Matrix Integrals and Knot Theory

Paul Zinn-Justin et Jean-Bernard Zuber

math-ph/9904019 (Discr. Math. 2001)
math-ph/0002020 (J.Kn.Th.Ramif. 2000)
P. Z.-J. (math-ph/9910010, math-ph/0106005)
J. L. Jacobsen & P. Z.-J. (math-ph/0102015, math-ph/0104009). . .
math-ph/0303049 (J.Kn.Th.Ramif. 2004)
G. Schaeffer and P. Z.-J., math-ph/0304034
P. Z.-J. and J.-B. Z., review article in preparation

Carghjese, March 25, 2009

Courtesy of *"U Rasaghiu"*

Main idea :

Use combinatorial tools of Quantum Field Theory in Knot Theory

Plan

I Knot Theory : a few definitions

II Matrix integrals and Link diagrams

$\int dM e^{N \mathrm{tr}\,(-\frac{1}{2}M^2+gM^4)}$ $\quad N\times N$ matrices, $N\to\infty$

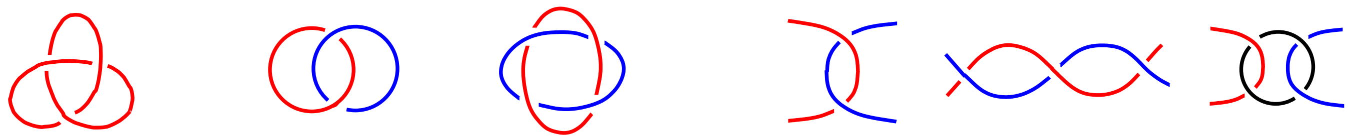
Removals of redundancies

$\Rightarrow$ reproduces recent results of Sundberg & Thistlethwaite (1998)

based on Tutte (1963)

III Virtual knots and links : counting and invariants

Basics of Knot Theory



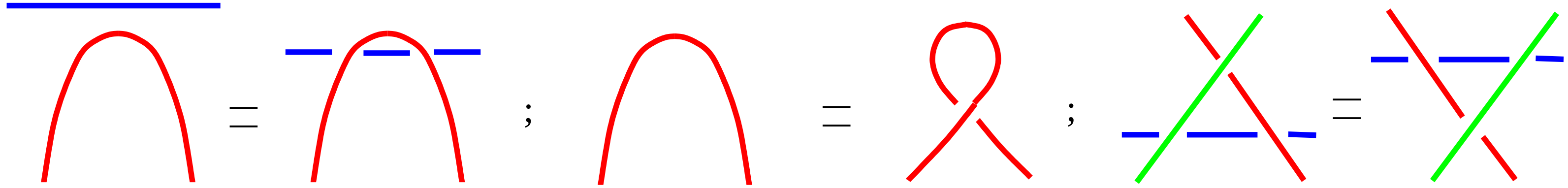
a knot , links and tangles

Equivalence up to isotopy

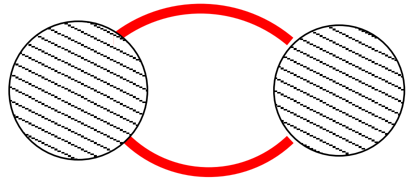
Problem : Count topologically inequivalent knots, links and tangles

Represent knots etc by their *planar projection* with minimal number of over/under-crossings

Theorem Two projections represent the same knot/link iff they may be transformed into one another by a sequence of Reidemeister moves :



Avoid redundancies by keeping only prime links (i.e. which cannot be factored)

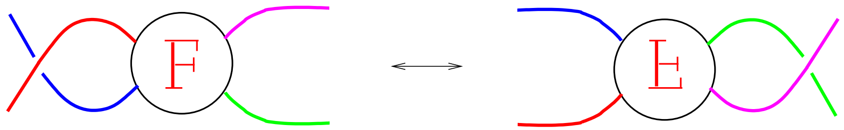


Consider the subclass of alternating knots, links and tangles, in which one meets alternatingly over- and under-crossings.

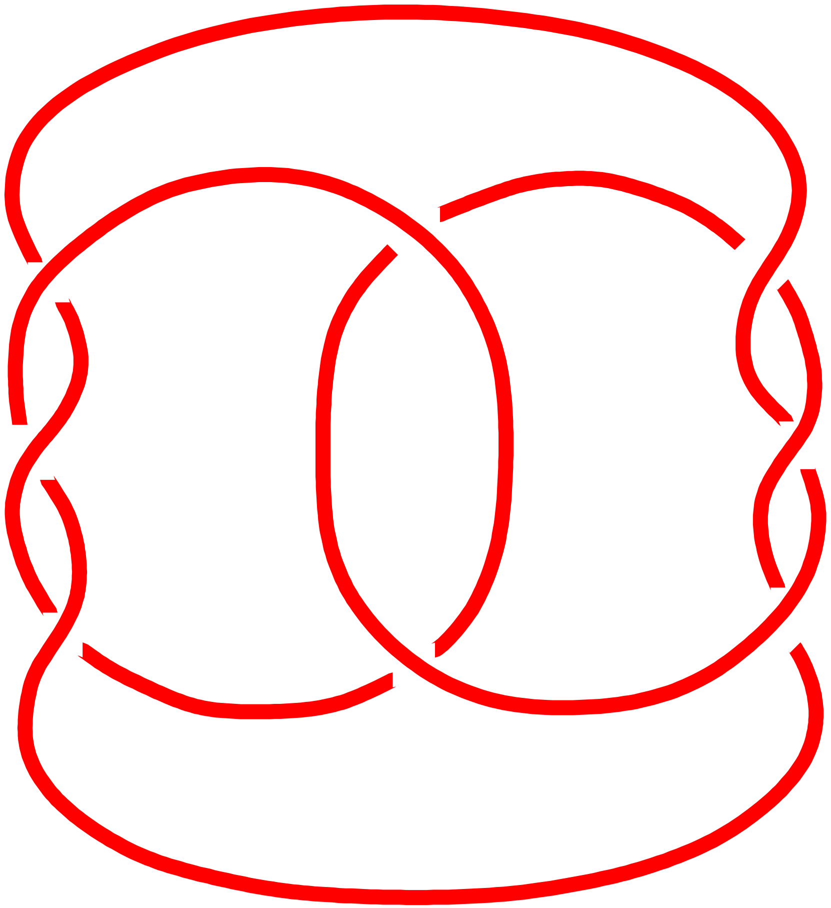
For $n \geq 8$ (resp. 6) crossings, there are knots (links) which cannot be drawn in an alternating form. Asymptotically, the alternating are subdominant.

Major result (Tait (1898), Menasco & Thistlethwaite, (1991))

Two alternating reduced knots or links represent the same object iff they are related by a sequence of "flypes" on tangles



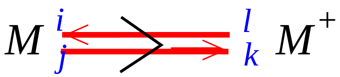
Problem Count alternating prime links and tangles



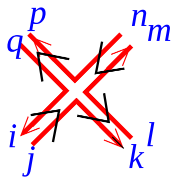
A 8-crossing non-alternating knot

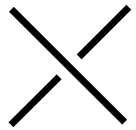
Matrix Feynman diagrams and link diagrams

Consider integral over complex (non Hermitean) matrices



$$\int dM \, \mathrm{e}^{N[-t\, \mathrm{tr}\, MM^\dagger + \frac{g}{2} \mathrm{tr}\, (MM^\dagger)^2]}$$



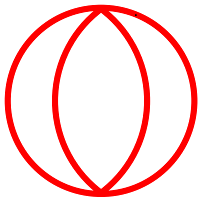


$\Rightarrow$ oriented (double) lines in propagators and vertices.

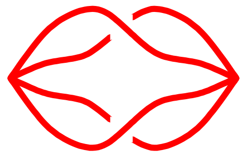
When $N \to \infty$, leading contribution from genus zero ("planar") diagrams :

$$\lim_{N\to\infty} \frac{1}{N^2} \log Z = \sum_{\substack{\text{planar diagrams} \\ \text{with } n \text{ vertices}}} \frac{g^n}{\text{symm.factor}}$$

for example, to second order

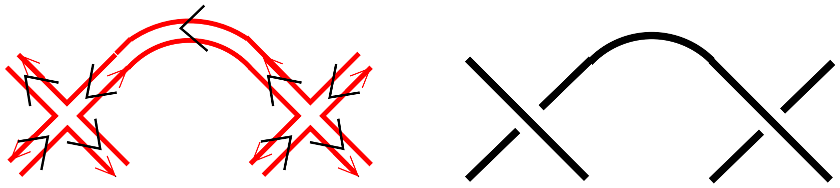


N^2



N^0

Moreover : Conservation of arrows $\Rightarrow$ alternating diagram !



But going from complex matrices to hermitian matrices doesn't affect the *planar limit* ... up to a global factor 2.

Moral After removing redundancies (incl. flypes), counting of Feynman diagrams of M^4 integral, (over hermitian matrices)

$$Z = \int dM\, \mathrm{e}^{N[-\frac{t}{2}\mathrm{tr}\, M^2 + \frac{g}{4}\mathrm{tr}\, M^4]}$$

for $N \to \infty$, yields the counting of alternating links and tangles.

Non perturbative results on M^4 integral, $N \to \infty$

Compute large N limit of integral $Z = \int dM\, e^{N[-\frac{t}{2}\mathrm{tr}\, M^2 + \frac{g}{4}\mathrm{tr}\, M^4]}$ by saddle point method, or orthogonal polynomials,or . . .

In the $N \to \infty$ limit, continuous distribution of eigenvalues λ with density $u(\lambda)$ of support $[-2a, 2a]$ (deformed semi-circle law)

$$u(\lambda) = \frac{1}{2\pi}(1 - 2\frac{g}{t^2}a^2 - \frac{g}{t^2}\lambda^2)\sqrt{4a^2 - \lambda^2}$$

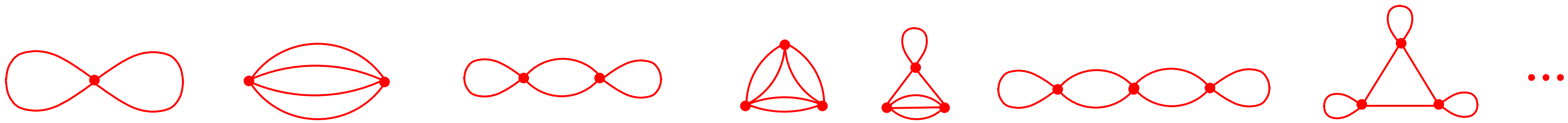
$$3\frac{g}{t^2}a^4 - a^2 + 1 = 0$$

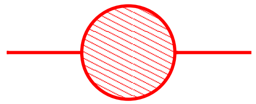
Thus "planar" limit of $\mathrm{tr}\, M^4$ integral

$$\lim_{N\to\infty} \frac{1}{N^2}\log\frac{Z(t,g)}{Z(t,0)} = F(t,g) = \frac{1}{2}\log a^2 - \frac{1}{24}(a^2-1)(9-a^2)$$

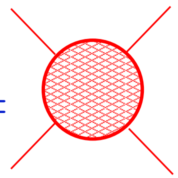
$$F(t,g) = \sum_{p=1} \left(\frac{3g}{t^2}\right)^p \frac{(2p-1)!}{p!(p+2)!}$$

As $p \to \infty$ $F_p \sim \text{const}(12)^p p^{-7/2}$



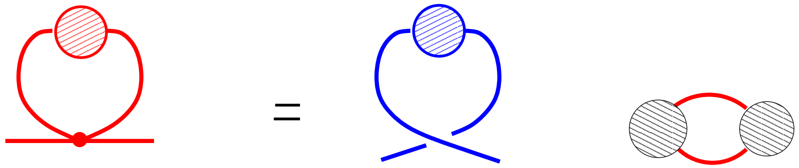
Also 2-point function $G_2 = \frac{1}{3t} a^2 (4 - a^2) =$ 

and (connected and "truncated") 4-point function

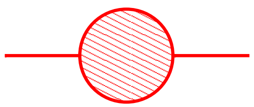
$\Gamma =$  $= \dfrac{(5-2a^2)(a^2-1)}{(4-a^2)^2}$

Counting Links and Tangles

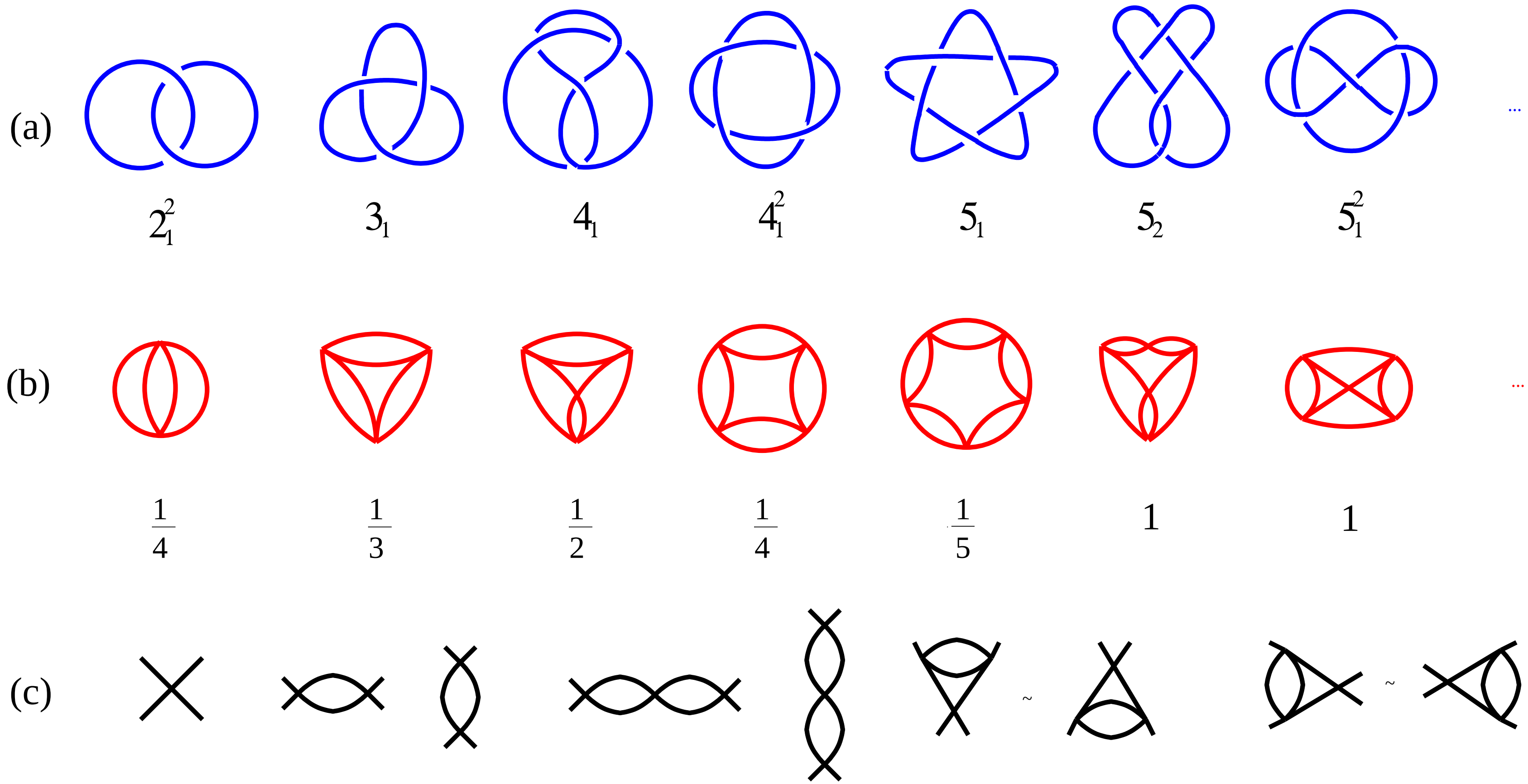
For the knot interpretation of previous counting, many irrelevant diagrams have to be discarded.



“Nugatory” and “non-prime” are removed by adjusting $t = t(g)$ so that

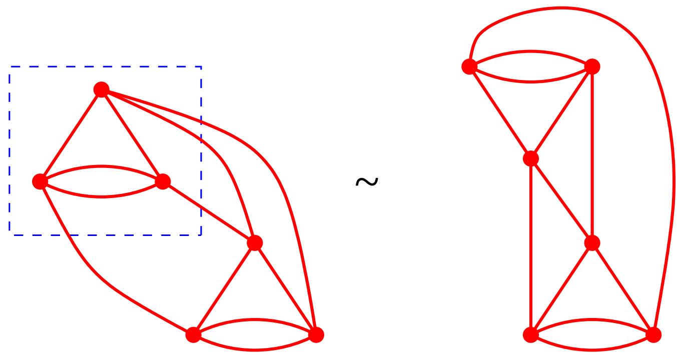
 $= 1$ (“wave function renormalisation”).

In that way, correct counting of links . . .up to 6 crossings !



Asymptotic behaviour $F_p \sim \mathrm{const}\,(27/4)^p\, p^{-7/2}$

What happens for $n \geq 6$ crossings ? Flypes !

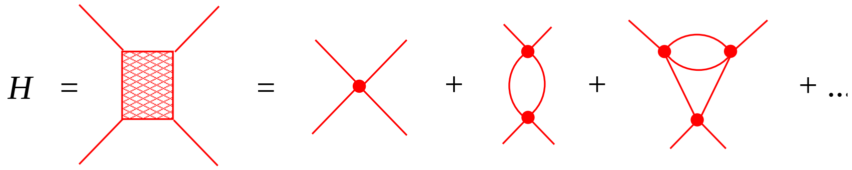


Must quotient by the flype equivalence ! Original combinatorial treatment (Sundberg & Thistlethwaite, Z-J&Z) rephrased and simplified by P. Z.-J. : it amounts to a coupling constant renormalisation $g \to g_0$! In other words, start from $N\mathrm{tr}\left(\frac{1}{2}tM^2 - \frac{g_0}{4}M^4\right)$, fix $t = t(g_0)$ as before. Then compute $\Gamma(g_0)$ and determine $g_0(g)$ as the solution of

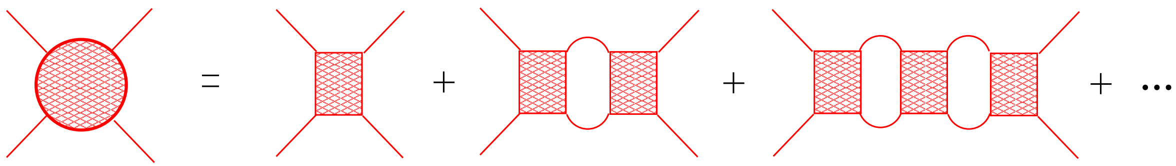
$$g_0 = g\left(-1 + \frac{2}{(1-g)(1+\Gamma(g_0))}\right) ,$$

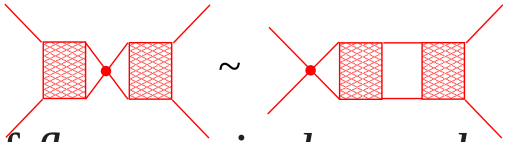
then the desired generating function is $\tilde{\Gamma}(g) = \Gamma(g_0)$.

Indeed let $H(g)$ be the generating function of "horizontally-two-particle-irreducible" diagrams (cannot separate the left part from the right by cutting two lines)

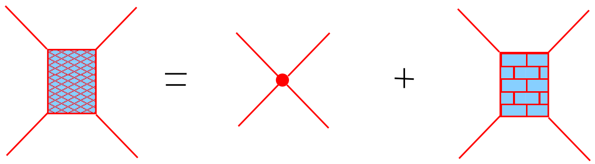


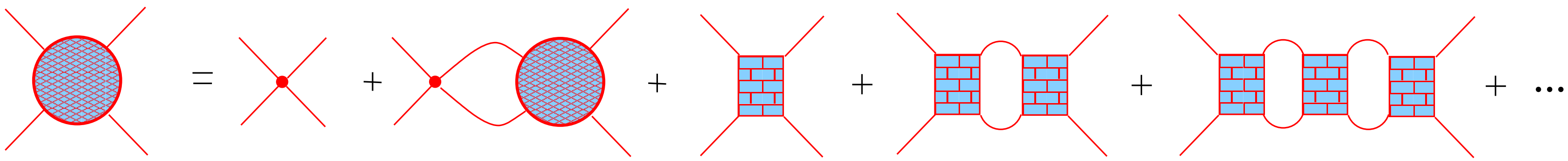
and then write $\Gamma = H/(1-H)$



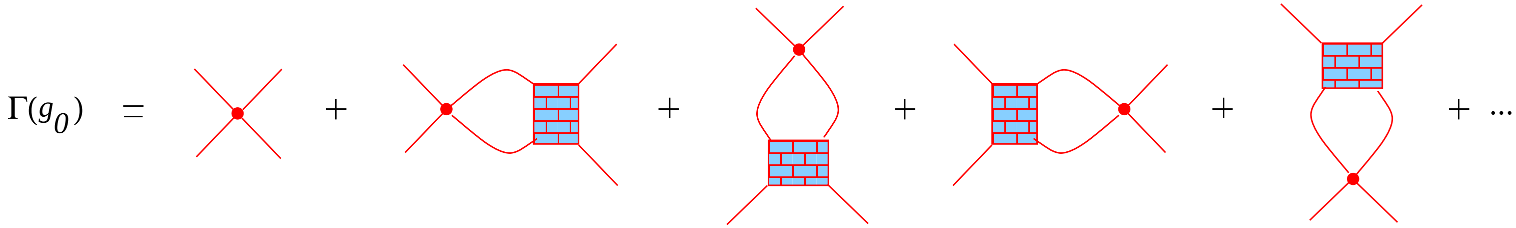
But  thus, with $\tilde{\Gamma}$, $\tilde{H}$ denoting generating functions of *flype equivalence classes* of prime tangles, resp 2PI tangles and if

$$\tilde{H} = g + \tilde{H}', \quad \tilde{\Gamma} = g + g\tilde{\Gamma} + \frac{\tilde{H}'}{1-\tilde{H}'}$$

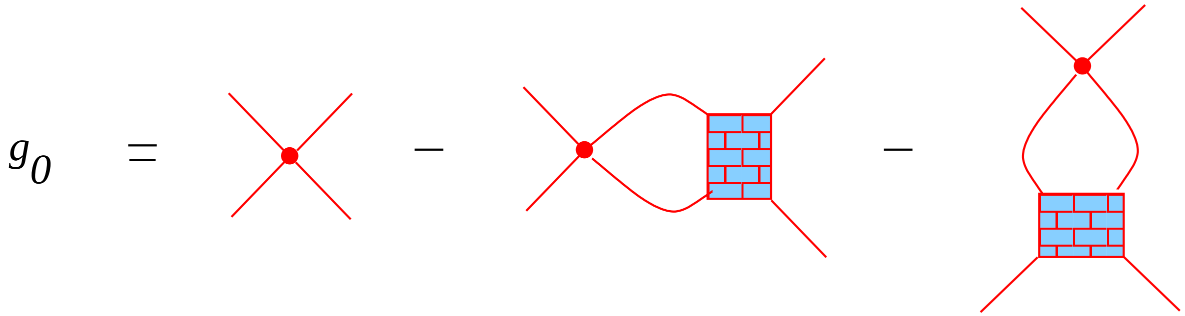




Return to $\Gamma(g_0)$



suggests to determine $g_0 = g_0(g)$ by demanding that $g_0 = g - 2g\tilde{H}'$



Three relations between g_0, $\tilde{H}'$, g and $\tilde{\Gamma}(g)$

Eliminating $\tilde{H}'$ and then

Eliminating g_0 gives $\tilde{\Gamma}(g) = \Gamma(g_0(g))$, the generating function of (flype-equivalence classes of) tangles.

Find

$\tilde{\Gamma} = g + 2g^2 + 4g^3 + 10g^4 + 29g^5 + 98g^6 + 372g^7 + 1538g^8 + 6755g^9 + \cdots$

Asymptotic behaviour $\tilde{\Gamma}_p \sim \text{const}\,\left(\frac{101+\sqrt{21001}}{40}\right)^p p^{-5/2}$

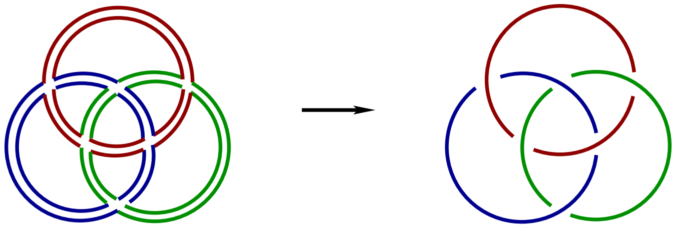
All this reproduces the results of Sundberg & Thistlethwaite.

⋆ Can we go further ? Control the number of connected components ? i.e. count *knots* rather than *links* ?

Coloured Links and Tangles

$$Z^{(N)}(n,g) = \int \prod_{a=1}^{n} dM_a \, \mathrm{e}^{N \, \mathrm{tr} \left(-\frac{1}{2} \sum_{a=1}^{n} M_a^2 + \frac{g}{4} \sum_{a,b=1}^{n} M_a M_b M_a M_b \right)}$$

Each connected component may come in n colours



If we write the free energy $F(n,g) = \sum_{k=1}^{\infty} F_k(g) n^k$, $F_k =$ generating function of diagrams with k loops. In particular, $F_1(g)$, that of knots.

Unfortunately this is computable only for $n = -2, 1, 2$

[P.Z.-J. 99, Z-J–Z 00]

⋆ Open and important problem to understand such integrals in the $n \to 0$ limit (replicas, combinatorics...)

Another direction : Virtual Links

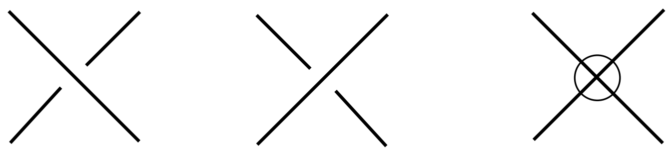
Higher genus contributions to matrix integral

What do they count ?

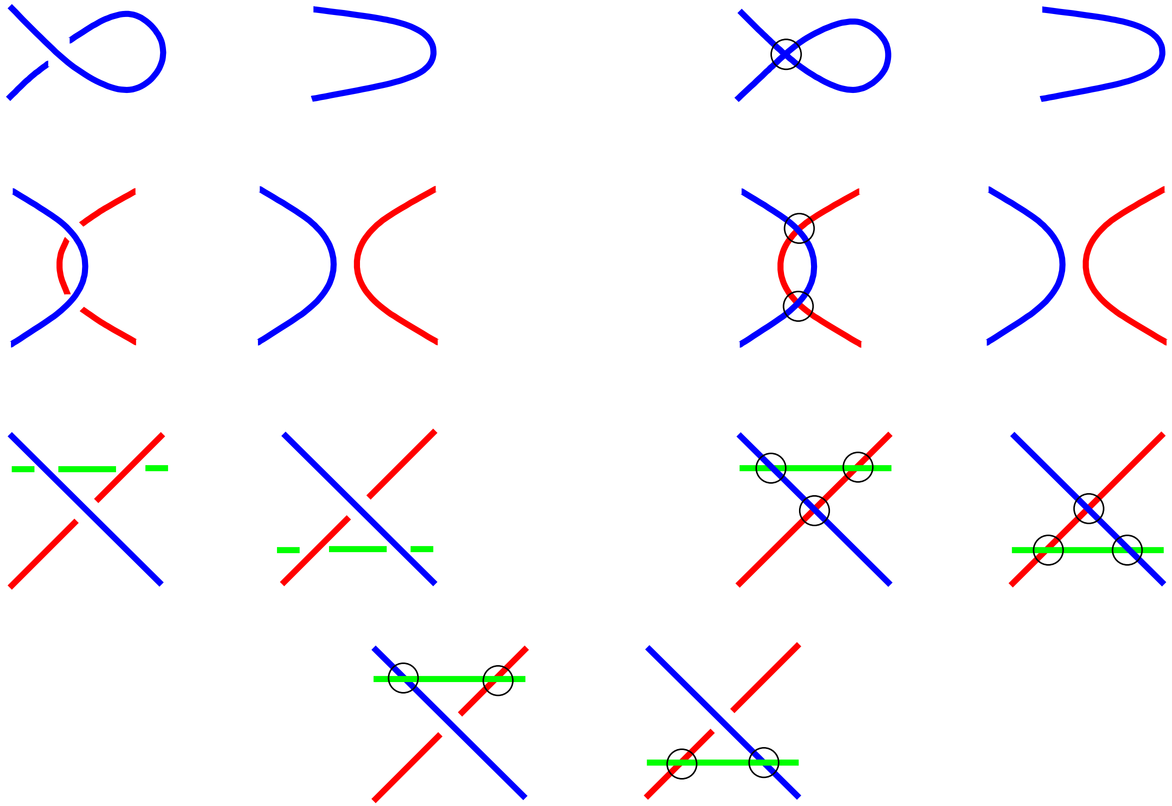
Suggested that knots/links live on other manifolds $\Sigma_h \times I$

Virtual knots and links [Kauffman] : equivalence classes of 4-valent diagrams with ordinary under- or over-crossings

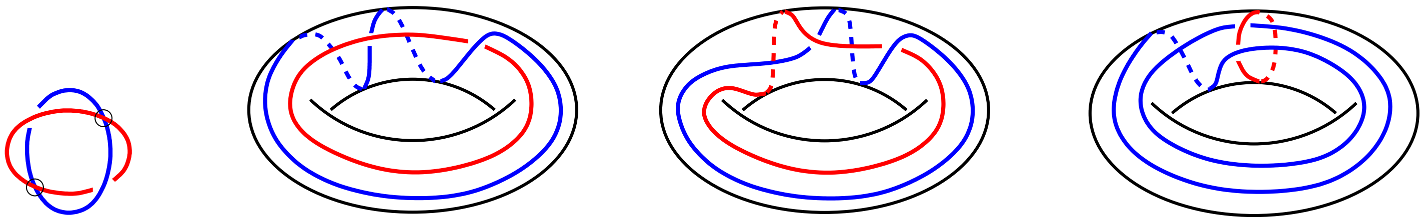
plus a new type of *virtual* crossing,



Equivalence w.r.t. generalized Reidemeister moves



From a different standpoint : Virtual knots (or links) seen as drawn in a "thickened" Riemann surface $\boldsymbol{\Sigma} := \Sigma \times [0,1]$, modulo isotopy in $\boldsymbol{\Sigma}$, **and** modulo orientation-preserving homeomorphisms of Σ, **and** addition or subtraction of empty handles.



But this is precisely what Feynman diagrams of the matrix integral do for us !

Thus return to the integral over complex matrices

$$Z(g,N) = \int dM\, \mathrm{e}^{N[-t\, \mathrm{tr}\, MM^\dagger + \frac{g}{2} \mathrm{tr}\, (MM^\dagger)^2]}$$

and compute $F(g,t,N) = \log Z$ beyond the leading large N limit ...

$$F(g,t,N) = \sum_{h=0}^{\infty} N^{2-2h} F^{(h)}(g,t)$$

$F^{(h)}(g)$: Feynman diagrams of genus h

$F^{(1)}$ computed by Morris (1991)
$F^{(2)}$ and $F^{(3)}$ by Akermann and by Adamietz (ca. 1997)

As before, determine $t = t(g,N)$ so as to remove the non prime diagrams.
Find the generating function of tangle diagrams $\Gamma(g) = 2\partial F/\partial g - 2$

$$\Gamma^{(0)}(g) \quad = g + 2g^2 + 6g^3 + 22g^4 + 91g^5 + 408g^6 + 1938g^7 + 9614g^8 + 49335g^9 + 260130g^{10} + O(g^{11})$$

$$\Gamma^{(1)}(g) \quad = g + 8g^2 + 59g^3 + 420g^4 + 2940g^5 + 20384g^6 + 140479g^7 + 964184g^8 + 6598481g^9 + 45059872g^{10} + O(g^{11}$$

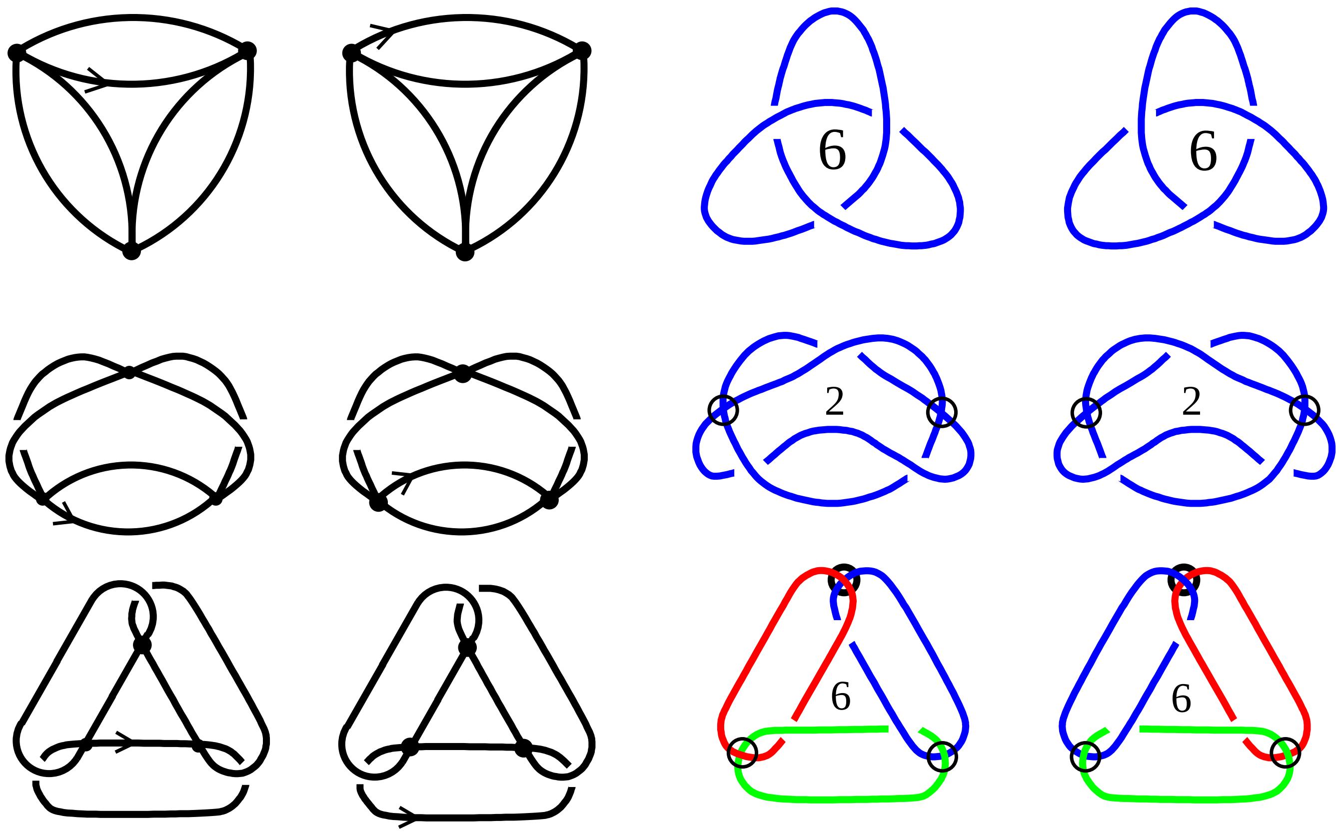
$$\Gamma^{(2)}(g) \quad = 17g^3 + 456g^4 + 7728g^5 + 104762g^6 + 1240518g^7 + 13406796g^8 + 135637190g^9 + 1305368592g^{10} + O(g^{1}$$

$$\Gamma^{(3)}(g) \quad = 1259g^5 + 62072g^6 + 1740158g^7 + 36316872g^8 + 627368680g^9 + 9484251920g^{10} + O(g^{11})$$

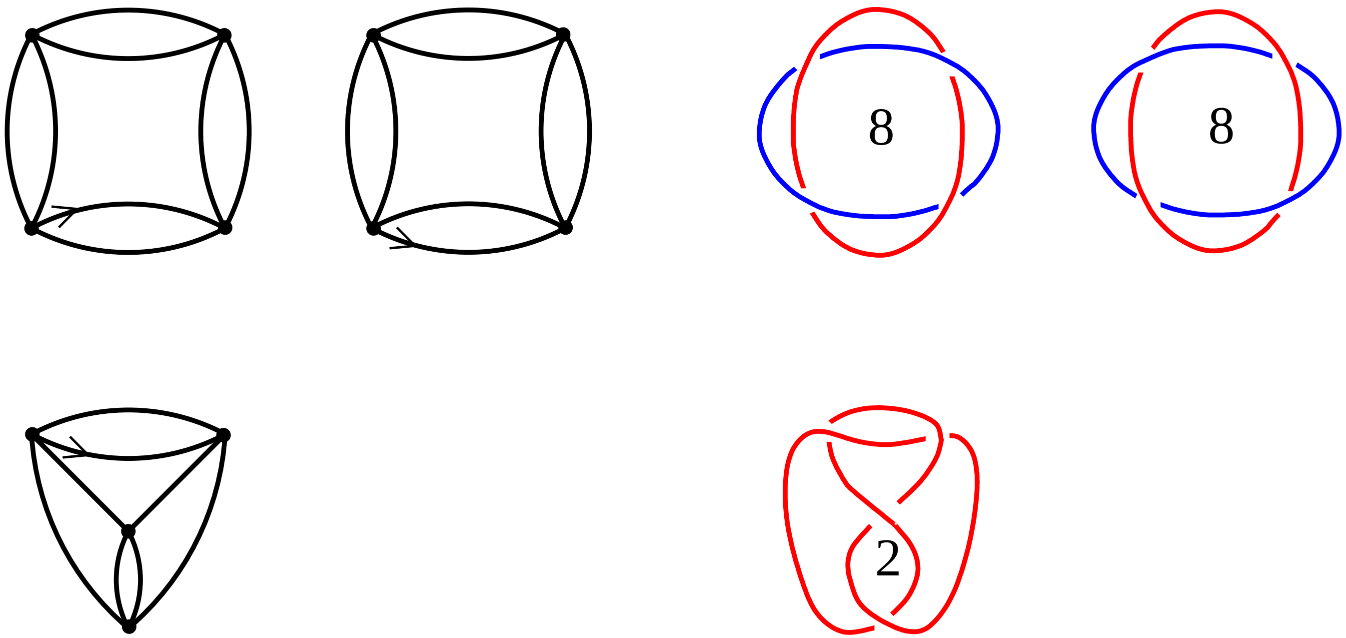
$$\Gamma^{(4)}(g) \quad = 200589\,g^7 + 14910216\,g^8 + 600547192\,g^9 + 17347802824g^{10} + O(g^{11})$$

$$\Gamma^{(5)}(g) \quad = 54766516\,g^9 + 5554165536g^{10} + O(g^{11})$$

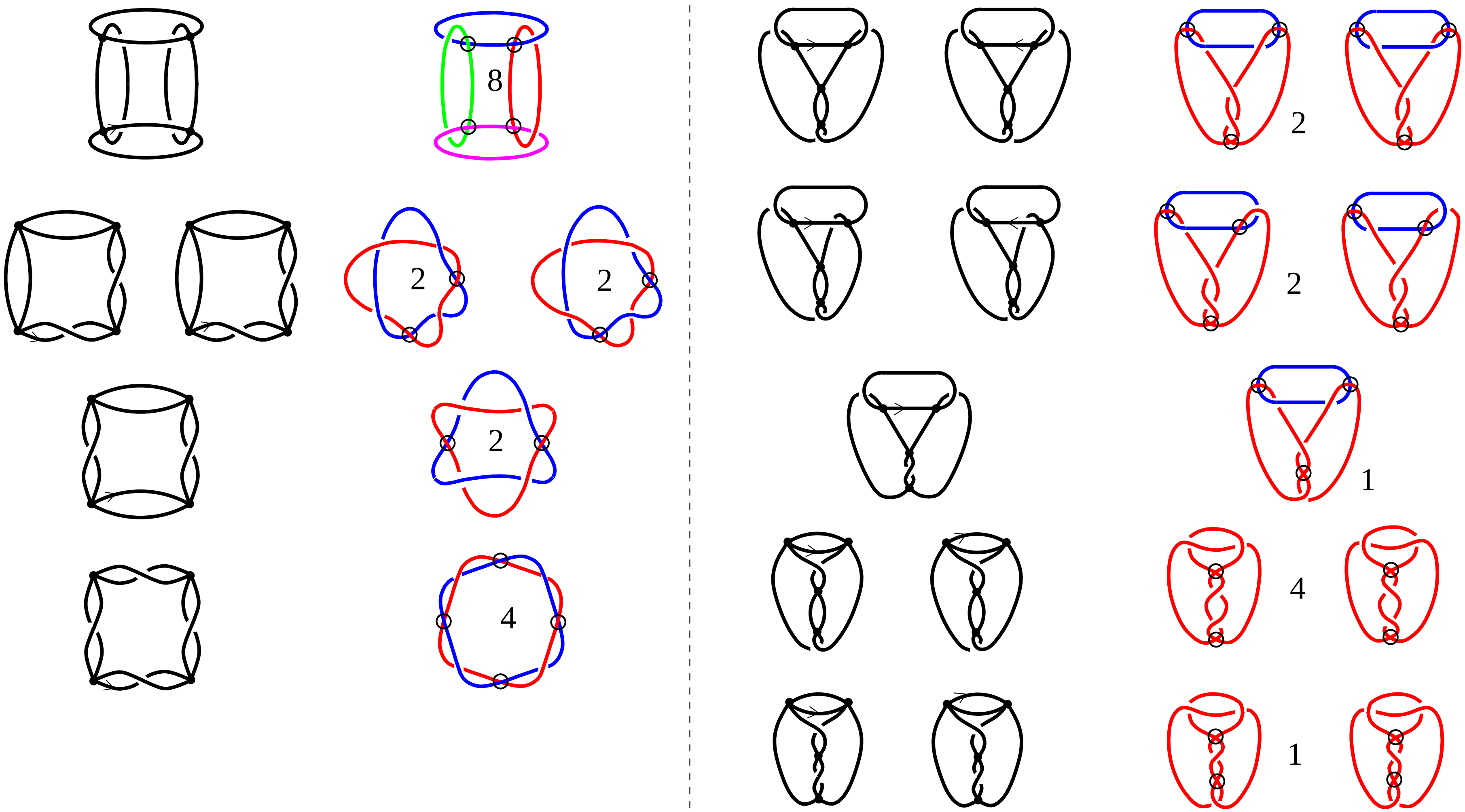
The genus 0 and 1 2-crossing alternating virtual link diagrams in the two representations, the Feynman diagrams on the left, the virtual diagrams on the right : for each, the inverse of the weight in F is indicated



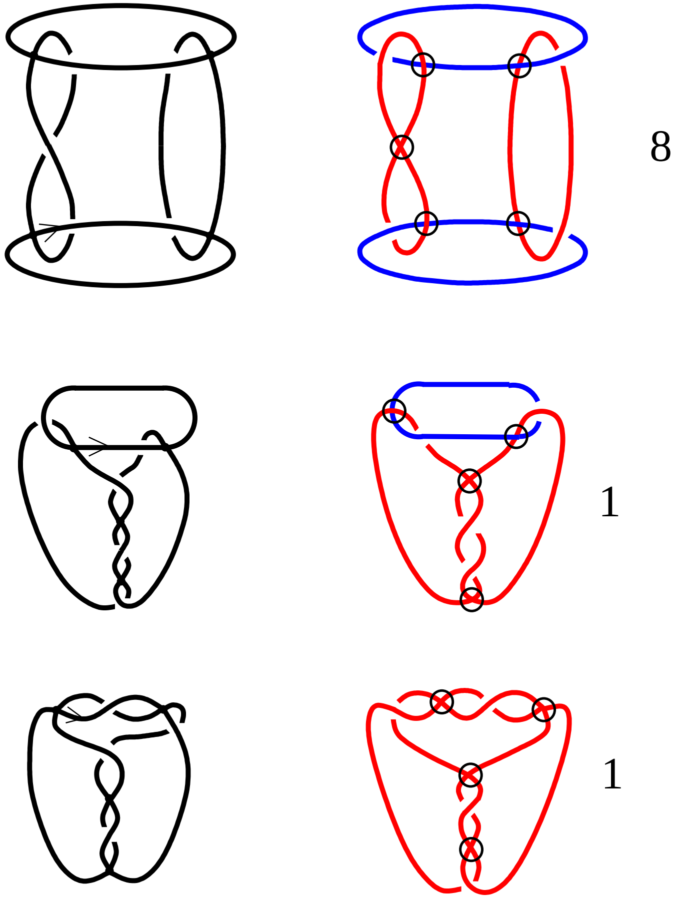
order 3, genus 0 and 1



order 4, genus 0

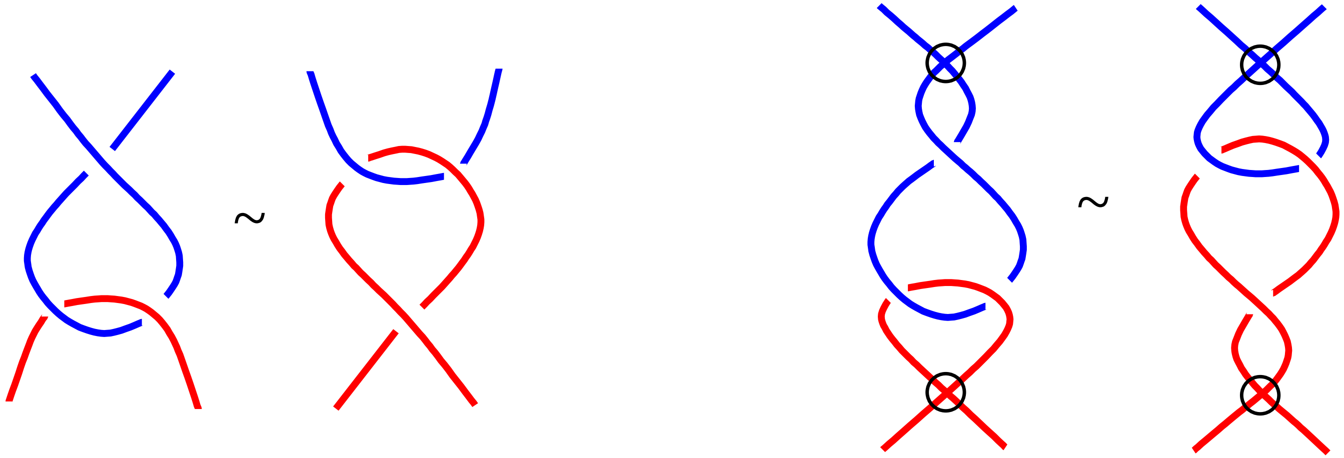


order 4, genus 1



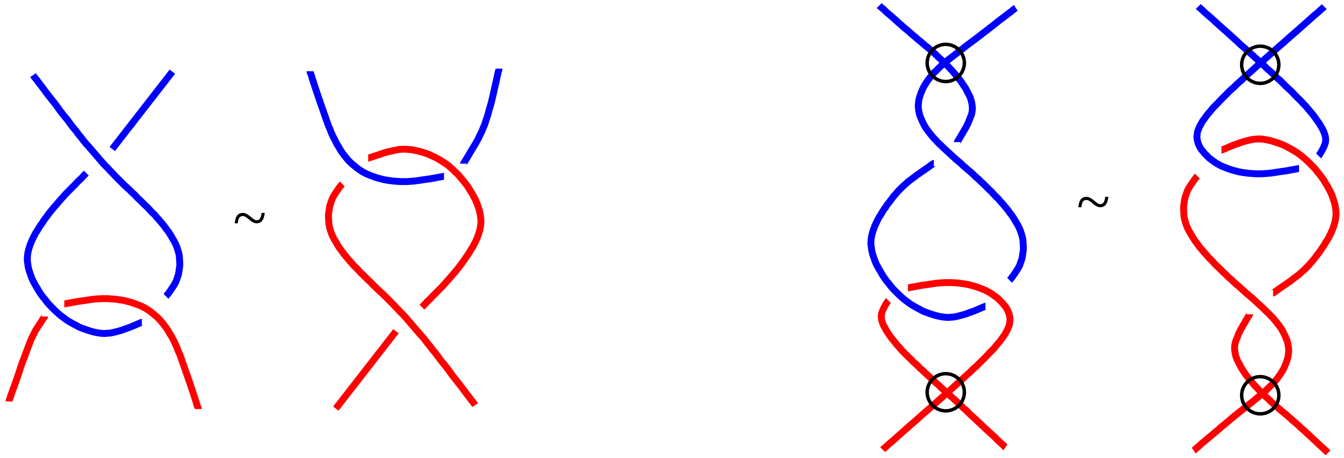
order 4, genus 2

Removing the flype redundancies.



First occurences of flype equivalence in tangles with 3 crossings

Removing the flype redundancies.



First occurences of flype equivalence in tangles with 3 crossings

It is suggested that it is (necessary and) sufficient to quotient by the planar flypes, thus to perform the same *renormalization* $g \to g_0(g)$ as for genus 0.

Generalized flype conjecture : For a given (minimal) genus h, $\widetilde{\Gamma}^{(h)}(g) = \Gamma^{(h)}(g_0)$ is the generating function of flype-equivalence classes of virtual alternating tangles. Then asymptotic behavior

$$\#\ \text{inequivalent tangles of order}\ p\ =\ \tilde{\Gamma}_p^{(h)} \sim \left(\frac{101+\sqrt{21001}}{40}\right)^p p^{\frac{5}{2}(h-1)}\ .$$

Test this generalized flype conjecture by computing invariants of virtual links

1 linking numbers

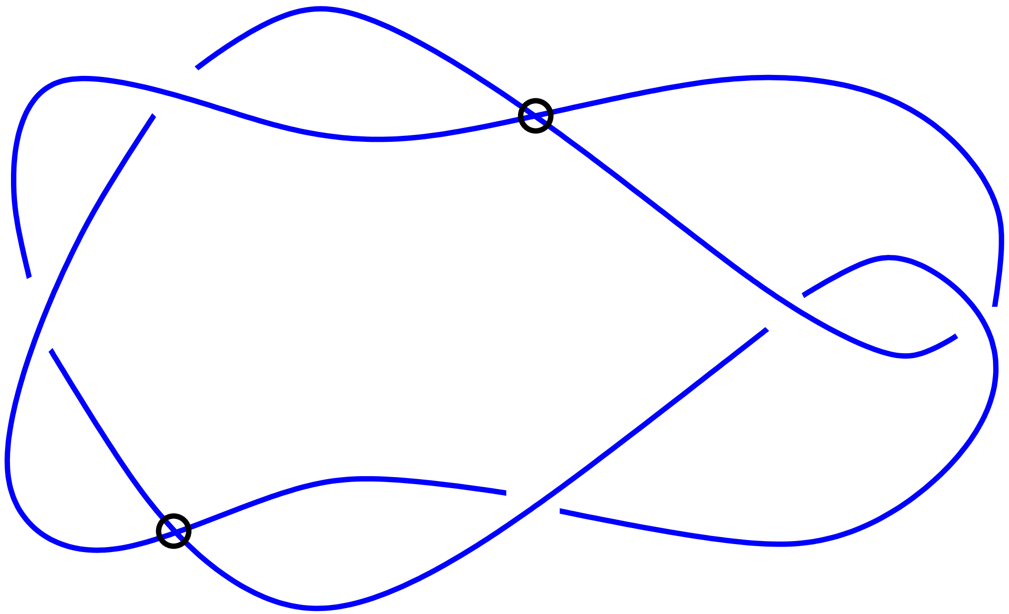
2 polynomials : Jones, cabled Jones, Kauffman,...

3 Alexander-Conway polynomials and their multi-variable extensions ...

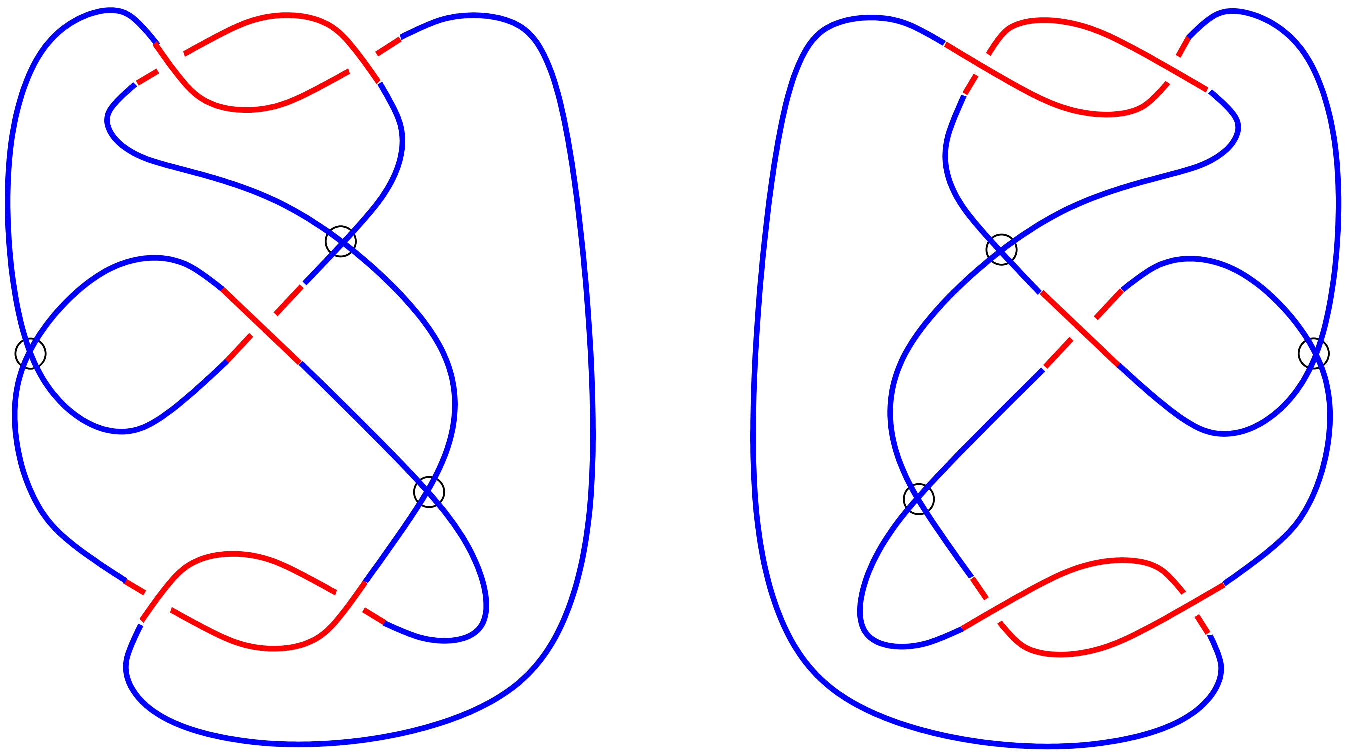
4 fundamental group π

Up to order 4 (4 real crossings), this suffices to distinguish all flype-equivalence classes : Conjecture $\surd$

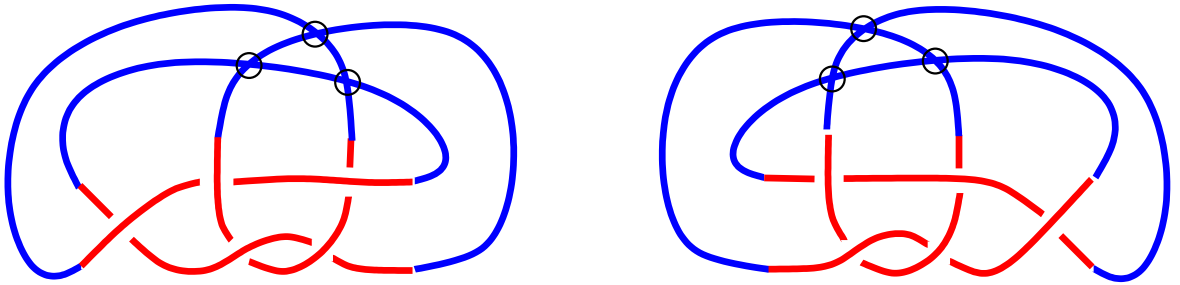
Higher orders : sometimes difficult to distinguish images under discrete symmetries (mirror, “global flip”=mirror $\times$ under-cr$\leftrightarrow$ over-cr.) ? . . .
Examples :



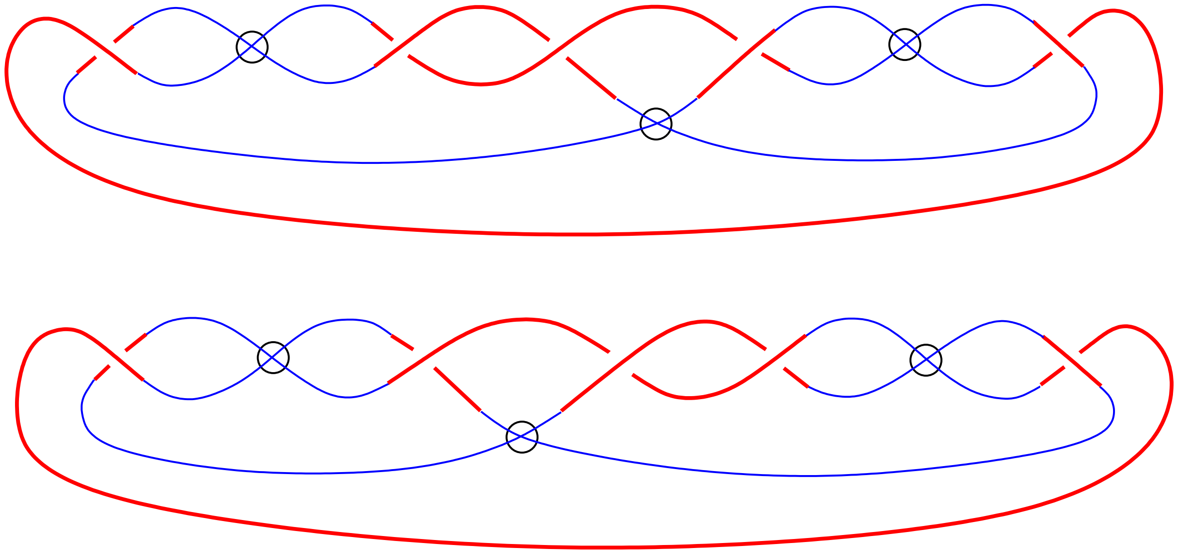
A genus-1 order-5 virtual diagram which is distinguished from its mirror image through the 2-cabled Jones polynomial



At order 5, a pair of virtual flipped knots of genus 1, distinguished by their Alexander-Conway polynomial.



A pair of virtual flipped knots of genus 1, conjectured to be non equivalent.



A pair of virtual flipped knots of genus 2, conjectured to be non equivalent.

Conclusions

Field theoretic methods : Feynman diagrams and matrix integrals, but also transfer matrix methods, offer new and powerful ways of handling the counting of links/tangles. Some progress, but still many open issues.

• Count knots (rather than links) ? $K_p = \#$ knots with p crossings.
Consider a n-colouring of links, then term linear in n ... ?

• Asymptotic behaviour of K_p as $p \to \infty$?

$K_p \sim C\tau^p p^{\gamma-3}, \gamma = -\frac{1+\sqrt{13}}{6}, \gamma - 3 \approx -3.7676$ [G. Schaeffer and P. Z.-J.]

• Non alternating Links and Knots ? ? ? ? ? ? ? ? ? ? ? ? ?