REGULAR COVERS OF OPEN RELATIVELY COMPACT SUBANALYTIC SETS

ADAM PARUSIŃSKI

Abstract. Let U be an open relatively compact subanalytic subset of a real analytic manifold. We show that there exists a finite linear cover (in the sense of Guillermou and Schapira) of U by subanalytic open subsets of U homeomorphic to a unit ball.

We also show that the algebra of open relatively compact subanalytic subsets of a real analytic manifold is generated by subsets subanalytically and bi-lipschitz homeomorphic to a unit ball.

Let M be a real analytic manifold of dimension n. In this paper we study the algebra $\mathscr{S}(M)$ of relatively compact open subanalytic subsets of M. As we show this algebra is generated by sets with Lipschitz regular boundaries. More precisely, we call a relatively compact open subanalytic subset $U \subset M$ *an open subanalytic Lipschitz ball* if its closure is subanalytically bi-Lipschitz homeomorphic to the unit ball of $\mathbb{R}^n$. Here we assume that M is equipped with a Riemannian metric. Any two such metrics are equivalent on relatively compact sets and hence the above definition is independent of the choice of a metric.

Theorem 0.1. *The algebra $\mathscr{S}(M)$ is generated by open subanalytic Lipschitz balls.*

That is to say if U is a relatively compact open subanalytic subset of M then its characteristic function 1_U is a linear combination of characteristic functions $1_{W_1}, ..., 1_{W_m}$, where the W_j are open subanalytic Lipschitz balls. Note that, in general, U cannot be covered by finetely many subanalytic Lipschitz balls, as it is easy to see for $\{(x,y) \in \mathbb{R}^2; y^2 < x^3, x < 1\}$, $M = \mathbb{R}^2$, due to the presence of cusps. Nevertheless we show the existence of a "regular" cover in the sense that we control the distance to the boundary.

Theorem 0.2. *Let $U \in \mathscr{S}(M)$. Then there exist a finite cover $U = \bigcup_i U_i$ by open subanalytic sets such that :*

1. *every U_i is subanalytically homeomorphic to an open n-dimensional ball;*
2. *there is $C > 0$ such that for every $x \in U$, $\operatorname{dist}(x, M \setminus U) \le C \max_i \operatorname{dist}(x, M \setminus U_i)$*

The proof of Theorem 0.1 is based on the classical cylindrical decomposition and the L-regular decomposition of subanalytic sets, cf. [4], [9], [10]. L-regular sets are natural multidimensional generalization of classical cusps. We recall them briefly in Subsection 1.6. For the proof of Theorem 0.2 we need also the regular projection theorem, cf. [7], [8], [9], that we recall in Subsection 1.4.

We also show the following strengthening of Theorem 0.2.

Theorem 0.3. *In Theorem 0.2 we may require additionally that all U_i are open L-regular cells.*

For an open $U \subset M$ we denote $\partial U = \overline{U} \setminus U$.

1. Proofs

1.1. **Reduction to the case $M = \mathbb{R}^n$.** Let $U \in \mathscr{S}(M)$. Choose a finite cover $\overline{U} \subset \bigcup_i V_i$ by open relatively compact sets such that for each V_i there is an open neighborhood of $\overline{V}_i$ analytically diffeomorphic to $\mathbb{R}^n$. Then there are finitely many open subanalytic U_{ij} such that $U_{ij} \subset V_i$ and 1_U is a combination of $1_{U_{ij}}$. Thus it suffices to show Theorem 0.1 for relatively compact open subanalytic subsets of $\mathbb{R}^n$.

Similarly, it suffices to show Theorems 0.2 and 0.3 for $M = \mathbb{R}^n$. Indeed, it follow from the observation that the function

$$x \to \max_i \operatorname{dist}(x, M \setminus V_i)$$

is continuous and nowhere zero on $\bigcup_i V_i$ and hence bounded from below by a nonzero constant $c > 0$ on $\overline{U}$. Then

$$\operatorname{dist}(x, M \setminus U) \le C_1 \le c^{-1} C_1 \max_i \operatorname{dist}(x, M \setminus V_i)$$

where C_1 is the diameter of $\overline{U}$ and hence, if $c^{-1}C_1 \ge 1$,

$$\operatorname{dist}(x, M \setminus U) \le c^{-1} C_1 \max_i (\min\{\operatorname{dist}(x, M \setminus U), \operatorname{dist}(x, M \setminus V_i)\}).$$

Now if for each $U \cap V_i$ we choose a cover $\bigcup_j U_{ij}$ satisfying the statement of Theorem 0.2 or 0.3 then for $x \in U$

$$\begin{aligned} \operatorname{dist}(x, M \setminus U) &\le c^{-1} C_1 \max_i (\min\{\operatorname{dist}(x, M \setminus U), \operatorname{dist}(x, M \setminus V_i)\} \\ &\le c^{-1} C_1 \max_i \operatorname{dist}(x, M \setminus U \cap V_i) \le C c^{-1} C_1 \max_{ij} \operatorname{dist}(x, M \setminus U_{ij}) \end{aligned}$$

Thus the cover $\bigcup_{i,j} U_{ij}$ satisfies the claim of Theorem 0.2, resp. of Theorem 0.3.

1.2. **Regular projections.** We recall after [8], [9] the subanalytic version of the regular projection theorem of T. Mostowski introduced originally in [7] for complex analytic sets germs.

Let $X \subset \mathbb{R}^n$ be subanalytic. For $\xi \in \mathbb{R}^{n-1}$ we denote by $\pi_\xi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the linear projection parallel to $(\xi, 1) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Fix constants $C, \varepsilon > 0$. We say that $\pi = \pi_\xi$ is *(C, ε)-regular at $x_0 \in \mathbb{R}^n$ (with respect to X)* if

(a) $\pi|_X$ is finite;

(b) the intersection of X with the open cone

$$\mathscr{C}_\varepsilon(x_0, \xi) = \{x_0 + \lambda(\eta, 1); |\eta - \xi| < \varepsilon, \lambda \in \mathbb{R} \setminus 0\} \tag{1.1}$$

is empty or a finite disjoint union of sets of the form

$$\{x_0 + \lambda_i(\eta)(\eta, 1); |\eta - \xi| < \varepsilon\},$$

where λ_i are real analytic nowhere vanishing functions defined on $|\eta - \xi| < \varepsilon$.

(c) the functions λ_i from (b) satisfy for all $|\eta - \xi| < \varepsilon$

$$\| \operatorname{grad} \lambda_i(\eta) \| \le C |\lambda_i(\eta)|,$$

We say that $\mathcal{P} \subset \mathbb{R}^{n-1}$ defines *a set of regular projections for X* if there exists $C, \varepsilon > 0$ such that for every $x_0 \in \mathbb{R}^n$ there is $\xi \in \mathcal{P}$ such that π_ξ is (C, ε)-regular at x_0.

Theorem 1.1. *[8], [9]] Let X be a compact subanalytic subset of $\mathbb{R}^n$ such that $\dim X < n$. Then the generic set of $n+1$ vectors $\xi_1, \dots, \xi_{n+1}$, $\xi_i \in \mathbb{R}^{n-1}$, defines a set of regular projections for X.*

Here by generic we mean in the complement of a subanalytic nowhere dense subset of $(\mathbb{R}^{n-1})^{n+1}$.

1.3. **Cylindrical decomposition.** We recall the first step of a basic construction called the cylindrical algebraic decomposition in semialgebraic geometry or the cell decomposition in o-minimal geometry, for details see for instance [2], [3].

Set $X = \overline{U} \setminus U$. Then X is a compact subanalytic subset of $\mathbb{R}^n$ of dimension $n-1$. We denote by $Z \subset X$ the set of singular points of X that is the complement in X of the set

$$Reg(X) := \{x \in X; (X,x) \text{ is the germ of a real analytic submanifold of dimension } n-1\}.$$

Then Z is closed in X, subanalytic and $\dim Z \leq n-2$.

Assume that the standard projection $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ restricted to X is finite. Denote by $\Delta_\pi \subset \mathbb{R}^{n-1}$ the union of $\pi(Z)$ and the set of critical values of $\pi|_{Reg(X)}$. Then Δ_π, called *the discriminant set of* π, is compact and subanalytic. It is clear that $\overline{\pi(U)} = \pi(U) \cup \Delta_\pi$.

Proposition 1.2. *Let $U' \subset \pi(U) \setminus \Delta_\pi$ be open and connected. Then there are finitely many bounded real analytic functions $\varphi_1 < \varphi_2 < \cdots < \varphi_k$ defined on U', such that $X \cap \pi^{-1}(U')$ is the union of graphs of φ_i's. In particular, $U \cap \pi^{-1}(U')$ is the union of the sets*

$$\{(x', x_n) \in \mathbb{R}^n; x' \in U', \varphi_i(x') < x_n < \varphi_{i+1}(x'), \}$$

and moreover, if U' is subanalytically homeomorphic to an open $(n-1)$-dimensional ball, then each of these sets is subanalytically homeomorphic to an open n-dimensional ball.

1.4. **The case of a regular projection.** Fix $x_0 \in U$ and suppose that $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is (C, ε)-regular at $x_0 \in \mathbb{R}^n$ with respect to X. Then the cone (1.1) contains no point of Z. By [9] Lemma 5.2, this cone contains no critical point of $\pi|_{Reg(X)}$, provided ε is chosen sufficiently small (for fixed C). In particular, $x_0' = \pi(x_0) \notin \Delta_\pi$.

In what follows we fix $C, \varepsilon > 0$ and suppose ε small. We denote the cone (1.1) by $\mathscr{C}$ for short. Then for $\tilde{C}$ sufficiently large, that depends only on C and ε, we have

$$\operatorname{dist}(x_0, X \setminus \mathscr{C}) \leq \tilde{C} \operatorname{dist}(x_0', \pi(X \setminus \mathscr{C})) \leq \tilde{C} \operatorname{dist}(x_0', \Delta_\pi). \tag{1.2}$$

The first inequality is obvious, the second follow from the fact that the singular part of X and the critical points of $\pi|_{Reg(X)}$ are both outside the cone.

1.5. **Proof of Theorem 0.2.** Induction on n. Set $X = \overline{U} \setminus U$ and let $\pi_{\xi_1}, \dots, \pi_{\xi_{n+1}}$ be a set of (C, ε)-regular projections with respect to X. To each of these projections we apply the cylindrical decomposition. More precisely, let us fix one of these projections that for simplicity we suppose standard and denote it by π. Then we apply the inductive assumption to $\pi(U) \setminus \Delta_\pi$. Thus let $\pi(U) \setminus \Delta_\pi = \bigcup U_i'$ be a finite cover satisfying the statement of Theorem 0.2. Applying to each U_i' Proposition 1.2 we obtain a family of cylinders that covers $U \setminus \pi^{-1}(\Delta_\pi)$. In particular they cover the set of those points of U at which π is (C, ε)-regular.

Lemma 1.3. *Suppose π is (C,ε)-regular at $x_0 \in U$. Let U' be an open subanalytic subset of $\pi(U) \setminus \Delta_\pi$ such that $x'_0 = \pi(x_0) \in U'$ and*

$$\operatorname{dist}(x'_0, \Delta_\pi) \le \tilde{C} \operatorname{dist}(x'_0, \partial U'), \tag{1.3}$$

with $\tilde{C} \ge 1$ for which (1.2) *holds. Then*

$$\operatorname{dist}(x_0, X) \le (\tilde{C})^2 \operatorname{dist}(x_0, \partial U_1), \tag{1.4}$$

where U_1 is the member of cylindrical decomposition of $U \cap \pi^{-1}(U')$ containing x_0.

Proof. We decompose ∂U_1 into two parts. The first one is vertical, i.e. contained in $\pi^{-1}(\partial U')$, and the second part is contained in X. If $\operatorname{dist}(x_0, \partial U_1) < \operatorname{dist}(x_0, X)$ then the distance to the vertical part realizes the distance of x_0 to ∂U_1 and $\operatorname{dist}(x_0, \partial U_1) = \operatorname{dist}(x'_0, \partial U')$. Hence

$$\operatorname{dist}(x_0, \partial U_1) = \min\{\operatorname{dist}(x_0, X), \operatorname{dist}(x'_0, \partial U')\}. \tag{1.5}$$

If $\operatorname{dist}(x_0, \partial U_1) = \operatorname{dist}(x_0, X)$ then (1.4) holds with $\tilde{C} = 1$, otherwise by (1.2) and (1.3)

$$\operatorname{dist}(x_0, X \setminus \mathscr{C}) \le \tilde{C} \operatorname{dist}(x'_0, \Delta_\pi) \le (\tilde{C})^2 \operatorname{dist}(x'_0, \partial U') = (\tilde{C})^2 \operatorname{dist}(x_0, \partial U_1). \tag{1.6}$$

□

Thus to complete the proof of Theorem 0.2 it suffices to show that (1.3) holds if U' is an element of the cover $\pi(U) \setminus \Delta_\pi = \bigcup U'_i$ for which $\operatorname{dist}(x'_0, \partial \pi(U)) \le \tilde{C} \operatorname{dist}(x'_0, \partial U')$. This follows from the inclusion $\partial \pi(U) \subset \Delta_\pi$ that gives $\operatorname{dist}(x'_0, \Delta_\pi) \le \operatorname{dist}(x'_0, \partial \pi(U))$. This ends the proof of Theorem 0.2.

1.6. **L-regular sets.** Let $Y \subset \mathbb{R}^n$ be subanalytic, $\dim Y = n$. Then Y is called *L-regular (with respect to given system of coordinates)* if

(1) if $n = 1$ then Y is a non-empty closed bounded interval;
(2) if $n > 1$ then Y is of the form

$$Y = \{(x', x_n) \in \mathbb{R}^n; f(x') \le x_n \le g(x'), x' \in Y'\}, \tag{1.7}$$

where $Y' \subset \mathbb{R}^{n-1}$ is L-regular, f and g are continuous subanalytic functions defined in Y'. It is also assumed that on the interior of Y', f and g are analytic, satisfy $f < g$, and have the first order partial derivatives bounded.

If $\dim Y = k < n$ then we say that Y is *L-regular (with respect to given system of coordinates)* if

$$Y = \{(y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k};\ z = h(y),\ y \in Y'\}, \tag{1.8}$$

where $Y' \subset \mathbb{R}^k$ is L-regular, $\dim Y' = k$, h is a continuous subanalytic map defined on Y', such that h is real analytic on the interior of Y', and has the first order partial derivatives bounded.

We say that Y is *L-regular* if it is L-regular with respect to a linear (or equivalently orthogonal) system of coordinates on $\mathbb{R}^n$.

We say that $A \subset \mathbb{R}^n$ is an *L-regular cell* if A is the relative interior of an L-regular set. That is, it is the interior of A if $\dim A = n$, and it is the graph of h restricted to $Int(Y')$ for an L-regular set of the form (1.8). By convention, every point is a zero-dimensional L-regular cell.

By [4], see also Lemma 2.2 of [9] and Lemma 1.1 of [5], L-regular sets and L-regular cells satisfy the following property, called in [4] quasi-convexity. We say that $Z \subset \mathbb{R}^n$ is *quasi-convex* if there is a constant $C > 0$ such that every two points x, y of Z can be connected in Z by a continuous subanalytic arc of length bounded by $C\|x-y\|$. It can be shown that for an L-regular set or cell Y of the form (1.7) or (1.8) the constant C depends only on n, the analogous constant for Y', and the bounds on first order partial derivatives of f and g, resp. h. By Lemma 2.2 of [9], an L-regular cell is subanalytically homeomorphic to the (open) unit ball.

Let Y be a subanalytic subset of a real analytic manifold M. We say that Y is *L-regular* if there exists its neighborhood V in M and an analytic diffeomorphism $\varphi : V \to \mathbb{R}^n$ such that $\varphi(Y)$ is L-regular. Similarly we define an L-regular cell in M.

1.7. **Proof of Theorem 0.3.** Fix a constant C_1 sufficiently large and a projection $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ that is assumed, for simplicity, to be the standard one. We suppose that π restricted to $X = \partial U$ is finite. We say that $x' \in \pi(U) \setminus \Delta_\pi$ is *C_1-regularly covered* if there is a neighborhood $\tilde{U}'$ of x' in $\pi(U) \setminus \Delta_\pi$ such that $X \cap \pi^{-1}(\tilde{U}')$ is the union of graphs of analytic functions with the first order partial derivatives bounded (in the absolute value) by C_1. Denote by $U'(C_1)$ the set of all $x' \in \pi(U) \setminus \Delta_\pi$ that are C_1 regularly covered. Then $U'(C_1)$ is open (if we use strict inequalities while defining it) and subanalytic. By Lemma 5.2 of [9], if π is a (C, ε)-regular projection at x_0 then x'_0 is C_1-regularly covered, for C_1 sufficiently big $C_1 \geq C_1(C, \varepsilon)$. Moreover we have the following result.

Lemma 1.4. *Given positive constants C, ε. Suppose that the constants $\tilde{C}$ and C_1 are chosen sufficiently big, $C_1 \geq C_1(C, \varepsilon)$, $\tilde{C} \geq \tilde{C}(C, \varepsilon)$. Let π be (C, ε)-regular at $x_0 \notin X$ and let*

$$V' = \{x' \in \mathbb{R}^{n-1}; \operatorname{dist}(x', x'_0) < (\tilde{C})^{-1} dist(x_0, X \cap \mathscr{C})\}.$$

Then $\pi^{-1}(V') \cap X \cap \mathscr{C}$ is the union of graphs of φ_i with all first order partial derivatives bounded (in the absolute value) by C_1. Moreover, then either $\pi^{-1}(V') \cap (X \setminus \mathscr{C}) = \emptyset$ or

$$\operatorname{dist}(x'_0, \Delta_\pi) = \operatorname{dist}(x'_0, \pi(X \setminus \mathscr{C})) \leq \operatorname{dist}(x'_0, \partial U'(C_1)).$$

Proof. We only prove the second part of the statement since the first part follows from Lemma 5.2 of [9]. If $\pi^{-1}(V') \cap X \setminus \mathscr{C} \neq \emptyset$ then any point of $\pi(X \setminus \mathscr{C})$ realizing $\operatorname{dist}(x'_0, \pi(X \setminus \mathscr{C}))$ must be in the discriminant set Δ_π. □

We now apply to $U'(C_1)$ the inductive hypothesis and thus assume that $U'(C_1) = \bigcup U'_i$ is a finite regular cover by open L-regular cells. Fix one of them U' and let U_1 be a member of the cylindrical decomposition of $U \cap \pi^{-1}(U')$. Then U_1 is an L-regular cell. Let $x_0 \in U_1$. We apply to x_0 Lemma 1.4.

If $\pi^{-1}(V') \cap (X \setminus \mathscr{C}) = \emptyset$ then

$$\operatorname{dist}(x_0, X) \leq \operatorname{dist}(x_0, X \cap \mathscr{C}) \leq \tilde{C} \operatorname{dist}(x'_0, \partial U'(C_1)) \leq \tilde{C}^2 \operatorname{dist}(x'_0, \partial U'),$$

where the second inequality follows from the first part of Lemma 1.4 and the last inequality by the induction hypothesis. Then $\operatorname{dist}(x_0, X) \leq \tilde{C}^2 \operatorname{dist}(x_0, \partial U_1)$ follows from (1.5).

Otherwise, $\operatorname{dist}(x'_0, \Delta_\pi) \leq \operatorname{dist}(x'_0, \partial U'(C_1)) \leq \tilde{C} \operatorname{dist}(x'_0, \partial U')$ and the claim follows from Lemma 1.3. This ends the proof.

1.8. **Proof of Theorem 0.1.** The proof is based on the following result.

Theorem 1.5. [Theorem A of [4]] *Let $Z_i \subset \mathbb{R}^n$ be a finite family of bounded subanalytic sets. Then there is be a finite disjoint collection $\{A_j\}$ of L-regular cells such that each Z_i is the disjoint union of some of A_j.*

Similar results in the (more general) o-minimal set-up are proven in [5] and [10].

Let U be a relatively compact open subanalytic subset of $\mathbb{R}^n$. By Theorem 1.5, U is a disjoint union of L-regular cells and hence it suffices to show the statement of Theorem 0.1 for an L-regular cell. We consider first the case of an open L-regular cell. Thus suppose that

$$U = \{(x', x_n) \in \mathbb{R}^n; f(x') < x_n < g(x'), x' \in U'\}, \tag{1.9}$$

where U' is a relatively compact L-regular cell, f and g are subanalytic and analytic functions on U' with the first order partial derivatives bounded. Then, by the quasi-convexity of U', f and g are Lipschitz. By an extension formula of [6], see also [11] and [1], we may suppose that f and g are restrictions of Lipschitz subanalytic functions, that we denote later also by f and g, defined everywhere on $\mathbb{R}^{n-1}$ and satisfying $f \le g$. Indeed, this extension of f is given by

$$\tilde f(p) = \sup_{q \in U'} f(q) - L\|p - q\|,$$

where L is the Lipschitz constant of f. Then $\tilde f$ is Lipschitz with the same constant as f and subanalytic. Therefore by the inductive assumption on dimension we may assume that U is given by (1.9) with U' a subanalytic Lipschitz ball. Denote U by $U_{f,g}$ to stress its dependence on f and g (with U' fixed). Then

$$1_{U_{f,g}} = 1_{U_{f-1,g}} + 1_{U_{f,g+1}} - 1_{U_{f-1,g+1}}$$

and $U_{f-1,g}$, $U_{f,g+1}$. and $U_{f-1,g+1}$ are open subanalytic Lipschitz balls.

Suppose now that

$$U = \{(y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k};\ z = h(y),\ y \in U'\}, \tag{1.10}$$

where U' is an open L-regular cell of $\mathbb{R}^k$, h is a subanalytic and analytic map defined on U' with the first order partial derivatives bounded. Hence h is Lipschitz. We may again assume that h is the restriction of a Lipschitz subanalytic map $h : \mathbb{R}^k \to \mathbb{R}^{n-k}$ and then, by the inductive hypothesis, that U' is a subanalytic Lipschitz ball. Let

$$U_\emptyset = \{(y, z) \in U' \times \mathbb{R}^{n-k};\ h_i(y) - 1 < z_i < h_i(y) + 1\,, i = 1, ..., n-k\}$$

For $I \subset \{1, ..., n-k\}$ we denote

$$U_I = \{(y, z) \in U_\emptyset;\ z_i \ne h_i(y) \text{ for } i \in I\}.$$

Note that each U_I is the disjoint union of $2^{|I|}$ of open subanalytic Lipschitz balls and that

$$1_U = \sum_{I \subset \{1,\dots,n-k\}} (-1)^{|I|} 1_{U_I}.$$

This ends the proof.

2. Remarks on the o-minimal case

It would be interesting to know whether the main theorems of this paper, Theorems 0.1, 0.2, 0.3, hold true in an arbitrary o-minimal structure in the sense of [3], i.e. if we replace the word "subanalytic" by "definable in an o-minimal strucure", and fix $M = \mathbb{R}^n$. This is the case for Proposition 1.2 and Theorem 1.5 by [3], resp. [5], [10], and therefore Theorem 0.1 holds true in the o-minimal set-up. But it is not clear whether the analog of Theorem 1.1 holds in an arbitrary o-minimal structure. Its proof in [8] uses Puiseux Theorem with parameters in an essential way. Thus we state the following questions.

Question 2.1. *Does the regular projections theorem, Theorem 1.1, hold true in an arbitrary o-minimal strucure?*

Question 2.2. *Do Theorems 0.2, 0.3, hold true in an arbitrary o-minimal strucure?*

One would expect the positive answers for the polynomially bounded o-minimal structures, though even this case in not entirely obvious.

References

[1] S. Banach, *Wstęp do teorii funkcji rzeczywistych*, Monografie Matematyczne, Warszawa-Wrocław, 1951, (in Polish)

[2] M. Coste, *An introduction to o-minimal geometry*, Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica, Istituti Editoriali e Poligrafici Internazionali, Pisa (2000), http://perso.univ-rennes1.fr/michel.coste/polyens/OMIN.pdf

[3] L. van den Dries, *Tame Topology and O-minimal Structures*, London Math. Soc. Lecture Note **248**. Cambridge Univ. Press 1998

[4] K. Kurdyka, *On a Subanalytic Stratification Satisfying a Whitney Property with Exponent* 1, in Real Algebraic Geometry, Proceedings, Rennes 1991, Lecture Notes in Math. **1524** , eds M. Coste et al., (1992), 316–323

[5] K. Kurdyka, A. Parusiński, *Quasi-convex decomposition in o-minimal structures. Application to the gradient conjecture.* Singularity theory and its applications, 137177, Adv. Stud. Pure Math., 43, Math. Soc. Japan, Tokyo, 2006.

[6] E. J. McShane, *Extension of range of functions*, Bull. Amer. Math. Soc. **40** (1934), 837–842

[7] T. Mostowski, *Lipschitz Equisingularity*, Dissertationes Math. **243** (1985).

[8] A. Parusiński, *Regular projection for sub-analytic sets*, C. R. Acad. Sci. Paris, Série I **307**, (1988), 343–347.

[9] A. Parusiński, *Lipschitz stratification of subanalytic sets*, Annales Sci. Éc. Norm. Sup. **27**, 6 (1994), 661–696

[10] W. Pawłucki, *Lipschitz cell decomposition in o-minimal structures.* I. Illinois J. Math. 52 (2008), no. 3, 1045–1763

[11] H. Whitney, *Analytic extensions of differentiable functions defined on closed sets.* Trans. Amer. Math. Soc. 36 (1934), no. 1, 63–89

Univ. Nice Sophia Antipolis, CNRS, LJAD, UMR 7351, 06108 Nice, France
E-mail address: adam.parusinski@unice.fr