RENORMALIZATION GROUP: AN INTRODUCTION

J. ZINN-JUSTIN*

CEA,IRFU et Institut de Physique Théorique, Centre de Saclay,
F-91191 Gif-sur-Yvette cedex, FRANCE.

Renormalization group has played a crucial role in 20th century physics in two apparently unrelated domains: the theory of fundamental interactions at the microscopic scale and the theory of continuous macroscopic phase transitions. In the former framework, it emerged as a consequence of the necessity of renormalization to cancel infinities that appear in a straightforward interpretation of quantum field theory and the possibility to define the parameters of the renormalized theory at different momentum scales.

*Email : jean.zinn-justin@cea.fr

In the statistical physics of phase transitions, a more general renormalization group, based on recursive averaging over short distance degrees of freedom, was latter introduced to explain the universality properties of continuous phase transitions.

The field renormalization group now is understood as the asymptotic form of the general renormalization group in the neighbourhood of the Gaussian fixed point.

Correspondingly, in the framework of statistical field theories relevant for simple phase transitions, we review here first the perturbative renormalization group then a more general formulation called functional or exact renormalization group.

As general references, *cf.* for example,

J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Clarendon Press 1989 (Oxford 4th ed. 2002),

J. Zinn-Justin, *Transitions de phase et groupe de renormalisation.* EDP Sciences/CNRS Editions, Les Ulis 2005

English version *Phase transitions and renormalization group*, Oxford Univ. Press (Oxford 2007).

In preparation, the introduction

Wilson-Fisher fixed point on the site www.scholarpedia.org

Statistical field theory

In the theory of continuous phase transitions, one is interested in the large distance behaviour or macroscopic properties of physical observables near the transition temperature $T = T_c$. At the critical temperature, the correlation length, which defines the scale on which correlations above T_c decay exponentially, diverges and the correlation functions decay only algebraically. This gives rise to non-trivial large distance properties that are, to a large extent, independent of the short distance structure, a property called universality.

Intuitive arguments indicate that even if the initial statistical model is defined in terms of random variables associated to the sites of a space lattice, and taking only a finite set of values (like, *e.g.*, the classical spins of the Ising model), the large distance behaviour can be inferred from a statistical field theory in continuum space.

Therefore, we consider a classical statistical system defined in terms of a random real field $\phi(x)$ in continuum space, $x \in \mathbb{R}^d$, and a functional measure on fields of the form $\mathrm{e}^{-\mathcal{H}(\phi)}/\mathcal{Z}$, where $\mathcal{H}(\phi)$ is called the Hamiltonian in statistical physics and $\mathcal{Z}$ is the partition function (a normalization) given by the field integral (*i.e.*, a sum over field configurations)

$$\mathcal{Z} = \int [\mathrm{d}\phi(x)]\,\mathrm{e}^{-\mathcal{H}(\phi)}.$$

The condition of short range interactions in the statistical system translates into the property of locality of the field theory: $\mathcal{H}(\phi)$ can be chosen as a space-integral over a linear combination of monomials in the field and its derivatives.

We assume also space translation and rotation invariance and, to discuss a specific example, $\mathbb{Z}_2$ reflection symmetry: $\mathcal{H}(\phi)=\mathcal{H}(-\phi)$. A typical form then is

$$\mathcal{H}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\big(\partial_\mu \phi(x)\big)^2 + \tfrac{1}{2} r \phi^2(x) + \frac{g}{4!}\phi^4(x) + \cdots\right].$$

Finally, the coefficients of $\mathcal{H}(\phi)$ are regular functions of the temperature T near the critical temperature T_c. In the low temperature phase $T < T_c$, the $\mathbb{Z}_2$ symmetry is spontaneously broken.

Correlation functions

Physical observables involve field correlation functions (generalized moments),

$$\langle \phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle \equiv \frac{1}{\mathcal{Z}}\int [\mathrm{d}\phi(x)]\phi(x_1)\phi(x_2)\dots\phi(x_n)\,\mathrm{e}^{-\mathcal{H}(\phi)}.$$

They can be derived by functional differentiation from the generating functional (generalized partition function) in an external field $H(x)$,

$$\mathcal{Z}(H)=\int [\mathrm{d}\phi(x)]\,\exp\left[-\mathcal{H}(\phi)+\int \mathrm{d}^d x\, H(x)\phi(x)\right],$$

as

$$\langle \phi(x_1)\phi(x_2)\dots\phi(x_n)\rangle = \frac{1}{\mathcal{Z}(0)}\,\frac{\delta}{\delta H(x_1)}\frac{\delta}{\delta H(x_2)}\dots\frac{\delta}{\delta H(x_n)}\mathcal{Z}(H)\bigg|_{H=0}\,.$$

Connected correlation functions

The more relevant physical observables are the connected correlation functions $W^{(n)}(x_1, x_2, \ldots, x_n)$ (generalized cumulants), which can be obtained by function differentiation from the free energy $\mathcal{W}(H) = \ln \mathcal{Z}(H)$ in the external field H:

$$W^{(n)}(x_1, x_2, \ldots, x_n) = \frac{\delta}{\delta H(x_1)} \frac{\delta}{\delta H(x_2)} \cdots \frac{\delta}{\delta H(x_n)} \mathcal{W}(H)\Big|_{H=0} .$$

Due to translation invariance,

$$W^{(n)}(x_1, x_2, \ldots, x_n) = W^{(n)}(x_1 + a, x_2 + a, \ldots, x_n + a)$$

for any vector a.

Connected correlation functions have the so-called cluster property: if one separates the points $x_1, \ldots, x_n$ in two non-empty sets, connected functions go to zero when the distance between the two sets goes to infinity. It is the large distance behaviour of connected correlation functions in the critical domain near T_c that may exhibit universal properties.

The renormalization group: General formulation

To construct an RG flow, the basic idea is to integrate in the field integral recursively over short distance degrees of freedom. This leads to the definition of an effective Hamiltonian $\mathcal{H}_\lambda$ function of a scale parameter $\lambda > 0$ (such that $\mathcal{H}_1 = \mathcal{H}$) and of a transformation $\mathcal{T}$ in the space of Hamiltonians such that

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{H}_\lambda = \mathcal{T}\left[\mathcal{H}_\lambda\right], \tag{1}$$

an equation called RG equation (RGE). The appearance of the derivative $\lambda \mathrm{d}/\mathrm{d}\lambda = \mathrm{d}/\mathrm{d}\ln\lambda$ reflects the multiplicative character of dilatations. The RGE thus defines a dynamical process in the "time" $\ln\lambda$. The denomination renormalization group (RG) refers to the property that $\ln\lambda$ belongs to the additive group of real numbers.

RG equation: General structure, fixed points

We will derive RGE that define a stationary Markov process, that is, $\mathcal{T}[\mathcal{H}_\lambda]$ depends on $\mathcal{H}_\lambda$ but not on the trajectory that has led from $\mathcal{H}_{\lambda=1}$ to $\mathcal{H}_\lambda$, and depends on λ only through $\mathcal{H}_\lambda$.

Universality is then related to the existence of fixed points, solution of

$$\mathcal{T}(\mathcal{H}^*) = 0\,.$$

We assume also that the mapping $\mathcal{T}$ is differentiable, so that near a fixed point the RG flow can be linearized,

$$\mathcal{T}(\mathcal{H}^* + \Delta\mathcal{H}_\lambda) \sim L^* \Delta\mathcal{H}_\lambda\,,$$

and is governed by the eigenvalues and eigenvectors of the linear operator L^*. Formally, the solution of the linearized equations can be written as

$$\mathcal{H}_\lambda = \mathcal{H}^* + \lambda^{L^*}\left(\mathcal{H}_{\lambda=1} - \mathcal{H}^*\right).$$

The Gaussian measures

In the spirit of the central limit theorem of probabilities, one could hope that the universal properties of phase transitions can be described by Gaussian or weakly perturbed Gaussian measures. The simplest form satisfying all conditions in d space dimensions (we assume $d \ge 2$), corresponds to the quadratic Hamiltonian ($\alpha_0 \ge 0$ constant)

$$\mathcal{H}^{(0)}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\big(\nabla_x \phi(x)\big)^2 + \tfrac{1}{2}\alpha_0 \phi^2(x)\right]. \qquad (2)$$

One sees immediately that the Gaussian model can only describe the high temperature phase $T \ge T_c$.

In a Gaussian model, all correlation functions can be expressed in terms of the two-point function with the help of Wick's theorem.

Regularization

However, with this Hamiltonian the Gaussian model has a problem: too singular fields contribute to the field integral in such a way that correlation functions at coinciding points are not defined. For example,

$$\langle \phi^2(x) \rangle = W^{(2)}(0,0) = \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d p}{p^2 + \alpha_0} \,,$$

which diverges in any space dimension $d \geq 2$. In particular, expectation values of perturbations to the Gaussian theory of the form $\phi^m(x)$ (powers of the field at the same point) are not defined.

Thus, it is necessary to modify the Gaussian measure to restrict the field integration to more regular fields, continuous to define powers of the field, satisfying differentiability conditions to define expectation values of the field and its derivatives taken at the same point, a procedure called regularization.

This can be achieved by adding to

$$\mathcal{H}^{(0)}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\big(\nabla_x \phi(x)\big)^2 + \tfrac{1}{2}\alpha_0 \phi^2(x)\right],$$

enough terms with more derivatives (here, periodic boundary conditions are assumed)

$$\mathcal{H}^{(0)}(\phi) \mapsto \mathcal{H}_{\mathrm{G}}(\phi) = \mathcal{H}^{(0)}(\phi) + \frac{1}{2}\sum_{k=2}^{2k_{\max}} \alpha_k \int \mathrm{d}^d x\, \phi(x) \nabla_x^{2k} \phi(x), \tag{3}$$

where the coefficient α_k are only constrained by the positivity of the Hamiltonian. For example, simple continuity requires $2k_{\max} > d$.

The Gaussian two-point function at large distance

The two-point function then reads

$$W^{(2)}(x,0) = \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d p \, \mathrm{e}^{ipx}}{\alpha_0 + p^2 + \sum_{k=2} \alpha_k p^{2k}} .$$

At large distance $|x| \to \infty$, for $\alpha_0 \neq 0$, correlations decrease exponentially as

$$W^{(2)}(x,0) \propto \frac{1}{|x|^{(d-1)/2}} \, \mathrm{e}^{-|x|/\xi} ,$$

where ξ is the correlation length and

$$\xi \propto 1/\sqrt{\alpha_0} \text{ for } \alpha_0 \to 0 \, .$$

At the critical point ($T = T_c$), the correlation length diverges, which implies $\alpha_0 = 0$ and, for $d > 2$, one finds the algebraic critical behaviour

$$W^{(2)}(x,0) \underset{|x| \to \infty}{\propto} \frac{1}{|x|^{d-2}} \, .$$

For $d = 2$, the Gaussian model is not defined at T_c.

The Gaussian fixed point

An RG can be constructed that reproduces the properties of the Gaussian models. The Hamiltonian flow can be implemented by the simple scaling

$$\phi(x) \mapsto \lambda^{(2-d)/2}\phi(x/\lambda). \tag{4}$$

After the change of variables $x' = x/\lambda$, one verifies that the Hamiltonian

$$\mathcal{H}_{\mathrm{G}}^{*}(\phi) = \frac{1}{2}\int \mathrm{d}^{d}x \bigl(\nabla_{x}\phi(x)\bigr)^{2}, \tag{5}$$

corresponding to the critical Gaussian model, is invariant. The RG has $\mathcal{H}_{\mathrm{G}}^{*}$ as a fixed point. The Hamiltonian flow (4) corresponds in fact to the linear approximation of the general RG near the Gaussian fixed point.

The linearized RG flow

The transformation (4) generates the linearized RG flow at the Gaussian fixed point. Eigenvectors of the linear flow (4) are monomials of the form

$$\mathcal{O}_{n,k}(\phi) = \int \mathrm{d}^d x \, O_{n,k}(\phi, x),$$

where $O_{n,k}(\phi, x)$ is a product of powers of the field and its derivatives at point x with $2n$ powers of the field (reflection $\mathbb{Z}_2$ symmetry) and $2k$ powers of ∂_μ.

Their RG behaviour under the transformation (4) is then given by a simple dimensional analysis. One defines the dimension of x as -1 and the (Gaussian) dimension of the field is $[\phi] = (d-2)/2$. The dimension $[\mathcal{O}_{n,k}]$ of $\mathcal{O}_{n,k}$ is then

$$[\mathcal{O}_{n,k}] = -d + n(d-2) + 2k \,. \tag{6}$$

It can be verified that $\mathcal{O}_{n,k}$ scales like $\lambda^{-[\mathcal{O}_{n,k}]}$, and the corresponding eigenvalue of L^* thus is $\ell_{n,k} = -[\mathcal{O}_{n,k}]$.

Discussion

When $\lambda \to +\infty$,

(i) for $\ell_{n,k} > 0$ the amplitude of $\mathcal{O}_{n,k}(\phi)$ increases; it is a direction of instability and in the RG terminology $\mathcal{O}_{n,k}(\phi)$ is a relevant perturbation;

(ii) for $\ell_{n,k} < 0$, the amplitude of $\mathcal{O}_{n,k}(\phi)$ decreases; it is a direction of stability and $\mathcal{O}_{n,k}(\phi)$ is an irrelevant perturbation;

(iii) in the special case $\ell_{n,k} = 0$, one speaks of a marginal perturbation and the linear approximation is no longer sufficient to discuss stability. Logarithmic behaviour in λ is then expected (we omit here unphysical redundant perturbations).

Since $\ell_{1,0} = 2$, $\int \mathrm{d}^d x\, \phi^2(x)$ corresponds always to a direction of instability: indeed it induces a deviation from the critical temperature and thus a finite correlation length.

For $d > 4$, no other perturbation is relevant and the Gaussian fixed point is stable on the critical surface ($\xi = \infty$).

Since $\ell_{2,0} = 4 - d$, at $d = 4$ one term becomes marginal: $\int \mathrm{d}^d x\, \phi^4(x)$, which below dimension four becomes relevant. In dimension $d = 4 - \varepsilon$, $\varepsilon > 0$ small (a notion we define later), it is the only relevant perturbation and one expects to be able to describe critical properties with a Gaussian theory to which this unique term is added.

To summarize, for systems with a $\mathbb{Z}_2$ or, more generally, with an $O(N)$ symmetry, one concludes that

(i) the Gaussian fixed point is stable above space dimension four;

(ii) from a next order analysis, one shows that it is marginally stable in dimension four;

(iii) it is unstable below dimension four.

Rescaling

If what follows, we assume that initially the statistical system is very close to the Gaussian fixed point. The RG flow is then first governed by the local linear flow. Therefore, we implement first the corresponding RG transformation. We introduce a parameter $\Lambda \gg 1$ and substitute

$$\phi(x) \mapsto \Lambda^{(2-d)/2}\phi(x/\Lambda).$$

After the change of variables $x' = x/\Lambda$, a monomial $\mathcal{O}_{n,k}(\phi)$ is multiplied by $\Lambda^{-[\mathcal{O}_{n,k}]}$, where $[\mathcal{O}_{n,k}]$ is the dimension in the sense of the linearized RG. In the quantum field theory language, this could be called a Gaussian renormalization. The Gaussian RG dimensions can then be expressed in terms of Λ: space coordinates x have dimension Λ^{-1}, derivatives dimension Λ and the field dimension $\Lambda^{(d-2)/2}$. The Hamiltonian is dimensionless.

In the context of quantum field theory, since the regularization has the effect, in Fourier representation, to suppress the contributions of momenta $|p| \gg \Lambda$ in Feynman graphs, Λ is called the cut-off.

Statistical scalar field theory: Perturbation theory

The Gaussian model

After rescaling, the Hamiltonian of the Gaussian model takes the form

$$\mathcal{H}_{\mathrm{G}}(\phi) = \frac{1}{2}\int \mathrm{d}^d x \left[\left(\nabla_x \phi(x)\right)^2 + \alpha_0 \Lambda^2 \phi^2(x) + \sum_{k=2} \alpha_k \Lambda^{2-2k} \phi(x) \nabla_x^{2k} \phi(x)\right], \tag{7}$$

where α_0 is the amplitude of the only relevant term for $d > 4$. Except for the two-point function at coinciding points, one can take the $\Lambda \to \infty$ limit. However, for $\alpha_0 \neq 0$, to obtain a non-trivial universal large distance behaviour, it is also necessary to compensate the RG flow by choosing α_0 infinitesimal, taking the $\Lambda \to \infty$ limit at $r = \alpha_0 \Lambda^2$ fixed (r is a Gaussian renormalized parameter in quantum field theory language). This defines the critical domain.

The perturbed Gaussian or quasi-Gaussian model

To allow for spontaneous symmetry breaking and, thus, to be able to describe physics below T_c, terms have necessarily to be added to the Gaussian Hamiltonian to generate a double-well potential for constant fields. The minimal addition, and the leading term from the RG viewpoint, is of ϕ^4 type. This leads to

$$\mathcal{H}(\phi) = \mathcal{H}_{\mathrm{G}}(\phi) + \frac{g}{4!}\Lambda^{4-d}\int \mathrm{d}^d x\, \phi^4(x), \quad g \geq 0\,.$$

The ϕ^4 term generates a shift of the critical temperature. To recover a critical theory ($T = T_c$), it is necessary to choose a ϕ^2 term with a specific g-dependent coefficient $\frac{1}{2}(\alpha_0)_c(g)$, a mass renormalization in quantum field theory terminology.

As we have explained, for $d > 4$ the ϕ^4 term then is an irrelevant contribution that does not invalidate the universal predictions of the Gaussian model.

Corrections to the Gaussian theory can be obtained by expanding in powers of the coefficient g.

Setting $u = g\Lambda^{4-d}$, the partition function, for example, is then given by

$$\mathcal{Z} = \sum_{k=0}^{\infty} \frac{(-u)^k}{(4!)^k k!} \left\langle \left(\int \mathrm{d}^d x \, \phi^4(x) \right)^k \right\rangle_{\mathrm{G}} .$$

The Gaussian expectations values $\langle \bullet \rangle_{\mathrm{G}}$ can then be evaluated in terms of the Gaussian two-point function with the help of Wick's theorem (Feynman graph expansion).

By contrast, for any $d < 4$, the ϕ^4 contribution is relevant: the Gaussian fixed point is unstable and no longer governs the large distance behaviour. The perturbative expansion of the critical theory ($T = T_c$) in powers of u contains so-called infra-red, that is, long distance, or zero momentum in Fourier space, divergences.

Renormalization group in dimension $d = 4 - \varepsilon$

For $d < 4$, to determine the large distance behaviour of correlation functions, it becomes necessary to construct a general renormalization group: this leads to functional equations that we describe later, but which, in general, unfortunately cannot be solved analytically.

However, a trick has been discovered to extend the definition of all terms of the perturbative expansion to arbitrary complex values of the dimension d in the form of meromorphic functions.

This allows replacing, in dimension $d = 4 - \varepsilon$ and in the framework of an expansion in powers of ε, the general renormalization group by a much simpler asymptotic form and studying the model analytically.

Dimensional continuation and regularization

To discuss dimensional continuation, it is convenient to introduce the Fourier representation of correlation functions. Taking into account translation invariance, one defines

$$(2\pi)^d \delta^{(d)}\left(\sum_{i=1}^{n} p_i\right) \tilde{W}^{(n)}(p_1,\ldots,p_n)$$

$$= \int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \, W^{(n)}(x_1,\ldots,x_n) \exp\left(i\sum_{j=1}^{n} x_j p_j\right), \tag{8}$$

where, in analogy with quantum mechanics, the Fourier variables p_i are called momenta (and have dimension Λ). We also introduce the Fourier representation of the two-point function (or propagator) $\Delta(x)$, corresponding to the Hamiltonian of the Gaussian model,

$$\Delta(x) \equiv \langle \phi(x)\phi(0)\rangle_{\mathrm{G}} = \frac{1}{(2\pi)^d}\int \mathrm{d}^d p \, \mathrm{e}^{-ipx}\tilde{\Delta}(p).$$

Dimensional continuation

A general representation useful for dimensional continuation of the Gaussian two-point function is the Laplace representation

$$\tilde{\Delta}(p) = \int_0^\infty \mathrm{d}s\, \rho(s\Lambda^2)\mathrm{e}^{-sp^2}, \qquad (9)$$

where the function $\rho(s) \to 1$ when $s \to \infty$.

Moreover, to reduce the field integration to continuous fields and, thus, to render the perturbative expansion finite, one needs at least $\rho(s) = O(s^q)$ with $q > (d-2)/2$ for $s \to 0$.

If, in addition, one wants the expectation values of all local polynomials to be defined, one must impose to $\rho(s)$ to converge to zero faster than any power.

A contribution to perturbation theory (represented graphically by a Feynman diagram) takes, in Fourier representation, the form of a product of propagators integrated over a subset of momenta. With the Laplace representation, all momentum integrations become Gaussian and can be performed, resulting in explicit analytic meromorphic functions of the dimension parameter d. For example, the contribution of order g to the two-point function is proportional to

$$\begin{aligned}\Omega_d &= \frac{1}{(2\pi)^d}\int \mathrm{d}p\,\tilde{\Delta}(p) = \frac{1}{(2\pi)^d}\int \mathrm{d}p\int_0^\infty \mathrm{d}s\,\rho(s\Lambda^2)\mathrm{e}^{-sp^2}\\ &= \frac{1}{(4\pi)^{d/2}}\int_0^\infty \mathrm{d}s\,s^{-d/2}\rho(s\Lambda^2),\end{aligned}$$

which, in the latter form, is holomorphic for $2 < \operatorname{Re} d < 2(1+q)$.

Dimensional regularization

For the theory of critical phenomena, dimensional continuation is sufficient since it allows exploring the neighbourhood of dimension four, determining fixed points and calculating universal quantities as $\varepsilon = (4-d)$-expansions.

However, for practical calculations, but then restricted to the leading large distance behaviour, an additional step is extremely useful. It can be verified that if one decreases $\mathrm{Re}\, d$ enough, so that by naive power counting all momentum integrals are convergent, one can, after explicit dimensional continuation, take the infinite Λ limit. The resulting perturbative contributions become meromorphic functions with poles at dimensions at which large momentum, and low momentum in the critical theory, divergences appear. This method of regularizing large momentum divergences is called dimensional regularization and is extensively used in quantum field theory. In the theory of critical phenomena, it has also been used to calculate universal quantities like critical exponents, as $\varepsilon = (4-d)$-expansions.

Perturbative renormalization group

The renormalization theorem

The perturbative renormalization group, as it has been developed in the framework of the perturbative expansion of quantum field theory, relies on the so-called renormalization theory. For the ϕ^4 field theory it has been first formulated in space dimension $d = 4$. For critical phenomena, a small extension is required that involves an additional expansion in powers of $\varepsilon = 4 - d$, after dimensional continuation.

To formulate the renormalization theorem, one introduces a momentum μ, called the renormalization scale, and a parameter g_{r} characterizing the effective ϕ^4 coefficient at scale μ, called the renormalized coupling constant.

One can then find two dimensionless functions $Z(\Lambda/\mu, g)$ and $Z_g(\Lambda/\mu, g)$ that satisfy (g and Λ/μ are the only two dimensionless combinations)

$$\Lambda^{4-d} g = \mu^{4-d} Z_g(\Lambda/\mu, g) g_{\mathrm{r}} = \mu^{4-d} g_{\mathrm{r}} + O(g^2), \; Z(\Lambda/\mu, g) = 1 + O(g), \quad (10)$$

calculable order by order in a double series expansion in powers of g and ε, such that all connected correlations functions

$$\tilde{W}_{\mathrm{r}}^{(n)}(p_i; g_{\mathrm{r}}, \mu, \Lambda) = Z^{-n/2}(g, \Lambda/\mu) \tilde{W}^{(n)}(p_i; g, \Lambda), \quad (11)$$

called renormalized, have, order by order in g_{r}, finite limits $\tilde{W}_{\mathrm{r}}^{(n)}(p_i; g_{\mathrm{r}}, \mu)$ when $\Lambda \to \infty$ at p_i, μ, g_{r} fixed.

The renormalization constant $Z^{1/2}(\Lambda/\mu, g)$ is the ratio between the renormalization in presence of the ϕ^4 interaction and the Gaussian field renormalization $\Lambda^{(d-2)/2}$.

Remarks

There is some arbitrariness in the choice of the renormalization constants Z and Z_g since they can be multiplied by arbitrary functions of g_{r}. The constants can be completely determined by imposing three renormalization conditions to the renormalized correlation functions, which are then independent of the specific choice of the regularization. This a first important result: since initial and renormalized correlation functions have the same large distance behaviour, this behaviour is to a large extent universal since it can, therefore, only depend at most on one parameter, the ϕ^4 coefficient g.

Critical RG equations

From the relation between initial and renormalized functions (equation (11)) and the existence of a limit $\Lambda \to \infty$, a new equation follows, obtained by differentiation of the equation with respect to Λ at μ, g_{r} fixed:

$$\Lambda \frac{\partial}{\partial \Lambda}\bigg|_{g_{\mathrm{r}},\mu\ \text{fixed}} Z^{n/2}(g,\Lambda/\mu)\tilde{W}^{(n)}(p_i;g,\Lambda) \to 0\,. \tag{12}$$

In agreement with the perturbative philosophy, one then neglects all contributions that, order by order, decay as powers of Λ. One defines asymptotic functions $\tilde{W}^{(n)}_{\text{as.}}(p_i;g,\Lambda)$ and $Z_{\text{as.}}(g,\Lambda/\mu)$ as sums of the perturbative contributions to the functions $\tilde{W}^{(n)}(p_i;g,\Lambda)$ and $Z(g,\Lambda/\mu)$, respectively, that do not go to zero when $\Lambda \to \infty$. Using the chain rule, one derives from equation (12)

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g,\Lambda/\mu)\frac{\partial}{\partial g} + \frac{n}{2}\eta(g,\Lambda/\mu)\right]\tilde{W}^{(n)}_{\text{as.}}(p_i;g,\Lambda) = 0\,.$$

The functions β and η are defined by

$$\beta(g,\Lambda/\mu) = \Lambda \frac{\partial}{\partial\Lambda}\bigg|_{g_{\mathrm{r}},\mu} g\,, \quad \eta(g,\Lambda/\mu) = -\Lambda \frac{\partial}{\partial\Lambda}\bigg|_{g_{\mathrm{r}},\mu} \ln Z_{\mathrm{as.}}(g,\Lambda/\mu).$$

Since the functions $\tilde{W}^{(n)}_{\mathrm{as.}}$ do not depend on μ, the functions β and η cannot depend on Λ/μ, and one finally obtains the RG equations (Zinn-Justin 1973):

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} + \frac{n}{2}\eta(g)\right)\tilde{W}^{(n)}_{\mathrm{as.}}(p_i;g,\Lambda) = 0\,. \tag{13}$$

From equation (10), one immediately infers that $\beta(g) = -\varepsilon g + O(g^2)$.

RG equations in the critical domain above T_c

Correlation functions may also exhibit universal properties near T_c when the correlation length ξ is large in the microscopic scale, here, $\xi\Lambda \gg 1$. To describe universal properties in the critical domain above T_c, one adds the ϕ^2 relevant term to the Hamiltonian:

$$\mathcal{H}_t(\phi) = \mathcal{H}(\phi) + \frac{t}{2}\int \mathrm{d}^d x\, \phi^2(x),$$

where t, the coefficient of ϕ^2, characterizes the deviation from the critical temperature: $t \propto T - T_{\mathrm{c}}$. The renormalization theorem leads to the appearance of a new renormalization factor $Z_2(\Lambda/\mu, g)$ associated with the parameter t. By arguments of the same nature as in the critical theory, one derives a more general RGE of the form (Zinn-Justin 1973)

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} + \frac{n}{2}\eta(g) - \eta_2(g)t\frac{\partial}{\partial t}\right]\tilde{W}^{(n)}_{\text{as.}}(p_i;t,g,\Lambda) = 0\,, \tag{14}$$

where a new RG function $\eta_2(g)$, related to $Z_2(\Lambda/\mu, g)$, appears.

These equations can be further generalized to deal with an external field (a magnetic field for magnetic systems) and the corresponding induced field expectation value (magnetization for magnetic systems).

Renormalized RG equations

For $d = 4 - \varepsilon$, if one is only interested in the leading scaling behaviour (and the first correction), it is technically simpler to use dimensional regularization and the renormalized theory in the so-called minimal (or modified minimal) subtraction scheme. The relation (11) between initial and renormalized correlation functions is asymptotically symmetric. One thus derives also (for the critical theory)

$$\left(\mu\frac{\partial}{\partial\mu} + \tilde{\beta}(g_{\mathrm{r}})\frac{\partial}{\partial g_{\mathrm{r}}} + \frac{n}{2}\tilde{\eta}(g_{\mathrm{r}})\right)\tilde{W}_{\mathrm{r}}^{(n)}(p_i, g_{\mathrm{r}}, \mu) = 0$$

with the definitions

$$\tilde{\beta}(g_{\mathrm{r}}) = \mu\frac{\partial}{\partial\mu}\bigg|_g g_{\mathrm{r}}\,, \quad \tilde{\eta}(g_{\mathrm{r}}) = \mu\frac{\partial}{\partial\mu}\bigg|_g \ln Z(g_{\mathrm{r}}, \varepsilon)\,.$$

In this scheme, the renormalization constants (10) are obtained by going to low dimensions where the infinite Λ limit, at g_{r} fixed, can be taken. For example,

$$\lim_{\Lambda\to\infty} Z(\Lambda/\mu, g)\big|_{g_{\mathrm{r}}\ \text{fixed}} = Z(g_{\mathrm{r}}, \varepsilon).$$

Then, order by order in powers of g_{r}, they have a Laurent expansion in powers of ε. In the minimal subtraction scheme, the freedom in the choice of the renormalization constants is used to reduce the Laurent expansion to the singular terms. For example, $Z(g_{\mathrm{r}}, \varepsilon)$ takes the form

$$Z(g_{\mathrm{r}}, \varepsilon) = 1 + \sum_{n=1}^{\infty} \frac{\sigma_n(g_{\mathrm{r}})}{\varepsilon^n} \text{ with } \sigma_n(g_{\mathrm{r}}) = O(g_{\mathrm{r}}^{n+1}).$$

A remarkable consequence is that the RG functions $\tilde{\eta}(g_{\mathrm{r}})$, and $\tilde{\eta}_2(g_{\mathrm{r}})$ when a ϕ^2 term is added, become independent of ε and $\tilde{\beta}(g_{\mathrm{r}})$ has the simple dependence $\tilde{\beta}(g_{\mathrm{r}}) = -\varepsilon g_{\mathrm{r}} + \tilde{\beta}_2(g_{\mathrm{r}})$, where $\tilde{\beta}_2(g_{\mathrm{r}}) = O(g_{\mathrm{r}}^2)$ is also independent of ε.

Solution of the RG equations: The epsilon-expansion

RG equations can be solved by the method of characteristics. In the simplest example of the critical theory, one introduces a scale parameter λ and two functions of $g(\lambda)$ and $\zeta(\lambda)$ defined by

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} g(\lambda) = -\beta\bigl(g(\lambda)\bigr),\ g(1) = g\,, \quad \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \zeta(\lambda) = -\eta\bigl(g(\lambda)\bigr),\ \zeta(1) = 1\,. \tag{15}$$

The function $g(\lambda)$ is the effective coefficient of the ϕ^4 term at the scale λ. One verifies that equation (13) is then equivalent to

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \left[\zeta^{n/2}(\lambda) \tilde{W}^{(n)}_{\text{as.}}\bigl(p_i; g(\lambda), \Lambda/\lambda\bigr)\right] = 0\,,$$

which implies ($\Lambda \mapsto \lambda\Lambda$)

$$\tilde{W}^{(n)}_{\text{as.}}\bigl(p_i; g, \lambda\Lambda\bigr) = \zeta^{n/2}(\lambda) \tilde{W}^{(n)}_{\text{as.}}\bigl(p_i; g(\lambda), \Lambda\bigr).$$

From its definition, one infers that $\tilde{W}^{(n)}_{\text{as.}}$ has dimension $(d-(d+2)n/2)$. Therefore,

$$\begin{aligned}\tilde{W}^{(n)}_{\text{as.}}\big(p_i/\lambda; g, \Lambda\big) &= \lambda^{(d+2)n/2-d}\tilde{W}^{(n)}_{\text{as.}}\big(p_i; g, \lambda\Lambda\big) \\ &= \lambda^{(d+2)n/2-d}\zeta^{n/2}(\lambda)\tilde{W}^{(n)}_{\text{as.}}\big(p_i; g(\lambda), \Lambda\big).\end{aligned}$$

These equations show that here the general Hamiltonian flow reduces here to the flow of $g(\lambda)$ and, thus, the large distance behaviour is governed by the zeros of the function $\beta(g)$. When $\lambda \to \infty$, since $\beta(g) = -\varepsilon g + O(g^2)$, if $g > 0$ is initially very small, it moves away from the unstable Gaussian fixed point, in agreement with the general RG analysis.

If one assumes the existence of another zero g^* with then $\beta'(g^*) > 0$, then $g(\lambda)$ will converge toward this fixed point. Since $g(\lambda)$ tends toward the fixed point value g^*, and if $\eta(g^*) \equiv \eta$ is finite, one finds the universal behaviour

$$\tilde{W}^{(n)}_{\text{as.}}(p_i/\lambda; g, \Lambda) \underset{\lambda\to\infty}{\propto} \lambda^{(d+2-\eta)n/2-d} \tilde{W}^{(n)}_{\text{as.}}(p_i; g^*, \Lambda).$$

For the connected correlation functions in space, this result translates into

$$W^{(n)}_{\text{as.}}(\lambda x_i; g, \Lambda) \underset{\lambda\to\infty}{\propto} \lambda^{-n(d-2+\eta)/2} W^{(n)}_{\text{as.}}(x_i; g^*, \Lambda),$$

for all x_i distinct.

The exponent $d_\phi = (d-2+\eta)/2$ is the dimension of the field ϕ, from the point of view of large distance properties.

Explicit calculations

From the perturbative calculation of the two- and four-point functions at one-loop order, one derives

$$\beta(g) = -\varepsilon g + \frac{3}{16\pi^2} g^2 + O(g^3, \varepsilon g^2).$$

In the sense of an ε-expansion, $\beta(g)$ has a zero g^* with a positive slope (Wilson–Fisher 1972)

$$g^* = \frac{16\pi^2 \varepsilon}{3} + O(\varepsilon^2), \quad \omega = \beta'(g^*) = \varepsilon + O(\varepsilon^2),$$

which governs the large momentum behaviour of correlation functions. In addition, the exponent ω governs the leading correction to the critical behaviour.

Generalization

The results obtained for models with a $\mathbb{Z}_2$ reflection symmetry can easily be generalized to N-vector models with $O(N)$ orthogonal symmetry, which belong to different universality classes. Their universal properties can then be derived from an $O(N)$ symmetric field theory with an N-component field $\boldsymbol{\phi}(x)$ and a $g(\boldsymbol{\phi}^2)^2$ quartic term. Further generalizations involve theories with N-component fields but smaller symmetry groups, such that several independent quartic ϕ^4 terms are allowed. The structure of fixed points may then be more complicate.

Finally, correlation functions of the $O(N)$ model can be evaluated in the large N limit explicitly and the predictions of the ε-expansion can then be verified.

Epsilon-expansion: A few results

From the simple existence of the fixed point and of the corresponding ε-expansion, universal properties of a large class of critical phenomena can be proved to all orders in ε: this includes relations between critical exponents, scaling behaviour of correlation functions or the equation of state.

Moreover, universal quantities can then be calculated as ε-expansions.

The scaling equation of state

An example of the general results that can be obtained is provided by the equation of state of magnetic systems, that is, the relation between applied magnetic field H, magnetization M and temperature T. In the relevant limit $|H| \ll 1$, $|T - T_c| \ll 1$, RG has proved Widom's conjectured scaling form

$$H = M^{\delta} f\big((T - T_c)/M^{1/\beta}\big),$$

where $f(z)$ is a universal function (up to normalizations).

Moreover, the exponents satisfy the relations

$$\delta = \frac{d+2-\eta}{d-2+\eta}, \quad \beta = \tfrac{1}{2}\nu(d-2+\eta),$$

where ν, the correlation length exponent, given by $\nu = 1/\big(\eta_2(g^*)+2\big)$, characterizes the divergence ξ of the correlation length at T_c:

$$\xi \propto |T-T_c|^{-\nu}.$$

Other relations can be derived, involving the magnetic susceptibility exponent γ characterizing the divergence of the two-point correlation function at zero momentum at T_c, or the exponent α characterizing the behaviour of the specific heat:

$$\gamma = \nu(2-\eta), \quad \alpha = 2-\nu d\,.$$

Note the relations involving the dimension d explicitly are not valid for the Gaussian fixed point.

Critical exponents as ε-expansions

As an illustration, we give here two successive terms of the ε-expansion of the exponents η, γ and ω for the $O(N)$ models, although the RG functions of the $(\phi^2)^2$ field theory are known to five-loop order and, thus, the critical exponents are known up to ε^5. In terms of the variable $v = N_d g$ where N_d is the loop factor

$$N_d = 2/(4\pi)^{d/2}\Gamma(d/2),$$

the RG functions $\beta(v)$ and $\eta_2(v)$ at two-loop order, $\eta(v)$ at three-loop order are

$$\beta(v) = -\varepsilon v + \frac{(N+8)}{6}v^2 - \frac{(3N+14)}{12}v^3 + O(v^4),$$
$$\eta(v) = \frac{(N+2)}{72}v^2\left[1 - \frac{(N+8)}{24}v\right] + O(v^4),$$
$$\eta_2(v) = -\frac{(N+2)}{6}v\left[1 - \frac{5}{12}v\right] + O(v^3).$$

The fixed point value solution of $\beta(v^*) = 0$ is then

$$v^*(\varepsilon) = \frac{6\varepsilon}{(N+8)}\left[1 + \frac{3(3N+14)}{(N+8)^2}\varepsilon\right] + O(\varepsilon^3).$$

The values of the critical exponents

$$\eta = \eta(v^*), \quad \gamma = \frac{2-\eta}{2+\eta_2(v^*)}, \quad \omega = \beta'(v^*),$$

follow

$$\eta = \frac{\varepsilon^2(N+2)}{2(N+8)^2}\left[1 + \frac{(-N^2+56N+272)}{4(N+8)^2}\varepsilon\right] + O(\varepsilon^4),$$

$$\gamma = 1 + \frac{(N+2)}{2(N+8)}\varepsilon + \frac{(N+2)}{4(N+8)^3}\left(N^2+22N+52\right)\varepsilon^2 + O(\varepsilon^3),$$

$$\omega = \varepsilon - \frac{3(3N+14)}{(N+8)^2}\varepsilon^2 + O(\varepsilon^3).$$

Though this may not be obvious on these few terms, the ε-expansion is divergent for any $\varepsilon > 0$, as large order estimates based on instanton calculus have shown. Extracting precise numbers from the known terms of the series requires a summation method. For example, adding simply the known successive terms for $\varepsilon = 1$ and $N = 1$ yields

$$\gamma = 1.000\ldots\,,\ 1.1666\ldots\,,\ 1.2438\ldots\,,1.1948\ldots\,,\ 1.3384\ldots\,,\ 0.8918\ldots\,,$$

while the best field theory estimate is $\gamma = 1.2396 \pm 0.0013$.

Summation of the ε-expansion and numerical values of exponents

We display below (Table 1) the results for some critical exponents of the $O(N)$ model obtained from Borel summation of the ε-expansion (Guida and Zinn-Justin 1998). Due to scaling relations like $\gamma = \nu(2-\eta)$, $\gamma + 2\beta = \nu d$, only two among the first four are independent, but the series are summed independently to check consistency. $N = 0$ corresponds to statistical properties of polymers (mathematically the self-avoiding random walk), $N = 1$, the Ising universality class, to liquid-vapour, binary mixtures or anisotropic magnet phase transitions. $N = 2$ describes the superfluid Helium transition, while $N = 3$ correspond to isotropic ferromagnets.

As a comparison, we also display (Table 2) the best available field theory results obtained from Borel summation of $d = 3$ perturbative series (Guida and Zinn-Justin 1998).

Table 1

Critical exponents of the $O(N)$ model, $d = 3$, obtained from the ε-expansion.

N	0	1	2	3
γ	1.1571 ± 0.0030	1.2355 ± 0.0050	1.3110 ± 0.0070	1.3820 ± 0.0090
ν	0.5878 ± 0.0011	0.6290 ± 0.0025	0.6680 ± 0.0035	0.7045 ± 0.0055
η	0.0315 ± 0.0035	0.0360 ± 0.0050	0.0380 ± 0.0050	0.0375 ± 0.0045
β	0.3032 ± 0.0014	0.3265 ± 0.0015	0.3465 ± 0.0035	0.3655 ± 0.0035
ω	0.828 ± 0.023	0.814 ± 0.018	0.802 ± 0.018	0.794 ± 0.018

Table 2

Critical exponents of the $O(N)$ model, $d = 3$, obtained from the $(\phi^2)^2_3$ field theory.

N	0	1	2	3
g^*_{Ni}	1.413 ± 0.006	1.411 ± 0.004	1.403 ± 0.003	1.390 ± 0.004
γ	1.1596 ± 0.0020	1.2396 ± 0.0013	1.3169 ± 0.0020	1.3895 ± 0.0050
ν	0.5882 ± 0.0011	0.6304 ± 0.0013	0.6703 ± 0.0015	0.7073 ± 0.0035
η	0.0284 ± 0.0025	0.0335 ± 0.0025	0.0354 ± 0.0025	0.0355 ± 0.0025
β	0.3024 ± 0.0008	0.3258 ± 0.0014	0.3470 ± 0.0016	0.3662 ± 0.0025
ω	0.812 ± 0.016	0.799 ± 0.011	0.789 ± 0.011	0.782 ± 0.0013

Functional (or exact) renormalization group

We now briefly describe a general approach to the RG close to ideas initially developed by Wegner and Wilson, and based on a partial integration over the large-momentum modes of fields. This RG takes the form of functional renormalization group (FRG) equations that express the equivalence between a change of a scale parameter related to microscopic physics and a change of the parameters of the Hamiltonian. Some forms of these equations are exact and one then also speaks of the exact renormalization group.

These FRG equations have been used to recover the first terms of the ε-expansion and later by Polchinski to give a new proof of the renormalizability of field theories, avoiding any argument based on Feynman diagrams and combinatorics.

From the practical viewpoint, several variants of these FRG equations have led to new approximation schemes no longer based on the standard perturbative expansion.

Technically, these FRG equations follow from identities that express the invariance of the partition function under a correlated change of the propagator and the other parameters of the Hamiltonian. We discuss these equations, in continuum space, in the framework of local statistical field theory. It is easy to verify that, except in the Gaussian case, these equations are closed only if an infinite number of local interactions are included.

It is then possible to infer various RGE satisfied by correlation functions. Depending on the chosen form, these RGE are either exact or only exact at large distance or small momenta, up to corrections decaying faster than any power of the dilatation parameter.

Here, we discuss only the Hamiltonian flow, the RGE for correlation functions requiring some additional considerations.

Partial field integration and effective Hamiltonian

Using identities that involve only Gaussian integrations, one first proves equality between two partition functions corresponding to two different Hamiltonians. This relation can then be interpreted as resulting from a partial integration over some components of the fields. One infers a sufficient condition for correlated modifications of the propagator and interactions, in a statistical field theory, to leave the partition function invariant.

In what follows, we assume that the field theory is translation invariant. Moreover, all Gaussian two-point functions, or propagators, $\Delta(x-y)$ are such that in the Fourier representation

$$\Delta(x)=\frac{1}{(2\pi)^d}\int \mathrm{d}^d p \, \mathrm{e}^{ipx}\,\tilde{\Delta}(p),$$

$\tilde{\Delta}(p)$ decreases faster than any power for $|p|\to\infty$ so that the Gaussian expectation value of any local polynomials in the field exists.

Partial integration

One first establishes a relation between partition functions corresponding to two different local Hamiltonians in d space dimensions.

The first Hamiltonian depends on a field ϕ and we write it in the form

$$\mathcal{H}_1(\phi) = \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,\phi(x)K_1(x-y)\phi(y) + \mathcal{V}_1(\phi), \tag{16}$$

where K_1 is a positive operator and the functional $\mathcal{V}_1(\phi)$ is expandable in powers of the field ϕ, local and translation invariant. To the explicit quadratic part is associated the propagator Δ_1,

$$\int \mathrm{d}z\,K_1(x-z)\Delta_1(z-y) = \delta(x-y).$$

The second Hamiltonian depends on two fields ϕ_1, ϕ_2 in the form

$$\mathcal{H}(\phi_1,\phi_2) = \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,\left[\phi_1(x)K_2(x-y)\phi_1(y) + \phi_2(x)\mathcal{K}(x-y)\phi_2(y)\right]$$
$$+\,\mathcal{V}_1(\phi_1+\phi_2).$$

Again, we define

$$\int \mathrm{d}z\, K_2(x-z)\Delta_2(z-y) = \delta(x-y), \quad \int \mathrm{d}z\, \mathcal{K}(x-z)\mathcal{D}(z-y) = \delta(x-y).$$

The kernels K_1, K_2 and $\mathcal{K}$ are positive, which is a necessary condition for the field integrals to exist, at least in a perturbative sense. Moreover, the properties of the propagators Δ_1, Δ_2 and $\mathcal{D}$ (thus also positive) ensures the existence of a formal expansion of the field integrals in powers of the interaction $\mathcal{V}_1$.

Then, if $\Delta_1 = \Delta_2 + \mathcal{D} \;\Rightarrow\; K_1 = K_2(K_2+\mathcal{K})^{-1}\mathcal{K}$, the ratio of the partition functions

$$\mathcal{Z}_1 = \int [\mathrm{d}\phi]\, \mathrm{e}^{-\mathcal{H}_1(\phi)} \text{ and } \mathcal{Z}_2 = \int [\mathrm{d}\phi_1\, \mathrm{d}\phi_2]\, \mathrm{e}^{-\mathcal{H}(\phi_1,\phi_2)} \tag{17}$$

does not depend on $\mathcal{V}_1$:

$$\mathcal{Z}_2 = \left(\frac{\det(\mathcal{D}\Delta_2)}{\det \Delta_1}\right)^{1/2} \mathcal{Z}_1\,. \tag{18}$$

Another form of the identity. In what follows, we use the compact notation

$$\int \mathrm{d}x\, \mathrm{d}y\, \phi(x) K(x-y) \phi(y) \equiv (\phi K \phi).$$

We define

$$\mathrm{e}^{-\mathcal{V}_2(\phi)} = (\det \mathcal{D})^{-1/2} \int [\mathrm{d}\varphi] \exp\left[-\tfrac{1}{2}(\varphi \mathcal{K} \varphi) - \mathcal{V}_1(\phi+\varphi)\right], \tag{19}$$

as well as $\mathcal{H}_2(\phi) = \frac{1}{2}(\phi K_2 \phi) + \mathcal{V}_2(\phi)$. Then, the equivalence takes the more interesting form

$$\int [\mathrm{d}\phi]\, \mathrm{e}^{-\mathcal{H}_2(\phi)} = \left(\frac{\det \Delta_2}{\det \Delta_1}\right)^{1/2} \int [\mathrm{d}\phi]\, \mathrm{e}^{-\mathcal{H}_1(\phi)}. \tag{20}$$

The left hand side can be interpreted as resulting from a partial integration over the field ϕ since the propagator $\mathcal{D}$ is positive and, thus, in the sense of operators, $\Delta_2 < \Delta_1$.

Differential form

We now assume that the propagator Δ is a function of a real parameter s: $\Delta \equiv \Delta(s)$. Moreover, $\Delta(s)$ is a smooth function with a negative derivative. We define

$$D(s) = \frac{\mathrm{d}\Delta(s)}{\mathrm{d}s} < 0\,,$$

where $D(s)$ is represented by the kernel $D(s; x-y)$.

For $s < s'$, we identify

$$\Delta_1 = \Delta(s), \quad \Delta_2 = \Delta(s') \quad \text{and thus} \quad \mathcal{D}(s,s') = \Delta(s) - \Delta(s') > 0\,. \quad (21)$$

Similarly,

$$K_1 = K(s) = \Delta^{-1}(s), \quad K_2 = K(s'), \quad \mathcal{K}(s,s') = [\mathcal{D}(s,s')]^{-1} > 0\,. \quad (22)$$

Since the kernels $K(s)$, $\mathcal{K}(s,s')$ are positive, all Gaussian integrals are defined.

Finally, we set

$$\mathcal{V}_1(\phi) = \mathcal{V}(\phi, s), \quad \mathcal{V}_2(\phi) = \mathcal{V}(\phi, s'), \quad \mathcal{H}_1(\phi) = \mathcal{H}(\phi, s), \quad \mathcal{H}_2(\phi) = \mathcal{H}(\phi, s').$$

The equivalence then takes the form

$$\int [\mathrm{d}\phi]\, \mathrm{e}^{-\mathcal{H}(\phi, s')} = \left(\frac{\det \Delta(s')}{\det \Delta(s)}\right)^{1/2} \int [\mathrm{d}\phi]\, \mathrm{e}^{-\mathcal{H}(\phi, s)},$$

where $\mathcal{V}(\phi, s')$ is given by

$$\mathrm{e}^{-\mathcal{V}(\phi, s')} = \big(\det \mathcal{D}(s, s')\big)^{-1/2} \int [\mathrm{d}\varphi] \exp\left[-\tfrac{1}{2}\big(\varphi \mathcal{K}(s, s')\varphi\big) - \mathcal{V}(\phi + \varphi, s)\right]. \tag{23}$$

Differential form

Setting $s' = s + \sigma$, $\sigma > 0$, one expands in powers of $\sigma \to 0$. Identifying the terms of order σ, after some algebraic manipulations one obtains the functional equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{V}(\phi, s) = -\frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\, D(s; x-y)\left[\frac{\delta^2\mathcal{V}}{\delta\phi(x)\delta\phi(y)} - \frac{\delta\mathcal{V}}{\delta\phi(x)}\frac{\delta\mathcal{V}}{\delta\phi(y)}\right]. \tag{24}$$

The equation expresses a sufficient condition for the partition function

$$\mathcal{Z}(s) = (\det\Delta(s))^{-1/2}\int[\mathrm{d}\phi]\,\mathrm{e}^{-\mathcal{H}(\phi,s)} \text{ with}$$
$$\mathcal{H}(\phi, s) = \tfrac{1}{2}(\phi K(s)\phi) + \mathcal{V}(\phi, s), \tag{25}$$

to be independent of the parameter s.

This property relates a modification of the propagator to a modification of the interaction, quite in the spirit of the RG.

Remark.

(i) A sufficient condition for $\mathcal{Z}(s)$ to be independent of s, is that the equation is satisfied as an expectation value with the measure $\mathrm{e}^{-\mathcal{H}(\phi,s)}$. One can thus add to the equation contributions with vanishing expectation value to derive other sufficient conditions (useful for correlation functions).

(ii) Let us set

$$\Sigma(\phi, s) = \mathrm{e}^{-\mathcal{V}(\phi,s)} \,.$$

Then, the functional equation reduces to

$$\frac{\mathrm{d}}{\mathrm{d}s}\Sigma(\phi, s) = -\frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,D(s; x - y)\frac{\delta^2\Sigma(\phi, s)}{\delta\phi(x)\delta\phi(y)}.$$

This a functional heat equation since the kernel $-D$ is positive. One may wonder why we do not consider this equation, which is linear and thus much simpler. The reason is that, in contrast to $\Sigma(\phi, s)$, $\mathcal{V}(\phi, s)$ is a local functional, a property that plays an essential role.

Hamiltonian evolution

From the evolution equation, for $\mathcal{V}(\phi, s)$ one then infers the evolution of the Hamiltonian

$$\mathcal{H}(\phi, s) = \tfrac{1}{2}(\phi K(s)\phi) + \mathcal{V}(\phi, s).$$

Introducing the operator $L(s)$, with kernel $L(s; x-y)$, defined by

$$L(s) \equiv D(s)\Delta^{-1}(s) = \frac{\mathrm{d}\ln\Delta(s)}{\mathrm{d}s}, \tag{26}$$

one finds

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{H}(\phi, s) = {}& -\frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\, D(s; x-y)\left[\frac{\delta^2\mathcal{H}}{\delta\phi(x)\delta\phi(y)} - \frac{\delta\mathcal{H}}{\delta\phi(x)}\frac{\delta\mathcal{H}}{\delta\phi(y)}\right] \\ & - \int \mathrm{d}x\,\mathrm{d}y\,\phi(x)L(s; x-y)\frac{\delta\mathcal{H}}{\delta\phi(y)} + \frac{1}{2}\,\mathrm{tr}\,L(s).\end{aligned} \tag{27}$$

Formal solution

Since the evolution equation is a first-order differential equation in s, the form of the functional $\mathcal{V}(\phi, s)$ for an initial value s_0 of the parameter s determines the solution for all $s \geq s_0$. From the very proof of equation (24), its solution can be easily inferred:

$$\mathrm{e}^{-\mathcal{V}(\phi,s)} = \left(\det \mathcal{D}(s_0, s)\right)^{-1/2} \int [\mathrm{d}\varphi] \exp\left[-\tfrac{1}{2}\left(\varphi \mathcal{K}(s_0, s)\varphi\right) - \mathcal{V}(\phi + \varphi, s_0)\right]. \tag{28}$$

The equation implies that $-\mathcal{V}(\phi, s)$ is the sum of the connected contributions of the diagrams constructed with the propagator $\mathcal{K}^{-1}(s_0, s) = \Delta(s) - \Delta(s_0)$ and interactions $\mathcal{V}(\phi + \varphi, s_0)$.

Only if initially $\mathcal{V}(\phi, s_0)$ is a quadratic form (a Gaussian model) it remains so. However, if one adds, for example, a quartic term, then an infinite number of terms of higher degrees will automatically be generated.

Field renormalization

In order to be able to find RG fixed-point solutions, it is necessary to introduce a field renormalization. To prove the corresponding identities, we set

$$\phi(x) = \sqrt{Z(s)}\phi'(x), \quad \mathcal{H}(\phi, s) = \mathcal{H}'(\phi', s),$$

where $Z(s)$ is an arbitrary differentiable function. We then define

$$\eta(s) = \frac{\mathrm{d} \ln Z(s)}{\mathrm{d}s} \,. \tag{29}$$

One then infers from the Hamiltonian flow equation the modified equation (omitting some constant term)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{H}(\phi, s) = -\frac{1}{2}\int \mathrm{d}x \, \mathrm{d}y \, D(x-y)\left[\frac{\delta^2\mathcal{H}}{\delta\phi(x)\delta\phi(y)} - \frac{\delta\mathcal{H}}{\delta\phi(x)}\frac{\delta\mathcal{H}}{\delta\phi(y)}\right] \\ - \int \mathrm{d}x \, \mathrm{d}y \, \phi(x) L(s; x-y)\frac{\delta\mathcal{H}}{\delta\phi(y)} + \frac{1}{2}\eta(s)\int \mathrm{d}x \, \phi(x)\frac{\delta\mathcal{H}}{\delta\phi(x)} \,. \end{aligned} \tag{30}$$

High-momentum mode integration and RGE

These equations can be applied to a situation where the partial integration over the field corresponds, in the Fourier representation, to a partial integration over its high-momentum modes, which in position space also corresponds to an integration over short-distance degrees of freedom.

In what follows, we specialize Δ to a critical (massless) propagator. A possible deviation from the critical theory is included in $\mathcal{V}(\phi)$.

Cut-off parameter and propagator. In the preceding formalism, we now identify $s \equiv -\ln \Lambda$, where Λ is a large-momentum cut-off, which also represents the inverse of the microscopic scale. A variation of s then corresponds to a dilatation of the parameter Λ.

We then choose a regularized propagator Δ_Λ of the form

$$\Delta_\Lambda(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \mathrm{e}^{-ikx}\, \tilde{\Delta}_\Lambda(k) \quad \text{with} \quad \tilde{\Delta}_\Lambda(k) = \frac{C(k^2/\Lambda^2)}{k^2}\,.$$

The function $C(t)$ is regular for $t \geq 0$, positive, decreasing, goes to 1 for $t \to 0$ and goes to zero faster than any power for $t \to \infty$. It suppresses the field Fourier components corresponding to momenta much higher than Λ in the field integral.

The Fourier transform of the derivative $D_\Lambda(x)$,

$$\tilde{D}_\Lambda(k) = -\Lambda \frac{\partial \tilde{\Delta}_\Lambda(k)}{\partial \Lambda} = \frac{2}{\Lambda^2} C'(k^2/\Lambda^2), \tag{31}$$

has an essential property: it has no pole at $k = 0$ and thus it is not critical.

The function

$$D_\Lambda(x) = -\Lambda \frac{\partial \Delta_\Lambda(x)}{\partial \Lambda} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, \mathrm{e}^{ikx} \, \tilde{D}_\Lambda(k) = \Lambda^{d-2} D_{\Lambda=1}(\Lambda x), \tag{32}$$

thus decays for $|x| \to \infty$ faster than any power if $C(t)$ is smooth, exponentially if $C(t)$ is analytical. The propagator $\mathcal{D}(\Lambda_0, \Lambda) = D_{\Lambda_0} - D_\Lambda$, $\Lambda_0 > \Lambda$, whose inverse now appears in the field integral solving the flow equation, shares this property.

RGE

With these assumptions and definitions, the flow equation for $\mathcal{V}(\phi, \Lambda)$ becomes

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \mathcal{V}(\phi, \Lambda) = \frac{1}{2} \int \mathrm{d}^d x \, \mathrm{d}^d y \, D_\Lambda(x-y) \left[\frac{\delta^2 \mathcal{V}}{\delta\phi(x)\delta\phi(y)} - \frac{\delta \mathcal{V}}{\delta\phi(x)} \frac{\delta \mathcal{V}}{\delta\phi(y)} \right]. \quad (33)$$

This equation being exact, one uses also the terminology exact renormalization group.

Remarks.

Since $D_\Lambda(x)$ decreases faster than any power, if $\mathcal{V}(\phi)$ is initially local, it remains local, a property that becomes more apparent when one expands the equation in powers of ϕ.

The equation differs from the abstract RGE by the property that the scale parameter Λ appears explicitly through the function D_Λ. We shall later eliminate this explicit dependence.

Field Fourier components

In terms of the Fourier components $\tilde{\phi}(k)$ of the field,

$$\phi(x) = \int \mathrm{d}^d k \; \mathrm{e}^{ikx} \, \tilde{\phi}(k) \;\Leftrightarrow\; \tilde{\phi}(k) = \int \frac{\mathrm{d}^d x}{(2\pi)^d} \, \mathrm{e}^{-ikx} \, \phi(x),$$

the equation becomes

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \mathcal{V}(\phi, \Lambda) = \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{D}_\Lambda(k) \left[\frac{\delta^2 \mathcal{V}}{\delta\tilde{\phi}(k) \delta\tilde{\phi}(-k)} - \frac{\delta \mathcal{V}}{\delta\tilde{\phi}(k)} \frac{\delta \mathcal{V}}{\delta\tilde{\phi}(-k)} \right]. \quad (34)$$

In the equation, locality translates into regularity of the Fourier components. If the coefficients of the expansion of $\mathcal{V}(\phi, \Lambda)$ in powers of $\tilde{\phi}$, after factorization of the δ-function, are regular functions for an initial value of $\Lambda = \Lambda_0$, they remain for $\Lambda < \Lambda_0$ because $\tilde{D}_\Lambda(k)$ is a regular function of k.

Finally, the flow equation for the complete Hamiltonian expressed in terms of Fourier components,

$$\mathcal{H}(\phi, \Lambda) = \frac{1}{2}(2\pi)^d \int \mathrm{d}^d k \, \tilde{\phi}(k) \tilde{\Delta}_\Lambda^{-1}(k) \tilde{\phi}(-k) + \mathcal{V}(\phi, \Lambda),$$

takes the form (omitting the term independent of ϕ)

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \mathcal{H}(\phi, \Lambda) = \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{D}_\Lambda(k) \left[\frac{\delta^2 \mathcal{H}}{\delta\tilde{\phi}(k) \delta\tilde{\phi}(-k)} - \frac{\delta \mathcal{H}}{\delta\tilde{\phi}(k)} \frac{\delta \mathcal{H}}{\delta\tilde{\phi}(-k)} \right]$$
$$+ \int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{L}_\Lambda(k) \frac{\delta \mathcal{H}}{\delta\tilde{\phi}(k)} \tilde{\phi}(k) \tag{35}$$

with (equation (26))

$$\tilde{L}_\Lambda(k) = \tilde{D}_\Lambda(k) / \tilde{\Delta}_\Lambda(k). \tag{36}$$

In equation (35), we have implicitly subtracted both from $\mathcal{H}(\phi, \Lambda)$ and from the equation their values at $\phi = 0$.

RGE: Standard form

After a field renormalization, required to be able to reach non-Gaussian fixed points, the RGE take the form (30) with $s = -\ln \Lambda$:

$$\begin{aligned}
&\Lambda \frac{\partial}{\partial \Lambda} \mathcal{H}(\phi, \Lambda) \\
&\quad = \frac{1}{2} \int \mathrm{d}^d x \, \mathrm{d}^d y \, D_\Lambda(x-y) \left[\frac{\delta^2 \mathcal{H}}{\delta \phi(x) \delta \phi(y)} - \frac{\delta \mathcal{H}}{\delta \phi(x)} \frac{\delta \mathcal{H}}{\delta \phi(y)} \right] \\
&\qquad + \int \mathrm{d}^d x \, \mathrm{d}^d y \, \phi(x) L_\Lambda(x-y) \frac{\delta \mathcal{H}}{\delta \phi(y)} + \frac{\eta}{2} \int \mathrm{d}^d x \, \phi(x) \frac{\delta \mathcal{H}}{\delta \phi(x)} \, . \qquad (37)
\end{aligned}$$

The function η is *a priori* arbitrary but with one restriction, it must depend on Λ only through $\mathcal{H}(\phi, \Lambda)$. It must be adjusted to ensure the existence of fixed points.

The latter equation does not have a stationary Markovian form since D_Λ and L_Λ depend explicitly on Λ:

$$L_\Lambda(x) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \ \mathrm{e}^{ikx} \, \tilde{L}_\Lambda(k) = \Lambda^d L_1(x), \quad D_\Lambda(x) = \Lambda^{d-2} D_1(\Lambda x).$$

To eliminate this dependence, we perform a Gaussian renormalization of the form $\phi \mapsto \phi'$ with

$$\phi'(x) = \Lambda^{(2-d)/2} \phi(x/\Lambda).$$

In what follows, we omit the primes. Moreover, we introduce the dilatation parameter $\lambda = \Lambda_0/\Lambda$ that relates the initial scale Λ_0 to the running scale Λ and thus

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} = -\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \, .$$

Then, the RGE take a form consistent with the general RG flow equation:

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{H}(\phi, \lambda) = \mathcal{T}\left[\mathcal{H}(\phi, \lambda)\right],$$

with

$$\begin{aligned} \mathcal{T}\left[\mathcal{H}\right] = &-\frac{1}{2} \int \mathrm{d}^d x \, \mathrm{d}^d y \, D(x-y) \left[\frac{\delta^2 \mathcal{H}}{\delta\phi(x)\delta\phi(y)} - \frac{\delta \mathcal{H}}{\delta\phi(x)} \frac{\delta \mathcal{H}}{\delta\phi(y)}\right] \\ &- \int \mathrm{d}^d x \, \frac{\delta \mathcal{H}}{\delta\phi(x)} \left[\frac{1}{2}(d-2+\eta) + \sum_\mu x^\mu \frac{\partial}{\partial x^\mu}\right] \phi(x) \\ &- \int \mathrm{d}^d x \, \mathrm{d}^d y \, L(x-y) \frac{\delta \mathcal{H}}{\delta\phi(x)} \phi(y). \end{aligned} \tag{38}$$

This is the form more suitable for looking for fixed points. At a fixed point, the right hand side vanishes for a suitably chosen renormalization of the field ϕ, which determines the value of the exponent η.

Expansion in powers of the field: RGE in component form

If one expands $\mathcal{H}(\phi,\lambda)$ (and then equation (38)) in powers of $\phi(x)$,

$$\mathcal{H}(\phi,\lambda)=\sum_{n=0}\frac{1}{n!}\int\prod_i \mathrm{d}^d x_i\,\phi(x_i)\mathcal{H}^{(n)}(x_1,\ldots,x_n;\lambda),$$

one derives equations for the components. For $n\neq 2$ ($D_1\equiv D$),

$$\begin{aligned}\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{H}^{(n)}(x_i;\lambda)&=\left(\frac{1}{2}n(d+2-\eta)+\sum_{j,\mu}x_j^\mu\frac{\partial}{\partial x_j^\mu}\right)\mathcal{H}^{(n)}(x_i;\lambda)\\&-\frac{1}{2}\int\mathrm{d}^d x\,\mathrm{d}^d y\,D(x-y)\Big[\mathcal{H}^{(n+2)}(x_1,x_2,\ldots,x_n,x,y;\lambda)\\&-\sum_I\mathcal{H}^{(l+1)}(x_{i_1},\ldots,x_{i_l},x;\lambda)\mathcal{H}^{(n-l+1)}(x_{i_{l+1}},\ldots,x_{i_n},y;\lambda)\Big],\end{aligned}$$

where the set $I\equiv\{i_1,i_2,\ldots,i_l\}$ describes all distinct subsets of $\{1,2,\ldots,n\}$.

In the Fourier representation, the equations take the form

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}\tilde{\mathcal{H}}^{(n)}(p_i;\lambda) = \left(d - \frac{1}{2}n(d-2+\eta) - \sum_{j,\mu} p_j^{\mu}\frac{\partial}{\partial p_j^{\mu}}\right)\tilde{\mathcal{H}}^{(n)}(p_i;\lambda)$$
$$- \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\tilde{D}(k)\tilde{\mathcal{H}}^{(n+2)}(p_1, p_2, \ldots, p_n, k, -k;\lambda)$$
$$+ \frac{1}{2}\sum_{I}\tilde{D}(p_0)\tilde{\mathcal{H}}^{(l+1)}(p_{i_1}, \ldots, p_{i_l}, p_0;\lambda)\tilde{\mathcal{H}}^{(n-l+1)}(p_{i_{l+1}}, \ldots, p_{i_n}, -p_0), \quad (39)$$

where the momentum p_0 is determined by total momentum conservation.

For $n = 2$, one finds an additional term in the equation:

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathcal{H}^{(2)}(x_1;\lambda) = \left(d + 2 - \eta + \sum_{\mu} x_1^{\mu} \frac{\partial}{\partial x_1^{\mu}} \right) \mathcal{H}^{(2)}(x_1;\lambda)$$

$$- \frac{1}{2} \int \mathrm{d}^d x \, \mathrm{d}^d y \, D(x-y) \left[\mathcal{H}^{(4)}(x_1, 0, , x, y; \lambda) - 2\mathcal{H}^{2)}(x - x_1;\lambda) \mathcal{H}^{(2)}(y;\lambda) \right]$$

$$- 2 \int \mathrm{d}^d y \, L(x_1 - y) \mathcal{H}^{(2)}(y;\lambda).$$

One verifies explicitly that, except in the Gaussian example, all functions $\mathcal{H}^{(n)}$ are coupled. In the Fourier representation,

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \tilde{\mathcal{H}}^{(2)}(p;\lambda) = \left(2 - \eta - \sum_{\mu} p^{\mu} \frac{\partial}{\partial p^{\mu}} \right) \tilde{\mathcal{H}}^{(2)}(p;\lambda) - 2\tilde{L}(p) \tilde{\mathcal{H}}^{(2)}(p;\lambda)$$

$$- \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{D}(k) \tilde{\mathcal{H}}^{(4)}(p, -p, k, -k; \lambda) + \tilde{D}(p) \left(\tilde{\mathcal{H}}^{(2)}(p;\lambda) \right)^2 . \quad (40)$$

Perturbative solution

The flow equations is any of the different forms, can be solved perturbatively. One first specifies the form of the Hamiltonian at the initial scale $\lambda = 1$, for instance,

$$\mathcal{H}(\phi) = \mathcal{H}_{\mathrm{G}}(\phi) + \frac{g}{4!} \int \mathrm{d}^d x \, \phi^4(x), \quad u \geq 0 \, .$$

In the finite form (28) of the RGE, one can then expand the field integral in powers of the constant g. Alternatively, in the differential form, one first expands in powers of the field ϕ. This leads to the infinite set of coupled integro-differential equations (39,40) that one can integrate perturbatively with the Ansatz that the terms in $\mathcal{H}(\phi; \lambda)$ quadratic and quartic in ϕ are of order g and the general term of degree $2n$ of order g^{n-1}.

It is also possible to further expand the equations in powers of $\varepsilon = 4 - d$ and to look for fixed points. The results of the perturbative RG are then recovered.

Fixed points and local flow

Once a fixed point $\mathcal{H}_*$ has been determined, one can expand equation (38) in the vicinity of the fixed point:

$$\mathcal{H}(\lambda) = \mathcal{H}_* + E(\lambda).$$

One then obtains the linearized RGE

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} E(\lambda) = \mathcal{L}_* E(\lambda),$$

where the linear operator $\mathcal{L}_*$, after an integration by parts, takes the form

$$\begin{aligned}\mathcal{L}_* = &\int \mathrm{d}^d x\, \phi(x) \left[\frac{1}{2}(d+2-\eta) + \sum_\mu x^\mu \frac{\partial}{\partial x^\mu}\right] \frac{\delta}{\delta\phi(x)} \\ &+ \int \mathrm{d}^d x\, \mathrm{d}^d y\, D(x-y) \left[-\frac{1}{2}\frac{\delta^2}{\delta\phi(x)\delta\phi(y)} + \frac{\delta\mathcal{H}_*}{\delta\phi(x)}\frac{\delta}{\delta\phi(y)}\right] \\ &- \int \mathrm{d}^d x\, \mathrm{d}^d y\, L(x-y)\phi(x) \frac{\delta}{\delta\phi(y)}. \end{aligned} \tag{41}$$

We denote by ℓ the eigenvalues and $E_\ell \equiv E_\ell(\lambda = 1)$ the eigenvectors of $\mathcal{L}_*$:

$$\mathcal{L}_* E_\ell = \ell E_\ell \,, \tag{42}$$

and thus

$$E_\ell(\lambda) = \lambda^\ell E_\ell(1).$$

Equation (42) can be written more explicitly in terms of the components $\tilde{E}_\ell^{(n)}(p_i)$ of E_ℓ as

$$\begin{aligned}
&\ell \tilde{E}_\ell^{(n)}(p_i) \\
&\quad = \left(d - \frac{1}{2} n (d - 2 + \eta) - \sum_j \tilde{D}(p_j) \tilde{\Delta}^{-1}(p_j) - \sum_{j,\mu} p_j^\mu \frac{\partial}{\partial p_j^\mu} \right) \tilde{E}_\ell^{(n)}(p_i) \\
&\qquad - \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{D}(k) \tilde{E}_\ell^{(n+2)}(p_1, p_2, \ldots, p_n, k, -k) \\
&\qquad + \sum_I \tilde{D}(p_0) \tilde{E}_\ell^{(l+1)}(p_{i_1}, \ldots, p_{i_l}, p_0) \tilde{\mathcal{H}}_*^{(n-l+1)}(p_{i_{l+1}}, \ldots, p_{i_n}, -p_0).
\end{aligned} \tag{43}$$

Gaussian fixed point

At the Gaussian fixed point, the Hamiltonian is quadratic and $\eta = 0$. The local flow at the fixed point is then governed by the operator

$$\mathcal{L}_* = \int \mathrm{d}^d x\, \phi(x) \left(\frac{1}{2}(d+2) + \sum_\mu x^\mu \frac{\partial}{\partial x^\mu} \right) \frac{\delta}{\delta \phi(x)} - \frac{1}{2} \int \mathrm{d}^d x\, \mathrm{d}^d y\, D(x-y) \frac{\delta^2}{\delta \phi(x) \delta \phi(y)} .$$

The eigenvectors are obtained by choosing $\mathcal{H}^{(n)}$ vanishing for all n larger than some value N. The coefficient of the term of degree N in ϕ then satisfies the homogeneous equation

$$\ell \tilde{E}_\ell^{(N)}(p_i) = \left(d - \frac{1}{2} N (d-2) - \sum_{j,\mu} p_j^\mu \frac{\partial}{\partial p_j^\mu} \right) \tilde{E}_\ell^{(N)}(p_i).$$

This is an eigenvalue equation identical to the one obtained in the perturbative RG. The solutions are homogeneous polynomials in the momenta.

If r is the degree in the variables p_i, the eigenvalue is given by

$$\ell = d - \tfrac{1}{2} N(d-2) - r \, .$$

The other coefficients $\mathcal{H}^{(n)}$, $n < N$, are then entirely determined by the equations

$$\left(\frac{1}{2}(N-n)(d-2) + r - \sum_{j,\mu} p_j^\mu \frac{\partial}{\partial p_j^\mu} \right) \tilde{E}_\ell^{(n)}(p_i) = \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{D}(k) \tilde{E}_\ell^{(n+2)}(p_1, p_2, \ldots, p_n, k, -k). \tag{44}$$

One might be surprised by the occurrence of these additional terms. In fact, one can verify that if one sets

$$E(\phi) = \exp\left[-\frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, \Delta(x-y)\frac{\delta^2}{\delta\phi(x)\delta\phi(y)}\right]\Omega(\phi),$$

the functional $\Omega(\phi)$ satisfies the simpler eigenvalue equation

$$\ell\,\Omega_\ell(\phi) = \int \mathrm{d}^d x\, \phi(x)\left[\frac{1}{2}(d+2) + \sum_\mu x^\mu \frac{\partial}{\partial x^\mu}\right]\frac{\delta}{\delta\phi(x)}\Omega_\ell(\phi),$$

whose solutions are the simple monomials $O_{n,k}(\phi, x)$ found in the perturbative analysis of the stability of the Gaussian fixed point. For example, for

$$\Omega_\ell(\phi) = \int \mathrm{d}^d x\, \phi^m(x),$$

after some algebra, one verifies

$$\ell = d - m(d-2)/2\,.$$

The linear operator that transforms $\Omega(\phi)$ into $E(\phi)$,

$$\exp\left[-\frac{1}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y\,\Delta(x-y)\frac{\delta^2}{\delta\phi(x)\delta\phi(y)}\right],$$

replaces all monomials in ϕ that contribute to Ω by their normal products.

We recall that the normal product of a $E^{(N)}(\phi)$ of a monomial of degree N in ϕ is a polynomial with the same term of order N and is such that, for all $n < N$, the Gaussian correlation functions

$$\left\langle \prod_{i=1}^{n}\phi(x_i)E^{(N)}(\phi)\right\rangle$$

with the measure $\mathrm{e}^{-\mathcal{H}_*}$ vanish. Let us point out that the definition of normal products depends explicitly on the choice of the Gaussian measure.

Beyond the Gaussian model: perturbative solution

In dimension 4 (and then in $d = 4 - \varepsilon$), a non-trivial theory can be defined and parametrized, for example, in terms of $g(\lambda)$, the value of $\tilde{\mathcal{H}}^{(4)}(p_i, \lambda)$ at $p_1 = \cdots = p_4 = 0$:

$$g(\lambda) \equiv \tilde{\mathcal{H}}^{(4)}(p_i = 0, \lambda). \tag{45}$$

One then introduces the function $\beta(g)$ defined by

$$\lambda \frac{\mathrm{d}g}{\mathrm{d}\lambda} = -\beta\big(g(\lambda)\big). \tag{46}$$

This allows substituting in the left hand side of the flow equation

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} = \beta(g) \frac{\mathrm{d}}{\mathrm{d}g}.$$

One then solves perturbatively the flow equations, with appropriate boundary conditions.

All other interactions are then determined perturbatively as functions of g, under the assumption that they are at least of order g^2. They become implicit functions of λ through $g(\lambda)$. One suppresses in this way all corrections due to irrelevant operators, keeping only the contributions due to the marginal operator. The Hamiltonian flow, like in the perturbative renormalization group is reduced the flow of $g(\lambda)$, but the fixed point Hamiltonian is much more complicate, since all subleading corrections to the leading behaviour are suppressed.

The function $\eta(g)$ is determined by the condition

$$\frac{\partial}{\partial p^2}\tilde{\mathcal{H}}^{(2)}(p;g)\bigg|_{p=0} = 1 \;\Rightarrow\; \frac{\partial}{\partial p^2}\tilde{\mathcal{V}}^{(2)}(p;g)\bigg|_{p=0} = 0\,, \qquad (47)$$

which suppresses the redundant operator that corresponds to a change of normalization of the field.

The two conditions (45), (47) replace the renormalization conditions of the usual renormalization theory.

Final remarks

In practice, the FRG has been used as a starting point for various non-perturbative approximations, reducing the functional equations to partial differential equations. Quite interesting results have been obtained. The main problem is that none of these approximation scheme has a systematic character, or when a systematic method is claimed, the next approximation is out of reach.

A further practical remark is that since the effective interaction $\mathcal{V}(\phi)$ appears as the generating functional of some kind of connected Feynman diagrams, an additional simplification has been obtained by introducing its Legendre transform, which contains only one-line irreducible diagrams.