Unified Credit-Equity Modeling

Rafael Mendoza-Arriaga
Based on joint research with: Vadim Linetsky and Peter Carr

The University of Texas at Austin
McCombs School of Business (IROM)

Recent Advancements in the Theory and Practice of Credit Derivatives
Nice, France
September 28-30, 2009
Research Projects

![Diagram of Research Projects]

- **Multi-Firm**
- **Single-Firm**

- **Calendar Time**
- **Time Changes**
Research Projects

The Constant Elasticity of Variance Model

- Multi-Firm
- Single-Firm
- Calendar Time
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- The Constant Elasticity of Variance Model
- Equity Default Swaps under the JDCEV process
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- Equity Default Swaps under the JDCEV process
- Time Changed Markov Processes in Unified Credit-Equity Modeling
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- The Constant Elasticity of Variance Model
- Equity Default Swaps under the JDCEV process
- Time Changed Markov Processes in Unified Credit-Equity Modeling
- Modeling Correlated Defaults by Multiple Firms (Future Research)
Literature Review

Stock Option Pricing Literature

Black-Scholes

(geometric Brownian motion)
Literature Review

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- Infinite lifetime process
Stock Option Pricing Literature

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  No possibility of Bankruptcy!
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  - No volatility smiles!
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- **Subsequent Generations of Models**
  - (modeling the volatility smile)

Rafael Mendoza (McCombs) Unified Credit-Equity Modeling Credit Risk 2009
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- Pure Jump Models
  Based on Levy processes
  (VG, NIG, CGMY, etc.)
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**Problem:**

- These models **ignore the possibility of bankruptcy** of the underlying firm.
- In real world, firms have a **positive probability of default** in finite time.
Literature Review

**Stock Option Pricing Literature**

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  - **Local Volatility**
    
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**Credit Risk Literature**

- **Reduced Form Framework**
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Credit Risk Literature

- **Reduced Form Framework**

  **Default Intensity Models**
  Since Duffie & Singleton, Jarrow, Lando & Turnbull:
  A vast amount of research has been developed
Literature Review

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- **Subsequent Generations of Models**
  - *(modeling the volatility smile)*
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    - Based on *Levy processes* *(VG, NIG, CGMY, etc.)*

**Credit Risk Literature**

- **Reduced Form Framework**
  - **Default Intensity Models**
  - *Since Duffie & Singleton, Jarrow, Lando & Turnbull:*
    - *A vast amount of research has been developed*
  - **Modeling Focus:**
    - Credit Default Events, Credit Spreads, Credit Derivatives, etc.
Literature Review

**Stock Option Pricing Literature**
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**Subsequent Generations of Models** (modeling the volatility smile)
- Local Volatility (CEV, Dupier, etc.)
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**Credit Risk Literature**

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**Problem:**
- These credit models ignore the information of stock option markets

**Equity Models and Credit Models**
- Disconnection between

**Default Intensity Models**

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**Disconnection between Equity Models and Credit Models**
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**Credit Risk Literature**

- Subsequent Generations of Models (modeling the volatility smile)
  - Local Volatility
    - (CEV, Dupier, etc.)
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**Modeling Focus:**

- Unified Credit-Equity Modeling
Motivating Example

2 weeks before bankruptcy (9/02/2008) Lehman Brothers (LEH) stock price was $16.13

The stock price drop of 72% from the high $62.19 to $16.13!

Open Interest on Put contracts with strike prices $K = 2.50 USD Maturing on 4/18/2009 (228 days) were 1529 contracts. Maturing on 1/16/2010 (501 days) were 2791 contracts.

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Put options provide default protection. Deep out-of-the-money puts are essentially credit derivatives which close the link between equity and credit products.
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- Pricing of equity derivatives should take into account the possibility of bankruptcy of the underlying firm.

- Possibility of default contributes to the implied volatility skew in stock options.
Research Goals

**Unified Credit –Equity Framework**

- Credit and equity derivatives on the same firm should be modeled within a unified framework
- Consistent pricing *across Credit and Equity assets*
- Consistent risk management and hedging
Research Goals

Unified Credit –Equity Framework

Credit and equity derivatives on the same firm should be modeled within a unified framework

- Consistent pricing across Credit and Equity assets
- Consistent risk management and hedging

Our Goal is to develop analytically tractable unified credit-equity models to improve pricing, calibration, and hedging

- Analytical tractability is desirable for fast computation of prices and Greeks, and calibration.
Our Contributions

- We introduce a new analytically tractable class of credit-equity models.
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- Our **model architecture** is based on applying random time changes to Markov diffusion processes to create new processes with desired properties.
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- For the first time in the literature, we present **state-dependent jumps** that exhibit the leverage effect:
  - As stock price falls $\Rightarrow$ arrival rates of large jumps increase
  - As stock price rises $\Rightarrow$ arrival rate of large jumps decrease
Our Contributions (cont.)

In our **model architecture**, time changes of diffusions have the following effects:
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- Lévy subordinator time change induces jumps with state-dependent Levy measure, including the possibility of a jump-to-default (stock drops to zero).
Our Contributions (cont.)

In our model architecture, time changes of diffusions have the following effects:

- **Lévy subordinator time change** induces jumps with state-dependent Levy measure, including the possibility of a jump-to-default (stock drops to zero).

- **Time integral of an activity rate process** induces stochastic volatility in the diffusion dynamics, the Levy measure, and default intensity.
Unifying Credit-Equity Models

The Jump to Default Extended Diffusions (JDED)

Before moving on to use time changes to construct models with jumps and stochastic volatility, we review the Jump-to-Default Extended Diffusion framework (JDED)
Jump to Default Extended Diffusions (JDED)

$$s_t = \begin{cases} \tilde{s}_t, & \zeta > t \\ 0, & \zeta \leq t \end{cases}$$

(\(\zeta\) default time)

We assume absolute priority: the stock holders do not receive any recovery in the event of default.
Jump to Default Extended Diffusions (JDED)

Defaultable Stock Price

\[ S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \leq t \end{cases} \]

(\( \zeta \) default time)

Model the pre-default stock dynamics under an EMM \( Q \) as:

\[ d\tilde{S}_t = [\mu + h(\tilde{S}_t)]\tilde{S}_tdt + \sigma(\tilde{S}_t)\tilde{S}_t dB_t \]
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\[ \Rightarrow \mu = r - q. \textbf{Drift}: \text{short rate } r \text{ minus the dividend yield } q \]
Jump to Default Extended Diffusions (JDED)

Defaultable Stock Price

\[ S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \leq t \end{cases} \]

(\( \zeta \) default time)

Model the pre-default stock dynamics under an EMM \( \mathbb{Q} \) as:

\[ d\tilde{S}_t = \left[ \mu + h(\tilde{S}_t) \right]\tilde{S}_t dt + \sigma(\tilde{S}_t) \tilde{S}_t dB_t \]

\[ \Rightarrow \sigma(S). \text{ State dependent volatility} \]
Jump to Default Extended Diffusions (JDED)

Defaultable Stock Price

$S_t = \begin{cases} 
\tilde{S}_t, & \zeta > t \\
0, & \zeta \leq t 
\end{cases}$

($\zeta$ default time)

Model the pre-default stock dynamics under an EMM $Q$ as:

$d\tilde{S}_t = [ \mu + h(\tilde{S}_t) ]\tilde{S}_t dt + \sigma(\tilde{S}_t) \tilde{S}_t dB_t$

$\Rightarrow h(S)$. State dependent default intensity
Jump to Default Extended Diffusions (JDED)

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Model the pre-default stock dynamics under an EMM \( \mathbb{Q} \) as:

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d\tilde{S}_t = \left[ \mu + h(\tilde{S}_t) \right] \tilde{S}_t \, dt + \sigma(\tilde{S}_t) \tilde{S}_t \, dB_t \]

\( \Rightarrow h(S) \). State dependent default intensity

- Compensates for the jump-to-default and ensures the discounted martingale property
Defaultable Stock Price

\[ S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \leq t \end{cases} \]  

(\zeta \text{ default time})

If the diffusion \( \tilde{S}_t \) can hit zero:
Jump to Default Extended Diffusions (JDED)

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(\zeta \text{ default time})

If the diffusion \( \tilde{S}_t \) can hit zero:
⇒ Bankruptcy at the first hitting time of zero,

\[ \tau_0 = \inf \left\{ t : \tilde{S}_t = 0 \right\} \]
Jump to Default Extended Diffusions (JDED)

**Defaultable Stock Price**

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(\( \zeta \) default time)

Prior to \( \tau_0 \) default could also arrive by a **jump-to-default** \( \tilde{\zeta} \) with default intensity \( h(\tilde{S}) \),

\[ \tilde{\zeta} = \inf \left\{ t \in [0, \tau_0] : \int_0^t h(\tilde{S}_u) \geq e \right\}, \quad e \approx \text{Exp}(1) \]
Jump to Default Extended Diffusions (JDED)

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\( \Rightarrow \) At time \( \tilde{\zeta} \) the stock price \( S_t \) jumps to zero and the firm defaults on its debt
Defaultable Stock Price

\[ S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \leq t \end{cases} \]

(\( \zeta \) default time)

The default time \( \zeta \) is the earliest of:

1. The stock hits level zero by diffusion: \( \tau_0 \)
2. The stock jumps to zero from a positive value: \( \tilde{\zeta} \)

\[ \zeta = \min(\tilde{\zeta}, \tau_0) \]
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Contingent Claims

Risk Neutral Survival Probability \((no \text{ default by time } T)\)

\[
Q(S, t; T) = \mathbb{E}\left[\mathbf{1}_{\{\zeta > T\}}\right]
\]

\[
= \mathbb{E}\left[e^{-\int_t^T h(S_u)du} \mathbf{1}_{\{\tau_0 > T\}}\right]
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Recall: Default time \(\zeta = \min\left(\tilde{\zeta}, \tau_0\right)\).
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\]

Recall: Default time \(\zeta = \min(\tilde{\zeta}, \tau_0)\).

1. No jump-to-default before maturity \(T\),
2. Diffusion does not hit zero before maturity \(T\).
Contingent Claims

Defaultable Zero Coupon Bond (at time $t$)

$$B(S, t; T) = e^{-r(T-t)} Q(S, t; T)$$

Disc. Dollar if No Default occurs prior to maturity

Recall: $Q(S, t; T)$ is the risk neutral survival probability
Contingent Claims

**Defaultable Zero Coupon Bond (at time $t$)**

\[
B(S, t; T) = e^{-r(T-t)} Q(S, t; T) + e^{-r(T-t)} R [1 - Q(S, t; T)]
\]

- **Disc. Dollar if No Default occurs prior to maturity**
- **Disc. recovery $R \in [0, 1]$ if Default occurs before maturity**

Recall: $Q(S, t; T)$ is the risk neutral survival probability

$R$ is a fraction of a dollar paid at maturity.
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Recall: $Q (S, t; T)$ is the risk neutral survival probability
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Defaultable bonds with coupons are valued as portfolios of zero-coupon bonds
Contingent Claims

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Defaultable bonds *with coupons* are valued as portfolios of zero-coupon bonds

Call Option

$$C(S, t; K, T) = e^{-r(T-t)} \mathbb{E} \left[ e^{-\int_t^T h(S_u) du} (S_T - K)^+ 1_{\{\tau_0 > T\}} \right]$$
Contingent Claims

Put Payoff (Strike Price $K > 0$)

$$(K - S_T)^+ 1_{\{\zeta > T\}}$$

Put Payoff given no default by time $T$
Contingent Claims

Put Payoff (Strike Price $K > 0$)

\[
(K - S_T)^+ \cdot 1_{\{\zeta > T\}} + K \cdot 1_{\{\zeta \leq T\}}
\]

- Put Payoff given no default by time $T$
- Recovery amount $K$ if default occurs before maturity $T$
Contingent Claims

**Put Payoff (Strike Price \( K > 0 \))**

\[
(K - S_T)^+ 1_{\{\zeta > T\}} + K 1_{\{\zeta \leq T\}}
\]

- Put Payoff given no default by time \( T \)
- Recovery amount \( K \) if default occurs before maturity \( T \)

**Put Option Price**

\[
P(S, t; K, T) = e^{-r(T-t)} \mathbb{E} \left[ e^{-\int_t^T h(S_u)du} (K - S_T)^+ 1_{\{\tau_0 > T\}} \right] + Ke^{-r(T-t)} [1 - Q(S, t; T)]
\]

**NOTE.** A default claim is embedded in the Put Option.
Jump-to-Default Extended Constant Elasticity of Variance (JDCEV) Model

The JDCEV process (Carr and Linetsky (2006))

\[
dS_t = \left[ \mu + h(S_t) \right] S_t \, dt + \sigma(S_t) S_t \, dB_t, \quad S_0 = S > 0
\]

\[
\sigma(S) = a S^\beta \quad \text{CEV Volatility (Power function of } S) \quad h(S) = b + c \sigma^2(S) \quad \text{Default Intensity (Affine function of Variance)}
\]

- \( a > 0 \) \Rightarrow \text{volatility scale parameter (fixing ATM volatility)}
- \( \beta < 0 \) \Rightarrow \text{volatility elasticity parameter}
- \( b \geq 0 \) \Rightarrow \text{constant default intensity}
- \( c \geq 0 \) \Rightarrow \text{sensitivity of the default intensity to variance}

For \( c = 0 \) and \( b = 0 \) the JDCEV reduces to the standard CEV process
The JDCEV process (Carr and Linetsky (2006))

\[
\begin{align*}
    dS_t &= [\mu + h(S_t)]S_t \, dt + \sigma(S_t)S_t \, dB_t, \quad S_0 = S > 0 \\
    \sigma(S) &= aS^\beta \\
    h(S) &= b + c \sigma^2(S)
\end{align*}
\]

CEV Volatility  
(Power function of \( S \)) 

Default Intensity  
(Affine function of Variance) 

The model is consistent with:

- leverage effect \( \Rightarrow S \downarrow \rightarrow \sigma(S) \uparrow \)
- stock volatility–credit spreads linkage \( \Rightarrow \sigma(S) \uparrow \leftrightarrow h(S) \uparrow \)
An Application of Jump to Default Extended Diffusions (JDED)

Equity Default Swaps under the JDCEV Model
Equity Default Swaps (EDS)

- Credit-Type Instrument to bring protection in case of a Credit Event

Credit Events:

1. Reference Entity Defaults
2. Reference Stock Price drops significantly ($L = 30\%$, $S_0$)
Equity Default Swaps (EDS)

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- Similar to CDS
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Credit Events:
1. Reference Entity Defaults
2. Reference Stock Price drops significantly ($L = 30\% S_0$)

- Similar to CDS
  - Protection Buyer makes periodic Premium Payments on exchange of protection in case of a Credit Event.
Equity Default Swaps (EDS)

- Credit-Type Instrument to bring protection in case of a Credit Event

Credit Events:

1. Reference Entity Defaults
2. Reference Stock Price drops significantly ($L = 30\%S_0$)

- Similar to CDS
  - Protection Buyer makes periodic Premium Payments on exchange of protection in case of a Credit Event.
  - Protection Seller pays a recovery amount $(1 - r)$ for each dollar of principal at credit event time, if the event occurs prior to Maturity.
Equity Default Swaps (EDS)

- **Equity Default Swap (EDS)**
  - Protection Seller
  - Protection Buyer
  - Hits Level (L)

![Diagram showing Equity Default Swap (EDS) concept]

- **Protection Payment**
- **Hitting Level (L)**
- **Time (yrs)**
- **S(t)**

Equity Default Swaps (EDS) are financial derivatives that allow parties to transfer default risk associated with equity investments. The diagram illustrates how protection sellers and buyers interact, with the protection seller receiving payments when the equity index (S(t)) falls below a specified level (L) up to a certain time (T).
Equity Default Swaps (EDS)

Equity Default Swap (EDS)

Protection Seller

Protection Buyer

Default Occurs

Hits Level (L)

Time (yrs)

S(t)

Hitting Level (L)

Protection Payment

Default Event (or)

Rafael Mendoza (McCombs)
Equity Default Swaps (EDS)

- Protection Seller
- Protection Buyer

- Hits Level \( L \)
- Default Occurs

- Premium Payments

- **Equity Default Swap (EDS)**

- **Default Event** (or)
- **Hitting Level** \( L \)

- Graph showing time \( t \), premium payment, protection payment, and hitting level \( L \).
Equity Default Swaps (EDS)

Equity Default Swap (EDS)

Protection Seller

Protection Buyer

Default Occurs

Hits Level (L)

Or

Premium Payments

Accrued Interest

Acc. Interest

Default Event

(or)

Hitting Level (L)

Premium Payment

Protection Payment

(Δt)

S(t)

L

Time (yrs)

0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0

0 20 40 60 80 100
Equity Default Swaps (EDS): Balance Equation

We want to obtain the EDS rate $\varrho^*$ that balances out:

$$\varrho^* = \{ \varrho \mid PV(\text{Protection Payment}) = PV(\text{Premium Payments} + \text{Accrued Interest}) \}$$

Define: Credit Event Time $\Rightarrow T_L^\Delta = \min \{ \text{first hitting time to } L, \text{ Default Time} \}$

\[
\begin{align*}
PV(\text{Protection Payment}) &= (1 - r) \cdot \mathbb{E} \left[ e^{-r \cdot T_L^\Delta} \mathbb{1}_{\{T_L^\Delta \leq T\}} \right] \\
PV(\text{Premium Payments}) &= \varrho \cdot \Delta_t \cdot \sum_{i=1}^{N} e^{-r \cdot t_i} \mathbb{E} \left[ \mathbb{1}_{\{T_L^\Delta \geq t_i\}} \right] \\
PV(\text{Accrued Interests}) &= \varrho \cdot \mathbb{E} \left[ e^{-r \cdot T_L^\Delta} \left( T_L^\Delta - \Delta_t \cdot \frac{T_L^\Delta}{\Delta_t} \right) \mathbb{1}_{\{T_L^\Delta \leq T\}} \right]
\end{align*}
\]

<table>
<thead>
<tr>
<th>$\Delta_t$</th>
<th>Time Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>Recovery</td>
</tr>
<tr>
<td>$T$</td>
<td>Maturity</td>
</tr>
<tr>
<td>$T_L^\Delta$</td>
<td>Credit Event Time</td>
</tr>
<tr>
<td>$\varrho$</td>
<td>EDS rate</td>
</tr>
<tr>
<td>$r$</td>
<td>Risk Free Rate</td>
</tr>
</tbody>
</table>
Advantages of EDS over CDS

- **Transparency** on which an EDS payoff is triggered. It is easy to know whether a firm stock price has crossed a lower threshold \((L)\).
Equity Default Swaps (EDS)

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Advantages of EDS over CDS

- **Transparency** on which an EDS payoff is triggered. It is easy to know whether a firm stock price has crossed a lower threshold ($L$).
- Using the Stock Price as the state variable to determine a credit event allows investors to have an Exposure to Firms for which CDS are not usually traded. (as in the case of firms with high yield debt)
- EDS closes the gap between equity and credit instruments since it is structurally similar to the credit default swap.
Under the jump-to-default extended diffusion framework (including JDCEV), the pre-default stock process evolves continuously and may experience a single jump to default.
Time-Changing the Jump to Default Extended Diffusions (JDED)

- Under the jump-to-default extended diffusion framework (including JDCEV), the pre-default stock process evolves continuously and may experience a single jump to default.

- Our contribution is to construct far-reaching extensions by introducing jumps and stochastic volatility by means of time-changes.
Time-Changing the Jump to Default Extended Diffusions (JDED)

“Time Changes of Markov Processes in Credit-Equity Modeling”
General Panorama

Continuous Markov Process w/ Default Intensity
General Panorama

Continuous Markov Process w/ Default Intensity

Time Changes

Bochner Levy Subordination

Absolute Continuous Time Changes
General Panorama

Continuous Markov Process w/ Default Intensity

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Absolute Continuous Time Changes

Levy Subordination & Absolute Continuous Time Changes

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General Panorama

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Analytical Unified Credit and Equity Option Pricing Formulas
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Analytical Unified Credit and Equity Option Pricing Formulas

\( f(x) \notin L^2 \)

Laplace Transform Approach
Analytical Unified Credit and Equity Option Pricing Formulas

- Continuous Markov Process w/ Default Intensity
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- Levy Subordination & Absolute Continuous Time Changes
- Time Changes
- Laplace Transform Approach
- Spectral Expansion Approach

$f(x) \not\in L^2$

$f(x) \in L^2$
Time-Changed Process $Y_t = X_{T_t}$

Time Changed Process Construction

$Y_t = X_{T_t}$

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**Random Clock \( \{ T_t, t \geq 0 \} \)**

Non-decreasing RCLL process starting at \( T_0 = 0 \) and \( \mathbb{E} [ T_t ] < \infty \).
- We are interested in T.C. with *analytically tractable Laplace Transform (LT)*:

\[
\mathcal{L}(t, \lambda) = \mathbb{E} \left[ e^{-\lambda T_t} \right] < \infty
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2. Absolutely Continuous (A.C.) time changes ⇒ induce stochastic volatility
3. Composite Time Changes ⇒ induce jumps & stochastic volatility
Illustration of Lévy Subordinators

\[ Y = X_{T_t} \text{ where } X_t = B_t \text{ and } T_t = t + \text{ Compound Poisson Process with Exponential Jumps} \]

Background Process \( X(t) \)

Time Process \( T(t) \)

Time Changed Process \( Y(t) = X(T(t)) \)

Jumps arriving at (expected) time intervals \( 1/\alpha = 1/4 \) yrs. of (expected) jump size
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\textit{with Exponential Jumps}

\[ T_t = t + \text{Compound Poisson Process with Exponential Jumps} \]

When jump in \( T(t) \) arrives, the clock skips ahead, and time-changed process is generated by cutting out the corresponding piece of the diffusion sample path in which \( T(t) \) skips ahead.

Jumps arriving at (expected) time intervals \( 1/\alpha = 1/4 \text{ yrs.} \) of (expected) jump size
Illustration of Lévy Subordinators

\[ Y = X_{T_t} \text{ where } X_t = B_t \text{ and } T_t = t^+ \]

Compound Poisson Process with Exponential Jumps

\[
\begin{align*}
\text{Background Process } X(t) & \\
\text{Time Process } T(t) & \\
\text{Time Changed Process } Y(t) = X(T(t)) &
\end{align*}
\]

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Examples of Lévy Subordinators

Three Parameter Lévy measure:

\[ \nu(ds) = Cs^{-Y-1}e^{-\eta s}ds \]

where \( C > 0, \ \eta > 0, \ Y < 1 \)

- \( C \) changes the time scale of the process (simultaneously modifies the intensity of jumps of all sizes)
- \( Y \) controls the small size jumps
- \( \eta \) defines the decay rate of big jumps
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Lévy-Khintchine formula

\[ \mathcal{L}(t, \lambda) = e^{-\phi(\lambda)t} \]

where

\[ \phi(\lambda) = \begin{cases} 
\gamma \lambda - C \Gamma(-Y)[(\lambda + \eta)^Y - \eta^Y], & Y \neq 0 \\
\gamma \lambda + C \ln(1 + \lambda / \eta), & Y = 0 
\end{cases} \]
Absolutely Continuous Time Changes

Absolutely Continuous Time Changes (A.C)

An A.C. Time change is the time integral of some positive function $V(z)$ of a Markov process $\{Z_t, t \geq 0\}$,

$$T_t = \int_0^t V(Z_u) du$$

We are interested in cases with Laplace Transform in closed form:

$$L_z(t, \lambda) = \mathbb{E}_z \left[ e^{-\lambda \int_0^t V(Z_u) du} \right]$$

Example: The Cox-Ingersoll-Ross (CIR) process:

$$dV_t = \kappa (\theta - V_t) dt + \sigma V_t \sqrt{V_t} dW_t$$

with $V_0 = v > 0$, rate of mean reversion $\kappa > 0$, long-run level $\theta > 0$, and volatility $\sigma > 0$. 
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Absolutely Continuous Time Changes

The Laplace Transform of the Integrated CIR process:

\[ \mathcal{L}_V(t, \lambda) = \mathbb{E}_V \left[ e^{-\lambda \int_0^t V_u du} \right] = A(t, \lambda) e^{-B(t, \lambda)V} \]

\[ A = \left( \frac{2\omega e^{(\omega+\kappa)t/2}}{(\omega + \kappa)(e^{\omega t} - 1) + 2\omega} \right)^{2\kappa\theta \sigma_V^2} , \quad B = \frac{2\lambda(e^{\omega t} - 1)}{(\omega + \kappa)(e^{\omega t} - 1) + 2\omega} , \quad \omega = \sqrt{2\sigma_V^2 \lambda + \kappa^2} \]
Absolutely Continuous Time Changes

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\]

This is the Zero Coupon Bond formula under the CIR interest rate

\[ r_t = \lambda V_t. \]
Illustration of Absolutely Continuous Time Changes

CIR parameters $\kappa = 7, \theta = 2, V_0 = 0.5$ and $\sigma_V = \sqrt{2}$
Illustration of Absolutely Continuous Time Changes

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Time speeds up or slows down based on the amount of new information arriving and the amount trading (trading time)
Illustration of Absolutely Continuous Time Changes

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A Composite Time Change induces both jumps and stochastic volatility

\[ T_t = T_t^1 T_t^2 \]

- \( T_t^1 \) is a Lévy Subordinator
- \( T_t^2 \) is and A.C time change
Composite Time Changes

A Composite Time Change induces both *jumps* and *stochastic volatility*

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- \( T_t^1 \) is a Lévy Subordinator
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Laplace Transform of the Composite Time Change

It is obtained by first conditioning w.r.t. the A.C. time change

\[ \mathbb{E}[e^{-\lambda T_t}] = \mathbb{E}[e^{-T_t^2 \phi(\lambda)}] = \mathcal{L}_z(t, \phi(\lambda)) \]
Quick Summary

We have:

1. A Jump-to-Default Extended Diffusion process:
   \[ E[ f(X_t)1_{\tau_0 > \tau} g] = E[e^{\int_0^t h(X_u)du} f(X_t)1_{\tau_0 > \tau} g] \]

2. A time-changed process \( Y_t = X_{T_t} \) with the Laplace transform for the time change \( T_t \) given in closed form,
   \[ E[e^{\lambda T_t}] = L(t, \lambda) \]

How do we evaluate contingent claims written on the time-changed process \( Y_t \)?
Quick Summary

We have:

1. A Jump-to-Default Extended Diffusion process:

\[
E \left[ f \left( X_t \right) 1_{\{\zeta > t\}} \right] = E \left[ e^{-\int_0^t h(X_u)du} f \left( X_t \right) 1_{\{\tau_0 > t\}} \right]
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Quick Summary

We have:

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\[
\mathbb{E} \left[ f(X_t) 1_{\{\zeta > t\}} \right] = \mathbb{E} \left[ e^{-\int_0^t h(X_u) \, du} f(X_t) 1_{\{\tau_0 > t\}} \right]
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How do we evaluate contingent claims written on the time-changed process \( Y_t \)?

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Contingent Claims for the Time-Changed Process

Valuing contingent claims written on $Y_t = X_{T_t}$

$$
\mathbb{E} \left[ 1_{\{\zeta > T_t\}} f(Y_t) \right] = \mathbb{E} \left[ \mathbb{E}_x \left[ 1_{\{\zeta > T_t\}} f(X_{T_t}) \mid T_t \right] \right]
$$

**Conditioning** since $X_t$ and $T_t$ are independent
Contingent Claims for the Time-Changed Process

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*Conditioning since $X_t$ and $T_t$ are independent*

Conditional Expectation

$$\mathbb{E} \left[ 1_{\{\zeta > T_t\}} f(X_{T_t}) \mid T_t \right]$$

It is equivalent to pricing a contingent claim written on the process $X_t$ maturing at time $T_t$
Contingent Claims for the Time-Changed Process

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We employ two methodologies to evaluate the expectations and do the pricing in closed form:
Valuing contingent claims written on $Y_t = X_{T_t}$

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Contingent Claims for the Time-Changed Process

Valuing contingent claims written on $Y_t = X_{T_t}$

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We employ two methodologies to evaluate the expectations and do the pricing in closed form:

1. **Resolvent Operator**: general methodology.
2. **Spectral Representation**: for square-integrable payoffs.
Resolvent Operator

Resolvent Operator:

The Laplace Transform of the Expectation Operator:

\[
(\mathcal{R}_\lambda f)(x) := \int_0^\infty e^{-\lambda t} \mathbb{E}_x \left[ 1_{\{\zeta > t\}} f(X_t) \right] dt
\]
Resolvent Operator

Resolvent Operator:

The Laplace Transform of the Expectation Operator:

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- We recover the Expectation via the Bromwich Laplace Inversion formula:
Resolvent Operator

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\[\mathbb{E}_x \left[1_{\{\zeta > t\}} f(X_t)\right] = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{\lambda t} (\mathcal{R}_\lambda f)(x) \, d\lambda\]
Resolvent Operator

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- NOTE. The time \(t\) enters in this expression only through the exponential \(e^{\lambda t}\)
Spectral Expansion

1. If the infinitesimal generator $G$ of the diffusion process $X$ is self-adjoint

\[ E[x(t)] = \sum_{n=1}^{\infty} e^{\lambda_n t} c_n \phi_n(x) \]

where $c_n = \langle f, \phi_n \rangle$ are the expansion coefficients and, $\lambda_n$ are the eigenvalues, $\phi_n(x)$ the eigenfunctions solving $G \phi_n(x) = \lambda_n \phi_n(x)$.

NOTE. The time $t$ enters in this expression only through the exponential $e^{\lambda_n t}$. 

Rafael Mendoza (McCombs) Unified Credit-Equity Modeling Credit Risk 2009 36 / 1
Spectral Expansion

1. If the infinitesimal generator $G$ of the diffusion process $X$ is self-adjoint
   - If $X$ is a 1D diffusion process then $G$ is self-adjoint in
     $\mathcal{H} = L^2((0, \infty), m)$ with respect to the speed measure $m(dx)$
Spectral Expansion

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2. If $f \in \mathcal{H}$
   $\Rightarrow$ Then we can use the Spectral Representation Theorem in order to obtain the Expectation

Eigenfunction Expansion (when the spectrum of $G$ is discrete):

$$\mathbb{E}_x \left[ \mathbf{1}_{\{\zeta > t\}} f(X_t) \right] = \sum_{n=1}^{\infty} e^{-\lambda_n t} c_n \varphi_n(x)$$

where $c_n = \langle f, \varphi \rangle$ are the expansion coefficients and, $\lambda_n$ are the eigenvalues, $\varphi_n(x)$ the eigenfunctions solving $G \varphi_n(x) = \lambda_n \varphi_n(x)$

NOTE. The time $t$ enters in this expression only through the exponential $e^{-\lambda_n t}$
Valuing contingent claims written on $Y_t = X_{T_t}$

Resolvent Operator

$$\mathbb{E} \left[ \mathbf{1}_{\{\zeta > T_t\}} f(Y_t) \right]$$

Spectral Expansion

$$\mathbb{E} \left[ \mathbf{1}_{\{\zeta > T_t\}} f(Y_t) \right] = \sum_{n=1}^{\infty} \mathbb{E} \left[ e^{\lambda_n T_t} c_n \phi_n(x) \right] = \sum_{n=1}^{\infty} L(t, \lambda_n) c_n \phi_n(x)$$
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\mathbb{E} \left[ \mathbf{1}_{\{\zeta > T_t\}} f(Y_t) \right]
= \mathbb{E} \left[ \mathbb{E}_X \left[ \mathbf{1}_{\{\zeta > T_t\}} f(X_{T_t}) \mid T_t \right] \right]
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Spectral Expansion

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= \mathbb{E} \left[ \int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{\lambda T_t} (R_\lambda f)(X) \frac{d\lambda}{2\pi i} \right]
\]

**Spectral Expansion**

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\mathbb{E} \left[ 1_{\{\zeta > T_t\}} f(Y_t) \right]
= \mathbb{E} \left[ \mathbb{E}_X \left[ 1_{\{\zeta > T_t\}} f(X_{T_t}) \mid T_t \right] \right]
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\]
\[
= \int_{\epsilon-i\infty}^{\epsilon+i\infty} \mathcal{L}(t, -\lambda)(\mathcal{R}_\lambda f)(x) \frac{d\lambda}{2\pi i}
\]

### Spectral Expansion

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\mathbb{E} \left[ \mathbf{1}_{\{\zeta > T_t\}} f(Y_t) \right] = \mathbb{E} \left[ \mathbb{E}_X \left[ \mathbf{1}_{\{\zeta > T_t\}} f(X_{T_t}) \mid T_t \right] \right]
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\]
A new class of Credit-Equity Models with state-dependent jumps, S.V. and default intensity

Model Architecture for the Defaultable Stock

\[ S_t = 1_{\{t < \tau_d\}} e^{\rho t} X_{T_t}. \]
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- \( X_t \Rightarrow \) Jump-to-Default Extended Diffusion; e.g. JDCEV Process:
  \[ dX_t = [\mu + h(X_t)] X_t \, dt + \sigma(X_t) X_t \, dB_t, \quad X_0 = x > 0, \]
  \[ \sigma(x) = ax^\beta, \quad h(x) = b + c \sigma^2(x) \]
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- \( T_t \Rightarrow \) Random Clock: Lévy Subordinator, A.C. Time Change, or Composite T.C.

- \( \rho \Rightarrow \) Compensation Parameter (discounted martingale)

- \( \tau_d \Rightarrow \) Default Time
  - If \( \zeta = \min(\tau_0, \tilde{\zeta}) \) is the lifetime of \( X \), then
    \[ \tau_d = \inf\{t \geq 0 : \zeta \leq T_t\} \]
  - At \( \tau_d \) the stock drops to zero (Bankruptcy)
Survival Probability and Defaultable Zero Bonds

Survival Probability

\[ Q(\tau_d > t) = Q(\zeta > T_t) \]
\[ = \sum_{n=0}^{\infty} \mathcal{L}(t, (b + \omega n)) \frac{\Gamma(1+c/|\beta|)\Gamma(n+1/(2|\beta|))}{\Gamma(\nu+1)\Gamma(1/(2|\beta|))n!} \]
\[ \times A^{1/2|\beta|} xe^{-Ax^{-2\beta}} \quad _1F_1(1 - n + c/|\beta|, \nu + 1, Ax^{-2\beta}) \]

Where \(_1F_1(a, b, z)\) is the Kummer Confluent Hypergeometric function; and \(\omega = 2|\beta|(\mu + b), \nu = \frac{1+2c}{2|\beta|}, \text{ and } A = \frac{\mu + b}{a^2|\beta|}.\)
Survival Probability and Defaultable Zero Bonds

**Survival Probability**

\[ \mathbb{Q}(\tau_d > t) = \mathbb{Q}(\zeta > T_t) \]

\[ = \sum_{n=0}^{\infty} \mathcal{L}(t, (b + \omega n)) \frac{\Gamma(1+c/|\beta|)\Gamma(n+1/(2|\beta|))}{\Gamma(\nu+1)\Gamma(1/(2|\beta|))n!} \times A \frac{1}{2|\beta|} x e^{-Ax^{-2\beta}} {}_1F_1(1 - n + c/|\beta|, \nu + 1, Ax^{-2\beta}) \]

Where \( {}_1F_1(a, b, z) \) is the Kummer Confluent Hypergeometric function;
and \( \omega = 2|\beta|(\mu + b), \nu = \frac{1+2c}{2|\beta|} \), and \( A = \frac{\mu+b}{a^2|\beta|} \).

**Defaultable Zero Coupon Bond**

\[ B_R(x, t) = e^{-rt}\mathbb{Q}(\tau_d > t) + Re^{-rt}[1 - \mathbb{Q}(\tau_d > t)] \]

Recovery fraction \( R \in [0, 1] \).
Put Options

Put Option

\[ P(x; K, t) = P_D(x; K, t) + P_0(x; K, t), \]

where:

\[ P_D(x; K, t) = Ke^{-rt}[1 - Q(\tau_d > t)], \quad \text{Default before } t \]

\[ P_0(x; K, t) = e^{-rt}\mathbb{E}_x [(K - e^{\rho t} X_T)^+ 1_{\{\tau_d > t\}}] \]

\[ = e^{-(r-\rho)t} \sum_{n=1}^{\infty} L(t, \lambda_n) c_n \varphi_n(x) \]

The default claim \( P_D(x; K, t) \) is directly calculated from the Survival Probability \( Q(\tau_d > t) \) previously computed.

The claim, \( P_0(x; K, t) \), is calculated by means of the Spectral Expansion since \( f(x) = (K - x)^+ \in L^2((0, \infty), m) \).
Put Options

Put Option

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where:

\[ P_D(x; K, t) = Ke^{-rt}[1 - Q(\tau_d > t)], \quad \text{Default before } t \]

\[ P_0(x; K, t) = e^{-rt} \mathbb{E}_x \left[(K - e^{\rho t}X_T)^+1_{\{\tau_d > t\}}\right] \quad \text{No Default before } t \]

\[ = e^{-(r-\rho)t} \sum_{n=1}^{\infty} \mathcal{L}(t, \lambda_n)c_n\varphi_n(x) \]

- The default claim \( P_D(x; K, t) \) is directly calculated from the Survival Probability \( Q(\tau_d > t) \) previously computed.

- The claim, \( P_0(x; K, t) \), is calculated by means of the Spectral Expansion since \( f(x) = (K - x)^+ \in L^2((0, \infty), \mu) \).
Put Options

Put claim conditional on *no default event* before maturity

\[ P_0(x; K, t) = e^{-(r-\rho)t} \sum_{n=1}^{\infty} \mathcal{L}(t, \lambda_n) c_n \varphi_n(x) \]

where \( k = K e^{-\rho t} \) and,

**Eigenvalues** \( \Rightarrow \lambda_n = \omega n + 2c(\mu + b) + b, \) with \( \omega = 2|\beta|(\mu + b) \)

**Eigenfunctions** \( \Rightarrow \varphi_n(x) = A^{\nu/2} \sqrt{\frac{(n-1)! (\mu+b)(2c+1)}{\Gamma(\nu+n)}} xe^{-Ax^{-2\beta}} L_{n-1}^{(\nu)}(Ax^{-2\beta}) \)

**Expansion Coefficients** \( \Rightarrow c_n = \frac{A^{\nu/2+1} k^{2c+1-2\beta} \sqrt{\Gamma(\nu+n)}}{\Gamma(\nu+1) \sqrt{(\mu+b)(2c+1)(n-1)!}} \times \left\{ \begin{array}{l}
\frac{|\beta|}{c+|\beta|} 2F_2 \left( \begin{array}{c}
1-n, \frac{c}{|\beta|} + 1 \\
\nu + 1, \frac{c}{|\beta|} + 2
\end{array} ; Ak^{-2\beta} \right) - \frac{\Gamma(\nu+1)(n-1)!}{\Gamma(\nu+n+1)} L_{n-1}^{\nu+1} (Ak^{-2\beta})
\end{array} \right\}, \)

\( 2F_2 \): generalized hypergeometric function, \( L_n^{(\nu)} \): generalized Laguerre polynomials.
Numerical Examples

- Assume a background process \( \{X_t, t > 0\} \) following a JDCEV, and a composite time change Inverse Gaussian Process & CIR:

<table>
<thead>
<tr>
<th>Parameters</th>
<th>JDCEV</th>
<th>CIR</th>
<th>IG</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a</td>
<td>10</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( \beta )</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>0.5</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>b</td>
<td>0.01</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>r</td>
<td>0.05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>q</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \gamma )</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>( \eta )</td>
<td></td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>C</td>
<td></td>
<td>0.5</td>
<td>2\sqrt{2/\pi}</td>
</tr>
</tbody>
</table>

Note that \( \gamma = 0 \), thus the time changed process is a pure jump process!
Infinitesimal Generator of the Time Changed Process \( (Y_t = X_{T_t}, V_t) \)

\[
\mathcal{G}f(x, v) = 
\gamma v \left( \frac{1}{2} a^2 x^{2\beta + 2} \frac{\partial^2 f}{\partial x^2} (x, v) + (b + ca^2 x^{2\beta}) x \frac{\partial f}{\partial x} (x, v) - (b + c^2 a^2 x^{2\beta}) f(x, v) \right) 
\]

\[
+ \nu \left( \int_{(0,\infty)} (f(y, v) - f(x, v)) \pi(x, y) dy - k(x) f(x, v) \right) 
\]

\[
+ \frac{\sigma_V^2}{2} \nu \frac{\partial^2 f}{\partial v^2} (x, v) + \kappa(\theta - v) \frac{\partial f}{\partial v} (x, v) 
\]

- State dependent jump measure \( \pi(x, y) = \int_{(0,\infty)} p(s; x, y) \nu(ds) \)
- Additional killing rate \( k(x) = \int_{(0,\infty)} P_s(x, \{0\}) \nu(ds) \)
Infinitesimal Generator of the Time Changed Process \((Y_t = X_{T_t}, V_t)\)

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Gf(x, v) = \gamma_v \left( \frac{1}{2} a^2 x^{2\beta + 2} \frac{\partial^2 f}{\partial x^2}(x, v) + (b + ca^2 x^{2\beta}) \frac{\partial f}{\partial x}(x, v) - (b + c^2 a^2 x^{2\beta}) f(x, v) \right)
\]

\(G_x f(x,v)\)  JDCEV's infinitesimal generator

\[
+ v \left( \int_{(0,\infty)} (f(y, v) - f(x, v)) \pi(x, y) dy - k(x) f(x, v) \right)
\]

\(\int_{(0,\infty)} (\mathcal{P}_s f - f) \nu(ds)\) Subordination component

\[
+ \frac{\sigma^2}{2} v \frac{\partial^2 f}{\partial v^2}(x, v) + \kappa(\theta - v) \frac{\partial f}{\partial v}(x, v)
\]

\(G_v f(x,v)\)  CIR's infinitesimal generator

- State dependent jump measure \(\pi(x, y) = \int_{(0,\infty)} p(s; x, y) \nu(ds)\)

- Additional killing rate \(k(x) = \int_{(0,\infty)} P_s(x, \{0\}) \nu(ds)\)
Numerical Examples (Cont.)

Implied Volatility

Time/Strike | 30  | 35  | 40  | 45  | 50  | 55  | 60  | 65  |
---|---|---|---|---|---|---|---|---|
1/4     | 62.04 | 47.94 | 35.52 | 26.19 | 21.41 | 20.09 | 20.28 | 20.88 |
1/2     | 51.94 | 41.47 | 32.72 | 26.39 | 22.64 | 20.72 | 19.84 | 19.46 |
1       | 45.74 | 38.24 | 32.14 | 27.53 | 24.30 | 22.12 | 20.65 | 19.64 |
2       | 43.03 | 37.68 | 33.23 | 29.61 | 26.72 | 24.45 | 22.66 | 21.25 |
3       | 42.80 | 38.34 | 34.55 | 31.34 | 28.64 | 26.39 | 24.52 | 22.96 |

Implied volatility smile/skew curves as functions of the strike price.
Credit spreads and default probabilities as functions of time to maturity for current stock price levels $S = 30, 40, 50, 60, 70$. 

Rafael Mendoza (McCombs)  
Unified Credit-Equity Modeling  
Credit Risk 2009
Summary of Features and Practical Benefits of Our Modeling Framework

- Our Stock price is a jump-diffusion process with stochastic volatility and default intensity,
Summary of Features and Practical Benefits of Our Modeling Framework

- Our Stock price is a jump-diffusion process with stochastic volatility and default intensity,

- The Default intensity explicitly depends on the stock price and volatility
Summary of Features and Practical Benefits of Our Modeling Framework

- Our **Stock price** is a jump-diffusion process with stochastic volatility and default intensity,

- The **Default intensity** explicitly depends on the stock price and volatility

- The **leverage effect** is introduced in the diffusion and in jumps components - as the stock falls, the diffusion volatility and arrival rates of large jumps increase
Summary of Features and Practical Benefits of Our Modeling Framework (cont.)

- Stochastic volatility affects the diffusion and jump components
Summary of Features and Practical Benefits of Our Modeling Framework (cont.)

- Stochastic volatility affects the diffusion and jump components

- Unified credit-equity framework ⇒ consistency in the pricing and hedging of credit and equity derivatives
Summary of Features and Practical Benefits of Our Modeling Framework (cont.)

- Stochastic volatility affects the diffusion and jump components

- Unified credit-equity framework \(\Rightarrow\) consistency in the pricing and hedging of credit and equity derivatives

- We obtain \textit{analytical solutions} \(\Rightarrow\) faster computation of prices and Greeks, and faster calibration
Questions?

Thank you!
Appendix
Lévy Subordinators

Lévy subordinator

Non-decreasing Lévy process \( \{ T_t, t \geq 0 \} \) with *positive jumps and non-negative drift*

Laplace Transform (LT):

\[
\mathcal{L}(t, \lambda) = \mathbb{E}[e^{-\lambda T_t}] = e^{-t\phi(\lambda)}
\]

The Laplace exponent \( \phi(\lambda) \) is given by the Lévy-Khintchine formula:

\[
\phi(\lambda) = \gamma \lambda + \int_{(0,\infty)} (1 - e^{-\lambda s}) \nu(ds)
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\( \gamma \geq 0 \quad \Rightarrow \) positive drift
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- \( \gamma \geq 0 \) \( \implies \) positive drift
- \( \nu(ds) \) \( \implies \) Lévy measure which satisfies \( \int_{(0,\infty)} (s \wedge 1) \nu(ds) < \infty \)
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- \( \nu(ds) \) \( \Rightarrow \) Lévy measure which satisfies \( \int_{(0,\infty)} (s \land 1) \nu(ds) < \infty \)

*transition probability* \( \pi_t(ds) \) is obtained by:

\[
\int_{[0,\infty)} e^{-\lambda s} \pi(ds) = e^{-t \phi(\lambda)}
\]
Examples of Lévy Subordinators (cont.)

- The processes $T_t$ is a *Compound Poisson processes* with gamma distributed jump sizes if $Y < 0$
  - Compound Poisson process with exponential jumps ($Y = -1$)
    \[
    \nu(ds) = \alpha \eta e^{-\eta s} ds, \quad \phi(\lambda) = \gamma \lambda + \frac{\alpha \lambda}{\lambda + \eta}
    \]

- *Tempered Stable Subordinators* ($Y \in (0, 1)$)
  - *Inverse Gaussian process* ($Y = 1/2$)
  - *Gamma process* ($Y \rightarrow 0$)

- The processes with $Y \in [0, 1)$ are of infinite activity.
Martingale Property

- Intensity $h(S)$ has to be added in the drift of $X$ to compensate for jump to zero, and $\rho$ and $\mu$ are parameters to be selected to make the discounted time-changed process into a martingale:

$$
\mathbb{E}[S_{t_2} | \mathcal{F}_{t_1}] = e^{(r-q)(t_2-t_1)} S_{t_1}, \quad t_1 \leq t_2,
$$

where $r$ and $q$ are the risk-free rate and dividend yield.

- If $T_t$ is a subordinator, then $\mu$ can be arbitrary and,

$$
\rho = r - q + \phi(-\mu).
$$

- If $T_t$ is an A.C. time change, then

$$
\mu = 0, \quad \rho = r - q.
$$
Survival Probability

1. Condition w.r.t the Random Clock $T_t$

$$Q(\tau_d > t) = Q(\zeta > T_t) = \mathbb{E} \left[ \mathbb{E} \left[ e^{-\int_0^u \lambda(S_v)dv} 1_{\{T_0 > u\}} \middle| T_t = u \right] \right]$$

2. Since the Function $f(x) = 1$ is NOT in $L^2(D, m)$, we use the resolvent operator $R_\lambda$

$$Q(\zeta > T_t) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \mathcal{L}(t, -\lambda)(R_\lambda 1)(x) d\lambda,$$

3. The resolvent is available in closed form

$$R_\lambda f(x) = \int_0^\infty G_\lambda(x, y) f(y) dy$$

$G_\lambda(x, y)$ is the Resolvent Kernel or Green’s Function
Survival Probability

4. $G_\lambda(x, y)$ is known in closed form ($\mu + b > 0$):

$$G_\lambda(x, y) = \frac{\Gamma\left(\frac{\nu}{2} + 1/2 - k(\lambda)\right)}{(\mu + b)\Gamma(1 + \nu)y} \left(\frac{x}{y}\right)^{c + 1/2 - \beta} e^{A(y^{-2\beta} - x^{-2\beta})}$$

$$\times M_{k(\lambda), \frac{\nu}{2}}(A(x \wedge y)^{-2\beta}) W_{k(\lambda), \frac{\nu}{2}}(A(x \vee y)^{-2\beta})$$

where $\nu = \frac{1+2c}{2|\beta|}$, $k(\lambda) = \frac{\nu-1}{2} - \frac{\lambda}{2|\beta|(\mu+b)}$, $A = \frac{\mu+b}{a^2|\beta|}$ and,

$M_{k,m}(z)$ and $W_{k,m}(z)$ are the first and second Whittaker functions.

5. Using the Cauchy Residue Theorem to invert the Resolvent we obtain the Survival Probability
Appendix
Spectral Expansion

Assume \( \exists \, m \) on \( D \) with full support (i.e. \( \text{SSup}(m) = D \)) s.t. the (bounded) contraction semigroup \( \mathcal{P}_t \) (e.g. \( \mathcal{P}_t f(x) = \mathbb{E}_x[f(X_t)1_{\{\zeta > t\}}] \)) are symmetric on \( \mathcal{H} = L^2(D, m) \)

\[
\langle \mathcal{P}_t f, g \rangle_m = \int_D \mathcal{P}_t f \, g \, dm = \int_D f \, \mathcal{P}_t g \, dm = \langle f, \mathcal{P}_t g \rangle_m
\]

Then the infinitesimal generator \( \mathcal{G} \) is (generally unbounded) self-adjoint operator in \( \mathcal{H} \), i.e., \( \mathcal{G} \) is symmetric,

\[
\langle \mathcal{G} f, g \rangle_m = \langle f, \mathcal{G} g \rangle_m, \quad \forall f, g \in \text{Dom}(\mathcal{G})
\]

The domains of \( \mathcal{G} \) and its adjoint \( \mathcal{G}^* \) coincide in \( \mathcal{H} \), i.e. \( \text{Dom}(\mathcal{G}) = \text{Dom}(\mathcal{G}^*) \subset \mathcal{H} \)

The infinitesimal operator \( \mathcal{G} \) is non-positive in \( \mathcal{H} \), i.e. \( \langle \mathcal{G} f, f \rangle_m < 0 \) for all \( f \in \text{Dom}(\mathcal{G}) \).
Let $\mathcal{H}$ be a separable real Hilbert space and let $\{\mathcal{P}_t, t \geq 0\}$ be a strongly continuous self-adjoint contraction semigroup in $\mathcal{H}$ with the non-positive self-adjoint infinitesimal generator $\mathcal{G}$. Then there exists a unique integral representation of $\{\mathcal{P}_t, t \geq 0\}$ of the form

$$
\mathcal{P}_t f = e^{t\mathcal{G}} f = \int_{[0, \infty)} e^{-\lambda t} E(d\lambda)f, \quad f \in \mathcal{H}, \quad t \geq 0,
$$

where $E$ is the spectral measure of the negative $-\mathcal{G}$ of the infinitesimal generator $\mathcal{G}$ of $\mathcal{P}$ with the support of the spectral measure (the spectrum of $-\mathcal{G}$) $\text{Supp}(E) \subset [0, \infty)$, namely,

$$
-\mathcal{G} f = \int_{[0, \infty)} \lambda E(d\lambda)f, \quad f \in \text{Dom}(\mathcal{G}),
$$

$$
\text{Dom}(\mathcal{G}) = \left\{ f \in \mathcal{H} : \int_{[0, \infty)} \lambda^2 (E(d\lambda)f, f) < \infty \right\}.
$$

Hille and Phillips (1957, Theorem 22.3.1) and Reed and Simon (1980, Theorem VIII.6)
Discrete Case Spectral Representation

Things simplify further when the generator has a purely discrete spectrum. Let $-G$ be a self-adjoint non-negative operator with purely discrete spectrum $\sigma_d(-G) \subset [0, \infty)$. Then the spectral measure can be defined by

$$E(B) = \sum_{\lambda \in B} P(\lambda),$$

where $P(\lambda)$ is the orthogonal projection onto the eigenspace corresponding to the eigenvalue $\lambda \in \sigma_d(-G)$. Then the spectral theorem for the self-adjoint semigroup takes the simpler form:

$$P_t f = e^{tG} f = \sum_{\lambda \in \sigma_d(-G)} e^{-\lambda t} P(\lambda)f, \quad t \geq 0, \quad f \in \mathcal{H},$$

$$-G f = \sum_{\lambda \in \sigma_d(-G)} \lambda P(\lambda)f, \quad f \in \text{Dom}(G).$$

(e.g. $P(\lambda)f = c(\lambda)\phi_\lambda = \langle f, \phi_\lambda \rangle_m \phi_\lambda$)
Exponential Lévy and jump-diffusion models correspond to incomplete market models

\[ \Rightarrow \text{No perfect hedges can be found} \]
\[ \Rightarrow \text{The (equivalent) martingale measure cannot be defined in a unique way} \]

Any arbitrage-free market prices of securities can be represented as discounted conditional expectations w.r.t. a risk-neutral measure \( Q \) under which discounted asset prices are martingales

\[ \Rightarrow \text{Model Calibration. Find a risk-neutral model } Q \text{ which matches the prices of the observed market prices } V_{\{i \in I\}}(S) \text{ of securities } i \in I \text{ at time } t = 0, \]
\[ \forall i \in I, \quad V_i(S) = e^{-r t_i} \mathbb{E}^Q[f(S_{t_i})] \]
Notes on Calibration and the Implied Measure
(Cont & Tankov, 2004)

- Least Square Calibration.

\[ \theta^* = \arg \min_{\mathbb{Q}_\theta \in \mathbb{Q}} \sum_{i \in I} \omega_i \left| V_i^\theta(S, t_i) - V_i(S) \right|^2 \]

where \( \mathbb{Q} \) is the set of martingale measures

\[ \Rightarrow \text{The objective functional is non-convex.} \]

\[ \Rightarrow \text{Since the number of observable prices is finite there are multiple Lévy measures giving the same error level (multiple local minimum)} \]

- To obtain a unique solution in a stable manner we need to introduce a penalty functional (regularization) \( F \)

\[ \theta^* = \arg \min_{\mathbb{Q}_\theta \in \mathbb{Q}} \sum_{i \in I} \omega_i \left| V_i^\theta(S, t_i) - V_i(S) \right|^2 + \alpha F(\mathbb{Q}_\theta | \mathbb{P}_0) \]

where \( \mathbb{P}_0 \) is the historical measure at \( t = 0 \) and \( F \) is a convex function which penalizes the objective if \( \mathbb{Q} \) deviates much from \( \mathbb{P}_0 \) and ensures uniqueness (v.g. \( F \) relative entropy)
Jump measure and killing rate

\[(Jump \ measure) \ \pi (x, y) = 2 |\beta| AC \left( \frac{y}{x} \right)^{c - \frac{1}{2}} y^{-(2\beta + 1)} \]

\[\times \int_{(0, \infty)} s^{-3/2} e^{\left( \frac{\omega \nu}{2} - \xi - \eta \right) s} \exp \left\{ -A \left( \frac{x^{-2\beta} e^{\omega s} + y^{-2\beta}}{e^{\omega s} - 1} \right) \right\} I_{\nu} \left( \frac{A(xy)^{-\beta}}{\sinh(\omega s/2)} \right) ds. \]

and

\[\text{(killing rate) } k(x) = \]

\[C \int_{(0, \infty)} \left( 1 - \frac{\Gamma \left( \frac{c}{|\beta|} + 1 \right) \tau(s) \frac{1}{2|\beta|^2} e^{-t(s) - bs} \ _1F_1 \left( \frac{c}{|\beta|} + 1 \ ; \tau(s) \right)}{\Gamma(\nu + 1)} \right) s^{-3/2} e^{-\eta s} ds \]

where \( \tau(s) := \frac{\omega x^{-2\beta}}{2|\beta|^2 a^2 (1 - e^{-\omega s})} \),
Appendix
Appendix
Protection Payment under JDCEV

\[ PV(\text{Protection Payment}) = (1 - r) \mathbb{E} \left[ e^{-r \cdot T^\Delta} 1_{\{T^\Delta \leq T\}} \right] \]

\[ = (1 - r) \left\{ \mathbb{E} \left[ e^{-r \cdot T^L - \int_0^{T^L} h(X_u) du} 1_{\{T^L \leq T\}} \right] + \int_0^T e^{-r \cdot u} \mathbb{E} \left[ e^{-\int_0^u h(X_v) dv} h(X_u) 1_{\{T^L > u\}} \right] du \right\} \]

Recall that the first hitting time to \( L \) is given by \( T^L = \inf \{ t : X_t = L \} \), and that the first jump time to \( \Delta \) is given by

\[ \zeta = \inf \left\{ t \in [0, \infty] : \int_0^t h(X_u) du \geq e \right\} \]

The default intensity is the power function:

\[ h(X_t) = b + ca^2 X_t^{2\beta} \]

**Notice.** Since \( e \) is an exponentially distributed r.v. with unit mean, then

\[ \mathbb{P}[\zeta > t] = e^{-\int_0^t h(X_u) du} \quad \text{and} \quad \mathbb{P}[\zeta < t] = \int_0^t h(X_v) e^{-\int_0^v h(X_u) du} dv \]
Premium Payment under JDCEV

\[
PV(\text{Premium Payment}) = \varrho \cdot \Delta t \cdot \sum_{i=1}^{N} e^{-r \cdot t_i} \mathbb{E}\left[1_{\{T^A \geq t_i\}}\right]
\]

\[
= \varrho \cdot \Delta t \cdot \sum_{i=1}^{N} e^{-r \cdot t_i} \mathbb{E}\left[e^{-\int_{0}^{t_i} h(X_u)du} 1_{\{T \geq t_i\}}\right]
\]

NO jump to default & NO hitting level

The premium is paid at times \( t_i \) conditional on No default and that the stock price did Not drop to level \( L \) by time \( t_i \)

The default intensity is the power function:

\[
h(X_t) = b + ca^2 X_t^{2\beta}
\]
Accrued Interests under JDCEV

\[ PV(\text{Acc. Interest}) = \varrho \cdot \mathbb{E} \left[ e^{-r \cdot T_{L}^\Delta} \left( T_{L}^\Delta - \Delta \cdot \left[ \frac{T_{L}^\Delta}{\Delta} \right] \right) 1_{\{ T_{L}^\Delta \leq T \}} \right] \]

\[ = \varrho \sum_{i=0}^{N-1} \mathbb{E} \left[ e^{-r \cdot T_{L}^\Delta} ( T_{L}^\Delta - \Delta \cdot i ) 1_{\{ T_{L}^\Delta \in (t_i, t_{i+1}) \}} \right] \]

Expressed in terms of Diffusion and Jump components:

\[ \begin{align*}
&= \varrho \cdot \left\{ \int_{0}^{T} u e^{-r \cdot u} \mathbb{E} \left[ e^{- \int_{0}^{u} h(X_v) dv} h(X_u) 1_{\{ T_{L} \geq u \}} \right] du \right. \\
&\quad + \mathbb{E} \left[ e^{-r \cdot T_{L} - \int_{0}^{T} h(X_u) du} T_{L} 1_{\{ T_{L} \leq T \}} \right] \\
&\quad - \sum_{i=1}^{N-1} (i \cdot \Delta \cdot t_i) \int_{t_i}^{t_{i+1}} u e^{-r \cdot u} \mathbb{E} \left[ e^{- \int_{0}^{u} h(X_v) dv} h(X_u) 1_{\{ T_{L} \geq u \}} \right] du \\
&\quad \left. - \sum_{i=1}^{N-1} (i \cdot \Delta \cdot t_i) \left( \mathbb{E} \left[ e^{-r \cdot T_{L} - \int_{0}^{T} h(X_u) du} 1_{\{ T_{L} \leq t_{i+1} \}} \right] - \mathbb{E} \left[ e^{-r \cdot T_{L} - \int_{0}^{T} h(X_u) du} 1_{\{ T_{L} \leq t_i \}} \right] \right) \right\} 
\end{align*} \]
Expectations to Solve: Jump Term and Diffusion Term

- Jump Term.

\[ \mathbb{E} \left[ e^{-\int_0^u h(X_v)dv} h(X_u) 1_{\{T_L > u\}} \right] \]

Since the default intensity is given by a power function, \( h(X_t) = b + c a^2 X_t^{2\beta} \), we can solve, more generally, for a given \( p \) the expectation which we name truncated \( p \)-Moment

\[ \mathbb{E} \left[ e^{-\int_0^u h(X_v)dv} (X_u)^p 1_{\{T_L > u\}} \right] \]

- Diffusion Term.\(^1\) This term can be seen as the Expected Discount (given no default) up to the first hitting time to level \( L \)

\[ \mathbb{E} \left[ e^{-r \cdot T_L - \int_0^{T_L} h(X_u)du} 1_{\{T_L \leq T\}} \right] \]
Solving the Expectations: the truncated \( p \)-Moment

The truncated \( p \)-Moment for \( L > 0 \) and \( \mu + b > 0 \) is given by

\[
\mathbb{E}_x \left[ e^{-\int_0^t h(X_u) du} 1_{\{T_L > t\}}(X_t)^p \right] = \sum_{n=0}^{\infty} \left( \frac{M_{\nu-1/2} + n - \left( \frac{2c+p}{2|\beta|} \right), \nu}{2} (Ax)^{2\beta} - \frac{M_{\nu-1/2} + n - \left( \frac{2c+p}{2|\beta|} \right), \nu}{2} (AL)^{-2\beta} \right) 
\times \left[ \frac{M_{\nu-1/2} + n - \left( \frac{2c+p}{2|\beta|} \right), \nu}{2} (AL)^{-2\beta} \right] W_{\nu-1/2} + n - \left( \frac{2c+p}{2|\beta|} \right), \nu \right) (Ax)^{2\beta} \right] 
\times \left[ \frac{1}{2} \frac{\Gamma(1-\nu) - \Gamma(1+\nu)}{\Gamma(1+\nu)} \left[ \frac{d}{d\nu} W_{\nu-1/2} + n - \left( \frac{2c+p}{2|\beta|} \right), \nu \right] \right]_{\nu = \kappa, \nu}
\right)
\]

\[
\times \left[ 2F_2 \left( \frac{1}{2+2c+p}, \frac{1+\nu}{2+2c+p}; A^{-2\beta} \right) \right)
\]

where \( \kappa = \left\{ \kappa \mid W_{\kappa, \nu/2} (AL^{-2\beta}) = 0 \right\} \)
Solving the Expectations: the truncated $p$-Moment

The truncated $p$-Moment for $L = 0$ (CDS case) and $\mu + b > 0$ is given by

$$\mathbb{E}_x \left[ e^{-\int_0^t h(X_u)du} 1\{T_L > t\} (X_t)^p \right]$$

$$= \sum_{n=0}^{\infty} \frac{A^{\frac{1-2c-2p}{4|\beta|}} - \frac{1}{2} \left( \frac{1-p}{2|\beta|} \right)_n \Gamma \left( 1 + \frac{2c+p}{2|\beta|} \right) }{n! \Gamma (1+\nu)} x^{\frac{1}{2} - c + \beta} e^{-\frac{A}{2} x^{-2\beta}} e^{(p(\mu+b)-(b+\omega n))t}$$

$$\times M^{\nu-1} + n - (\frac{2c+p}{2|\beta|} , \nu) \left( Ax^{-2\beta} \right)$$

where $\kappa_n = \{ \kappa | W_{\kappa}, \frac{\nu}{2} (AL^{-2\beta}) = 0 \}$
Solving the Expectations: Diffusion Term

The Diffusion Term

\[
\mathbb{E}_x \left[ e^{-r \cdot T_L - \int_0^{T_L} h(X_u) du} 1_{\{T_L \leq T\}} \right] = \left( \frac{x}{L} \right)^{\frac{1}{2} - c + \beta} e^{\frac{A}{2} (x^{-2\beta} - L^{-2\beta})} \times
\]

\[
\begin{bmatrix}
W e^{\frac{1-\nu}{2} - \frac{r+\xi}{\omega}} (Ax^{-2\beta}) \\
W e^{\frac{1-\nu}{2} - \frac{r+\xi}{\omega}} (AL^{-2\beta})
\end{bmatrix}
+ \sum_{n=1}^{\infty} \frac{\omega e^{-\left( \frac{\omega}{\kappa_n - \frac{1-\nu}{2}} + r + \xi \right) T}}{\left( \omega \left( \kappa_n - \frac{1-\nu}{2} \right) + r + \xi \right)}
\begin{bmatrix}
W_{\kappa_n, \frac{\nu}{2}} (Ax^{-2\beta}) \\
\frac{\partial}{\partial \kappa} W_{\kappa, \frac{\nu}{2}} (AL^{-2\beta})
\end{bmatrix}_{\kappa = \kappa_n}
\]
Numerical Example 1: the effect of the sensitivity to variance “c”

Default Intensity function: \( h(X_t) = b + ca^2X_t^{2\beta} \). We choose \( a = \sigma/S_0^\beta = 10 \)

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
\Delta_t & r & L & S_0 & r & q & b & c & \beta & \sigma \\
0.25 & 0.5 & \{0, 15, 25\} & 50 & 0.05 & 0 & \{0, 0.02\} & \{0, 1, 2\} & -1 & 0.20 \\
\end{array}
\]
Numerical Example 2: the effect of volatility “$\sigma$”

Default Intensity function: $h(X_t) = b + ca^2X_t^{2/\beta}$. We choose $a = \sigma / S_0^{\beta} = 20$. 

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<th>$r$</th>
<th>$L$</th>
<th>$S_0$</th>
<th>$q$</th>
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