Matrix Integrals and Feynman diagrams

Jean-Bernard Zuber

Lectures 1 and 2

Cargèse March 24, 2009

Plan of the three lectures

- Introduction : Matrix integration, why and how ?
- Lecture 1 : Feynman diagrams and large N limit of matrix integrals
- Lecture 2 : Actual computation of (large N limit) of matrix integrals
- Lecture 3 : Applications : counting of alternating links and tangles

Why matrix integrals ?

– Random matrices, from statistics to physics (heavy nuclei, disordered mesoscopic systems,...), from Wishart to Wigner, Dyson, Mehta, ...
– Feynman diagrams ['t Hooft] (these lectures)
– Unexpected connections with combinatorics (these lectures)
– with Riemann ζ function, with algebraic geometry etc, etc ...

Matrix integrals, how ? what ?

Pick a set of matrices, for example $N \times N$ Hermitian matrices (these lectures) or symmetric, or unitary, etc, and consider integrals of the form

$$Z = \int dM \exp -N \mathrm{tr} V(M)$$

typically, V a polynomial $V = \frac{1}{2}M^2 + \cdots$, and accordingly

$$\langle F(.)\rangle = Z^{-1} \int dM F(M) e^{-N \mathrm{tr} V(M)}.$$

Basics of Feynman diagrams

Consider a Gaussian integral over n real variables x_i, $A = A^T > 0$ def. matrix

$$\int d^n x\, e^{-\frac{1}{2}\sum x_i A_{ij} x_j} = \frac{(2\pi)^{n/2}}{\det^{\frac{1}{2}} A}$$

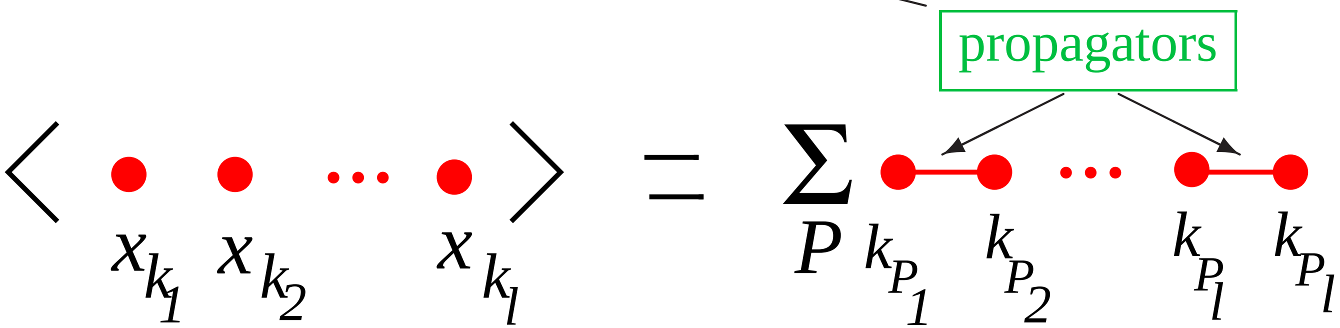
$$\int d^n x\, \mathrm{e}^{-\frac{1}{2}\sum x_i A_{ij} x_j + \sum b_i x_i} = \frac{(2\pi)^{n/2}}{\det^{\frac{1}{2}} A}\, \mathrm{e}^{\frac{1}{2}\sum b_i A^{-1}_{ij} b_j}$$

Differentiate w.r.t. b_i

$$\langle x_{k_1} x_{k_2} \cdots x_{k_\ell} \rangle := \frac{\int d^n x\, x_{k_1} x_{k_2} \cdots x_{k_\ell} e^{-\frac{1}{2} x.A.x}}{\int d^n x\, e^{-\frac{1}{2} x.A.x}} = \frac{\partial}{\partial b_{k_1}} \cdots \frac{\partial}{\partial b_{k_\ell}} \mathrm{e}^{\frac{1}{2} b.A^{-1}.b}\Big|_{b=0}$$

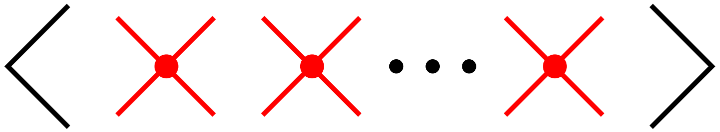
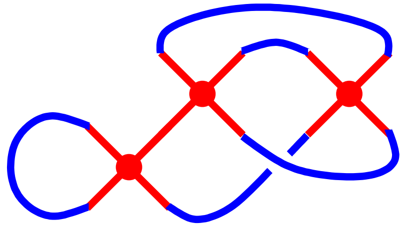
$$= \sum_{\substack{\text{all distinct} \\ \text{pairings } P \text{ of the } k}} A^{-1}_{k_{P_1} k_{P_2}} \cdots A^{-1}_{k_{P_{\ell-1}} k_{P_\ell}}$$

Wick theorem



Wick theorem also applies to monomials ($n = 1$ variable for simplicity) :

p vertices propagator A^{-1}

$\langle (x^4)^p \rangle =$  $= \sum_{\text{graphs}}$ 

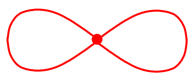
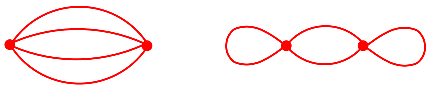
Non Gaussian integrals ($g < 0$) : power series "perturbative" expansions

$$Z = \int dx\, e^{-\frac{1}{2}Ax^2 + \frac{g}{4!}x^4} = \left(\frac{2\pi}{A}\right)^{\frac{1}{2}} \sum_{p=0}^{\infty} \frac{g^p}{p!} \int dx \left(\frac{x^4}{4!}\right)^p e^{-\frac{1}{2}Ax^2}$$

$$= \left(\frac{2\pi}{A}\right)^{\frac{1}{2}} \sum_{p=0}^{\infty} \sum_{\substack{\text{graphs } \mathcal{G} \text{ with } 2p \text{ lines} \\ \text{and } p \text{ 4-valent vertices}}} \frac{g^p}{|\mathrm{Aut}\,\mathcal{G}|} A^{-2p}$$

$\log Z =$ connected Feynman diagrams

$$= \frac{g}{8A^2} + \frac{g^2}{A^4}\left(\frac{1}{2.4!} + \frac{1}{2^4}\right) + \cdots$$

$=$   $+ \cdots$

Notes :

(i) sum over all *topologically distinct diagrams*

(ii) symmetry factor $= |\mathrm{Aut}\,\mathcal{G}\,|$ is the order of the automorphism group of the diagram, *i.e.* of the group of permutations of vertices and *internal* lines that leave the diagram invariant.

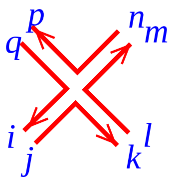
Matrix Integrals : Feynman Rules

$N \times N$ Hermitean matrices M, $dM = \prod_i dM_{ii} \prod_{i<j} d\Re e M_{ij}\, d\Im m M_{ij}$
(measure invariant under $M \to UMU^\dagger$, $U \in \mathrm{U}(N)$)

$$Z =: e^F = \int dM\, \mathrm{e}^{N[-\frac{1}{2}\mathrm{tr}\, M^2 + \frac{g}{4}\mathrm{tr}\, M^4]}$$

Feynman rules : propagator 

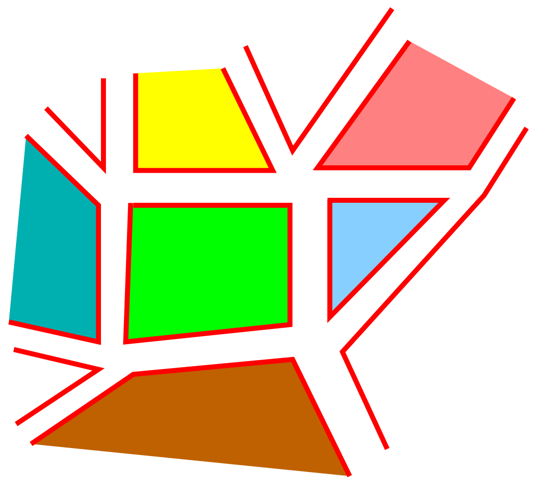
$= \frac{1}{N}\delta_{i\ell}\delta_{jk}$ ['t Hooft]

4-valent vertex : 

$= gN\delta_{jk}\delta_{\ell m}\delta_{np}\delta_{qi}$

For each connected diagram contributing to $\log Z$: fill each closed index loop with a disk $\Rightarrow$ discretized closed 2-surface Σ

Thus : index loop $\leftrightarrow$ face of Σ

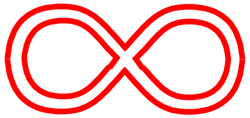


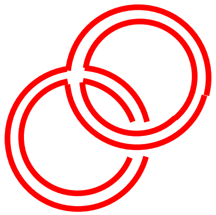
Power of N in a connected diagram

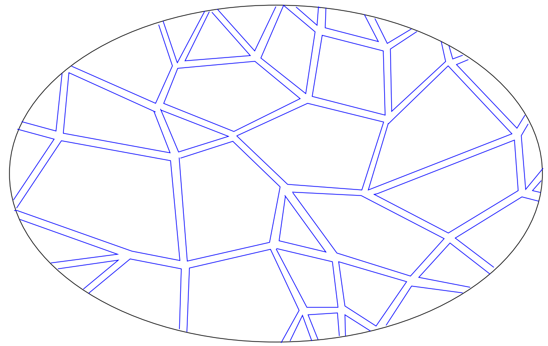
- each vertex $\to N$;
- each double line $\to N^{-1}$;
- each index loop $\to N$.

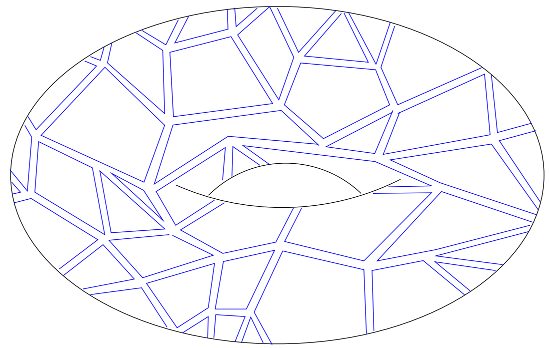
Thus $N^{\#\text{vert.}-\#\text{lines}+\#\text{loops}} = N^{\chi_{\text{Euler}}(\Sigma)}$

['t Hooft (1974)]. For example, compare


gN^2


gN^0





A topological expansion :

$$F = \log Z = \sum_{\text{conn. surf.}\Sigma} N^{2-2\text{genus}(\Sigma)} \frac{g^{\#\text{vert.}(\Sigma)}}{\text{symm. factor}}$$

$$= \sum_{n,h} g^n N^{2-2h} F^{(n,h)} = \text{“}\sum_{h=0}^{\infty} N^{2-2h} F^{(h)}(g).\text{”}$$

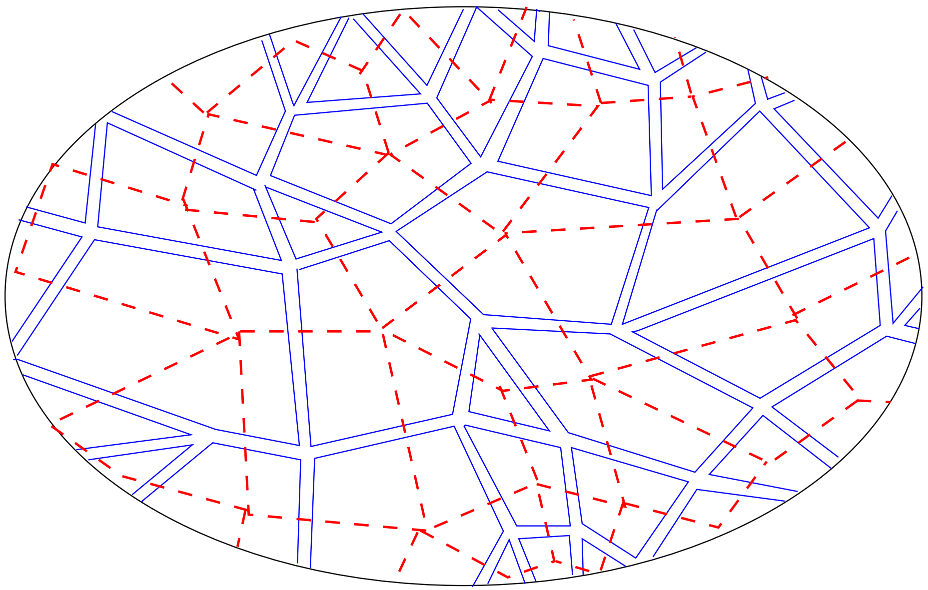
Thus large N limit of matrix integral $\int DMe^{-N\mathrm{tr}\,(M^2+\frac{g}{4}M^4)}$ = generating function of planar 4-valent graphs…(cf census of planar maps by Tutte)

[Brézin, Itzykson, Parisi, Z. 1978]

$$\text{“}\lim_{N\to\infty}\frac{1}{N^2}\log Z\text{”} = \sum_{n=0}^{\infty} g^n F^{(n,0)} = \sum_{\substack{\text{planar diagrams}\\ \text{with } n \text{ 4-vertices}}} \frac{g^n}{\text{symm.factor}}$$

or in a dual way, of *quadrangulations* of 2D surfaces of genus 0

[Kazakov ; David ; Kazakov-Kostov-Migdal ; Ambjørn-Durhuus-Fröhlich ’85]

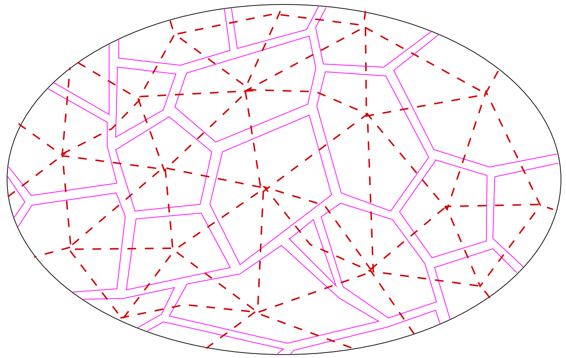


Thus large N limit of matrix integral $\int DMe^{-N\mathrm{tr}\,(M^2+\frac{g}{3}M^3)}$ = generating function of planar 3-valent graphs. . . [Brézin, Itzykson, Parisi, Z. 1978]

or in a dual way, of *triangulations* of 2D surfaces of genus 0

[Kazakov ; David ; Kazakov-Kostov-Migdal ; Ambjørn-Durhuus-Fröhlich '85]

Triangulated surfaces and discrete 2D gravity



Thus : Large N limit of matrix integrals $\Rightarrow$ Counting of planar objects : maps, triangulations, "alternating" knots and links [P Z-J & J-B Z], etc, or of objects of higher topology . . .

Computational techniques

Consider integral over $N \times N$ Hermitian matrices

$$Z = \int dM e^{-N \operatorname{tr} V(M)} ,$$

$V(M)$ a polynomial of degree $d+1$. For ex. $V_3(M) = (\frac{1}{2}M^2 + \frac{g}{3}M^3)$ and $V_4(M) = (\frac{1}{2}M^2 + \frac{g}{4}M^4)$. Note that multi-traces are excluded, for example $(\operatorname{tr} M^2)^2$.

Integrand and measure are invariant under $U(N)$ transformations $M \to UMU^\dagger$. Express both in terms of *eigenvalues* $\lambda_1, \cdots, \lambda_N$ of M :

$$Z = \int \prod_{i=1}^{N} d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{-N \sum_{i=1}^{N} V(\lambda_i)} ,$$

Several ways to treat this integral : saddle point approximation ; orthogonal polynomials ; “loop equation” (aka Schwinger-Dyson equation)...

1. Saddle point approximation

Rewrite

$$Z = \int \prod_{i=1}^{N} d\lambda_i \exp\left(2 \sum_{i<j} \log|\lambda_i - \lambda_j| - N \sum_{i=1}^{N} V(\lambda_i) \right)$$

In the large N limit, if $\lambda \sim O(1)$, both terms in exponential are of order N^2. Look for the stationary point, i.e. the solution of

$$\frac{2}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = V'(\lambda_i) \ . \qquad (*)$$

To solve this problem, introduce the resolvent

$$G(x) = \frac{1}{N} \left\langle \mathrm{tr} \frac{1}{x - M} \right\rangle = \left\langle \frac{1}{N} \sum_{i=1}^{N} \frac{1}{x - \lambda_i} \right\rangle \ .$$

Computing its square leads after some algebra to

$$G^2(x) = \frac{1}{N^2}\left\langle \sum_{i,j=1,\cdots,N} \frac{1}{(x-\lambda_i)(x-\lambda_j)} \right\rangle = \cdots = -\frac{1}{N}G'(x) + V'(x)G(x) - P(x)$$

with $P(x) := \frac{1}{N}\left\langle \sum_{i=1}^{N} \frac{V'(x)-V'(\lambda_i)}{x-\lambda_i} \right\rangle$ a *polynomial* in x of degree $d-1$, *i.e.*

$$G^2(x) - V'(x)G(x) + \frac{1}{N}G'(x) + P(x) = 0 \; .$$

(Beware ! Not exact for N finite !) For N large, neglect the $1/N$ term $\Rightarrow$ quadratic equation for $G(x)$, with yet unknown polynomial P, hence

$$G(x) = \frac{1}{2}\left(V'(x) - \sqrt{V'(x)^2 - 4P(x)}\right)$$

(minus sign in front of $\sqrt{}$ dictated by the requirement that for large $|x|$, $G(x) \sim 1/x$.)

In that large N limit, the λ's form a continuous distribution with density

$\rho(\lambda)$ on a support S , $\int_S d\lambda\,\rho(\lambda)=1$, and $G(x)=\int_S \frac{d\mu\,\rho(\mu)}{x-\mu}$.
For a purely Gaussian potential $V(\lambda)=\frac{1}{2}\lambda^2$, Wigner's "semi-circle law" :
$\rho(\lambda)=\frac{1}{2\pi}\sqrt{4-\lambda^2}$ on the segment $\lambda\in[-2,2]$.
For more general potentials, assume first S to be still a finite segment $[-2a',2a'']$, in such a way that (*) becomes

$$2\,\mathrm{P.P.}\int_{-2a'}^{2a''}\frac{d\mu\,\rho(\mu)}{\lambda-\mu}=V'(\lambda)\qquad \text{if } \lambda\in[-2a',2a'']\ .$$

(P.P.= principal part), expressing that, along its cut,

$$G(x\pm i\varepsilon)=\frac{1}{2}V'(\lambda)\mp i\pi\rho(x)\qquad x\in[-2a',2a'']\ .$$

Thus

$$G(x)=\frac{1}{2}V'(x)-Q(x)\sqrt{(x+2a')(x-2a'')}$$

where the coefficients of the polynomial $Q(x)$ and a', a'' are determined by the condition that $G(x)\sim 1/x$ for large $|x|$. Q is of degree $d-1$. The solution is unique (under the one-cut assumption).

Example For the quartic potential $V(\lambda) = \frac{1}{2}\lambda^2 + \frac{g}{4}\lambda^4$, by symmetry $a' = a'' =: a$,

$$G(x) = \frac{1}{2}(x+gx^3) - (\frac{1}{2} + \frac{g}{2}x^2 + ga^2)\sqrt{x^2 - 4a^2}$$

with a^2 the solution of

$$3ga^4 + a^2 - 1 = 0 \qquad (EQa^2)$$

which goes to 1 as $g \to 0$ (a limit where we recover Wigner's semi-circle law). From this we extract

$$\rho(\lambda) = \frac{1}{\pi}(\frac{1}{2} + \frac{g}{2}\lambda^2 + ga^2)\sqrt{4a^2 - \lambda^2}$$

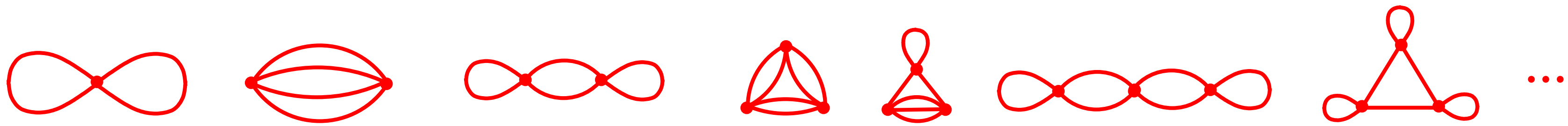
and we may compute all invariant quantities like the free energy or the moments

$$G_{2p} := \left\langle \frac{1}{N}\mathrm{tr}\, M^{2p} \right\rangle = \int d\lambda\, \lambda^{2p}\rho(\lambda)\ .$$

For example $G_2 = (4-a^2)a^2/3$, $G_4 = (3-a^2)a^4$, etc. All these functions of a^2 are singular as functions of g at the point $g_c = -\frac{1}{12}$ where the two roots of (EQa^2) coalesce. For example the genus 0 free energy

$$F^{(0)}(g) := \lim_{N\to\infty}(1/N^2)\log\left(\frac{Z(g)}{Z(0)}\right) = \frac{1}{2}\log a^2 - \frac{1}{24}(a^2-1)(9-a^2)$$

$$= \sum_{p=1}(3g)^p \frac{(2p-1)!}{p!(p+2)!} \qquad \text{[Tutte 62, BIPZ 78]}$$



has a power-law singularity

$$F^{(0)}(g) \underset{g\to g_c}{\approx} |g-g_c|^{5/2}$$

which reflects on the large order behaviour of its series expansion

$$F^{(0)}(g) = \sum_{n=0}^{\infty} f_n g^n \quad , \qquad f_n \underset{n\to\infty}{\approx} \text{const}|g_c|^{-n} n^{-7/2} \; .$$

Comments

i) Nature of the $1/N^2$ and of the g expansions, algebraic singularity at finite g_c

ii) “Universal” singular behavior at g_c

iii) Extension to several cuts, the rôle of the algebraic curve (cf Eynard).

2. Orthogonal polynomials

$$\int d\lambda\, P_m(\lambda) P_n(\lambda) e^{-NV(\lambda)} = h_n \delta_{mn}$$

Express Z and F in terms of the h_n's, their large n limit, etc.

[Mehta, Bessis, . . .]

3. Loop (or Schwinger-Dyson) equations

$$\int dM \frac{\partial}{\partial M_{ij}} \{\cdots e^{-N \operatorname{tr} V(M)}\} = 0$$

and make use of factorization property

$$\langle \tfrac{1}{N} \operatorname{tr} P_1 \tfrac{1}{N} \operatorname{tr} P_2 \rangle = \underbrace{\langle \frac{1}{N} \operatorname{tr} P_1 \rangle \langle \frac{1}{N} \operatorname{tr} P_2 \rangle}_{\text{disconnected diagrams}} + O(\tfrac{1}{N^2})$$

$\Rightarrow$ Recover algebraic equation satisfied by $G(x)$, etc.