One way, as in [9], is to use the Lagrangian specialization principle to define $\mathrm{CC}(V)$. Then for arbitrary α the claim follows from additivity (10). A good exercise for the reader is to use our definition and show that $\mathrm{CC}(V)$ is a cycle for a subanalytic curve $V \subset \mathbb{R}^2$.

Let $\Lambda \in H_n^{BM}(T^*M;\mathbb{Z})$ be a Lagrangian cycle. Write $\sum i(\Lambda_j^o)[\Lambda_j^o]$, where Λ_j^o are connected real analytic $\mathbb{R}^+$-conical Lagrangian submanifolds and $i(\Lambda_j^o) \neq 0$. By *the support* of Λ we mean the union of $\overline{\Lambda_j^o}$. It will be denoted by $|\Lambda|$. Let $S = \pi(|\Lambda|)$. Let $Reg(S) = \cup S_i$ be the decomposition of $Reg(S)$ into connected components. Since Λ is a cycle, it can be showed that there are integers $n_i \neq 0$ such over S_i, Λ equals $n_i[T^*_{S_i}M]$.

Let $\Lambda' = \Lambda - \sum_i n_i[T^*_{S_i}M]$ and let $S' = \pi(|\Lambda'|)$. Then $S' \subset Sing(S)$. Thus to show that CC is surjective we may argue by induction on $\dim \pi(|\Lambda|)$. If $\Lambda' = \mathrm{CC}(\alpha')$ then $\Lambda = \mathrm{CC}(\alpha)$, where $\alpha = \alpha' + \sum n_i 1\!\!1_{S_i}$.

There is another, direct description of the operation inverse to CC. Given a Lagrangian cycle $\Lambda \in \mathcal{L}(M)$ and let $p \in M$. Let $f : (M,p) \to (\mathbb{R},0)$ be a Morse function of index 0 (for instance $f(x) = x_1^2 + \cdots + x_n^2$ in local coordinates). Then

$$\alpha(p) = (\Lambda.Gr(df))_{(p,df(p))}\,. \tag{14}$$

Note that this formula follows from (9). The right-hand side of (14) is called *the Euler obstruction of* Λ *at* p.

4. Chern-Schwartz-MacPherson classes and characteristic cycles

In this section we recall some basic results and constructions on Chern-Schwartz-MacPherson classes and characteristic cycles of singular complex algebraic varieties. For more details the reader is refered to [2, 14, 20, 23, 26, 29].

For a variety X we denote by $F(X)$ the group of integer-valued *constructible functions on* X i.e. finite sums

$$\alpha = \sum_i n_i 1\!\!1_{V_i}$$

where n_i are integers and V_i are subvarieties of X.

Based on Grothendieck's ideas for modifying Grothendieck's conjecture on a Riemann-Roch type formula concerning the constructible étale sheaves and Chow rings (see [16], Part II), Deligne made the following conjecture and R. MacPherson [23] proved it affirmatively:

Theorem 4.1. *There is a unique transformation $c_* : F(\cdot) \to H_*(\cdot)$, from constructible functions on complex algebraic varieties to their homologies with integral coefficients and closed supports, satisfying:*

1. $f_*c_*(\alpha) = c_*f_*(\alpha)$ *for a proper morphism* $f : X \to Y$.
2. $c_*(\alpha + \beta) = c_*(\alpha) + c_*(\beta)$,
3. $c_*(1\!\!1_X) = c(TX) \cap [X]$ *for X nonsingular,*

where $c(TX) \in H^*(X, \mathbb{Z})$ *denote the Chern class of the tangent bundle.*

It turned out that the classes constructed by MacPherson are, by the Alexander duality isomorphism, equal to the characteristic classes introduced before by M.-H. Schwartz, cf. [32, 3]

4.1. **Constructible functions.** There are many interesting operations on constructible functions: sum, product, pull-back, push-forward, specialization, duality, and Euler integral, inherited from sheaf theory by taking the index of a constructible complex of sheaves. Recall that for a constructible complex of sheaves $\mathcal{F}_\bullet$ on X its index is the stalk-wise Euler characteristic $p \to \chi(\mathcal{F}_\bullet)(p) = \sum (-1)^i \dim H^i(\mathcal{F}_\bullet)_p$. It is a constructible function. Note that this definition is purely local so the global properties of $\mathcal{F}_\bullet$ are lost. The operations on constructible functions can be defined independently by means of Euler integral, see also [35], [27], [19].

Let $\alpha = \sum_i n_i 1\!\!1_{V_i}$. Recall that the *Euler integral* of α is defined as the weighted Euler characteristic:

$$\int \alpha \, d\chi := \sum_i n_i \chi^c(V_i).$$

For a proper map $f : X \to Y$ the *proper push-forward* $f_* : F(X) \to F(Y)$ is given by

$$(f_*\alpha)(y) := \int_{f^{-1}(y)} \alpha \, d\chi.$$

It follows from the existence of Whitney stratification for complex algebraic varieties and Thom-Mather Isotopy Lemma that $f_*\alpha$ is a constructible function on Y.

Let $f : X \to S$ be a morphism to a curve and let s_0 be a nonsingular point of S. Denote $X_0 = f^{-1}(s_0)$. The *specialization* homomorphism $\mathrm{sp} : F(X) \to F(X_0)$, or the *nearby Euler characteristic*, is the Euler integral on the Milnor fibre of f. That is, at $p \in X_0$ and for α as above

$$\mathrm{sp}\,(\alpha)(p) = \int_{F_p} \alpha \, d\chi = \sum_i n_i \chi(F_p \cap V_i), \tag{15}$$

where F_p is the Milnor fibre of f at p. That is, $F_p = f^{-1}(s) \cap B(p, \varepsilon)$, where $B(p, \varepsilon)$ denotes the ball centered at p of radius ε and s is chosen so that $0 < |s - s_0| \ll \varepsilon \ll 1$.

4.2. **Characteristic cycles.** Let M be a nonsingular complex algebraic variety of dimension n and let T^*M denote the cotangent bundle of M. An algebraic subvariety Λ of T^*M is called *Lagrangian* if it is of pure dimension n and if the symplectic form ω vanishes identically on the set $Reg(\Lambda) = \Lambda \setminus Sing(\Lambda)$ of regular points of Λ. We say that Λ is *conical* if for any $\lambda \in \mathbb{C}^*$ and any $(x, \xi) \in \Lambda$ we have $(x, \lambda\xi) \in \Lambda$.

Let V be a closed subvariety of M. The *conormal space* to V in M

$$T^*_V M := Closure \left\{ (x, \xi) \in T^*M \ \mid \ x \in Reg(V), \ \xi|_{T_x Reg(V)} \equiv 0 \right\},$$

is a conical Lagrangian subvariety of T^*M. If V is smooth then it coincides with the conormal bundle of V. By a complex algebraic version of Theorem 3.3 each irreducible conical Lagrangian subvariety Λ of T^*M is the conormal space of $\pi(\Lambda)$ that is an irreducible subvariety of M.

Remark 4.2. Since the singularities of Λ are of real codimension ≥ 2 it is automatically an integral cycle. Thus in the complex case there are important simplifications comparing to the real analytic (or algebraic) case described in the previous chapter. All irreducible conical Lagrangian subvarieties are conormal spaces and all irreducible conical Lagrangian subvarieties are Lagrangian cycles. Moreover all smooth varieties are canonically oriented as real manifolds by the complex orientation. Thus if Λ is a subvariety of T^*M then the complex orientation on $Reg(\Lambda)$ induces without a unique element of $C_*^{BM}(T^*M,\mathbb{Z})$ that we denote also by Λ.

Ler $\mathcal{L}(M)$ denote the free abelian group generated by the set of conical Lagrangian subvarieties of T^*M. Thus each element of $\mathcal{L}(M)$ is an integral combination of irreducible Lagrangian subvarieties. More generally, for a subvariety $X \subset M$ let $\mathcal{L}(X)$ denote the subgroup of $\mathcal{L}(M)$ given by the conical Lagrangian subvarieties of T^*M over X. We call an element of $\mathcal{L}(X)$ a *conical Lagrangian cycle over* X.

To a constructible function $\alpha \in F(X)$ one associates its *characteristic cycle* $\mathrm{CC}(\alpha) \in \mathcal{L}(X)$ so that we get a group isomorphism $\mathrm{CC} : F(X) \to \mathcal{L}(X)$. For instance, for a smooth irreducible subvariety V, $\mathrm{CC}(\mathbb{1}_V)$ can be defined by means of the characteristic cycle of a sheaf, cf. for instance [4], by

$$\mathrm{CC}(\mathbb{1}_V) = \mathrm{CC}(i_*\mathbb{C}_V) = (-1)^{\dim V} T_V^*M, \tag{16}$$

where $i : V \hookrightarrow M$ is the inclusion. Then, for singular V,

$$T_V^*M = (-1)^{\dim V}\,\mathrm{CC}(Eu_V),$$

where Eu_V denotes MacPherson's Euler obstruction [23].

Remark 4.3. In literature there are two sign conventions in the definition of CC that differ by $(-1)^{\dim M}$. We are following that of [20].

Let $f : (M,p) \to (\mathbb{C},0)$ be the germ of a holomorphic function and let $\alpha = \sum_i n_i \mathbb{1}_{V_i}$ be a constructible function on M. Let $\mathrm{sp}\,\alpha(p)$ be the specialization of α to the zero fibre of f as defined in (4.5). Let $\mathcal{S} = \{S_j\}$ be a Whitney stratification of M such that each V_i is a union of strata. Suppose that the graph $Gr(df)$ of df, considered as a section of T^*M, intersects $\bigcup T_{S_j}^*M$ only at $(p, df(p))$. Then by the index formula for the sheaf vanishing cycles due to Lê, Dubson, and Sabbah, cf. [6] and (4.5) and (4.6) of [26], the local intersection number of the cycles $\mathrm{CC}(\alpha)$ and $Gr(df)$ equals

$$(\mathrm{CC}(\alpha).Gr(df))_{(p,df(p))} = \alpha(p) - \mathrm{sp}\,\alpha(p). \tag{17}$$

Thus one may interpret $\mathrm{CC}(\alpha)$ as the set of such covectors $(p,\xi) \in T^*M$ that the Euler integral of the fibers of functions $f : (M,p) \to (C,0)$ with $df(p) = \xi$ changes at p. Clearly, by Thom-Mather theory, there is no change of topology of fibers of $f|_{V_i}$ if $(p,\xi) \notin \bigcup T^*_{S_j}M$. Thus

$$\mathrm{CC}(\alpha) = \sum_i n_i T^*_{S_j} M$$

with integer coefficients n_i. In general these coefficients may be zero or negative. By (17), they are determined by the vanishing Euler characteristic of such f that $Gr(df)$ intersects $T^*_{S_i}M$ at a generic point.

Example 4.4. Let $p \in X \subset M$. The coefficient of T^*_pM in $\mathrm{CC}(\mathbb{1}_X)$ equals

$$1 - \chi(lk_{\mathbb{C}}(X,p))$$

where $lk_{\mathbb{C}}(X,p)$ is the complex link of X at p (in local coordinates the intersection of X with a generic hyperplane near p).

There are operations of proper push-forward and specialization on conical Lagrangian cycles defined geometrically. CC is a natural transformation in the sense that it commutes with these operations and the corresponding operations on constructible functions, cf. [14], [20], [26],

(1) $f_*\,\mathrm{CC}(\alpha) = \mathrm{CC}\,f_*(\alpha)$ for proper morphisms $f : X \to Y$
(2) $\mathrm{CC}(\alpha + \beta) = \mathrm{CC}(\alpha) + \mathrm{CC}(\beta)$
(3) $\mathrm{CC}(\mathrm{sp}(\alpha)) = \mathrm{Sp}(\mathrm{CC}(\alpha))$.

We describe below the specialization.

4.3. **Lagrangian specialization.** The principle of Lagrangian specialization, valid both in the real and the complex set-ups, says that "the limit" of conical Lagrangian cycles is a conical Lagrangian cycle, cf. [19], [26], [21], [22], [17]. Here is a possible precise statement.

Theorem 4.5. *Let $\Omega \subset T^*M \times \mathbb{C}$ be a subvariety such that for $s \neq 0$ and small, $\Omega_s = \Omega \cap (T^*M \times \{s\})$ is conical Lagrangian. Suppose $\Omega_0 = \Omega_0 \cap \overline{\Omega \setminus \Omega_0}$. Then Ω_0 is conical Lagrangian.*

Proof. Clearly Ω_0 is conical. By an argument of the proof of Theorem 3.3, the Liouville form α satisfies $\alpha|_{Reg(\Omega_s)} \equiv 0$, for $s \neq 0$ and small. It suffices to show that this is also the case for $s = 0$.

We work in a neighborhood $U \subset \Omega$ of a generic point $p \in \Omega_0$. Clearly p is a smooth point of Ω_0 and assume that it is a singular point of Ω. Then, if U is chosen sufficiently small, $U \cap \Omega_0 = Sing(U)$, and the normalization $\widetilde{U}$ of U is smooth. The normalization map, $\Psi : \widetilde{U} \to U$, is an analytic covering branched along $U \cap \Omega_0$.

Denote the projection on the second factor $T^*M \times \mathbb{C} \to \mathbb{C}$ by φ. By genericity of the choice of p, we may assume that $\varphi \circ \Psi = u^k$, where u is a local coordinate at a point of $\Psi^{-1}(p) \in \widetilde{U}$ and $k \in \mathbb{N}$. By assumption $\Psi^*\alpha$ vanishes on the nonzero levels of u. Hence it vanishes on the zero level, that is on $\Psi^{-1}(\Omega_0)$. SInce $\Psi^{-1}(\Omega_0) \to \Omega_0$ is an (unbranched) covering this gives that $\alpha|_{U\cap\Omega_0} \equiv 0$. Finally, if α vanishes on a dense subset of $Reg(\Omega_0)$ then it vanishes on $Reg(\Omega_0)$. □

Remark 4.6. Let $\Omega \subset T^*M \times \mathbb{C}$ be an irreducible variety of dimension $n+1$ such that $\Omega_0 = \Omega_0 \cap \overline{\Omega \setminus \Omega_0}$. Then each irreducible component $Y \subset \Omega_0$ can be given a multiplicity as follows. Let p be a generic point of Y and let $\Psi : \widetilde{U} \to U$ be the normalization map as in the above proof. To each point $\tilde{p} \in \Psi^{-1}(p)$ we associate a number $k = k_{\tilde{p}}$, as above. Then the multiplicity assigned to Y is $\sum_{\tilde{p}\in\Psi^{-1}(p)} k_{\tilde{p}}$. Thus, in Theorem 4.5, $\lim_{s\to 0}\Omega_s$ can be understood as a Lagrangian conical cycle supported in Ω_0.

Let X be a subvariety of $M \times \mathbb{C}$ and let $X_0 = \overline{X \cap (M \times \mathbb{C}^*)} \cap (M \times \{0\})$. We define *the specialization morphism*

$$\mathrm{Sp} : \mathcal{L}(X) \to \mathcal{L}(X_0)$$

as follows. Let $\Lambda = T^*_S(M \times \mathbb{C}) \in \mathcal{L}(X)$, where S is an iirreducible subvariety of X and let $f : S \to \mathbb{C}$ be induced by the projection onto the second factor $S \subset M \times \mathbb{C} \to \mathbb{C}$. If $f(S)$ is a point then we put $\mathrm{Sp}(\Lambda) = 0$. Thus suppose that this is not the case. Then $Reg(f) = \{x \in Reg(S) : df_x \neq 0\}$ is dense in S. For $s \in \mathbb{C}$ we consider $f^{-1}(s)$ as a subvariety of $M \times \{s\}$ and hence $T(f^{-1}(s)) \subset T(M \times \{s\})$. Consider the *relative conormal*

$$T^*_f(M\times\mathbb{C}) = Closure\left\{(x,s,\xi,\eta) \in T^*(M\times\mathbb{C}) \mid (x,s) \in Reg(f),\ \xi|_{T_x Reg(f^{-1}(s))} \equiv 0\right\},$$

and its image $\Omega \subset T^*M \times \mathbb{C}$

$$\Omega = Closure\left\{(x,s,\xi) \in T^*(M\times\mathbb{C}) \mid (x,s) \in Reg(f),\ \xi|_{T_x Reg(f^{-1}(s))} \equiv 0\right\}.$$

Then Ω satisfies the assumptions of Theorem 4.5. We define $\mathrm{Sp}(\Lambda)$ the conical Lagrangian cycle of $\mathcal{L}(X_0)$ given by Remark 4.6 multiplied by -1. This sign comes from the sign in (16).

4.4. **Chern-Schwartz-MacPherson classes.** Let $X \subset M$ be a subvariety, $\dim X < n$, and let $\alpha \in F(X)$. Since $\mathrm{CC}(\alpha)$ is $\mathbb{C}^*$-conical, its projectivization $\mathbb{P}\,\mathrm{CC}(\alpha)$ is a well-defined integral cycle of $\mathbb{P}T^*M$. Folowing Sabbah, [26], (1.2.1), we define the Chern-Schwartz-MacPherson class (the CSM class) of α by

$$c_*(\alpha) := (-1)^{n-1} c\left(TM|_X\right) \cap \pi_*\left(c(\mathcal{O}(1))^{-1} \cap [\mathbb{P}\,\mathrm{CC}(\alpha)]\right), \tag{18}$$

where $\mathcal{O}(1)$ is the canonical line bundle on $\mathbb{P}T^*M$ and $\pi : \mathbb{P}T^*M|_X \to X$ denotes the projection. Sabbah showed that this class coincides with the call defined by MacPherson [23] and showed how to deduce the basic properties of the CSM class from the conormal geometry. Using Sabbah's own words "cela montre que la théorie des classes de Chern de [23] se ramène à une théorie de Chow sur T^*M, qui ne fait intervenir que des classes fondamentales".

For X smooth irreducible and $\alpha = 1\!\!1_X$ the formula (18) follows from the classical formulas for Chern end Segre class of the conormal bundle $T^*_X M$ of X. Indeed, in this case, by (16), the formula (18) reads

$$c(TX) \cap [X] = (-1)^{n-\dim X-1} c\left(TM|_X\right) \cap \pi_*\left(c(\mathcal{O}(1))^{-1} \cap [\mathbb{P}T^*_X M]\right), \tag{19}$$

The (cohomological) Segre class of $T^*_X M$ equals

$$s(T^*_X M) = \pi_*\left(c(\mathcal{O}(-1))^{-1} \cap [\mathbb{P}T^*_X M]\right),$$

see [12] Section 3.1, and hence

$$\begin{aligned} c(T^*X) \cap [X] &= \left(c\left(T^*M|_X\right) \cup s(T^*_X M)\right) \cap [X] \\ &= c\left(T^*M|_X\right) \cap \pi_*\left(c(\mathcal{O}(-1))^{-1} \cap [\mathbb{P}T^*_X M]\right), \end{aligned}$$

that implies (19).

The operation inverse to CC is related to MacPherson's Euler obstruction as follows. Let M be a complex manifold and let V be a subvariety of M. Consider V as a subanalytic subset of M and M itself as an oriented real analytic manifold. Then $T^*_V M$ is a real Lagrangian cycle. Let $p \in V$ and $f : (M,p) \to (\mathbb{R}, 0)$ be a real Morse function of index 0. Then $(Gr(df).T^*_V M))_{(p,df(p))} = (-1)^{\dim_{\mathbb{C}} V} Eu_V(p)$. This formula for the Euler obstruction is essentially the definition of MacPherson, where the intersection $Gr(df)$ is replaced by the intersection with the section given by the radial vector field.

The Chern-Mather class of V, defined originally in terms of Nash blowing-up cf. [23], equals

$$c_M(V) = c_*(Eu_V) = (-1)^{n-1-\dim V} c\left(TM|_V\right) \cap \pi_*\left(c(\mathcal{O}(-1)) \cap [\mathbb{P}T^*_V M]\right). \tag{20}$$

Let V be an irreducible subvariety of M. Write

$$\mathrm{CC}(V) = (-1)^{\dim V} T^*_V M + \sum_i n_i T^*_{V_i} M,$$

where V_i are proper subvarieties of V. In terms of constructible functions it translates to

$$1\!\!1_V = Eu_V + \sum (-1)^{\dim V - \dim V_i} n_i Eu_{V_i}.$$

Therefore

$$c_*(V) = c_M(V) + \sum (-1)^{\dim V - \dim V_i} n_i c_M(V_i).$$

4.5. **Verdier specialization of CSM classes.** Let $X \subset M$ and let $f : X \to \mathbb{C}$ be a proper morphism. Let us denote $f^{-1}(s)$ by X_s. we denote by D_r a disc of radius r. It is known that if $r > 0$ is sufficiently small then $X_0 \hookrightarrow f^{-1}(D_r)$ is a homotopy equivalence. For small nonzero $s \in \mathbb{C}$ we define the *specialization map on homology* $\mathrm{sp} : H_*(X_s) \to H_*(X_0)$ as the composition

$$H_*(X_s) \to H_*(f^{-1}(D_r)) \approx H_*(X_0).$$

Let $\alpha \in F(X)$. For small nonzero $s \in \mathbb{C}$ we denote α_s the restriction $\alpha_s = \alpha|_{X_s} \in F(X_s)$. Recall after Section 4.1 that $\mathrm{sp} : F(X) \to F(X_0)$ is the specialization morphism on constructible functions.

Theorem 4.7 (cf. Verdier [34], see also [26], [21], [8])**.** *For small nonzero $s \in \mathbb{C}$, we have the following identity in $H_*(X_0)$*

$$c_*(\mathrm{sp}\,\alpha) = \mathrm{sp}\, c_*(\alpha_s).$$

4.6. **Stiefel-Whitney classes.** In 1935 Stiefel defined a characteristic class $w_i(X) \in H_i(X; \mathbb{Z}_2)$ for any smooth compact manifold. He conjectured that $w_i(X)$ is represented by the sum of all the i-simplices of the first barycentric subdivision of a triangulation of X. Stiefel's Conjecture was proved by Whitney in 1939. In 1969 Sullivan observed that Stiefel's definition can be applied to real analytic spaces since they are (mod 2) Euler spaces, that is to say, the link of each point has even Euler characteristic. Then, for a triangulated Euler space, the sum of all the i-simplices of the first barycentric subdivision is a $\mathbb{Z}_2$-cycle.

It was noticed in [11] that the Stiefel-Whitney classes of subanalytic sets can be defined via the characteristic cycles using a real analog of Sabbah's formula. We give below just a short account, for details the reader is refered to [11].

Let M be an oriented real analytic manifold. Verdier Duality on sheaves, cf. [27], [19]), induces a duality on (subanalytically) constructible functions. This duality can be written as

$$D\alpha(p) = \alpha(p) - \int_{S_p^\varepsilon} \alpha \, d\chi,$$

where S_p^ε is a small sphere centered at p. The corresponding duality on the $\mathbb{R}^+$-conical Lagrangian cycles is given by the antipodal map that is by the multiplication by (-1) in the fibres of T^*M. We denote this duality as $D : \mathcal{L}(M) \to \mathcal{L}(M)$. Thus, after [11],

$$\mathrm{CC} \circ D = D \circ \mathrm{CC} \tag{21}$$

Note that in the complex case the duality on constructible functions and the one on conical Lagrangian cycles are the identity maps.

A constructible function $\alpha \in F(M)$ is called *(mod 2) Euler* if it is self dual modulo 2 (equivalently its Euler integral along any small sphere is even). By [33] every real analytic set is Euler. By (21) the constructible function α is Euler if and only if its characteristic cycle is mod 2 $\mathbb{R}^*$-conical. Hence, for such a function, the projectivization of its characteristic cycle

$$\mathbb{P}Ch(\alpha) \subset \mathbb{P}T^*M$$

is well-defined as a (mod 2)-cycle.

Let $\alpha \in F(M)$ be an Euler constructible function with support contained in compact subanalytic $X \subset M$, $\dim X < \dim M$. Then we define the ith Stiefel-Whitney class of

α by a formula corresponding to (18)

$$w_i(\alpha) = \pi_*(\gamma_M^{n-i-1} \cap [\mathbb{P}Ch(\alpha)]), \tag{22}$$

where $\pi : \mathbb{P}T^*M \to M$ is the projection and

$$\gamma_M^k = \sum_j \pi^*(w^i(TM)) \cap \zeta_M^{k-j},$$

where $\zeta_M \in H^1(\mathbb{P}T^*M; \mathbb{Z}_2)$ is the first Stiefel-Whitney class of the tautological line bundle on $\mathbb{P}T^*M$. Defined this way, Stiefel-Whitney homological classes satisfy the axioms analogous to the Deligne-Grothendieck axioms for the CSM-classes and the Verdier specialization property.

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Laboratoire J.A. Dieudonné, UMR CNRS 7351, Université de Nice - Sophia Antipolis, Parc Valrose, 06108 Nice Cedex 02, France

E-mail address: adam.parusinski@unice.fr