Modeling Counterparty Credit Exposure in the Presence of Margin Agreements

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Recent Advancements in the Theory and Practice of Credit Derivatives

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Discussion Plan

- Margin agreements as a means of reducing counterparty credit exposure
- Collateralized exposure and the margin period of risk
- Semi-analytical method for calculating collateralized EE
Margin agreements as a means of reducing counterparty credit exposure
Introduction

- **Counterparty credit risk** is the risk that a counterparty in an *OTC* derivative transaction will default prior to the expiration of the contract and will be unable to make all contractual payments.
  - *Exchange-traded* derivatives bear no counterparty risk.

- The primary feature that distinguishes counterparty risk from lending risk is the uncertainty of the exposure at any future date.
  - **Loan**: exposure at any future date is the outstanding balance, which is certain (not taking into account prepayments).
  - **Derivative**: exposure at any future date is the replacement cost, which is determined by the market value at that date and is, therefore, uncertain.

- **Counterparty risk is bilateral** because
  - derivative values can be both positive and negative
  - both counterparties can default
Exposure at Contract Level

- Assume that no netting or margin agreement is in place.

- Market value of contract $i$ with a counterparty is known only for current date $t = 0$. For any future date $t$, this value $V_i(t)$ is uncertain and should be assumed random.

- If a counterparty defaults at time $\tau$ prior to contract maturity, economic loss is equal to the replacement cost of the contract
  
  - If $V_i(t) > 0$, we do not receive anything from defaulting counterparty, but have to pay $V_i(t)$ to another counterparty to replace the contract.
  
  - If $V_i(t) < 0$, we receive $|V_i(t)|$ from another counterparty, but have to forward this amount to the defaulting counterparty.

- Combining these two scenarios, we can specify contract-level exposure $E_i(t)$ at time $t$ according to

$$E_i(t) = \max \{V_i(t), 0\}$$
Future value and exposure are uncertain!

Only positive outcomes (in the shaded area) result in exposure

Future Exposure

MTM (Current Exposure)

Future

5th Percentile of Value (concern of market risk)

95th Percentile of Value & Exposure

Expected Exposure

Future Exposure

Past

Today

$t = 1$ yr.
Exposure at Counterparty Level

- **Counterparty-level exposure** at future time $t$ can be defined as the loss experienced by the bank if the counterparty defaults at time $t$ under the assumption of no recovery.

- If counterparty risk is not mitigated in any way, **counterparty-level** exposure equals the sum of **contract-level** exposures

  $$ E(t) = \sum_{i} E_i(t) = \max \left\{ V_i(t), 0 \right\} $$

- If there are **netting agreements**, derivatives with positive value at the time of default offset the ones with negative value within each netting set $\text{NS}_k$, so that **counterparty-level exposure** is

  $$ E(t) = \max_{k} E_{\text{NS}_k}(t) = \max_{k} \sum_{i \in \text{NS}_k} \max \left\{ V_i(t), 0 \right\} $$

  - Each non-nettable trade represents a netting set.
Margin Agreements

- **Margin agreements** allow for further reduction of counterparty-level exposure.

- Margin agreement is a legally binding contract between two counterparties that requires one or both counterparties to post collateral under certain conditions:
  - A threshold is defined for one (unilateral agreement) or both (bilateral agreement) counterparties.
  - If the difference between the net portfolio value and already posted collateral exceeds the threshold, the counterparty must provide collateral sufficient to cover this excess (subject to minimum transfer amount).

- The threshold value depends primarily on the credit quality of the counterparty.
Exposure with Margin Agreements

Assuming that some netting sets may be covered by margin agreements, we can write bank’s exposure to the counterparty:

\[ E_C(t) = \max_{k} \max_{i \in \text{NS}_k} V_i(t) - C_k(t), \ 0 \]

where \( C_k(t) \) is the market value of collateral available to the bank for netting set \( k \) at time \( t \).

- If \( \text{NS}_k \) is not covered by a margin agreement, then \( C_k(t) \neq 0 \)

We assume the following sign convention:

- \( C_k(t) > 0 \) : at time \( t \) the bank holds collateral in the amount \(|C_k(t)|\)
- \( C_k(t) < 0 \) : at time \( t \) the bank has posted collateral in the amount \(|C_k(t)|\)
- \( C_k(t) = 0 \) : at time \( t \) the bank neither holds nor has posted collateral
Collateralized exposure and the margin period of risk
To simplify the notations, we will consider a single netting set:

\[ E_C(t) = \max\{ V(t) - C(t), 0 \} \]

where \( V(t) \) is the portfolio value for the netting set at time \( t \):

\[ V(t) = \sum_{i} V_i(t) \]

Let’s consider a **unilateral** margin agreement (in bank’s favor) with threshold \( H_{\text{cpt}} \geq 0 \) and minimum transfer amount MTA.

It is difficult to model collateral subject to MTA exactly because that would require *daily* simulation time points.

In practice, the actual threshold \( H_{\text{cpt}} \) is often replaced with the effective threshold \( H_{\text{cpt}}^{(e)} \) defined as

\[ H_{\text{cpt}}^{(e)} = H_{\text{cpt}} + \text{MTA} \]
Naive Approach

- Collateral covers excess of portfolio value $V(t)$ over threshold $H^{(e)}_{\text{cpt}}$
  \[ C(t) = \max\{ V(t) - H^{(e)}_{\text{cpt}}, 0 \} \]

- Therefore, collateralized exposure is
  \[ E_C(t) = \max\{ V(t) - C(t), 0 \} = \begin{cases} 0 & \text{if} & V(t) \leq 0 \\ V(t) & \text{if} & 0 < V(t) < H^{(e)}_{\text{cpt}} \\ H^{(e)}_{\text{cpt}} & \text{if} & V(t) \geq H^{(e)}_{\text{cpt}} \end{cases} \]

- Thus, any scenario of collateralized exposure is limited by the threshold from above and by zero from below.

- The problem with this approach is that it implicitly assumes that
  - collateral is delivered immediately
  - procedures of settling and replacing of trades start immediately when the required collateral is not posted
Margin Period of Risk

Even with daily margin call frequency, there is a significant delay $\delta t$, known as the *margin period of risk (MPR)*, between a margin call that the counterparty does not respond to and the start of the default procedures.

- Margin calls can be disputed, and it may take several days for the bank to realize that the counterparty is defaulting rather than disputing the call.
- There is a grace period after the bank issues a notice of default. During this grace period the counterparty may still post collateral.

Thus, collateral available at time $t$ is determined by portfolio value at time $t - \delta t$.

While $\delta t$ is not known with certainty, it is usually assumed to be a fixed number.
- Assumed value of $\delta t$ depends on margin call frequency and trade liquidity.
- Typical assumption for daily calls and liquid trades is $\delta t = 2$ weeks.
Including MPR in the Model

- Suppose that at time $t - \delta t$ we have collateral $C(t - \delta t)$ and portfolio value is $V(t - \delta t)$

- Then, the amount $\Delta C(t)$ that should be posted by time $t$ is

$$\Delta C(t) = \max \left\{ V(t - dt) - C(t - dt) - H^{(e)}_{\text{cpt}}, - C(t - dt) \right\}$$

- Negative $\Delta C(t)$ means that the bank will return collateral

- Collateral $C(t)$ available at time $t$ is

$$C(t) = C(t - dt) + \Delta C(t) = \max \left\{ V(t - dt) - H^{(e)}_{\text{cpt}}, 0 \right\}$$

- For comparison, collateral under the “naive” model is

$$C_{\text{naive}}(t) = \max \left\{ V(t) - H^{(e)}_{\text{cpt}}, 0 \right\}$$

- Thus, to determine collateralized exposure at time $t$, we need to simulate portfolio value both at $t - \delta t$ and at $t$. 
Full Monte Carlo Method

- Simulating exposure for collateralized counterparty
  - Collateralized exposure can go above the threshold due to MPR
Bilateral Margin Agreement

- Under a **bilateral** margin agreement, both the counterparty and the bank have to post collateral.

- Two thresholds are defined: $H_{\text{cpt}} = 0$ and $H_{\text{bnk}} = 0$
  - $H_{\text{bnk}}$ is negative because we value trades from the bank’s perspective
  - Bank posts collateral when portfolio value falls below $H_{\text{bnk}}$
  - Recall that we treat collateral posted by bank as a negative amount

- Two effective thresholds are specified:
  
  $$H_{\text{cpt}}^{(e)} = H_{\text{cpt}} + \text{MTA} \quad \quad \quad \quad \quad H_{\text{bnk}}^{(e)} = H_{\text{bnk}} - \text{MTA}$$

- After effective thresholds are defined, the bilateral margin agreement is treated as if it had zero MTA.
Collateral and Exposure for Bilateral MA

- Collateral available to bank at time $t$ is given by
  
  $$C(t) = \max\left\{ V(t - dt) - H^{(e)}_{\text{cpt}}, 0 \right\} + \min\left\{ V(t - dt) - H^{(e)}_{\text{bnk}}, 0 \right\}$$

- The two terms above describe two types of future scenarios:
  - **First term**: the bank receives collateral $C(t) > 0$
  - **Second term**: the bank posts collateral $C(t) < 0$

- Note that both terms cannot be non-zero simultaneously!

- Bank’s exposure to counterparty is still given by
  
  $$E_C(t) = \max\left\{ V(t) - C(t), 0 \right\}$$

- If the counterparty defaults when the bank has posted collateral, is there any credit exposure for the bank?
Exposure from Posting Collateral

- When the bank posts collateral, it can experience loss if the portfolio value increases by more than $|H_{bnk}^{(e)}|$ over the MPR $\delta t$.

\[ E_C(t) = \max\{V(t) - C(t), 0\} \]
Semi-analytical method for collateralized EE
Let us assume that we have run simulation only for primary time points $t$ and obtained portfolio value distribution in the form of $M$ quantities $V^{(j)}(t)$, where $j$ (from 1 to $M$) designates different scenarios.

From the set $\{V^{(j)}(t)\}$ we can estimate the unconditional expectation $\mu(t)$ and standard deviation $\sigma(t)$ of the portfolio value, as well as any other distributional parameter.

Can we estimate collateralized EE profile without simulating portfolio value at the look-back time points $\{V^{(j)}(t - dt)\}$?
Collateralized EE Conditional on Scenario

- Collateralized EE can be represented as
  \[
  EE_C(t) = E[EE_C^{(j)}(t)]
  \]
  where \( EE_C^{(j)}(t) \) is the collateralized EE conditional on \( V^{(j)}(t) \):
  \[
  EE_C^{(j)}(t) = E \max \{ V_C^{(j)}(t), 0 \} \bigg| V^{(j)}(t)
  \]
  where \( V_C^{(j)}(t) \) is the collateralized portfolio value defined as
  \[
  V_C^{(j)}(t) = V^{(j)}(t) - C^{(j)}(t)
  \]

- If we can calculate \( EE_C^{(j)}(t) \) analytically, the unconditional collateralized EE can be obtained as the simple average of \( EE_C^{(j)}(t) \) over all scenarios \( j \):
  \[
  EE_C(t) = \frac{1}{M} \sum_{j=1}^{M} EE_C^{(j)}(t)
  \]
Let us assume that portfolio value $V(t)$ at time $t$ is normally distributed with expectation $\mu(t)$ and standard deviation $\sigma(t)$.

Then, we can construct Brownian bridge from $V(0)$ to $V^{(j)}(t)$.

Conditionally on $V^{(j)}(t)$, $V^{(j)}(t - dt)$ has normal distribution with expectation

$$a^{(j)}(t) = \frac{dt}{t} V(0) + \frac{t - dt}{t} V^{(j)}(t)$$

and standard deviation

$$b(t) = s(t) \sqrt{\frac{dt(t - dt)}{t^2}}$$

Conditional collateralized EE can be obtained in closed form by integrating over a single normal variable!
Brownian bridge from $V(0)$ to $V^{(j)}(t)$

Conditionally on $V^{(j)}(t)$, the distribution of $V^{(j)}(t - dt)$ is normal with mean $a^{(j)}(t)$ and standard deviation $b(t)$
Arbitrary Portfolio Value Distribution

- We will keep the assumption that, conditionally on \( V^{(j)}(t) \), the distribution of \( V^{(j)}(t - dt) \) is normal, but will replace \( \sigma(t) \) with a local quantity \( \sigma_{\text{loc}}(t) \).

- Let us describe portfolio value \( V(t) \) at time \( t \) as
  \[
  V(t) = v(t, Z)
  \]
  where \( v(t, Z) \) is a monotonically increasing function of a standard normal random variable \( Z \).

- Let us also define a *normal equivalent* portfolio value as
  \[
  W(t) = w(t, Z) = m(t) + s(t)Z
  \]

- To obtain \( \sigma_{\text{loc}}(t) \), we will scale \( \sigma(t) \) by the ratio of probability densities of \( W(t) \) and \( V(t) \).
Scaled Standard Deviation

- Let us denote probability density of quantity $X$ via $f_X(\cdot)$ and scale the standard deviation according to

$$S_{loc}(t,Z) = \frac{f_{w(t)}[w(t,Z)]}{f_{v(t)}[v(t,Z)]} s(t)$$

- Changing variables from $W(t)$ and $V(t)$ to $Z$, we have

$$f_{v(t)}[v(t,Z)] = \frac{f(Z)}{v(t,Z)/Z}$$
$$f_{w(t)}[w(t,Z)] = \frac{f(Z)}{s(t)}$$

- Substitution to the definition of $\sigma_{loc}(t,Z)$ above gives

$$S_{loc}(t,Z) = \frac{v(t,Z)}{Z}$$
Estimating CDF

- Value of $Z^{(j)}$ corresponding to $V^{(j)}(t)$ can be obtained from
  \[ Z^{(j)} = F^{-1}\left(F_{V(t)}[V^{(j)}(t)]\right) \]

- Let us sort the array $V^{(j)}(t)$ in the increasing order so that
  \[ V^{[j(k)]}(t) = V^{(k)}_{\text{sorted}}(t) \]
  where $j(k)$ is the sorting index.

- From the sorted array we can build a piece-wise constant CDF that jumps by $1/M$ as $V(t)$ crosses any of the simulated values:
  \[ F_{V(t)}[V^{[j(k)]}(t)] = \frac{1}{2} \frac{k - 1}{M} + \frac{1}{2} \frac{k}{M} = \frac{2k - 1}{2M} \]
Estimating Derivative

- Now we can obtain $Z^{(j)}$ corresponding to $V^{(j)}(t)$ as

$$Z^{[j(k)]} = F^{-1} \frac{2k - 1}{2M}$$

- Local standard deviation $s_{\text{loc}}^{(j)}(t)$ can be estimated as:

$$s_{\text{loc}}^{[j(k)]}(t) = s_{\text{loc}}(t, Z^{[j(k)]}) = \frac{V^{[j(k+Dk)]}(t) - V^{[j(k-Dk)]}(t)}{Z^{[j(k+Dk)]} - Z^{[j(k-Dk)]}}$$

- Offset $k$ should not be too small (too much noise) or too large (loss of “locality”). This range seems to work very well:

$$20 \quad Dk \quad 0.05M$$
We assume that, conditionally on $V^{(j)}(t)$, $V^{(j)}(t - dt)$ has normal distribution with expectation

$$a^{(j)}(t) = \frac{dt}{t} V(0) + \frac{t - dt}{t} V^{(j)}(t)$$

and standard deviation

$$b^{(j)}(t) = s^{(j)}_{\text{loc}}(t) \sqrt{\frac{dt(t - dt)}{t^2}}$$

Collateralized exposure depends on $dV^{(j)}(t) = V^{(j)}(t) - V^{(j)}(t - dt)$ which is also normal conditionally on $V^{(j)}(t)$ with the same standard deviation $b^{(j)}(t)$ and expectation $da^{(j)}(t)$ given by

$$da^{(j)}(t) = V^{(j)}(t) - a^{(j)}(t) = \frac{dt}{t} V^{(j)}(t) - V(0)$$
Unilateral MA: Conditional Exposure

- Collateral available at time $t$ conditional on scenario $j$ is
  \[ C^{(j)}(t) = \max \left\{ V^{(j)}(t - dt) - H_{\text{cpt}}^{(e)}, 0 \right\} \]

- Conditional collateralized portfolio value at time $t$ is
  \[ V_C^{(j)}(t) = V^{(j)}(t) - C^{(j)}(t) = \min \left\{ V^{(j)}(t), H_{\text{cpt}}^{(e)} + dV^{(j)}(t) \right\} \]

- Conditional collateralized exposure at time $t$ is
  \[ E_C^{(j)}(t) = \max \min \left\{ V^{(j)}(t), H_{\text{cpt}}^{(e)} + dV^{(j)}(t) \right\}, 0 \]
  \[ = 1_{\{V^{(j)}(t) > 0\}} \min \left\{ V^{(j)}(t), \left[ H_{\text{cpt}}^{(e)} + dV^{(j)}(t) \right]^+ \right\} \]
Unilateral MA: Conditional EE

- Evaluating the conditional expectation, we obtain:

\[ \text{EE}_{c}^{(j)}(t) = 1_{\{V^{(j)}(t) > 0\}} \left\{ H^{(e)}_{\text{cpt}} + da^{(j)}(t) \cdot F \left( d^{(2)}_{\text{cpt}} \right) - F \left( d^{(1)}_{\text{cpt}} \right) \right. \\
+ b^{(j)}(t) \cdot f \left( d^{(2)}_{\text{cpt}} \right) - f \left( d^{(1)}_{\text{cpt}} \right) \right. \\
\left. + V^{(j)}(t) \cdot F \left( d^{(1)}_{\text{cpt}} \right) \right\} \]

where \( F() \) and \( f() \) are the CDF and the density of the standard normal distribution, respectively.

- Quantities \( d_{a}^{(1)} \) and \( d_{a}^{(2)} \) (where \( a \) can be either \( \text{cpt} \) or \( \text{bnk} \)) are defined according to

\[ d_{a}^{(1)} = \frac{H^{(e)}_{a} + da^{(j)}(t) - V^{(j)}(t)}{b^{(j)}(t)} \quad \quad d_{a}^{(2)} = \frac{H^{(e)}_{a} + da^{(j)}(t)}{b^{(j)}(t)} \]
Bilateral MA: Conditional Exposure

- Collateral available at time $t$ conditional on scenario $j$ is
  \[ C^{(j)}(t) = \max \left\{ V^{(j)}(t - dt) - H^{(e)}_{\text{cpt}}, 0 \right\} + \min \left\{ V^{(j)}(t - dt) - H^{(e)}_{\text{bnk}}, 0 \right\} \]

- Conditional collateralized portfolio value at time $t$ is
  \[ V^{(j)}_{C}(t) = \begin{cases} 
  H^{(e)}_{\text{cpt}} + dV^{(j)}(t) & \text{if } dV^{(j)}(t) < V^{(j)}(t) - H^{(e)}_{\text{cpt}} \\
  V^{(j)}(t) & \text{if } V^{(j)}(t) - H^{(e)}_{\text{cpt}} \leq dV(t) \leq V^{(j)}(t) - H^{(e)}_{\text{bnk}} \\
  H^{(e)}_{\text{bnk}} + dV^{(j)}(t) & \text{if } dV^{(j)}(t) > V^{(j)}(t) - H^{(e)}_{\text{bnk}} 
  \end{cases} \]

- Conditional collateralized exposure at time $t$ is
  \[ E^{(j)}_{C}(t) = \max \left\{ V^{(j)}_{C}(t), 0 \right\} \]
Bilateral MA: Conditional EE

Evaluating the conditional expectation, we obtain:

\[ EE_C^{(j)}(t) = 1_{V^{(j)}(t)>0} EE_C^{(j^+)}(t) + 1_{V^{(j)}(t) \leq 0} EE_C^{(j^-)}(t) \]

where \( EE_C^{(j^+)}(t) \) and \( EE_C^{(j^-)}(t) \) are given by

\[
EE_C^{(j^+)}(t) = H^{(e)}_{cpt} + da^{(j)}(t) + b^{(j)}(t) \left( d^{(2)}_{cpt} - d^{(1)}_{cpt}\right) + V^{(j)}(t) \left( d^{(1)}_{cpt} - d^{(1)}_{bnk}\right) + H^{(e)}_{bnk} + da^{(j)}(t) \left( d^{(1)}_{bnk}\right) + b^{(j)}(t) f \left( d^{(1)}_{bnk}\right) \]

and

\[
EE_C^{(j^-)}(t) = H^{(e)}_{bnk} + da^{(j)}(t) f \left( d^{(2)}_{bnk}\right) + b^{(j)}(t) f \left( d^{(2)}_{bnk}\right) \]
Example 1: 5-Year IR Swap Starting in 5 Years

Uncollateralized EE and the two thresholds we will consider
Forward Starting Swap and Small Threshold

- **Collateralized EE** when threshold is **0.5%**

![Graph showing expected exposure as a percentage of notional over time for different scenarios.]
Forward Starting Swap and Large Threshold

**Collateralized EE when threshold is 2.0%**

![Graph showing expected exposure over time for different scenarios.](image-url)

- **MPR = 0**
- **Full MC (ul)**
- **S-A (ul)**
- **Full MC (bl)**
- **S-A (bl)**
Example 2: 5-Year IR Swap Starting Now

**Uncollateralized EE** and the **two thresholds** we will consider

![Graph showing Expect exposure vs Time](image-url)
Swap Starting Now and Small Threshold

Collateralized EE when threshold is 0.5%
Swap Starting Now and Large Threshold

• **Collateralized EE** when threshold is **2.0%**
Conclusion

- **Margin agreements** are important risk mitigation tools that need to be modeled accurately.

- **Collateral** available at a primary time point depends on the portfolio value at the corresponding look-back time point.

- **Full Monte Carlo** is the most flexible approach, but it requires simulating trade values at both primary and look-back time points:
  - Simulation time is doubled in comparison to non-margined counterparties.

- We have developed a **semi-analytical** method of calculating collateralized EE that avoids doubling the simulation time:
  - Portfolio value is simulated only at primary time points.
  - For each portfolio value scenario at a primary time point, conditional collateralized EE is calculated in closed form.
  - Unconditional collateralized EE at a primary time point is obtained by averaging the conditional collateralized EE over all scenarios.