Large-N Matrix Models : Some Algebraic Aspects

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Based on

Work of physicists on Yang-Mills theory, matrix models & large-N limits: 't Hooft, Wilson, Migdal, Makeenko, Polyakov, Witten, Cvitanovic, Gambini, Trias, Rajeev, Tavares etc.

1. *Schwinger-Dyson operators as invariant vector fields on a matrix-model analogue of the group of loops*, GSK, J.Math.Phys.49:062303,2008.

2. *Schwinger-Dyson operator of Yang-Mills matrix models with ghosts and derivations of the graded shuffle algebra*, GSK, J.Phys.A41:145402,2008.

3. *Non-anomalous Ward identities to supplement large-N multi-matrix loop equations for correlations*, L. Akant and GSK, JHEP 02 (2007) 073.

4. *Algebraic Structure and Approximations for Multi-matrix Loop Equations*, GSK, JHEP 08 (2006) 035.

5. *Variational ansatz for gaussian + Yang-Mills two matrix model compared with Monte- Carlo simulations in 't Hooft limit*, GSK, hep-th/0310110.

6. *Collective potential for large-N Hamiltonian matrix models and free Fisher information*, A. Agarwal, L. Akant, GSK, S. G. Rajeev, Int. J. Mod. Phys. A **18**, 917 (2003).

7. *Entropy of Operator-Valued Random Variables: A Variational Principle for Large N Matrix Models*, L. Akant, GSK, S. G. Rajeev, Int.J.Mod.Phys.A17:2413, (2002).

Abstract

Matrix models are quantum theories whose correlations are basis independent averages over the entries of several $N \times N$ matrices. They are toy-models for non-abelian gauge theories. In the 'classical' limit as N becomes large, various algebraic structures may be exploited to understand these models. For instance, the Schwinger-Dyson operators are invariant derivations of the shuffle-deconcatenation Hopf algebra. This suggests an approximation method based on deformation theory. On the other hand, solving the Schwinger-Dyson equations involve computing the Legendre transform of a non-trivial one cocycle of the automorphism group of the free algebra. This leads to a method of variational approximations.

I will describe work which was done in part in collaboration with S. G. Rajeev, L. Akant and A. Agarwal.

Significance of large-N Yang-Mills theory

• QCD describes physics of quarks and gluons to form hadrons, so we must make every effort to solve it.

• The limit as the number of colours $N \to \infty$ is a 'classical' limit different from $\hbar \to 0$ and a promising approach.

• Yang-Mills theory is as central to physics today as Newtonian mechanics was in the 18^{th} & 19^{th} centuries.

• Newtonian mechanics $\leftrightarrow$ ordinary calculus as Yang-Mills theory $\leftrightarrow$ calculus of infinite dimensional spaces.

• A detailed theory of hadronic structure will tell us the right way to look at quantum field theory.

Yang-Mills Theory

• Dynamical variable is gauge/gluon field, connection 1-form $[A_\mu(x)]^a_b dx^\mu$ in principal $SU(N)$ bundle over space time M. $a, b = 1, \cdots N$, colours, $N = 3$ in nature.

• Think of $A_\mu(x)$ as an $N \times N$ hermitian matrix at each point x.

• $A(x)$ is not a physical observable. **Observables must be gauge invariant**

$$A(x) \to g(x)A(x)g^{-1}(x) + idg\, g^{-1}; \quad \text{where} \quad g(x) \in SU(N)$$

• **Example** of a gauge invariant observable is Wilson loop, *trace* of parallel transport around closed curve γ on space time. $W(\gamma)$ is a typical function on $Loop(M)$

$$\text{Holonomy around } \gamma : \quad W(\gamma) = \frac{1}{N} \,\text{tr}\, P e^{i \oint_\gamma A_\mu(x) \frac{dx^\mu}{dt} dt}$$

$$\text{Square of Curvature} \qquad \text{tr}\, F_{\mu\nu} F^{\mu\nu}(x), \qquad \text{where} \qquad F = dA + A \wedge A$$

Path ordered exponential: Younger ones to the right

- The path ordered exponential for a matrix $A(t)$

$$U(t) = Pe^{\int_0^t A(t')dt'}$$

is defined in one of several equivalent ways.

- As the infinite series of 'Chen' iterated integrals

$$U(t) = 1 + \int_0^t dt' A(t') + \int_0^t dt_1 \int_0^{t_1} dt_2 A(t_1)A(t_2) + \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 A(t_1)A(t_2)A(t_3) + \cdots$$

- As the unique solution of the ODE with initial condition $U(0) = 1$

$$\frac{dU}{dt} = A(t)U(t)$$

- As the limit of the product $(\epsilon = t/N, t_n = t - n\epsilon)$

$$U(t) = \lim_{N\to\infty} e^{\epsilon A(t_0)} e^{\epsilon A(t_1)} \cdots e^{\epsilon A(t_N)}$$

(Euclidean) Quantum Yang-Mills Theory

• Calculate **average values of gauge invariant observables** over all $A_\mu(x)$ with respect to a weight specified by the Yang-Mills action

$$S_{\rm YM} = \frac{1}{4g^2}\,{\rm tr}\,\int d^4x F_{\mu\nu}F^{\mu\nu} \quad \text{and} \quad \langle W(\gamma)\rangle = \frac{\int DA\; e^{-\frac{1}{\hbar}\,S}\;\; W(\gamma)}{\int DA\; e^{-\frac{1}{\hbar}\,S}}$$

• Perturbation theory around $\hbar \to 0$ where flat connections or instantons dominate is a successful approximation at short distances due to **asymptotic freedom**.

• Loop expansion in $\hbar$ or α_s is not adequate at moderate and large distances to find spectrum of particles, structure of bound states, confinement of quarks, mass gap, mass of the proton etc. Other approaches, such as large-N limit needed, but the problem is very hard and we are still far from experimentally relevant predictions.

Large-N (multi-color) limit of Yang-Mills theory

• 't Hooft: $1/N$ expansion holding $\hbar,\ g^2N$ fixed: non-perturbative approximation.

• As $\hbar \to 0$ **all variables, quarks, gluons stop fluctuating**.

• As $N \to \infty$, **only gauge-invariants stop fluctuating**, behave classically due to **factorization**:

$$\langle W(\gamma)W(\gamma')\rangle = \langle W(\gamma)\rangle\ \langle W(\gamma')\rangle + \mathcal{O}(\frac{1}{N^2})$$

• Many indications that large-N limit should be a good approximation. Phenomenology of planar diagrams; numerical evidence from lattice gauge theory.

• Need to solve large-N Yang-Mills before doing a $1/N$ expansion.

Difficulties are encountered in every viewpoint

1. Sum infinite classes of Feynman diagrams of planar topology

2. Solve Makeenko-Migdal equations for Wilson Loops.

3. Solve factorized Schwinger-Dyson equations for gluon correlations

Goals for this talk

• Find (approximation) methods to solve large-N limit of theories with several $N \times N$ matrix degrees of freedom.

• Identify mathematical structures of the equations which may lead to a better understanding.

• Postpone questions of renormalization.

Matrix models : Simplified versions of Yang-Mills theory

• Matrix field theories arise from dimensional reduction of gauge-fixed YM_{3+1} to $1+1$ dimensions: a theory of adjoint scalars (transverse polarization states of the gluon).

• To avoid divergences and to focus on matrix nature of fields, regularize space-time to have Λ points.

• Consider matrix models with Λ hermitian $N \times N$ matrices $[A_i]^a_b \longrightarrow$ gluon field at 'position' $i = 1, 2, \cdots, \Lambda$.

• Gauge invariance simplifies to invariance of action and observables under global adjoint action of $U(N)$: $A_i \mapsto U A_i U^\dagger$.

Examples of Matrix Models

- Action $\rightarrow$ polynomial $\text{tr}\, S(A) = \text{tr}\, S^I A_I$, $\quad S^I \rightarrow$ cyclic 'coupling tensors'. eg.

$$S(A) = \text{tr}\, [S^{ij} A_i A_j + S^{ijk} A_i A_j A_k + S^{ijkl} A_i A_j A_k A_l]$$

- $I = i_1 \cdots i_n \rightarrow$ multi-indices, repeated indices summed. $A_I = A_{i_1} A_{i_2} A_{i_3} \cdots A_{i_n}$.

- Interesting examples: Zero-momentum limits of field theories

$$S_{Gauss} = \frac{1}{2}\,\text{tr}\, C^{ij} A_i A_j; \qquad S_{CS} = \frac{2i\kappa}{3}\,\text{tr}\, C^{ijk} A_i [A_j, A_k];$$
$$S_{YM} = -\frac{1}{4\alpha}\,\text{tr}\, [A_i, A_j][A_k, A_l] g^{ik} g^{jl}.$$

- Gauge-fixed Yang-Mills action is a sort of grand limiting case

$$S = \int d^4x\, \text{tr}\, \{\frac{1}{2}\partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) - ig\partial_\mu A_\nu [A^\mu, A^\nu]$$
$$-\frac{g^2}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + \partial_\mu \bar{c}\, \partial^\mu c - ig\partial_\mu \bar{c}\, [A^\mu, c]\}.$$

$U(N)$ Invariants and Gluon Correlations

- Partition function $\longrightarrow Z = \int dA e^{-N \text{ tr } S(A)}$
- Observables $\longrightarrow U(N)$ invariants

$$\Phi_{k_1 \cdots k_n} = \frac{1}{N} \text{ tr } A_{k_1} \cdots A_{k_n}$$

- Aim: Calculate correlations: expectation values of products of invariants

$$\langle \Phi_{K_1} \cdots \Phi_{K_n} \rangle = \frac{1}{Z} \int dA e^{-N \text{ tr } S(A)} \; \Phi_{K_1} \cdots \Phi_{K_n}$$

- $N \to \infty \Rightarrow$ invariants don't fluctuate ('classical' limit though $\hbar = 1$): factorization

$$\langle \Phi_{K_1} \cdots \Phi_{K_n} \rangle = \langle \Phi_{K_1} \rangle \cdots \langle \Phi_{K_n} \rangle + \mathcal{O}(\frac{1}{N^2})$$

- In $N \to \infty$ limit restrict to single trace correlations $G_I = \lim_{N \to \infty} \langle \Phi_I \rangle = \langle \frac{1}{N} \text{ tr } A_I \rangle$

Rough summary of results: Algebra

• Generator of correlations $G(\xi) = G_I \xi^I$ live in the *shuffle-deconcatenation Hopf algebra*.

• Identified a finitely generated analogue of the group of loops on space time, the spectrum $\mathbf{G}_\Lambda$ of this Hopf algebra. Lie algebra of $\mathbf{G}_\Lambda$ is the FLA_Λ

• $G(\xi)$ is a function on $\mathbf{G}_\Lambda$. It satisfies **quadratic equations in convolution** product on the group: factorized SD equations $\mathcal{S}^i\, G(\xi) \;=\; G(\xi)\; \xi^i\; G(\xi)$.

• SD operators $\mathcal{S}^i$ of Yang-Mills, Chern-Simons and Gaussian models are **right-invariant vector fields on** $\mathbf{G}_\Lambda$, i.e., invariant derivations of the Hopf algebra.

• fSDE can be transformed into linear equations by replacing convolution (concatenation) by shuffle. To approximately solve: Expand concatenation as a deformation series around shuffle.

Rough summary of results: Probability and algebra

• Probabilistic interpretation of the configuration space of correlations: it is the space of non-commutative probability distributions.

• Produced a variational principle which implies the factorized Schwinger-Dyson equations: non-trivial due to a cohomological obstruction.

• Avoided cohomological obstruction by expressing configuration space as a coset space of automorphism group of free associative algebra in Λ generators.

• Variational principle: Extremize entropy of operator-valued random variables while holding correlations conjugate to coupling tensors fixed $\Rightarrow$ variational approximations.

• Showed that the entropy is a non-trivial 1-cocycle of the Automorphism group of the free associative algebra.

Configuration space of large-N 'classical' limit

• $G_I = \lim_{N\to\infty}\langle \frac{\text{tr}}{N} A_I\rangle$ not the moments of any probability distribution $\rho(\mathbf{x})$ on $\mathbf{R}^\Lambda$, since they are not symmetric tensors as $A_1, \cdots, A_\Lambda$ don't commute

• $\{G_I\} = \mathcal{P}_\Lambda =$ Space of Non-commutative Probability Distributions.

• As $N \to \infty$, can ignore relations between the traces of various products of matrices and treat G_I as almost independent variables.

• Conditions on G_I (coordinates on configuration space $\mathcal{P}_\Lambda$)

$$G_\phi = \langle \frac{\text{tr}}{N} 1\rangle = 1 \quad \text{NORMALIZED}$$

$$G_{i_1 i_2 \cdots i_k} = G_{i_2 i_3 \cdots i_k i_1} \quad \text{CYCLIC}$$

$$A_i^\dagger = A_i \quad \Rightarrow \quad G^*_{i_1 i_2 \cdots i_n} = G_{i_n i_{n-1} \cdots i_2 i_1} \quad \text{HERMITIAN}$$

$$f(A) = f^I A_I \quad \text{polynomial} \Rightarrow \langle \frac{\text{tr}}{N} f^\dagger(A) f(A)\rangle \geq 0 \quad \Rightarrow G_{IJ} f^{\bar{I}*} f^J \ \geq \ 0 \ \ \text{POSITIVE}$$

Matrix Model Loop equations for Gluon Correlations

- At $N = \infty$, G_I satisfy factorized Schwinger-Dyson or Loop equations

$$S^{J_1 i J_2}\, G_{J_1 I J_2} \;=\; \delta_I^{I_1 i I_2}\;\; G_{I_1}\, G_{I_2} \;\;\text{for all}\;\;\text{words}\;\; I \;\;\text{and letters}\; i$$

- Aim: Solve the factorized Schwinger-Dyson equations.

- Analogue of Makeenko-Migdal eqn. for Wilson loops of large-N Yang-Mills theory

$$\delta_\mu^x \frac{\delta}{\delta\sigma_{\mu\nu}(x)} W(C) = \lambda \oint_C dy_\nu \delta^{(4)}(x-y) W(C_{yx}) W(C_{xy}) \quad \forall \;\; \text{curves}\; C, \quad \text{points}\; x$$

- "word $I \leftrightarrow$ curve C" and "letter $i \leftrightarrow$ point x"

- Correlation tensors give an algebraic way of doing calculus on $Loop(M)$.

- For a single matrix $G_k = \langle \frac{\mathrm{tr}}{N} A^k \rangle$, the loop equations are

$$\sum_{l=1}^{m} \; l\; S_l G_{k+l} = \sum_{\substack{r+s=k \\ r,s\geq 0}} G_r G_s, \qquad \text{for} \qquad k = -1, 0, 1, 2, \cdots$$

Obtaining the factorized Schwinger Dyson equations

- Matrix integrals are invariant under infinitesimal non-linear changes of integration variable encoded in the vector fields L_v (infinitesimal automorphisms of free algebra)

$$L_v : A_i \mapsto A_i + v_i^I A_I \quad \text{leaves} \quad Z \ = \ \int dA \ \ e^{-N \text{ tr } S(A)} \quad \text{unchanged.}$$

- Change in action S and change in measure (divergence of vector field)

$$e^{-N \text{ tr } S^J A_J} \ \mapsto \ e^{-N \text{ tr } S^J A_J}(1 \ - \ N^2 \ v_i^I \ S^{J_1 i J_2} \ \Phi_{J_1 I J_2}) + \mathcal{O}(v^2),$$
$$\det\left(\frac{\partial [A_i']^a_b}{\partial [A_j]^c_d}\right) \ = \ 1 + N^2 \ v_i^I \ \delta_I^{I_1 i I_2} \ \Phi_{I_1} \ \Phi_{I_2} + \mathcal{O}(v^2)$$

- Invariance of $Z \Rightarrow v_i^I S^{J_1 i J_2} \langle \Phi_{J_1 I J_2} \rangle = v_i^I \delta_I^{I_1 i I_2} \langle \Phi_{I_1} \Phi_{I_2} \rangle$.

- Using factorization at large-N, the Loop equations are quadratic in G_I

$$S^{J_1 i J_2} \ G_{J_1 I J_2} = \delta_I^{I_1 i I_2} \ \ G_{I_1} \ G_{I_2} = \eta_I^i.$$

- LHS, change in action, linear in G_I. RHS, change in measure is quadratic, ‘anomaly’.

Structure of factorized Schwinger-Dyson equations (fSDE)

- Generating series of correlations $G(\xi) = G_I \xi^I$ in non-comuting generators ξ^i.

- Then fSDE $S^{J_1 i J_2} G_{J_1 I J_2} = \delta_I^{I_1 i I_2} G_{I_1} G_{I_2}$ become $\mathcal{S}^i \; G(\xi) = G(\xi) \; \xi^i \; G(\xi)$ where

$$\text{Schwinger} - \text{Dyson operators} \quad \mathcal{S}^i = \sum_{n \geq 0} (n+1) S^{i j_1 \cdots j_n} D_{j_n} \cdots D_{j_1}$$

- Left annihilation D_j defined as $D_j \xi^{i_1 \cdots i_n} = \delta_j^{i_1} \xi^{i_2 \cdots i_n}$ or equivalently, $[D_j G]_I = G_{jI}$.

- Juxtaposition $G(\xi) \xi^i G(\xi)$ denotes concatenation $(\xi^{I_1} \xi^i \xi^{I_2} = \xi^{I_1 i I_2})$

- But left annihilation D_j doesn't satisfy the Leibniz rule with respect to concatenation

- So fSDE are not differential equations in the usual sense.

- But D_j satisfies Leibniz rule with respect to shuffle product of correlations!

Hopf algebra of functions on $Loop(M)$

- Wilson loops $W(\gamma) = \text{ tr } P \exp \oint A_\mu(\gamma(t))\dot{\gamma}^\mu(t)dt$ are functions on $Loop(M)$.

- Based oriented loops γ on M (upto backtracking) form a non-abelian group.

- Successive traversal $\gamma_1\gamma_2$ is product and reversed orientation $\bar{\gamma}$ is inverse.

- Functions on $Loop(M)$ form a commutative but non-cocommutative Hopf algebra

- Point-wise product $(W_1W_2)(\gamma) = W_1(\gamma)W_2(\gamma)$; Coproduct $(\Delta W)(\gamma_1, \gamma_2) = W(\gamma_1\gamma_2)$.

- Antipode $(SW)(\gamma) = W(\bar{\gamma})$ encodes inverse.

- Makeenko-Migdal equations $\delta^x_\mu \frac{\delta}{\delta\sigma_{\mu\nu}(x)} W(C) = \lambda \oint_C dy_\nu \delta^{(4)}(x-y)W(C_{yx})W(C_{xy})$ are equations for a function on this group.

Shuffle-deconcatenation Hopf algebra of Gluon correlations

• Is there an analogue of the group of loops for a matrix model? Can the correlations G_I be regarded as functions on this group? Yes! But the group is not obvious.

• But Hopf algebra of correlations $G(\xi) = G_I \xi^I$ is identified by rewriting

$$\langle G(\gamma) \rangle = \sum_{m=0}^{\infty} i^m \int_{0 \le s_m \le \cdots \le s_1 \le 1} G_{\nu_1 \cdots \nu_m}(x(s_1), \cdots, x(s_m)) \frac{dx^{\nu_1}}{ds_1} \cdots \frac{dx^{\nu_m}}{ds_m} ds_1 \cdots ds_m$$

• As a vector space it is $\mathbf{C}\langle\langle \xi^1, \cdots, \xi^\Lambda \rangle\rangle$, linear span of words in alphabet $\xi^1, \cdots, \xi^\Lambda$

• Pointwise product $\langle (FG)(\gamma) \rangle = \langle F(\gamma) \rangle \langle G(\gamma) \rangle \Rightarrow$ shuffle product of correlations

$$(F \circ G)(\xi) = \sum_I (F \circ G)_I \xi^I \quad \text{where} \quad (F \circ G)_I = \sum_{I = J \sqcup K} F_J G_K.$$

• Sum over all complementary order-preserving sub-strings J and K of I. Eg.

$$(F \circ G)_{ijk} = F_\emptyset G_{ijk} + F_i G_{jk} + F_j G_{ik} + F_k G_{ij} + F_{ij} G_k + F_{ik} G_j + F_{jk} G_i + F_{ijk} G_\emptyset.$$

Hopf algebra of Gluon correlations

- Coproduct on Loop space ⇒ deconcatenation coproduct on correlations

$$\Delta\xi^I = \delta^I_{JK}\xi^J \otimes \xi^K \ \text{ extended linearly to } \ \Delta F(\xi) = \sum_{J,K} F_{JK}\xi^J \otimes \xi^K$$

- Unit is $G(\xi) = 1$ and co-unit picks out constant term $\epsilon(G(\xi)) = G_\emptyset$

- Antipode reverses the indices in a correlation upto a sign $S(\xi^{i_1i_2i_3}) = -\xi^{i_3i_2i_1}$

$$S(\xi^I) = (-1)^{|I|}\xi^{\bar{I}}, \quad \text{or} \ \ [S(G)]_I = (-1)^{|I|}G_{\bar{I}}$$

- Δ, S are homomorphisms of shuffle, and $sh(S \otimes 1)\Delta = sh(1 \otimes S)\Delta = 1\epsilon$

- $(sh, \Delta, 1, \epsilon, S) \rightarrow$ commutative, non-cocommutative Hopf algebra. Must be algebra of functions on some non-abelian group $\mathbf{G}_\Lambda$, a matrix model analogue of $Loop(M)$. But it is not any group built from $U(N)$ or the free group on Λ generators.

$\mathbf{G}_\Lambda$: Matrix model analogue of group $Loop(M)$

- Interesting since the fSDE would be equations for a function $G(\xi)$ on $\mathbf{G}_\Lambda$.

- $\mathbf{G}_\Lambda$ is group of characters (spectrum or dual) of the Hopf algebra. Characters are linear homomorphisms from shuffle algebra to $\mathbf{R}$ or $\mathbf{C}$

$$\chi(F \circ G) = \chi(F)\chi(G) \quad \text{and for } \; a, b \in \mathbf{C}, \quad \chi(aF + bG) = a\chi(F) + b\chi(G).$$

- Character χ is determined by $\chi^I = \chi(\xi^I)$ which are assembled as a series $\chi = \chi^I \xi_I$

- Characters form a group. Multiplication is concatenation $(\chi\psi)^I = \delta^I_{JK}\chi^J\psi^K$, coming from co-product, unit element is the co-unit $\epsilon^I = \delta^I_\emptyset$, and inverse is $(\chi^{-1})^I = (-1)^{|I|}\chi^{\bar{I}}$.

- The correlations $G(\xi) = G_I\xi^I$ are functions on $\mathbf{G}_\Lambda$, value at χ is $G(\chi) = G_I\chi^I$

Group of Characters $\mathbf{G}_\Lambda$

- But not every series $\chi^I \xi_I$ is a character, rather it must be a homomorphism of shuffle

$$\sum_{I \sqcup J = K} \chi^K = \chi^I \chi^J \quad \text{for all } I, J.$$

- These conditions are called shuffle relations elsewhere (paper of Rimhak Ree).

$$\begin{aligned} \chi^\emptyset = 1, \quad \chi^{ij} + \chi^{ji} = \chi^i \chi^j, \quad \chi^{ijk} + \chi^{jik} + \chi^{jki} &= \chi^i \chi^{jk}, \\ \chi^{ijkl} + \chi^{ikjl} + \chi^{iklj} + \chi^{kijl} + \chi^{kilj} + \chi^{klij} &= \chi^{ij} \chi^{kl}, \\ \chi^{ijkl} + \chi^{jikl} + \chi^{jkil} + \chi^{jkli} &= \chi^i \chi^{jkl}, \quad \text{e.t.c.} \end{aligned}$$

- What are the characters of Shuffle-deconcatenation Hopf algebra?

- For $F \in Sh(M) = \mathcal{T}(\Lambda^1(M))$ a loop $\gamma(t)$ defines a character $\gamma(F) = \int_\gamma F$. Eg. if $F = \alpha \otimes \beta$ for 1-forms α and β

$$\gamma(F) = \int_0^1 dt_1 \ \int_0^{t_1} dt_2 \ \alpha_i(\gamma(t_1)) \ \beta_j(\gamma(t_2)) \ \dot{\gamma}^i(t_1) \ \dot{\gamma}^j(t_2).$$

Characters of Shuffle-deconcatenation Hopf Algebra

• If $\Lambda = 1$, characters form a 1-parameter abelian group $\mathbf{G}_1 = \{\chi(\xi) = e^{\chi_1 \xi} | \chi_1 \in \mathbf{R}\}$

$$\text{More generally,} \quad e^{\chi^1 \xi_{i_1}} \, e^{\chi^2 \xi_{i_2}} \, \cdots \, e^{\chi^n \xi_{i_n}} \quad \text{are characters}$$

• If $\Lambda > 1$, the free product $\mathbf{G}_1 * \mathbf{G}_1 * \cdots * \mathbf{G}_1$ is a proper subgroup of the group of characters $\mathbf{G}_\Lambda$. It is a finitely generated analogue of the free group generated by the based loops on space time.

• This free product is physically not adequate, doesn't behave as a Lie group.

• Results of Ree and Friedrichs imply that $\mathbf{G}_\Lambda$ is the exponential of the Free Lie algebra. A Lie element is a linear combination of iterated commutators of ξ^i

$$\chi(\xi) \; = \; e^{\text{Lie element}} \; = \; \exp\{C^i \xi_i + C^{ijk}[\xi_i, [\xi_j, \xi_k]] + C^{ijkl}[[\xi_i, \xi_j], [\xi_k, \xi_l]] + \cdots\}$$

• $\log \chi$ is a Lie element $\Rightarrow$ Free Lie algebra is the Lie algebra of $\mathbf{G}_\Lambda$.

Functions and (invariant) Vector fields on $\mathbf{G}_\Lambda$

- Functions on $\mathbf{G}_\Lambda$: $G(\xi) \in Sh_\Lambda$, evaluated at character χ is $G(\chi) = G_I \chi^I$

- Vector fields on $\mathbf{G}_\Lambda$ are derivations of the Shuffle algebra (satisfy Leibniz rule)

- Left annihilation $[D_j G]_I = G_{jI}$ is a derivation: $D_i(F \circ G) = D_i F \circ G + F \circ D_i G$

- Iterated commutators of D_i span FLA$_\Lambda$, basis labelled by Lyndon words $D_{(L)}$

- General vector field on $\mathbf{G}_\Lambda$ with non-constant coefficients is $V = V^L(\xi) D_{(L)}$

- Moreover D_i are right invariant derivations of the Hopf algebra

$$\Delta D_i G = (D_i \otimes 1)\Delta G = G_{iJK} \xi^J \otimes \xi^K \quad \text{for all} \quad G(\xi)$$

- So linear combinations of iterated commutators of D_i with constant coefficients $V_\emptyset^L D_{(L)}$ are the right-invariant vector fields on $\mathbf{G}_\Lambda$ (same as Lie algebra of $\mathbf{G}_\Lambda$).

fSDE as equations for a function on $\mathbf{G}_\Lambda$

- Matrix model with action $S(A) = \mathrm{tr}\ S^I A_I$ has fSDE $\mathcal{S}^i\ G(\xi)\ =\ G(\xi)\ \xi^i\ G(\xi)$,

- fSDE are quadratic equations for a function on $\mathbf{G}_\Lambda$, the generator of correlations $G(\xi) = G_I \xi^I$.

- Concatenation appearing on RHS of fSDE $G(\xi)\ \xi^i\ G(\xi)$ is the convolution product in Hopf algebra dual to sh-deconc, i.e. conc-desh, which is the group algebra of $\mathbf{G}_\Lambda$

- Given a group, there are two dual Hopf algebras, the commutative algebra of functions and the non-commutative group algebra with convolution product of functions.

- The SD operators $\mathcal{S}^i\ =\ \Sigma_{n\geq 0}(n+1)S^{ij_1\cdots j_n}D_{j_n}\cdots D_{j_1}$ are expressed in terms of left annihilation $[D_j G]_I = G_{jI}$.

fSDE as equations for a function on $\mathbf{G}_\Lambda$

- Generically $\mathcal{S}^i = \Sigma_{n\geq 0}(n+1)S^{ij_1\cdots j_n}D_{j_n}\cdots D_{j_1}$ not a Lie element.

- But for many physically interesting models,

$$S_G = \frac{1}{2}\,\mathrm{tr}\; C^{ij}A_iA_j, \quad S_{CS} = \frac{2\sqrt{-1}\kappa}{3}\,\mathrm{tr}\;\epsilon^{ijk}A_iA_jA_k, \quad S_{YM} = g^{ik}g^{jl}[A_i, A_j][A_k, A_l].$$

- Schwnger-Dyson operators are indeed Lie elements, so focus on them

$$\mathcal{S}^i_G = C^{ij}D_j, \quad \mathcal{S}^i_{CS} = \sqrt{-1}\kappa\epsilon^{ijk}[D_k, D_j], \quad \mathcal{S}^i_{YM} = 4g^{ik}g^{jl}[D_j, [D_k, D_l]].$$

- This is true also for the full continuum Yang-Mills theory in $3+1$ dimensions

$$\mathcal{S}^\mu(x) = \partial_\nu\partial^{[\mu}D^{\nu]} + ig\{\partial_\nu[D^\mu, D^\nu] + [\partial^{[\nu}D^{\mu]}, D_\nu]\} - g^2[D^\nu, [D^\mu, D_\nu]].$$

where left annihilation $(D_\mu(x)G)_{\mu_1\cdots\mu_n}(x_1,\cdots,x_n) = G_{\mu\mu_1\cdots\mu_n}(x, x_1, \cdots, x_n)$.

- For these models, SD operators $\mathcal{S}^i$ are right-invariant vector fields on $\mathbf{G}_\Lambda$

fSDE in terms of Hopf algebra associated to group G_Λ

• There is one fSDE $\mathcal{S}^i\ G(\xi) \ = \ G(\xi)\ \xi^i\ G(\xi)$ for each letter ξ^i

• Linear combinations of ξ^i are precisely the primitive elements, $\Delta\xi^i = 1 \otimes \xi^i + \xi^i \otimes 1$

• So one fSDE for each linearly independent primitive of sh-deconc Hopf algebra.

• Which right-invariant vector field $\mathcal{S}^i$ to associate to a given primitive is determined by the action $S(A)$ of the matrix model.

• Except for the action $S(A)$ we formulated fSDE in terms of general concepts applicable to any group.

• Open issue: Generalize fSDE to more familiar groups and get insight into their solutions.

Idea for an approximation method that exploits these algebraic structures

• Want to solve factorized Schwinger-Dyson equations $\mathcal{S}^i G(\xi) = G(\xi)\xi^i G(\xi)$. Find some dimensionless expansion parameter over and above $1/N$

• Though classical ($N = \infty$), involve non-commutative but associative $conc$ product.

• Idea from Deformation Quantization

Regard a non-commutative but associative algebra as a deformation or quantization of a commutative algebra equipped with a Poisson bracket

• E.g. Associative algebra of operators in quantum mechanics approximated by commutative algebra of functions on phase space, equipped with Poisson bracket

• Can we take a further 'classical' limit of the factorized Schwinger-Dyson equations?

Approximate Concatenation by Shuffle: Deformation Quantization

- fSDE $\mathcal{S}^i G(\xi) = G(\xi) *_1 \xi^i *_1 G(\xi)$ where $\mathcal{S}^i = \Sigma_{n\geq 0}(n+1)S^{j_1\cdots j_n i} D_{j_n}\cdots D_{j_1}$

- fSDE fail to be PDEs since left annihilation D_i isn't a derivation of concatenation product $*_1$ on the RHS.

- But D_i are derivations of shuffle product.

- Approximate non-commutative conc by commutative shuffle, a 2^{nd} classical limit!

- Deformation parameter q interpolates from $*_1 = conc$ to $*_0 = shuffle$. (See also work of M. Rosso; G. Duchamp, A. Klyachko, D. Krob, J-Y. Thibon)

- Physical value is $q = 1$, measures amount by which fSDE are not PDEs.

Reduction to Linear System at $\mathcal{O}(q^0)$

- We will expand $conc = *_1$ around $shuffle = *_0$ in powers of $q = 1$: $*_q = *_0 + \mathcal{O}(q)$

- At order $\mathcal{O}(q^0)$, just replace $conc$ by $shuffle$: $\mathcal{S}^i G(\xi) = G(\xi) \circ \xi^i \circ G(\xi)$

$$\mathcal{S}^i \;=\; \sum_{n\geq 0} (n+1) S^{j_1 \cdots j_n i} D_{j_n} \cdots D_{j_1}$$

- Since D_i is derivation of $shuffle = \circ$, these really are non-linear PDEs.

- Now use the fact that $\mathcal{S}^i$ for Gaussian, Yang-Mills, Chern-Simons models are Lie elements, i.e. derivations of shuffle product.

- If $\mathcal{S}^i$ is a derivation of $shuffle$, can linearize by passage to shuffle reciprocal of $G(\xi)$

$$F(\xi) \circ G(\xi) = 1 \;\Rightarrow\; \mathcal{S}^i(F(\xi) \circ G(\xi)) = 0$$

$$F \circ \mathcal{S}^i G = -\mathcal{S}^i F \circ G \;\Rightarrow\; \mathcal{S}^i G = -G \circ \mathcal{S}^i F \circ G.$$

- Loop equations at $\mathcal{O}(q^0)$ become linear $\mathcal{S}^i F(\xi) = -\xi^i$. A major simplification.

- $shuffle$ preserves cyclicity and hermiticity $\Rightarrow$ F_I are cyclic and hermitian just like G_I

Linear equations for shuffle reciprocal $F(\xi)$

- Transformation to linear equations only works for theories with derivation property

$$\begin{aligned} Gaussian \quad & C^{ij}\, D_j F(\xi) \;=\; -\,\xi^i \\ Chern-Simons \quad & i\kappa\, \epsilon^{ijk}[D_k, D_j]F(\xi) \;=\; -\,\xi^i \\ Yang-Mills \quad & -\frac{1}{\alpha}\, g^{ik}g^{jl}[D_j,[D_k,D_l]]F(\xi) \;=\; -\,\xi^i. \end{aligned}$$

- Solve linear equations for $F(\xi)$. Then invert shuffle reciprocal to get back gluon correlations G_I

$$G_I = \sum_{n=1}^{|I|}(-1)^n \sum_{\substack{I=I_1\sqcup I_2\sqcup\cdots\sqcup I_n \\ I_k \neq \emptyset\ \forall\ k}} F_{I_1}F_{I_2}\cdots F_{I_n} \quad \text{for} \quad I \neq \emptyset.$$

$I = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_n \Leftrightarrow I_1, \cdots, I_n$ are complementary order-preserving subwords of I

- For example, $G_i = -F_i$,

$$\begin{aligned} G_{ij} &= -F_{ij} + 2F_iF_j \\ G_{ijk} &= -F_{ijk} + 2(F_iF_{jk} + F_jF_{ik} + F_kF_{ij}) - 6F_iF_jF_k \end{aligned}$$

- Shuffle reciprocal is one-to-one provided $G_\emptyset \neq 0$

- Remains to solve linear equations for $F(\xi)$! Unfortunately, they are under determined in general.

Zeroth order Approximation for Gaussian

- For Gaussian, $C^{ij}\ D_j F(\xi) \ = \ -\xi^i$ have unique soln. Inverting shuffle reciprocal, $(S(A) = \frac{1}{2\alpha}\,\mathrm{tr}\ A^2)$

Moments	exact	$\mathcal{O}(q^0)$
G_2	α	α
G_4	$2\alpha^2$	$6\alpha^2$
G_6	$5\alpha^3$	$90\alpha^3$
G_8	$14\alpha^4$	$2520\alpha^4$
$G_{2n}, n \to \infty$	$\frac{(4\alpha)^n}{\sqrt{\pi n^3}}$	$(\frac{\alpha}{2})^n (2n)!$

- Gives over-estimate of correlations.

- Get under-estimate by deforming $D_i \to \mathbf{D}_i$ fixing $conc$.

Moments	exact	$\mathcal{O}(p^0)$	$\mathcal{O}(p)$
G_2	α	0.5α	0.75α
G_4	$2\alpha^2$	$0.25\alpha^2$	$0.75\alpha^2$
G_6	$5\alpha^3$	$0.125\alpha^3$	$0.646\alpha^3$
G_8	$14\alpha^4$	$0.0625\alpha^4$	$0.490\alpha^4$
$G_{2n}, n \to \infty$	$\frac{(4\alpha)^n}{\sqrt{\pi n^3}}$	$(\frac{\alpha}{2})^n$	$(\frac{\alpha}{2})^n (2n \log n)$

Associative q-products interpolating between $shuffle$ and $conc$

- To go beyond zeroth order, we need a q-series for concatenation around shuffle

- 1-parameter family of associative products interpolate between conc $(q = 1)$ and shuffle $(q = 0)$ (see also work of Thibon et. al.)

$$[F *_q G]_I \equiv \sum_{J \sqcup K = I} (1-q)^{\chi(I,J,K)} F_J G_K.$$

- Crossing number $\chi(I; J, K)$: min $\#$ of transpositions of j_i, k_l to transform $JK \to I$

- For example, $\chi(ijk; i, jk) = 0, \quad \chi(ijk; ik, j) = 1, \quad \chi(ijk; jk, i) = 2$

- Take $q \to 0$ get a Poisson bracket on shuffle algebra

$$\{F, G\}_I = -\lim_{q \to 0} \frac{1}{q}([F, G]_q)_I = \sum_{I = J \sqcup K} \chi(I; J, K)(F_J G_K - G_J F_K).$$

- Still not enough, want an explicit q-series for $*_q$ in terms of $*_0$ and D_i

Associativity of $*_q$ products

- Associativity follows from the interesting formula

$$((F *_q G) *_q H)_I \quad = \quad (F *_q (G *_q H))_I \quad = \quad \sum_{I=J\sqcup K\sqcup L} p^{\chi(I;J,K,L)} F_J G_K H_L$$

- Here $I = J \sqcup K \sqcup L$ is the condition that J, K, L are complementary order-preserving sub-words of I.

- $p = 1 - q$.

- $\chi(I; J, K, L)$ is the smallest number of transpositions needed to transform JKL into I. It is the three word crossing number and also equals

$$\chi(I; J \sqcup K, L) + \chi(J \sqcup K; J, K) \quad = \quad \chi(I; J, K \sqcup L) + \chi(K \sqcup L; K, L)$$

Single Matrix q-product Interpolating between Shuffle and Concatenation

- For Single Matrix reduces to Gauss q-binomial coefficients

$$(F *_q G)_n = \sum_{r=0}^{n} \binom{n}{r}_{1-q} F_r G_{n-r}.$$

$$\binom{n}{r}_q = \frac{[n]_q!}{[r]_q![n-r]_q!} \quad \text{where} \quad [n]_q! = [1]_q[2]_q \cdots [n]_q \quad \text{and} \quad [n]_q = \frac{1-q^n}{1-q}.$$

- For $q = 0$ get ordinary binomial coefficients and shuffle product

$$\binom{n}{r}_1 = \binom{n}{r}; \qquad (F *_q G)_n = \sum_{r=0}^{n} \binom{n}{r} F_r G_{n-r}.$$

- For $q = 1$ get concatenation product

$$\binom{n}{r}_0 = 1; \qquad (F *_q G)_n = \sum_{r=0}^{n} F_r G_{n-r}.$$

Classical Action Principle for Loop Equations

- $N \to \infty$ 'classical' limit, $\Rightarrow$ Loop equations are classical equations of motion.

$$S^{J_1 i J_2}\, G_{J_1 I J_2} = \delta_I^{I_1 i I_2}\, G_{I_1}\, G_{I_2} = \eta_I^i$$

- What is the **classical action or variational principle** from which they follow?

- $S(A)$ won't do. It's variation won't give anomaly η_I^i coming from change in measure.

- Look for classical action $\Omega(G)$ whose extrema are loop equations

$$L_I^i\, \Omega(G) = -S^{J_1 i J_2}\, G_{J_1 I J_2} \;+\; \delta_I^{I_1 i I_2}\, G_{I_1}\, G_{I_2} = 0.$$

- Differentiate $\Omega(G)$ along the vector fields L_I^i

Lie Algebra of Vector Fields L^i_I on Configuration space $\mathcal{P}_\Lambda$ for Λ matrices

- For several matrices, $L_v : A_i \to A_i + v_i^I A_I$ are infinitesimal change of variables

$$L^i_I \quad : \quad A_j \to A_j + \epsilon \delta^i_j A_I \quad \longrightarrow \quad \text{Monomial vector fields}$$

- $L_v = v_i^I L^i_I$ infinitesimal automorphisms of tensor algebra in Λ generators $\mathcal{T}_\Lambda$ (free associative algebra).

- Action on coordinates $[L^i_I G]_J = \delta_J^{J_1 i J_2} G_{J_1 I J_2}$

- $L^i_I \to 1^{\text{st}}$ order differential operators on config. space $\quad L^i_I = G_{J_1 I J_2} \frac{\partial}{\partial G_{J_1 i J_2}}$

- L^i_I form a Lie algebra generalizing Witt algebra $[L^i_I, L^j_J] = \delta_J^{J_1 i J_2} L^j_{J_1 I J_2} - \delta_I^{I_1 j I_2} L^i_{I_1 J I_2}$

- $L^i_I \to$ Lie algebra of $\mathcal{G} = Aut(\mathcal{T}_\Lambda) =$ Diff(non-commutative manifold).

Lie Algebra of Vector Fields L_k on Configuration space for one matrix - omit

- For 1 matrix, infinitesimal change of variables is $L_k : A \rightarrow A + \epsilon A^{k+1}$.

- Taking expectation values $L_k G_p = p G_{k+p}$, is action on coordinate functions G_p

- $L_k = \Sigma_j \, G_{j+k} \frac{\partial}{\partial G_j}$ are 1^{st} order partial differential operators on configuration space, i.e. vector fields on $\mathcal{P}_\Lambda$.

- L_k satisfy the same Lie algebra as polynomial vector fields on $\mathbf{R}$ (Witt algebra)

$$[L_m, L_n] = (n-m) L_{m+n} \; , \qquad n, m = 0, 1, 2, \ldots$$

- Powers $1, A, A^2, A^3, \cdots$ generate the tensor algebra in one generator $\mathcal{T}_1$, which is also the algebra of polynomials on the real line.

Lie Algebra of Vector Fields L_k on Configuration space for 1-matrix - omit

• L_k can be regarded as infinitesimal automorphisms of the tensor algebra $\mathcal{T}_1$ in one generator A. So $\{L_k\}$ span the Lie algebra of $\mathcal{G}_1 = \mathrm{Aut}(\mathcal{T}_1)$

$$A \mapsto \phi(A) = \phi_1 A + \phi_2 A^2 + \phi_3 A^3 + \cdots \quad \longrightarrow \quad \mathrm{Aut}(\mathcal{T}_1)$$

• Automorphisms of the algebra of functions on a manifold can be regarded as a replacement for the diffeomorphism group of a manifold.

• So can think of L_k as spanning Lie algebra of group of formal diffeomorphisms of $\mathbf{R}$.

$$x \to x + \epsilon x^{k+1}; \qquad L_k = x^{k+1}\frac{\partial}{\partial x}$$

Loop equations for a single matrix

- For a single matrix A, action is $\mathrm{tr}\, S(A) = \mathrm{tr}\, \Sigma_{n=1}^{m} S_n A^n$ and loop equations

$$\sum_l \; l \; S_l \; G_{k+l} = \sum_{p+q=k} \; G_p G_q := \eta_k, \quad k = -1, 0, 1, 2, \cdots$$

- Can also formulate as Mehta-Dyson equation for $\rho(x)$ where $G_k = \int \rho(x) \; x^k \; dx$

$$S'(x) = 2\mathcal{P} \int dy \; \frac{\rho(y)}{x-y} \qquad \text{to go back,} \quad \times\, x^{k+1} \;\; \text{and} \quad \int dx$$

- Mehta-Dyson equation follows from a variational principle, extremize $\Omega = \chi - S$

$$\Omega(\rho) = \mathcal{P} \int dx \; dy \; \rho(x)\rho(y) \log|x-y| - \int dx \; S(x) \; \rho(x)$$

- But ρ does not generalize to several matrices, though G_k do.

- And χ can't be expressed in terms of G_k since $\log|x-y|$ is not a power series in x and y simultaneously.

Search for Classical Action for Several Matrices

- For several matrices want $\Omega(G)$ whose extremum is loop equations,

$$L_I^i \, \Omega(G) = -S^{J_1 i J_2} \, G_{J_1 I J_2} \; + \; \delta_I^{I_1 i I_2} \, G_{I_1} \, G_{I_2}$$

- Action dependent term comes from variation of expectation value of original action

$$L_I^i \; (\; S^J G_J \;) \quad = \quad S^{J_1 i J_2} G_{J_1 I J_2}.$$

- So let $\Omega(G) = \chi(G) - S^J \, G_J$ where $L_I^i \, \chi \; = \; \eta_I^i$ or $d\chi = \eta$

- Extremization of Ω is the (partial) Legendre transform of χ, (think thermodynamics).

- $\chi(G)$: entropy of the non-commutative probability distribution$\{G_I\}$. Maximize entropy holding moments G_I conjugate to couplings S^I fixed (Lagrange multipliers).

Motivation for calling χ ENTROPY

- Restrict observables: matrix elements $\rightarrow U(N)$ invariants

$$[A_i]^a_b \longrightarrow \frac{\mathrm{tr}}{N} A_{i_1 i_2 \cdots i_n}$$

- When observables of a system are restricted there is an entropy.

- In statistical mechanics: Don't measure positions, velocities of individual gas molecules.

Only measure macroscopic observables such as P, V, U, T, S.

- Strong interactions: Confinement of color degrees of freedom should lead to an entropy.

- Entropy $=$ Log(volume of microstates) with same values of macroscopic observables.

- It will turn out that $\chi(G)$ is the entropy in this sense.

Necessary Integrability Conditions for existence of χ

- $L^i_I\,\chi \;=\; \eta^i_I \;=\; \delta_I^{I_1 i I_2}\, G_{I_1}\, G_{I_2} \quad \Leftrightarrow \quad d\chi = \eta$

- System of $1^{\rm st}$ order linear PDEs on configuration space; $L^i_I = G_{J_1 I J_2}\ \frac{\partial}{\partial G_{J_1 i J_2}}$

- Integrability condition requires $d\eta = 0$

- Integrability conditions $\quad (d\eta)^{ij}_{IJ} = L^i_I(L^j_J\chi) - L^j_J(L^i_I\chi) - [L^i_I, L^j_J]\chi = 0$

$$\Rightarrow \quad L^i_I\eta^j_J - L^j_J\eta^i_I \;=\; \delta_J^{J_1 i J_2}\eta^j_{J_1 I J_2} - \delta_I^{I_1 j I_2}\eta^i_{I_1 J I_2}$$

- Look complicated: L^i_I non-commuting basis unlike $\frac{\partial}{\partial G_{iI}}$

- Calculation: Checked that integrability conditions are satisfied. $\eta \longrightarrow$ closed 1-form!

Is $\chi(G)$ expressible in terms of moments?

- Is there χ with $L_I^i \chi = \eta_I^i$? i.e. $d\chi = \eta$?

- Is η an exact 1-form? **Answer: No!**

- No formal series $\chi(G)$ on configuration space with $d\chi = \eta$

- Not possible even for 1-matrix $\chi = -\int dx\, dy\, \log|x-y|\, \rho(x)\, \rho(y)$

- $\chi(G)$ essentially involves 'logarithmic moment'. G_k are only polynomial moments!

- η is closed but not exact: **Cohomological obstruction** to finding χ!

χ and Lie algebra Cohomology

- $\eta \rightarrow$ element of 1^{st} cohomology of $\underline{\mathcal{G}}$ twisted by its representation on the vector space of formal power series in G_I.

- $\underline{\mathcal{G}} =$ Lie algebra of vector fields L_I^i and $\mathcal{G}$ is automorphism group of tensor algebra.

- η is infinitesimal version of χ.

- Expect $\chi \in 1^{\mathrm{st}}$ cohomology of group $\mathcal{G}$.

- χ should be a non-trivial 1-cocycle of $\mathcal{G} = Aut(\mathcal{T})$.

- η was got from infinitesimal change of variables.

- Suggests $\rightarrow$ find formula for χ via finite change of variable.

Group cohomology - skip

Given a group G and a G-module V (i.e., a representation of G on a vector space V), we can define a cohomology theory. The r-cochains are functions

$$f : G^r \to V.$$

The coboundary d is

$$df(g_1, g_2, \cdots g_{r+1}) = g_1 f(g_2, \cdots g_{r+1}) + \sum_{s=1}^{r} (-1)^s f(g_1, g_2, \cdots g_{s-1}, g_s g_{s+1}, g_{s+2}, \cdots g_{r+1}) + (-1)^{r+1} f(g_1, \cdots, g_r).$$

$d^2 f = 0$ for all f. A cochain c is a *cocycle* or is *closed* if $df = 0$; a cocycle is *exact* or is a *coboundary* if $b = df$ for some f; The rth cohomology of G twisted by the module V, $H^r(G, V)$ is the space of closed cochains modulo exact cochains. $H^0(G, V)$ is the space of invariant elements in V; i.e., the space of v satisfying $gv - v = 0$ for all $g \in G$. A 1-cocycle is a function $c : G \to V$ satisfying

$$c(g_1 g_2) = g_1 c(g_2) + c(g_1).$$

Solutions to this equation modulo 1-coboundaries (which are of the form $b(g) = (g - 1)v$ for some $v \in V$) is the first cohomology $H^1(G, V)$. If G acts trivially on V, a cocycle is just a homomorphism of G to the additive group of V: $c(g_1 g_2) = c(g_2) + c(g_1)$.

New Parametrization of Configuration Space $\mathcal{P}$

- To find χ need new way of describing functions on configuration space.

- Power series in G_I inadequate: cohomological obstruction.

- Another way: Change of variable $\phi : \Gamma_I \mapsto G_I$ where $\Gamma_I \longrightarrow$ reference probability distribution.

$$A_i \mapsto \phi_i(A) = \phi_i^j A_j + \phi_i^{j_1 j_2} A_{j_1} A_{j_2} + \cdots \qquad \det \phi_j^i > 0$$

$$G_{i_1 \cdots i_n} = [\phi_* \Gamma]_{i_i \cdots i_n} = \phi_{i_1}^{J_1} \cdots \phi_{i_n}^{J_n} \, \Gamma_{J_1 \cdots J_n}$$

- $\phi \longrightarrow$ automorphism of the tensor algebra! $\phi \in Aut(\mathcal{T}) = \mathcal{G}$

New Parametrization of Configuration Space $\mathcal{P}$

- Configuration space carries action of automorphism group $\mathcal{G} = \{\phi\}$

$$\phi \ : \ \Gamma \ \mapsto \ G$$

- But more than one change of variable $\phi : \Gamma_I \mapsto G_I$.

- Let $\mathcal{SG}$ be isotropy subgroup: changes of variable fixing Γ. $\phi : \ \Gamma \ \rightarrow \ \Gamma$

- $\mathcal{P} = \{G_I\}$ is the quotient $\mathcal{G}/\mathcal{SG}$ coset space

How does this solve problem of finding χ?

- New way to think of functions on config space $\mathcal{P} = \mathcal{G}/\mathcal{SG}$

- Functions on group $\mathcal{G}$ invariant under subgroup $\mathcal{SG}$

- Power series in G_I can be expressed as power series in ϕ_i^I. Just substitute $G_I = [\phi_* \Gamma]_I = \phi_{i_1}^{J_1} \cdots \phi_{i_n}^{J_n} \Gamma_{J_1} \cdots \Gamma_{J_n}$.

- But $\exists$ power series in ϕ_i^I not expressible as series in G_I! χ is one such function!

- $\eta \longrightarrow$ infinitesimal change in integration measure: $\det\left(\frac{\partial A'}{\partial A}\right)$

- $\chi \longrightarrow$ Jacobian $\det \frac{\partial \phi(A)}{\partial A}$ for finite change of variable ϕ

$$\chi(G) = \chi(\phi, \Gamma) = \chi(\Gamma) + c(\phi, \Gamma) = \chi(\Gamma) + \langle \frac{1}{N^2} \log \det \frac{\partial \phi(A)}{\partial A} \rangle$$

Formula for χ: cocycle of Automorphism Group

- Put $\phi_i(A) = \phi_i^j A_j + \phi_i^{jk} A_j A_k + \cdots$ in $\log\det$ of Jacobian

$$\chi(\phi, \Gamma) \;=\; \chi(\Gamma) + \log\det\phi_i^j + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \tilde{\phi}_{i_1}^{K_1 i_2 L_1} \tilde{\phi}_{i_2}^{K_2 i_3 L_2} \cdots \tilde{\phi}_{i_n}^{K_n i_1 L_n} \Gamma_{K_1 \cdots K_n} \Gamma_{L_n \cdots L_1}.$$

where $\tilde{\phi}_i^I = [\phi^{-1}]_i^j \phi_j^I$

- Multiplicativity of $\det \Rightarrow$ relative entropy $\chi(\phi, \Gamma) - \chi(\Gamma)$ is a cocycle

$$c(\phi\psi, \Gamma) = c(\phi, \psi_*(\Gamma)) + c(\psi, \Gamma)$$

$\chi(\phi, \Gamma) - \chi(\Gamma)$ is a 1-cocycle of automorphism group $Aut(\mathcal{T}_M)$.

- Can show that $\chi(\phi, \Gamma) = \chi(G)$ is actually a function on $\mathcal{P} = \mathcal{G}/\mathcal{SG}$, i.e. is invariant under $\mathcal{SG}$

- If $\psi_*\Gamma = \Gamma$, then can show $c(\phi\psi, \Gamma) = c(\phi, \Gamma)$. i.e. $\psi \in \underline{\mathcal{SG}}$ leaves χ unchanged.

Variational Principle is Legendre Transform of χ

- Solved problem of finding action for large N matrix models $\Omega = -S^I G_I + \chi$

$$\Omega(\phi,\Gamma) = -\sum_{n=1}^{\infty} S^{i_1\cdots i_n}\phi_{i_1}^{J_1}\cdots\phi_{i_n}^{J_n}\Gamma_{J_1\cdots J_n} + \chi(\Gamma) + \log\det\phi_j^i$$

$$+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\tilde{\phi}_{i_2}^{K_1 i_1 L_1}\tilde{\phi}_{i_3}^{K_2 i_2 L_2}\cdots\tilde{\phi}_{i_1}^{K_n i_n L_n}\Gamma_{K_n\cdots K_1}\Gamma_{L_1\cdots L_n}$$

- Variational principle $\rightarrow$ Legendre transform of χ

- Maximize χ holding G_I conjugate to S^I fixed $\rightarrow$ determines optimal ϕ

- Once $\phi_{optimal}$ is found, correlations, Free energy are

$$G_{i_1\cdots i_n} = \phi_{i_1}^{J_1}\cdots\phi_{i_n}^{J_n}\Gamma_{J_1\cdots J_n} \qquad F(S) = -\chi(\phi_{\text{optimal}},\Gamma)$$

expressed in terms of $\phi_{optimal}$ and reference moments Γ_I

- Coefficients of ϕ are variational parameters.

χ as Entropy

- Microstates = matrices; Macroscopic observables = invariants $\Phi_I = \frac{\text{tr}}{N} A_I$

- Contrast with Thermodynamics

1. Infinite number of macroscopic observables Φ_I
2. Concept of thermal equilibrium not relevant
3. Microstates matrices don't commute: Entropy is non-commutative probability

- $\chi \longrightarrow$ log of change in volume measure $\longrightarrow$ entropy

$$\chi(G) = \chi(\phi, \Gamma) = \chi(\Gamma) + c(\phi, \Gamma) = \chi(\Gamma) + \langle \frac{1}{N^2} \log \det J \rangle$$

- Coincides with entropy of non-commutative probability theory (Voiculescu)

Entropy for 1-Matrix Models

- Interpreting χ as entropy transparent for single matrix integral $Z = \int dA\, e^{-N \,\mathrm{tr}\, S(A)}$

- Diagonalize $N \times N$ hermitian matrix A: $A \longrightarrow UDU^\dagger, \quad D = \mathrm{diag}\,(\lambda_1, \cdots, \lambda_N)$

- Jacobian $=$ Vol(hermitian matrices with common spectrum) $\lambda_1 < \lambda_2 < \cdots < \lambda_N$

$$dA = Vol(U(N))\, \Delta^2 \prod_a d\lambda_a$$

- Vandermonde determinant $\Delta = \Pi_{a<b}(\lambda_a - \lambda_b) \Rightarrow \chi = \log \Delta^2 = 2\,\Sigma_{a<b} \log |\lambda_a - \lambda_b|$

- If eigenvalue density is $\rho(x) = \frac{1}{N} \Sigma_a\, \delta(x - \lambda_a)$, as $N \to \infty$

$$\chi = \mathcal{P} \int \rho(x)\, \rho(y)\, \log |x - y|\, dx\, dy$$

- This is Entropy of single operator-valued random variable.

- Contrast with Boltzmann entropy of one real-valued random variable $\int \rho(x) \log \rho(x)\, dx$

Entropy and Change of Variables for 1-matrix

- Γ reference prob. distribution. $G \to$ distribution of interest

$$G_k = \int \rho_G(x)\, x^k\, dx; \qquad \text{and} \quad \Gamma_k = \int \rho_\Gamma(x)\, x^k\, dx$$

- $\phi \to$ change of variables relates the two

$$\rho_\Gamma(x) = \rho_G(\phi(x))\, \phi'(x) \qquad \text{and} \quad G_k = \int \rho_\Gamma(y)\, \phi^k(y)\, dy$$

- Entropy $\chi(G) = \mathcal{P} \int dxdy\, \rho_G(x)\rho_G(y) \log|x-y|$ becomes

$$\chi(G) \;=\; \chi(\Gamma) + \mathcal{P} \int dxdy\, \rho_\Gamma(x)\rho_\Gamma(y) \log|\frac{\phi(x)-\phi(y)}{x-y}|$$

- 2^{nd} term on right: entropy of G relative to Γ

- If ϕ is an invertible power series $\phi(x) = \phi_1[x + \tilde{\phi}_2 x^2 + \tilde{\phi}_3 x^3 + \cdots] \quad \phi_1 > 0$

- Then entropy for a single matrix agrees with earlier formula for cocycle

$$\chi(G) = \chi(\Gamma) + \log\phi_1 \; + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{k_i+l_i>0} \tilde{\phi}_{k_1+1+l_1} \cdots \tilde{\phi}_{k_n+1+l_n} \Gamma_{k_1+\cdots k_n} \Gamma_{l_1+\cdots+l_n}$$

String Theoretic Interpretation – skip

- We sought the action $\Omega(G)$ of a model for closed string field theory.

- In a gauge-fixed toy-model for strings on a space time with Λ points

- $\Phi_I \to$ **'closed string field'**, G_I its vacuum expectation value.

- Closed String Field theory is a dynamical system (reminiscent of the Wess-Zumino-Witten model) on coset space $\mathcal{G}/\mathcal{SG}$, where $\mathcal{G} = \mathrm{Aut}(\mathcal{T}_\Lambda)$.

- We find a formula for classical action, and an approximation method to solve it.

- Includes term representing an **anomaly**, a *non-trivial one-cocycle* of $\mathcal{G}$.

Variational Approximations

- Aim: Given action $S(A)$ find G_I and free energy F in the large N limit.

- Fix a reference distribution Γ, eg. Wigner distribution, or other solved model

$$\Omega(\phi,\Gamma) = \chi(\phi,\Gamma) - S^I G_I(\phi)$$

- Exact maximum of entropy $\longrightarrow$ exact change of variable ϕ $\longrightarrow$ exact G_I, Free energy.

- For a variational approximation, take polynomial

$$\phi_i = \phi_i^j A_j + \phi_i^{j_1 j_2} A_{j_1} A_{j_2} + \cdots \phi_i^{j_1 \cdots j_n} A_{j_1} \cdots A_{j_n}$$

- Coefficients ϕ_i^J are variational parameters

- Fix variational parameters by maximizing entropy

Mean Field Theory

- Simplest possibility $\longrightarrow$ linear change of variable $A_i \mapsto \phi_i(A) = \phi_i^j A_J$

- For this case entropy is $\chi = \text{ tr } \log \phi_i^j$

- For eg. consider a quartic multi-matrix model $S(M) = \text{ tr } [\frac{1}{2} K^{ij} A_{ij} + \frac{1}{4} g^{ijkl} A_{ijkl}]$

- Variational principle: maximize $\Omega[\phi] = \text{ tr } \log[\phi_i^j] - \frac{1}{2} K^{ij} G_{ij} - \frac{1}{4} g^{ijkl} G_{ijkl}$

- Reference distribution $\longrightarrow$ Wigner distribution $\Rightarrow$ all correlations can be expressed in terms of 2-point correlation, $\alpha,\ \beta \rightarrow$ variational parameters

$$G_{ij} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

- Condition for extremum of Ω: Non-linear equation for $G_{ij} \longrightarrow$ Mean Field Theory

$$\frac{1}{2} K^{pq} + \frac{1}{4} [g^{pqkl} G_{kl} + g^{ijpq} G_{ij} + g^{pjqk} G_{jk} + g^{ipql} G_{il}] = \frac{1}{2} [G^{-1}]^{pq}$$

Example: Quartic One Matrix Model

$$S(A) = \frac{1}{2}A^2 + gA^4$$

- Linear Change of variable

$$\phi(x) = \phi_1 x$$

- Cubic change of variable

$$\phi(x) = \phi_1 x + \phi_3 x^3$$

- Compare eigenvalue distributions with exact result known from work of Brezin et. al.

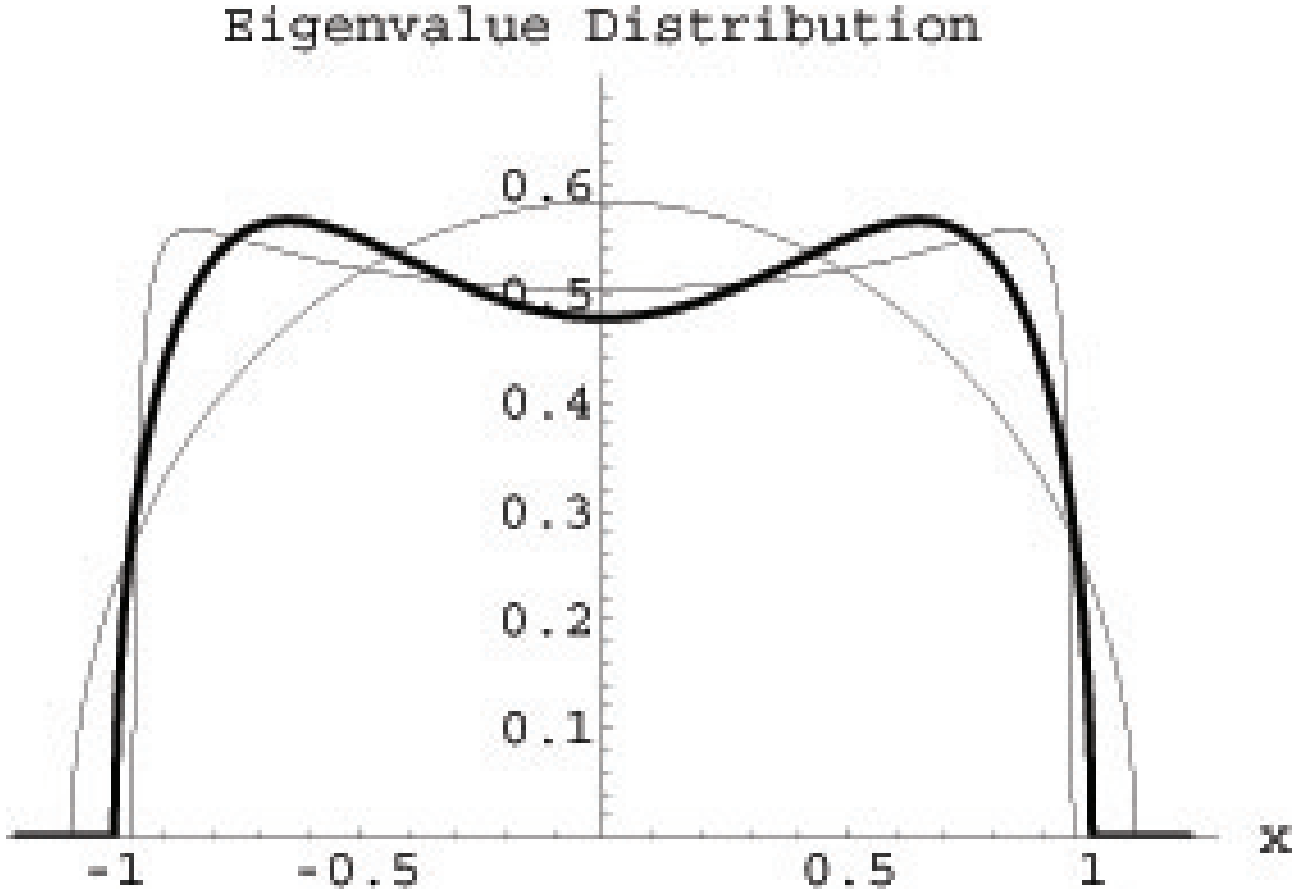

Figure 1: Eigenvalue Distribution. Dark curve is exact, semicircle is mean field and bi-modal light curve is cubic ansatz at 1-iteration.

Example: Mehta's 2 Matrix Model

$$S(A,B) = \text{ tr } [\tfrac{1}{2}(A^2 + B^2 - cAB - cBA) + \tfrac{g}{4}(A^4 + B^4)]$$

- Take reference distribution as Gaussian and linear change of variable. $G_{ij} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$

- Maximum of Ω occurs at (α, β) with $\beta = \frac{c\alpha}{1+2g\alpha}$ and

$$4g^2\alpha^3 + 4g\alpha^2 + (1 - c^2 - 2g)\alpha - 1 = 0$$

- Solve and get variational free energy and all correlations.

- Compare with Mehta's analytical results for some specific observables

$$\begin{aligned} E^{ex}(g, \frac{1}{2}) &= -.144 + 1.78g - 8.74g^2 + \cdots \\ E^{var}(g, \frac{1}{2}) &= -.144 + 3.56g - 23.7g^2 + \cdots \end{aligned}$$

$$\begin{aligned} G^{ex}_{AB}(g, \frac{1}{2}) &= \frac{2}{3} - 4.74g + 53.33g^2 + \cdots \\ G^{var}_{AB}(g, \frac{1}{2}) &= \frac{2}{3} - 4.74g + 48.46g^2 + \cdots \end{aligned}$$

$$
\begin{aligned}
G^{ex}_{AAAA}(g,\frac{1}{2}) &= \frac{32}{9} - 34.96g + \cdots \\
G^{var}_{AAAA}(g,\frac{1}{2}) &= \frac{32}{9} - 31.61g + 368.02g^2 + \cdots
\end{aligned}
$$

For strong coupling and arbitrary c:

$$
\begin{aligned}
E^{ex}(g,c) &= \frac{1}{2}\log g + \frac{1}{2}\log 3 - \frac{3}{4} + \cdots \\
E^{var}(g,c) &= \frac{1}{2}\log g + \frac{1}{2}\log 2 + \frac{1}{\sqrt{8g}} + \mathcal{O}(\frac{1}{g})
\end{aligned}
$$

$$
\begin{aligned}
G^{ex}_{AB}(g,c) &\rightarrow 0 \text{ as } g \rightarrow \infty \\
G^{var}_{AB}(g,c) &= \frac{c}{2g} - \frac{c}{(2g)^{\frac{3}{2}}} + \mathcal{O}(\frac{1}{g^2})
\end{aligned}
$$

$$
\begin{aligned}
G^{ex}_{AAAA}(g,c) &= \frac{1}{g} + \cdots. \\
G^{var}_{AAAA}(g,c) &= \frac{1}{g} - \frac{2}{(2g)^{\frac{3}{2}}} + \mathcal{O}(\frac{1}{g^2})
\end{aligned}
$$

- Both for strong and weak coupling, variational approx. gives good estimates.
- Mean Field Theory does not do well near phase transitions.

Conclusions

• Regarded large-N multi-matrix models as regularizations of Yang-Mills theory.

• Found a variational principle for fSDE for invariant observables.

• Non-trivial due to cohomological obstruction.

• Problem solved by expressing configuration space as a coset space of non-commutative analogue of diffeomorphism group.

• Variational principle interpreted as Legendre transform of entropy of operator-valued random variables.

• Led to variational approximations for matrix models.

• Brings together cohomology of $Aut(\mathcal{T}_\Lambda)$, non-commutative probability and physics.

Summary: Hopf algebraic formulation

• $G(\xi)$ of large-N matrix models live in the *shuffle-deconcatenation Hopf algebra*.

• Identified a finitely generated matrix model analogue of the group of loops on space time, the spectrum $\mathbf{G}_\Lambda$ of this Hopf algebra. Lie algebra of $\mathbf{G}_\Lambda$ is the FLA_Λ

• $G(\xi)$ is a function on $\mathbf{G}_\Lambda$. It satisfies **quadratic equations in convolution** product on the group: factorized SD equations $\mathcal{S}^i\, G(\xi) \;=\; G(\xi)\, \xi^i\, G(\xi)$.

• SD operators $\mathcal{S}^i$ of Yang-Mills, Chern-Simons and Gaussian models are **right-invariant vector fields on** $\mathbf{G}_\Lambda$, i.e., invariant derivations of the Hopf algebra.

• **fSDE can be transformed into linear equations** if we replace convolution (concatenation) by shuffle. To approximately solve: Expand concatenation as a deformation series around shuffle.