Matrix Integrals and Knot Theory

Paul Zinn-Justin et Jean-Bernard Zuber

G. Schaeffer and P. Z.-J., math-ph/0304034
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Main idea:
Use combinatorial tools of Quantum Field Theory in Knot Theory

Plan

I  Knot Theory: a few definitions
II  Matrix integrals and Link diagrams
   \[ \int dM e^{N\text{tr} \left(-\frac{1}{2}M^2 + gM^4\right)} \]
   \(N \times N\) matrices, \(N \to \infty\)
   Removals of redundancies
   \(\Rightarrow\) reproduces recent results of Sundberg & Thistlethwaite (1998)
   based on Tutte (1963)
III  Virtual knots and links: counting and invariants
Basics of Knot Theory

a knot, links and tangles

Equivalence up to isotopy

Problem: Count topologically inequivalent knots, links and tangles

Represent knots etc by their planar projection with minimal number of over/under-crossings

**Theorem** Two projections represent the same knot/link iff they may be transformed into one another by a sequence of **Reidemeister moves**:
Avoid redundancies by keeping only prime links (i.e. which cannot be factored).

Consider the subclass of alternating knots, links and tangles, in which one meets alternatingly over- and under-crossings.

For $n \geq 8$ (resp. 6) crossings, there are knots (links) which cannot be drawn in an alternating form. Asymptotically, the alternating are subdominant.

Major result (Tait (1898), Menasco & Thistlethwaite, (1991))

Two alternating reduced knots or links represent the same object iff they are related by a sequence of “flypes” on tangles.

**Problem** Count alternating prime links and tangles.
A 8-crossing non-alternating knot
Matrix Feynman diagrams and link diagrams

Consider integral over complex (non Hermitean) matrices

\[ \int dM e^{N[-t \text{ tr} MM^\dagger + \frac{g}{2} \text{ tr} (MM^\dagger)^2]} \]

⇒ oriented (double) lines in propagators and vertices.

When \( N \to \infty \), leading contribution from genus zero ("planar") diagrams:

\[ \lim_{N \to \infty} \frac{1}{N^2} \log Z = \sum_{\text{planar diagrams with } n \text{ vertices}} \frac{g^n}{\text{symm.factor}} \]

for example, to second order \( N^2 \)
Moreover: Conservation of arrows ⇒ alternating diagram!

But going from complex matrices to hermitian matrices doesn’t affect the planar limit . . . up to a global factor 2.

**Moral** After removing redundancies (incl. flypes), counting of Feynman diagrams of $M^4$ integral, (over *hermitian* matrices)

$$Z = \int dM e^{N\left[-\frac{t}{2} \text{tr} M^2 + \frac{g}{4} \text{tr} M^4\right]}$$

for $N \to \infty$, yields the counting of alternating links and tangles.
Non perturbative results on $M^4$ integral, $N \to \infty$

Compute large $N$ limit of integral $Z = \int dM e^{N[-\frac{t}{2} \text{tr} M^2 + \frac{g}{4} \text{tr} M^4]}$ by saddle point method, or orthogonal polynomials, or . . .

In the $N \to \infty$ limit, continuous distribution of eigenvalues $\lambda$ with density $u(\lambda)$ of support $[-2a, 2a]$ (deformed semi-circle law)

$$u(\lambda) = \frac{1}{2\pi} \left(1 - 2 \frac{g}{t^2} a^2 - \frac{g}{t^2} \lambda^2\right) \sqrt{4a^2 - \lambda^2}$$

$$3 \frac{g}{t^2} a^4 - a^2 + 1 = 0$$

Thus “planar” limit of $\text{tr} M^4$ integral

$$\lim_{N \to \infty} \frac{1}{N^2} \log \frac{Z(t, g)}{Z(t, 0)} = F(t, g) = \frac{1}{2} \log a^2 - \frac{1}{24} (a^2 - 1)(9 - a^2)$$
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\[ F(t, g) = \sum_{p=1} \left( \frac{3g}{t^2} \right)^p \frac{(2p-1)!}{p!(p+2)!} \]

As \( p \to \infty \) \( F_p \sim \text{const}(12)^p p^{-7/2} \)

Also 2-point function \( G_2 = \frac{1}{3t} a^2 (4 - a^2) = \)

and (connected and “truncated”) 4-point function

\[ \Gamma = \frac{(5 - 2a^2)(a^2 - 1)}{(4 - a^2)^2} \]
Counting Links and Tangles

For the knot interpretation of previous counting, many irrelevant diagrams have to be discarded.

\[ \text{“Nugatory” and “non-prime” are removed by adjusting } t = t(g) \text{ so that} \]

\[ = 1 \text{ (“wave function renormalisation”).} \]
In that way, correct counting of links ...up to 6 crossings!

(a) \( \begin{array}{cccc}
2_1^2 & 3_1 & 4_1 & 4_1^2 \\
\end{array} \)

(b) \( \begin{array}{cccc}
\frac{1}{4} & \frac{1}{3} & \frac{1}{2} & \frac{1}{4} \\
\end{array} \)

(c) \( \begin{array}{cccc}
\bigotimes & \bigotimes & \bigotimes & \bigotimes \\
\end{array} \)

Asymptotic behaviour \( F_p \sim \text{const} \ (27/4)^p \ p^{-7/2} \)
What happens for $n \geq 6$ crossings? **Flypes!**

Must quotient by the flype equivalence! Original combinatorial treatment (**Sundberg & Thistlethwaite, Z-J&Z**) rephrased and simplified by P. Z.-J.: it amounts to a coupling constant renormalisation $g \to g_0$! In other words, start from $Ntr \left( \frac{1}{2} t M^2 - \frac{g_0}{4} M^4 \right)$, fix $t = t(g_0)$ as before. Then compute $\Gamma(g_0)$ and determine $g_0(g)$ as the solution of

$$g_0 = g \left( -1 + \frac{2}{(1-g)(1+\Gamma(g_0))} \right),$$

then the desired generating function is $\tilde{\Gamma}(g) = \Gamma(g_0)$. 
Indeed let $H(g)$ be the generating function of “horizontally-two-particle-irreducible” diagrams (cannot separate the left part from the right by cutting two lines)

and then write $\Gamma = H/(1 - H)$
But \( \sim \) thus, with \( \tilde{\Gamma}, \tilde{H} \) denoting generating functions of flype equivalence classes of prime tangles, resp. 2PI tangles and if

\[
\tilde{H} = g + \tilde{H}', \quad \tilde{\Gamma} = g + g\tilde{\Gamma} + \frac{\tilde{H}'}{1 - \tilde{H}'}
\]
Return to $\Gamma(g_0)$

\[
\Gamma(g_0) = \begin{array}{c}
\times \\
+ \\
\times \\
+ \\
\times \\
+ \\
\times \\
+ \ldots
\end{array}
\]

suggests to determine $g_0 = g_0(g)$ by demanding that $g_0 = g - 2g\tilde{H}'$

\[
g_0 = \begin{array}{c}
\times \\
- \\
\times \\
- \end{array}
\]

Three relations between $g_0, \tilde{H}', g$ and $\tilde{\Gamma}(g)$

Eliminating $\tilde{H}'$ and then
Eliminating \( g_0 \) gives \( \tilde{\Gamma}(g) = \Gamma(g_0(g)) \), the generating function of (flype-equivalence classes of) tangles.

Find

\[
\tilde{\Gamma} = g + 2g^2 + 4g^3 + 10g^4 + 29g^5 + 98g^6 + 372g^7 + 1538g^8 + 6755g^9 + \cdots
\]

Asymptotic behaviour \( \tilde{\Gamma}_p \sim \text{const} \left( \frac{101 + \sqrt{21001}}{40} \right)^p p^{-5/2} \)

All this reproduces the results of Sundberg & Thistlethwaite.

\[* Can we go further? Control the number of connected components? i.e. count *knots* rather than *links*? \]
Coloured Links and Tangles

\[ Z^{(N)}(n, g) = \int \prod_{a=1}^{n} dM_a \ e^{N \text{tr} \left( -\frac{1}{2} \sum_{a=1}^{n} M_a^2 + \frac{g}{4} \sum_{a,b=1}^{n} M_a M_b M_a M_b \right)} \]

Each connected component may come in \( n \) colours

If we write the free energy \( F(n, g) = \sum_{k=1}^{\infty} F_k(g)n^k \), \( F_k = \) generating function of diagrams with \( k \) loops. In particular, \( F_1(g) \), that of knots.

Unfortunately this is computable only for \( n = -2, 1, 2 \)

[P.Z.-J. 99, Z-J–Z 00]

★ Open and important problem to understand such integrals in the \( n \to 0 \) limit (replicas, combinatorics…)
Another direction: Virtual Links

Higher genus contributions to matrix integral
What do they count?
Suggested that knots/links live on other manifolds $\Sigma_h \times I$

Virtual knots and links [Kauffman]: equivalence classes of 4-valent diagrams with ordinary under- or over-crossings

plus a new type of virtual crossing,
Equivalence w.r.t. generalized Reidemeister moves
From a different standpoint: Virtual knots (or links) seen as drawn in a “thickened” Riemann surface $\Sigma := \Sigma \times [0, 1]$, modulo isotopy in $\Sigma$, and modulo orientation-preserving homeomorphisms of $\Sigma$, and addition or subtraction of empty handles.

But this is precisely what Feynman diagrams of the matrix integral do for us!

Thus return to the integral over complex matrices

$$Z(g, N) = \int dM e^{N[-t \operatorname{tr} MM^\dagger + \frac{g}{2} \operatorname{tr} (MM^\dagger)^2]}$$

and compute $F(g, t, N) = \log Z$ beyond the leading large $N$ limit . . .
\[ F(g, t, N) = \sum_{h=0}^{\infty} N^{2-2h} F^{(h)}(g, t) \]

\( F^{(h)}(g) \): Feynman diagrams of genus \( h \)

\( F^{(1)} \) computed by Morris (1991)

\( F^{(2)} \) and \( F^{(3)} \) by Akermann and by Adamietz (ca. 1997)

As before, determine \( t = t(g, N) \) so as to remove the non prime diagrams.

Find the generating function of tangle diagrams \( \Gamma(g) = 2\partial F / \partial g - 2 \)

\[ \Gamma^{(0)}(g) = g + 2g^2 + 6g^3 + 22g^4 + 91g^5 + 408g^6 + 1938g^7 + 9614g^8 + 49335g^9 + 260130g^{10} + O(g^{11}) \]

\[ \Gamma^{(1)}(g) = g + 8g^2 + 59g^3 + 420g^4 + 2940g^5 + 140479g^6 + 964184g^7 + 6598481g^8 + 45059872g^9 + O(g^{11}) \]

\[ \Gamma^{(2)}(g) = 17g^3 + 456g^4 + 7728g^5 + 104762g^6 + 1240518g^7 + 13406796g^8 + O(g^{11}) \]

\[ \Gamma^{(3)}(g) = 1259g^5 + 62072g^6 + 1740158g^7 + 36316872g^8 + 627368680g^9 + O(g^{11}) \]

\[ \Gamma^{(4)}(g) = 200589g^7 + 14910216g^8 + 600547192g^9 + 17347802824g^{10} + O(g^{11}) \]

\[ \Gamma^{(5)}(g) = 54766516g^9 + 5554165536g^{10} + O(g^{11}) \]
The genus 0 and 1 2-crossing alternating virtual link diagrams in the two representations, the Feynman diagrams on the left, the virtual diagrams on the right: for each, the inverse of the weight in $F$ is indicated.
order 3, genus 0 and 1
order 4, genus 0
order 4, genus 1
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order 4, genus 2
Removing the flype redundancies.

First occurrences of flype equivalence in tangles with 3 crossings
Removing the flype redundancies.

First occurrences of flype equivalence in tangles with 3 crossings
It is suggested that it is (necessary and) sufficient to quotient by the planar flypes, thus to perform the same renormalization $g \to g_0(g)$ as for genus 0.

**Generalized flype conjecture:** For a given (minimal) genus $h$, $\tilde{\Gamma}^{(h)}(g) = \Gamma^{(h)}(g_0)$ is the generating function of flype-equivalence classes of virtual alternating tangles. Then asymptotic behavior

$$\# \text{ inequivalent tangles of order } p = \tilde{\Gamma}_p^{(h)} \sim \left(\frac{101 + \sqrt{21001}}{40}\right)^p p^{\frac{5}{2}(h-1)}.$$

Test this *generalized flype conjecture* by computing invariants of virtual links

1. linking numbers
2. polynomials: Jones, cabled Jones, Kauffman, . . .
4. fundamental group $\pi$
Up to order 4 (4 real crossings), this suffices to distinguish all flype-equivalence classes:

Conjecture \(\checkmark\)

Higher orders: sometimes difficult to distinguish images under discrete symmetries (mirror, “global flip” = mirror \(\times\) under-cr ↔ over-cr.)? …

Examples:

A genus-1 order-5 virtual diagram which is distinguished from its mirror image through the 2-cabled Jones polynomial
At order 5, a pair of virtual flipped knots of genus 1, distinguished by their Alexander-Conway polynomial.
A pair of virtual flipped knots of genus 1, conjectured to be non equivalent.

A pair of virtual flipped knots of genus 2, conjectured to be non equivalent.
Conclusions

Field theoretic methods: Feynman diagrams and matrix integrals, but also transfer matrix methods, offer new and powerful ways of handling the counting of links/tangles. Some progress, but still many open issues.

• Count knots (rather than links)? $K_p = \text{# knots with } p \text{ crossings}$. Consider a $n$-colouring of links, then term linear in $n$ . . . ?

• Asymptotic behaviour of $K_p$ as $p \to \infty$?

$K_p \sim C \tau^p p^{\gamma-3}, \gamma = -\frac{1+\sqrt{13}}{6}, \gamma - 3 \approx -3.7676$ [G. Schaeffer and P. Z.-J.]