Dynamics hedging of CDO tranches in Markovian set-ups

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Recent Advancements in the Theory and Practice of Credit Derivatives
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Presentation related to the papers:

- *Hedging default risks of CDOs in Markovian contagion models* (2008), to appear in *Quantitative Finance*, with Jean-Paul Laurent and Jean-David Fermanian
- *Hedging CDO tranches in a Markovian environment* (2009), book chapter with Monique Jeanblanc and Jean-Paul Laurent
In this presentation, we address the hedging issue of CDO tranches in a market model where pricing is connected to the cost of the hedge.

In credit risk market, models that connect pricing to the cost of the hedge have been studied quite lately.

Discrepancies with the interest rate or the equity derivative market.

Model to be presented is not new, require some stringent assumptions, but the hedging can be fully described in a dynamical way.
Presentation related to the papers:

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Contents

1 Theoretical framework

2 Homogeneous Markovian contagion model

3 Empirical results
Compared with previous presentations of this research

- The theoretical framework about replication of loss derivatives is presented in more details
- We provide a comparison analysis of hedging ratios computed in alternative models and using different methods
- We propose a natural extension of the model where individual deltas can be discriminated by the level of CDS spreads

- The replication of CDO tranches has also been investigated in a similar framework by Bielecki, Jeanblanc and Rutkowski (2007), Frey and Backhaus (2007, 2008)
Default times

- $n$ credit references
- $\tau_1, \ldots, \tau_n$: default times defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$
- $N^i_t = 1\{\tau_i \leq t\}, i = 1, \ldots, n$: default indicator processes
- $\mathcal{H}^i = (\mathcal{H}^i_t)_{t \geq 0}, \mathcal{H}^i_t = \sigma(N^i_s, s \leq t), i = 1, \ldots, n$: natural filtration of $N^i$
- $\mathcal{H} = \mathcal{H}^1 \vee \cdots \vee \mathcal{H}^n$: global filtration of default times
Default times

- No simultaneous defaults: \( \mathbb{P}(\tau_i = \tau_j) = 0, \forall i \neq j \)

- Default times admit \( \mathbb{H} \)-adapted default intensities

  - For any \( i = 1, \ldots, n \), there exists a non-negative \( \mathbb{H} \)-adapted process \( \alpha^{i,\mathbb{P}} \) such that
    \[
    M^{i,\mathbb{P}}_t := N^i_t - \int_0^t \alpha^{i,\mathbb{P}}_s ds
    \]
    is a \( (\mathbb{P}, \mathbb{H}) \)-martingale.

  - \( \alpha^{i,\mathbb{P}}_t = 0 \) on the set \( \{ t > \tau_i \} \)

  - \( M^{i,\mathbb{P}}, \ i = 1, \ldots, n \) will be referred to as the fundamental martingales
Market Assumption

- Instantaneous digital CDS are traded on the names $i = 1, \ldots, n$
- Instantaneous digital CDS on name $i$ at time $t$ is a stylized bilateral agreement
  - Offer credit protection on name $i$ over the short period $[t, t + dt]$
  - Buyer of protection receives 1 monetary unit at default of name $i$
  - In exchange for a fee equal to $\alpha_t^i dt$

\[
\begin{align*}
0 & \quad \quad 1 - \alpha_t^i dt : \text{ default of } i \text{ between } t \text{ and } t + dt \\
& \quad \quad -\alpha_t^i dt : \text{ survival of name } i
\end{align*}
\]

- Cash-flow at time $t + dt$ (buy protection position) : $dN_t^i - \alpha_t^i dt$
- $\alpha_t^i = 0$ on the set $\{t > \tau_i\}$ (Contrat is worthless)
Market Assumption

- **Credit spreads are driven by defaults**: $\alpha^1, \ldots, \alpha^n$ are $\mathbb{H}$-adapted processes
- **Payoff of a self-financed strategy**

\[
V_0 e^{rT} + \sum_{i=1}^n \int_0^T \delta^i e^{r(T-s)} \left( dN^i_s - \alpha^i_s ds \right).
\]

- $r$: default-free interest rate
- $V_0$: initial investment
- $\delta^i$, $i = 1, \ldots, n$, $\mathbb{H}$-predictable process
Theoretical framework
Homogeneous Markovian contagion model
Empirical results

Hedging and martingale representation theorem

Theorem (Predictable representation theorem)

Let \( A \in \mathcal{H}_T \) be a \( \mathbb{P} \)-integrable random variable. Then, there exists \( \mathbb{H} \)-predictable processes \( \theta^i, i = 1, \ldots, n \) such that

\[
A = \mathbb{E}_\mathbb{P}[A] + \sum_{i=1}^n \int_0^T \theta^i_s (dN^i_s - \alpha^i_s, \mathbb{P} \, ds)
\]

\[
= \mathbb{E}_\mathbb{P}[A] + \sum_{i=1}^n \int_0^T \theta^i_s dM^i_s, \mathbb{P}
\]

and \( \mathbb{E}_\mathbb{P} \left( \int_0^T |\theta^i_s| \alpha^i_s, \mathbb{P} \, ds \right) < \infty \).
Theorem (Predictable representation theorem)

Let $A \in \mathcal{H}_T$ be a $\mathbb{Q}$-integrable random variable. Then, there exists $\mathbb{H}$-predictable processes $\hat{\theta}^i_s$, $i = 1, \ldots, n$ such that

\[
A = \mathbb{E}[A] + \sum_{i=1}^{n} \int_{0}^{T} \hat{\theta}^i_s \left( dN^i_s - \alpha^i_s ds \right)
\]

\[
= \mathbb{E}[A] + \sum_{i=1}^{n} \int_{0}^{T} \hat{\theta}^i_s dM^i_s
\]

and $\mathbb{E} \left( \int_{0}^{T} |\theta^i_s| \alpha^i_s, \mathbb{P} ds \right) < \infty$. 

Dynamics hedging of CDO tranches in Markovian set-ups
Building a change of probability measure

Describe what happens to default intensities when the original probability is changed to an equivalent one.

From the PRT, any Radon-Nikodym density $\zeta$ (strictly positive $(\mathbb{P}, \mathbb{H})$-martingale with expectation equal to 1) can be written as

$$d\zeta_t = \zeta_t - \sum_{i=1}^{n} \pi^i_t dM^i_t, \mathbb{P}, \zeta_0 = 1$$

where $\pi^i, i = 1, \ldots, n$ are $\mathbb{H}$-predictable processes.
Conversely, the (unique) solution of the latter SDE is a local martingale (Doléans-Dade exponential)

\[ \zeta_t = \exp \left( - \sum_{i=1}^{n} \int_0^t \pi_s^i \xi_s^i \, ds \right) \prod_{i=1}^{n} (1 + \pi_{\tau_i}^i)^{N_{\tau_i}} \]

- The process \( \zeta \) is non-negative if \( \pi_i > -1 \), for \( i = 1, \ldots, n \)
- The process \( \zeta \) is a true martingale if \( \mathbb{E}_P[\zeta_t] = 1 \) for any \( t \) or if \( \pi_i \) is bounded, for \( i = 1, \ldots, n \)
Theorem (Change of probability measure)

Define the probability measure $Q$ as

$$dQ|_{\mathcal{H}_t} = \zeta_t dP|_{\mathcal{H}_t}.$$  

where

$$\zeta_t = \exp \left( - \sum_{i=1}^{n} \int_0^t \pi_i^s \alpha_{i,P}^s ds \right) \prod_{i=1}^{n} (1 + \pi_i^s \tau_i^s)^{N_i^s}$$

Then, for any $i = 1, \ldots, n$, the process

$$M_t^i := M_t^{i,P} - \int_0^t \pi_i^s \alpha_{i,P}^s ds = N_t^i - \int_0^t (1 + \pi_i^s) \alpha_{i,P}^s ds$$

is a $Q$-martingale. In particular, the $(Q, \mathbb{H})$-intensity of $\tau_i$ is $(1 + \pi_t^i) \alpha_t^{i,P}$. 

Hedging and martingale representation theorem

Dynamics hedging of CDO tranches in Markovian set-ups
From the absence of arbitrage opportunity

\[ \{ \alpha^i_t > 0 \} \Rightarrow_{\text{P-a.s.}} \{ \alpha^i_{t,P} > 0 \} \]

For any \( i = 1, \ldots, n \), the process \( \hat{\pi}^i \) defined by:

\[
\hat{\pi}^i_t = \left( \frac{\alpha^i_t}{\alpha^i_{t,P}} - 1 \right) (1 - N^i_{t-})
\]

is an \( \mathbb{H} \)-predictable process such that \( \hat{\pi}^i > -1 \)

The process \( \zeta \) defined with \( \pi^1 = \hat{\pi}^1, \ldots, \pi^n = \hat{\pi}^n \) is an admissible Radon-Nikodym density

Under \( \mathcal{Q} \), credit spreads \( \alpha^1, \ldots, \alpha^n \) are exactly the intensities of default times
Hedging and martingale representation theorem

- The predictable representation theorem also holds under $Q$
- In particular, if $A$ is an $\mathcal{H}_T$ measurable payoff, then there exists $\mathbb{H}$-predictable processes $\hat{\theta}^i$, $i = 1, \ldots, n$ such that
  \[
  A = \mathbb{E}_Q [A | \mathcal{H}_t] + \sum_{i=1}^{n} \int_t^T \hat{\theta}^i_s dM_s^i.
  \]
- Starting from $t$ the claim $A$ can be replicated using the self-financed strategy with
  - the initial investment $V_t = \mathbb{E}_Q \left[ e^{-r(T-t)}A | \mathcal{H}_t \right]$ in the savings account
  - the holding of $\delta^i_s = \hat{\theta}^i_s e^{-r(T-s)}$ for $t \leq s \leq T$ and $i = 1, \ldots, n$ in the instantaneous CDS
- As there is no charge to enter a CDS, the replication price of $A$ at time $t$ is $V_t = \mathbb{E}_Q \left[ e^{-r(T-t)}A | \mathcal{H}_t \right]$
Hedging and martingale representation theorem

- $A$ depends on the default indicators of the names up to time $T$
  - includes the cash-flows of CDO tranches or basket credit default swaps, given deterministic recovery rates

- In principle, the dynamics of a traditional CDS can also be described in terms of the dynamics of instantaneous CDS

- Can be used to replicate a CDO tranche with traditional CDS
  - Involve the inversion of a linear system
Hedging and martingale representation theorem

- Risk-neutral measure can be explicitly constructed
  - We exhibit a continuous change of probability measure

- Predictable representation theorem implies completeness of the credit market
  - Perfect replication of claims which depend only upon the default history with CDS on underlying names and default-free asset
  - Provide the replication price at time $t$

- But does not provide any practical way of constructing hedging strategies

- Need of a Markovian assumption to effectively compute hedging strategies
Markovian contagion model

- Pre-default intensities only depend on the current status of defaults

\[ \alpha^i_t = \tilde{\alpha}^i(t, N_t^1, \ldots, N_t^n) 1_{t<\tau_i}, \ i = 1, \ldots, n \]


\[ \tilde{\alpha}^i(t, N_t^1, \ldots, N_t^n) = a_i + \sum_{j \neq i} b_{i,j} N_t^j \]

- Ex: Lopatin (2008)

\[ \tilde{\alpha}^i(t, N_t) = a_i(t) + b_i(t)f(t, N_t) \]

- Connection with continuous-time Markov chains
  - \((N_t^1, \ldots, N_t^n)\) Markov chain with possibly \(2^n\) states
  - Default times follow a multivariate phase-type distribution
Homogeneous Markovian contagion model

- Pre-default intensities only depend on the current number of defaults
- All names have the same pre-default intensities

\[ \alpha^i_t = \tilde{\alpha}(t, N_t) \mathbf{1}_{t<\tau_i}, \; i = 1, \ldots, n \]

where

\[ N_t = \sum_{i=1}^{n} N^i_t \]

- The model is also referred to as the local intensity model
Homogeneous Markovian contagion model

- No simultaneous default, the intensity of $N_t$ is equal to
  \[ \lambda(t, N_t) = (n - N_t)\tilde{\alpha}(t, N_t) \]

- $N_t$ is a continuous-time Markov chain (pure birth process) with generator matrix:
  \[
  \Lambda(t) = \begin{pmatrix}
  -\lambda(t, 0) & \lambda(t, 0) & 0 & 0 \\
  0 & -\lambda(t, 1) & \lambda(t, 1) & 0 \\
  0 & 0 & -\lambda(t, n-1) & \lambda(t, n-1) \\
  0 & 0 & 0 & 0
  \end{pmatrix}
  \]

- Model involves as many parameters as the number of names
Homogeneous Markovian contagion model

- **Replication price of a European type payoff**

\[
V(t, k) = \mathbb{E}_Q \left[ e^{-r(T-t)} \Phi(N_T) \mid N_t = k \right]
\]

- \( V(t, k), \ k = 0, \ldots, n - 1 \) solve the backward Kolmogorov differential equations:

\[
\frac{\delta V(t, k)}{\delta t} = rV(t, k) - \lambda(t, k) \left( V(t, k + 1) - V(t, k) \right)
\]

Homogeneous Markovian contagion model

- **Computation of credit deltas...**
  - $V(t, N_t)$, price of a CDO tranche (European type payoff)
  - $V^I(t, N_t)$, price of the CDS index (European type payoff)

$$ V(t, N_t) = \mathbb{E}_Q \left[ e^{-r(T-t)} \Phi(N_T) \mid N_t \right] $$

$$ V^I(t, N_t) = \mathbb{E}_Q \left[ e^{-r(T-t)} \Phi^I(N_T) \mid N_t \right] $$

- Using standard Itô’s calculus

$$ dV(t, N_t) = \left( V(t, N_t) - \delta^I(t, N_t)V^I(t, N_t) \right) rdt + \delta^I(t, N_t)dV^I(t, N_t) $$

where

$$ \delta^I(t, N_t) = \frac{V(t, N_t + 1) - V(t, N_t)}{V^I(t, N_t + 1) - V^I(t, N_t)}. $$

- Perfect replication with the index and the risk-free asset

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**Theoretical framework**

Homogeneous Markovian contagion model

**Empirical results**

Dynamics hedging of CDO tranches in Markovian set-ups
Pricing and hedging in a binomial tree

- Binomial tree: discrete version of the homogeneous contagion model

\[ \Lambda(t) = \begin{pmatrix} -\lambda(t,0) & \lambda(t,0) & 0 & 0 \\ 0 & -\lambda(t,1) & \lambda(t,1) & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & -\lambda(t,n-1) & \lambda(t,n-1) \end{pmatrix} \]

- Calibration of loss intensities \( \lambda(t, k) \) on a loss surface by forward induction

- CDO tranches and index price computed by backward induction
Empirical results

- **Calibration of loss intensities**: Number of names: 125, risk-free interest rate: $r = 3\%$, recovery rate: $R = 40\%$, 5Y credit spreads: 20bps

<table>
<thead>
<tr>
<th>[0-3%]</th>
<th>[0-6%]</th>
<th>[0-9%]</th>
<th>[0-12%]</th>
<th>[0-22%]</th>
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<tbody>
<tr>
<td>18%</td>
<td>28%</td>
<td>36%</td>
<td>42%</td>
<td>58%</td>
</tr>
</tbody>
</table>

- **Time-homogeneous intensities**: $\lambda(t, k) = \lambda(k), k = 0, \ldots, 125$

- **Comparison with loss intensities calibrated on a flat correlation structure**

Dynamics hedging of CDO tranches in Markovian set-ups
Empirical results

- Dynamics of CDS index spreads in the Markov chain

<table>
<thead>
<tr>
<th>Nb Defaults</th>
<th>Weeks</th>
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<tbody>
<tr>
<td></td>
<td>0</td>
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<tr>
<td>0</td>
<td>20</td>
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<tr>
<td>1</td>
<td>0</td>
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<tr>
<td>2</td>
<td>0</td>
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<tr>
<td>3</td>
<td>0</td>
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<tr>
<td>4</td>
<td>0</td>
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<td>5</td>
<td>0</td>
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<tr>
<td>6</td>
<td>0</td>
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<tr>
<td>7</td>
<td>0</td>
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<td>8</td>
<td>0</td>
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<tr>
<td>9</td>
<td>0</td>
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<tr>
<td>10</td>
<td>0</td>
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</tbody>
</table>

- Explosive behavior associated with upward base correlation curve
Empirical results

- Dynamics of credit deltas **equity tranche** [0, 3%]

<table>
<thead>
<tr>
<th>Nb Defaults</th>
<th>Outstanding Nominal</th>
<th>Weeks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>3.00%</td>
<td>0.541</td>
</tr>
<tr>
<td>1</td>
<td>2.52%</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2.04%</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1.56%</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1.08%</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0.60%</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0.12%</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0.00%</td>
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</table>

- Deltas are between 0 and 1
- Gradually decrease with the number of defaults (concave payoff)
- Increase with time (consistent with a decrease of time value)
Empirical results

- **Market and theoretical deltas at inception**
- Market deltas computed under the Gaussian copula model
  - Uniform bump of index spreads
  - Market delta = Change in PV of the tranche / Change in PV of the CDS index
  - Base correlation is unchanged when shifting spreads
  - Standard way of computing CDS index hedges in trading desks

<table>
<thead>
<tr>
<th>Tranches</th>
<th>[0-3%]</th>
<th>[3-6%]</th>
<th>[3-9%]</th>
<th>[9-12%]</th>
<th>[12-22%]</th>
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</thead>
<tbody>
<tr>
<td>Market deltas</td>
<td>27</td>
<td>4.5</td>
<td>1.25</td>
<td>0.6</td>
<td>0.25</td>
</tr>
<tr>
<td>Model deltas</td>
<td>21.5</td>
<td>4.63</td>
<td>1.63</td>
<td>0.9</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Dynamics hedging of CDO tranches in Markovian set-ups
Empirical results

- **Smaller equity tranche deltas in the Markov chain model**
  - How would we explain this?
- **Contagion effect**: default is associated with a dynamic increase in dependence

- Increasing correlation leads to a decrease in the PV of the equity tranche
Empirical results

- Comparison with results provided by Arnsdorf and Halperin (2007): *BSLP*: Markovian bivariate spread-loss model for portfolio credit derivatives

<table>
<thead>
<tr>
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<th>[0-3%]</th>
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<th>[3-9%]</th>
<th>[9-12%]</th>
<th>[12-22%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market deltas</td>
<td>26.5</td>
<td>4.5</td>
<td>1.25</td>
<td>0.65</td>
<td>0.25</td>
</tr>
<tr>
<td>BSLP model deltas</td>
<td>21.9</td>
<td>4.81</td>
<td>1.64</td>
<td>0.79</td>
<td>0.38</td>
</tr>
</tbody>
</table>

- Computed in March 2007 on the iTraxx tranche
- Two dimensional Markov chain, shift in credit spreads, deltas not related to replication strategies

<table>
<thead>
<tr>
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<th>[0-3%]</th>
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<th>[3-9%]</th>
<th>[9-12%]</th>
<th>[12-22%]</th>
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<td>1.25</td>
<td>0.6</td>
<td>0.25</td>
</tr>
<tr>
<td>Model deltas</td>
<td>21.5</td>
<td>4.63</td>
<td>1.63</td>
<td>0.9</td>
<td>0.6</td>
</tr>
</tbody>
</table>

- Note that our results are quite similar
- Equity tranche deltas are smaller in contagion models
Empirical results

- Consistent with results provided by Frey and Backhaus (2007): *Dynamic hedging of synthetic CDO tranches with spread risk and default contagion*

<table>
<thead>
<tr>
<th>Tranche</th>
<th>[0,3]</th>
<th>[3,6]</th>
<th>[6,9]</th>
<th>[9,12]</th>
<th>[12,22]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spread</td>
<td>26 %</td>
<td>84 bp</td>
<td>24 bp</td>
<td>14 bp</td>
<td>11 bp</td>
</tr>
<tr>
<td>Tranche Correlation</td>
<td>17.30 %</td>
<td>3.22 %</td>
<td>9.93 %</td>
<td>15.81 %</td>
<td>27.46 %</td>
</tr>
<tr>
<td>Gauss Cop.</td>
<td>Δ</td>
<td></td>
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<td></td>
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<tr>
<td></td>
<td>0.61</td>
<td>0.23</td>
<td>0.06</td>
<td>0.03</td>
<td>0.07</td>
</tr>
</tbody>
</table>

- VOD: Value-on-default

<table>
<thead>
<tr>
<th>Tranche</th>
<th>VOD in the Markov model</th>
<th>VOD in the Copula model</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0,3]</td>
<td>0.344</td>
<td>1.002</td>
</tr>
<tr>
<td>[3,6]</td>
<td>0.138</td>
<td>0.171</td>
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<tr>
<td>[6,9]</td>
<td>0.058</td>
<td>0.023</td>
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<tr>
<td>[9,12]</td>
<td>0.039</td>
<td>0.008</td>
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<tr>
<td>[12,22]</td>
<td>0.107</td>
<td>0.010</td>
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</table>

- Much smaller delta in the contagion model than in the Gaussian copula model
Empirical results

- Comparison with results provided by Eckner (2007)
- Deltas computed in a Duffie and Garleanu (2001) reduced-form model
  - Model calibrated on December 2005 CDX data
  - Spread sensitivity deltas

<table>
<thead>
<tr>
<th>Tranches</th>
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<th>[3-7%]</th>
<th>[7-10%]</th>
<th>[10-15%]</th>
<th>[15-30%]</th>
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<tr>
<td>AJD model deltas</td>
<td>21.7</td>
<td>6.0</td>
<td>1.1</td>
<td>0.4</td>
<td>0.1</td>
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<tr>
<td>Market deltas</td>
<td>18.5</td>
<td>5.5</td>
<td>1.5</td>
<td>0.8</td>
<td>0.4</td>
</tr>
<tr>
<td>Contagion model deltas</td>
<td>17.9</td>
<td>6.3</td>
<td>2.5</td>
<td>1.3</td>
<td>0.8</td>
</tr>
</tbody>
</table>

- Deltas go in opposite direction when comparing with the contagion model
Empirical results

- Consistent with **Feldhütter (2008)** empirical study of the affine intensity model
- Comparison of hedging performance with the Gaussian copula model
- Back-test study: Use information at time $t + 1$ to compute hedge ratios at time $t$
- Higher deltas for the equity tranche in the affine model compared with the 1F Gaussian copula

Prediction of (equity tranche) MTM change are better in the intensity model
- But results are pre-crisis...
Empirical results

- The recent crisis is associated with joint upward shifts in credit spreads
- And an increase in base correlations
- Dependence parameters and credit spreads may be highly correlated

- Should go in favour of the contagion model but...
Empirical results

- Cont and Kan (2008) perform a similar study for various hedging strategies
  - Comparison of spread-sensitivity deltas and jump-to-default deltas
  - Computed using several market models calibrated to the same data set
  - Back-test the strategies before and during the crisis
- Spread-deltas are very similar across models (5Y Europe iTraxx on 20 September 2006)

- Gaussian copula model
- Local intensity (contagion model)
- BSLP (Arnsdorf and Halperin (2007))
- GPL : generalized Poisson loss model (Brigo et al. (2006))

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Gauss</th>
<th>Local</th>
<th>BSLP</th>
<th>GPL</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 - 3</td>
<td>24.48</td>
<td>24.52</td>
<td>24.79</td>
<td>24.48</td>
</tr>
<tr>
<td>3 - 6</td>
<td>5.54</td>
<td>5.45</td>
<td>5.30</td>
<td>5.54</td>
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<tr>
<td>6 - 9</td>
<td>1.79</td>
<td>1.80</td>
<td>1.80</td>
<td>1.79</td>
</tr>
<tr>
<td>9 - 12</td>
<td>0.87</td>
<td>0.85</td>
<td>0.88</td>
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<tr>
<td>12 - 22</td>
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<td>0.35</td>
<td>0.32</td>
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<tr>
<td>22 - 100</td>
<td>0.08</td>
<td>0.08</td>
<td>0.09</td>
<td>0.08</td>
</tr>
</tbody>
</table>
Empirical results

- However, hedging performance may significantly depend on the calibration method.
- Identification problem: several specification of loss intensities may be compatible with the same set of market data.

Left: Laurent et al. (2007), Right: Cont and Minca (2008)

Figure 2: Dependence of default intensity on number of defaults for \( t = 1 \) year: ITRAXX Europe Series 6, March 15 2007.
Empirical results

- And computed deltas are rather sensitive to the calibration of contagion parameters on quoted CDO tranches
- **Cont, Deguest and Kan (2009)**: Computation of jump-to-default deltas using different calibration methods (5Y Europe iTraxx on 25 March 2008)

**QP**: Quadratic programming method

**Para**: Parametric method (piecewise constant default intensities proposed by Herbertsson (2007))

**EM**: Entropy minimization method (**Cont and Minca (2008)**)
Empirical results

- Limited hedging performance of the contagion model may be related to absence of specific spread risk
- But incorporating additional risks will create incompleteness
- Introducing some Brownian risks on top of jump-to-default risks brings unclear practical issues
  - It is not clear how defaults would drive the volatility of credit spreads
  - Regarding the hedging issue, one can think of using CDS with two different maturities for each name to cope both with default and credit spread risks
- or using local risk minimization techniques as in Frey and Backhaus (2008)
Empirical results

- Hedging with individual CDS may perform a better hedge (than hedging with the index)
  - Heterogeneous portfolio where some individual spreads are suddenly widening
  - Equity tranche very sensitive to idiosyncratic risk
- Obviously, beyond the scope of a pure top model
- Individual spread-ratios may be very different across names when computed in a bottom-up approach
Empirical results

- One natural extension of the Markovian contagion model...
- CDO Tranches on a portfolio composed with two disjoint sub-groups

\[ n_1 + n_2 = n, \quad N_t = N_t^1 + N_t^2 \]
Empirical results

- \((N^1, N^2)\) is a bivariate Markov chain, simultaneous defaults are precluded
  - Markovian contagion model

\[
\begin{align*}
(k_1, k_2) &\quad \lambda_1(k_1, k_2) \\
1 - \lambda_1(k_1, k_2) - \lambda_2(k_1, k_2) &\quad (k_1 + 1, k_2) \\
\lambda_2(k_1, k_2) &\quad (k_1, k_2 + 1)
\end{align*}
\]

- Dynamics of sub-index loss processes can be described in a trinomial tree
- As in the previous approaches, replication is theoretically feasible
- Loss intensities \(\lambda_1(k_1, k_2)\) and \(\lambda_2(k_1, k_2)\):

\[
\begin{align*}
\lambda_1(N^1_t, N^2_t) &= (n^1 - N^1_t)\alpha_1(N^1_t, N^2_t) \\
\lambda_2(N^1_t, N^2_t) &= (n^2 - N^2_t)\alpha_2(N^1_t, N^2_t)
\end{align*}
\]

- \(\alpha_1\) pre-default individual intensity of names in sub-portfolio 1
- \(\alpha_2\) pre-default individual intensity of names in sub-portfolio 2