The Prym-Hitchin connection

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Moduli of vector bundles

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C smooth projective curve / $\mathbb{C}$, $\quad g =$ genus of C, $\quad g \geq 2$

$\mathcal{M}_C(r, d) =$ moduli space of semi-stable vector bundles of rank r and degree d over C / $\sim$

$\mathcal{M}_C(r, d)$ is a projective (in general, singular) variety.

Note that $\mathcal{M}_C(1, d) = \mathrm{Pic}^d(C)$ is the Picard variety of C.

We will need the *fixed determinant* moduli space. Let $L \in \mathrm{Pic}^d(C)$.
$\mathcal{M}_C(r, L) = \{E \text{ semi-stable}, \ \mathrm{rank}\ E = r, \ \det\ E = L\}$
$\mathcal{M}_C(r, L) \subset \mathcal{M}_C(r, d)$
étale Galois covering $\mathcal{M}_C(r, L) \times \mathrm{Pic}^0(C) \to \mathcal{M}_C(r, d)$

$$(E, M) \mapsto E \otimes M$$

Generalized theta divisor

Generalized theta divisor:

$\Theta = \{E \mid h^0(C,E) = h^1(C,E) > 0\} \subset \mathcal{M}_C(r, r(g-1)) \cong \mathcal{M}_C(r,0)$

For $r = 1$ we see that $\Theta \subset \mathrm{Pic}^{g-1}(C)$ is the Riemann theta divisor.

We denote by $\mathcal{L}$ the restriction of the line bundle $\mathcal{O}(\Theta)$ to

$$\mathcal{M}_C(r) := \mathcal{M}_C(r, \mathcal{O}) \subset \mathcal{M}_C(r,0).$$

The line bundle $\mathcal{L}$ is independent of the choice of translation and

$$\mathrm{Pic}(\mathcal{M}_C(r)) = \mathbb{Z} \cdot \mathcal{L}.$$

Generalized theta functions

Space of generalized (or non-abelian) theta functions of level l and rank r

$$V_C(r,l) = H^0(\mathcal{M}_C(r), \mathcal{L}^{\otimes l})$$

Remarks:

1) Why are these sections called "theta functions" ?

By analogy with the case $r = 1, d = 0$ "full moduli space" $H^0(\mathrm{Pic}^0(C), \mathcal{O}(l\Theta))$ is the space of abelian theta functions of level l, i.e. holomorphic functions on $\mathbb{C}^g$ quasi-periodic with respect to the lattice Λ. Note that $\mathrm{Pic}^0(C) = \mathrm{Jac}(C) = \mathbb{C}^g/\Lambda$.

2) There exist a canonical isomorphism (1992) between the Verlinde space $V_C(r, l)$ and the space of *conformal blocks* associated to C, the simple Lie algebra $\mathfrak{sl}(r)$ and marked point $p \in C$ labelled with the trivial representation of $\mathfrak{sl}(r)$, introduced (in much greater generality) in some rational conformal field theories.

As a consequence:

$\dim V_C(r, l) =$ Verlinde formula (1990).

How does $V_C(r,l)$ change when C moves in a family ?

Consider a flat family of smooth genus g curves

$$\begin{array}{ccc} \mathcal{C}_s & \subset & \mathcal{C} \\ \downarrow & & \downarrow \pi \\ \{s\} & \subset & S \end{array}$$

Then one can construct the relative moduli space of semi-stable vector bundles of rank r and of trivial determinant

$$p : \mathcal{M}_{\mathcal{C}/S}(r) \to S$$

and a relatively ample line bundle $\mathcal{L}$, such that $p^{-1}(s) = \mathcal{M}_{\mathcal{C}_s}(r)$.

The Hitchin connection

We define the *Verlinde bundle* as the vector bundle $\mathcal{V}(r, l) = p_*(\mathcal{L}^{\otimes l})$.

Then $\mathcal{V}(r, l)_s = H^0(\mathcal{M}_{\mathcal{C}_s}(r), \mathcal{L}^{\otimes l})$ for all $s \in S$.

Theorem (Hitchin 1990)

The Verlinde bundle $\mathcal{V}(r, l)$ is equipped with a projective connection ∇^{Hit}. This projective connection is projectively flat.

The projective connection induces a projective monodromy representation for any rank $r \geq 2$ and level $l \geq 1$

$$\rho^{Hit} : \pi_1^{top}(S, s_0) \to \mathbb{P}\mathrm{GL}(\mathcal{V}(r, l)_{s_0}).$$

The Hitchin connection

Hitchin's connection is inspired by Welters' work on heat operators in the context of abelian schemes:

$\pi : \mathcal{A} \to S$ abelian scheme, $\mathcal{L} =$ relatively ample line bundle over $\mathcal{A}$,
$\mathcal{V} := \pi_* \mathcal{L} =$ vector bundle over S such that $(\pi_* \mathcal{L})_s = H^0(\mathcal{A}_s, \mathcal{L}_s)$.

Theorem (Mumford - Welters)

There exists a natural projective flat connection ∇^{MW} on the bundle $\mathcal{V}$ given by a heat operator.

This projective connection induces a projective monodromy representation

$$\rho^{MW} : \pi_1^{top}(S, s_0) \to \mathbb{P}\mathrm{GL}(H^0(\mathcal{A}_{s_0}, \mathcal{L}_{s_0})).$$

Important fact: $H^0(\mathcal{A}_{s_0}, \mathcal{L}_{s_0})$ is an irreducible linear representation of the Mumford group $\mathcal{G}(\mathcal{L}_{s_0})$, a central extension

$$0 \to \mathbb{C}^* \to \mathcal{G}(\mathcal{L}_{s_0}) \to K(\mathcal{L}_{s_0}) \to 0.$$

and $K(\mathcal{L}_{s_0}) \subset \mathcal{A}_{s_0}$ is a finite group.

$$\mathrm{im}(\rho^{MW}) = \mathrm{Aut}^0(\mathcal{G}(\mathcal{L}_{s_0}))$$

In particular $\mathrm{im}(\rho^{MW})$ is a *finite* group.

Monodromy of Hitchin connection

Theorem (Laszlo-Sorger-P (2013), Andersen-Ueno (2014), Masbaum (1998), Funar(1997))

Let $r = 2$. If $l \neq 1, 2, 4, 8$, then there exists a family of smooth curves $\pi : \mathcal{C} \to S$ such that $\mathrm{im}(\rho^{Hit})$ is infinite.

Remarks:

1) This result was known in TQFT (Masbaum '98, Funar '97):
$\Gamma_{g,n}$ = mapping class group, $Z_{r,l}$ = vector space contructed in knot theory
"quantum representation" $\rho : \Gamma_{g,n} \to \mathrm{GL}(Z_{r,l})$.

Andersen-Ueno (2014):
"Conformal Field Theory" = "Quantum Field Theory"

Monodromy of Hitchin connection

2) Idea of proof:

degenerating conformal blocks: from smooth curves to nodal curves, then to $\mathbb{P}^1$ with 4 marked points. On $\mathbb{P}^1$ with 4 marked points monodromy = respresentation of braid group (Tsuchiya-Kanie).

Exceptional cases

1) $l = 1$, any $r \geq 2$. Strange duality

$$m : \mathcal{M}_C(r) \times \mathrm{Pic}^{g-1}(C) \to \mathcal{M}_C(r, r(g-1))$$

Then $m^*\mathcal{O}(\Theta) = \mathcal{L} \boxtimes \mathcal{O}(r\Theta)$. The generalized theta divisor induces a projectively flat isomorphism

$$SD : (H^0(\mathrm{Pic}^{g-1}(C), \mathcal{O}(r\Theta))^*, \nabla^{MW}) \to (H^0(\mathcal{M}_C(r), \mathcal{L}), \nabla^{Hit}).$$

Consequence: SD projectively flat $\Rightarrow \rho^{Hit} = \rho^{MW}$ has finite image

Exceptional cases

2) $l = 2$, any $r \geq 2$: not known whether $\mathrm{im}(\rho^{Hit})$ finite or infinite

Observation: the multiplication map

$$\mathrm{Sym}^2 H^0(\mathcal{M}_C(r), \mathcal{L}) \to H^0(\mathcal{M}_C(r), \mathcal{L}^{\otimes 2})$$

is *not* projectively flat (for the natural connections induced by Hitchin connections).

Exceptional cases

3) $l = 8$, $r = 2$: Jorgensen (2014) $\mathrm{im}(\rho^{Hit})$ is infinite

4) $l = 4$, $r = 2$:
Decompose $H^0(\mathcal{M}_C(2), \mathcal{L}^4)$ with respect to the action of $Jac(C)[2]$

$$H^0(\mathcal{M}_C(2), \mathcal{L}^4) = \bigoplus_{\alpha \in Jac(C)[2]} H^0(\mathcal{M}_C(2), \mathcal{L}^4)_\alpha$$

For $\alpha \in Jac(C)[2] \setminus \{0\}$ consider double étale cover associated to α

$$\pi_\alpha : \tilde{C} \to C, \qquad (\pi_\alpha)_*(\mathcal{O}_{\tilde{C}}) = \mathcal{O}_C \oplus \alpha.$$

$\sigma : \tilde{C} \to \tilde{C}$ is the sheet involution.

Prym varieties

Define Prym variety of covering $\tilde{C}/C$ as σ-*anti-invariant* part.

$P_\alpha^{even} = \{L \in \mathrm{Pic}^{\tilde{g}-1}(\tilde{C}) \mid \sigma^* L = K_{\tilde{C}} L^{-1}, \text{ dim } H^0(\tilde{C}, L) \text{ even}\}$

$\qquad \cong \mathrm{Prym}(\tilde{C}/C) = \{L \in \mathrm{Jac}(\tilde{C}), \ \sigma^* L = L^{-1}\}_0$

Fact: $\tilde{\Theta}_{|P_\alpha^{even}} = 2\Xi$, with Ξ principal polarization on P_α^{even}

Theorem (Oxbury- P 1997)

For any $\alpha \in Jac(C)[2]$ there is a natural isomorphism

$$\Phi_\alpha : H^0(P_\alpha^{even}, \mathcal{O}(3\Xi))_+ \to H^0(\mathcal{M}_C(2), \mathcal{L}^4)_\alpha$$

Question: Is Φ_α projectively flat (for ∇^{MW} on LHS and ∇^{Hit} on RHS) ?

If yes, then $\rho^{Hit} = \rho^{MW}$ has finite image.

Non-abelian Prym varieties

In order to study Φ_α we introduce "higher rank" Prym varieties

$$\mathcal{M}_{\tilde{C}/C}(r) := \{E \in \mathcal{M}_{\tilde{C}}(r) \mid \sigma^* E = E^*\}.$$

Theorem (Zelaci 2017)

$\mathcal{L}_{|\mathcal{M}_{\tilde{C}/C}(r)} = \mathcal{P}^{\otimes 2}$, $\mathcal{P}$ = *pfaffian line bundle of cohomology*

There is also an analogue of strange duality:

$$m : \mathcal{M}_{\tilde{C}/C}(r) \times P^{even} \to \mathcal{M}_{\tilde{C}/C}(r, r(\tilde{g}-1)) \subset \mathcal{M}_{\tilde{C}}(r, r(\tilde{g}-1))$$

with $m^*\mathcal{O}(\frac{1}{2}\tilde{\Theta}) = \mathcal{P} \boxtimes \mathcal{O}(r\Xi)$.

$$PSD : H^0(P^{even}, \mathcal{O}(r\Xi))^* \to H^0(\mathcal{M}_{\tilde{C}/C}(r), \mathcal{P}) \ \text{ is an isomorphism.}$$

Non-abelian Prym varieties

Observation: Φ_α factorizes through PSD.

$$\Phi_\alpha : H^0(P_\alpha^{even}, \mathcal{O}(3\Xi))_+ \to H^0(\mathcal{M}_{\tilde{C}/C}(3), \mathcal{P})_+ \to^{f_\alpha^*} H^0(\mathcal{M}_C(2), \mathcal{L}^4)_\alpha$$

with $f_\alpha : \mathcal{M}_C(2) \to \mathcal{M}_{\tilde{C}/C}(3)$, $E \mapsto \pi_\alpha^* End_0(E)$.

Theorem (Prym-Hitchin connection)

For a family of étale double covers $\tilde{\mathcal{C}} \to \mathcal{C} \to S$ the vector bundle $\mathcal{W}(r,l)$ over S such that $\mathcal{W}(r,l)_s = H^0(\mathcal{M}_{\tilde{\mathcal{C}}_s/\mathcal{C}_s}(r), \mathcal{P}^{\otimes l})$ is equipped with a projective flat connection $\nabla^{Prym-Hit}$.

Known facts:

1) PSD is projectively flat.

2) $f_0^* : H^0(\mathcal{M}_C(3), \mathcal{L})_+ \to H^0(\mathcal{M}_C(2), \mathcal{L}^4)_0$ is projectively flat (Belkale).

So it remains to show that f_α^* are projectively flat for $\alpha \neq 0$...

Construction of Hitchin connection

Hitchin connection comes from a heat operator = certain second order differential operator

set-up: $p : \mathcal{X} \to S$ and $\mathcal{L}$ line bundle over $\mathcal{X}$, we choose local coordinates: horizontal $t_1, \ldots, t_r$ on S and vertical $x_1, \ldots, x_n$ on $\mathcal{X}_s$

A *heat operator* D on $\mathcal{L}$ is an $\mathcal{O}_{\mathcal{X}}$-module homomorphism

$$D : p^*(T_S) \to \mathrm{Diff}^{(2)}_{\mathcal{X}}(\mathcal{L}), \text{ such that for } \frac{\partial}{\partial t} \in T_S$$

$$D(\frac{\partial}{\partial t}) = f + \frac{\partial}{\partial t} + \sum_i f_i \frac{\partial}{\partial x_i} + \sum_{i,j} f_{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

with f, f_i, f_{ij} local sections of $\mathcal{O}_{\mathcal{X}}$.

Construction of Hitchin connection

Fact: If D is a heat operator on $\mathcal{L}$, then D induces a connection ∇ on $p_*(\mathcal{L})$ by

$$\nabla_\partial(s) := D(p^{-1}(\partial))(s),$$

with s local section of $\mathcal{L}$ and ∂ local vector field on S.

How to construct heat operators ?

In some cases it suffices to contruct its symbol:

$$symb : p^*(T_S) \to \mathrm{Diff}^{(2)}_{\mathcal{X}}(\mathcal{L}) \to \mathrm{Sym}^2(T_{\mathcal{X}}),$$

Note that the image of $symb$ is contained in $\mathrm{Sym}^2(T_{\mathcal{X}/S})$, so we can view

$$symb : T_S \to p_*(\mathrm{Sym}^2(T_{\mathcal{X}/S})).$$

Under certain assumptions on p and $\mathcal{L}$ the map can be "lifted" to a heat operator D using Welters' deformation theory. (This can be done for $\mathcal{X} = \mathcal{M}_{\mathcal{C}/S}(r)$ and $\mathcal{X} = \mathcal{A}$ abelian scheme.)

The Hitchin symbol

For $\mathcal{X} = \mathcal{M}_{\mathcal{C}/S}(r)$ the Hitchin symbol at $s \in S$ is given by

$$symb_s : T_s S \to H^1(\mathcal{C}_s, T_{\mathcal{C}_s}) \to H^0(\mathcal{X}_s, \mathrm{Sym}^2 T_{\mathcal{X}_s}).$$

Note that for $E \in \mathcal{X}_s = \mathcal{M}_{\mathcal{C}_s}(r)$, we have $T_E = H^1(\mathcal{C}_s, End_0(E))$, so it is enough to give for any E a map

$$H^1(\mathcal{C}_s, T_{\mathcal{C}_s}) \to \mathrm{Sym}^2 H^1(\mathcal{C}_s, End_0(E)).$$

After applying Serre duality this is the dual of the natural map

$$\mathrm{Sym}^2 H^0(\mathcal{C}_s, End_0(E) \otimes K) \to H^0(\mathcal{C}_s, K^2), \quad \phi \otimes \psi \mapsto Tr(\phi \circ \psi).$$

Note that the latter map is also the degree 2 part of the Hitchin map.

Welters' deformation theory

Put $X = \mathcal{X}_s$, $C = \mathcal{C}_s$, $L = \mathcal{L}_{|\mathcal{X}_s}$, $s \in H^0(X, L)$.

$$
\begin{array}{cccl}
H^0(X, \mathrm{Sym}^2 T_X) & \to & \mathbb{H}^1(d^1 s) & = \mathrm{Defo}(X, L, s) \\
 & & \downarrow & \\
\uparrow \; symb & \searrow \mu_L & H^1(X, \mathrm{Diff}^{(1)}(L)) & = \mathrm{Defo}(X, L) \\
 & & \downarrow & \\
H^1(C, T_C) & \to^{ks} & H^1(X, T_X) & = \mathrm{Defo}(X)
\end{array}
$$

Compatibility condition (up to some constant): $\mu_L \circ symb = ks$.

Theorem (Construction of Prym-Hitchin connection)

1) $\mu_L = \cup([L] + \frac{1}{2}[K_X])$ *with* $[L], [K_X] \in H^1(X, \Omega^1_X)$.
2) If $Y = \mathcal{M}_{\tilde{C}/C}(r) \subset X$, *then* $L_{|Y} = P^{\otimes 2}$ *and* $K_{X|Y} = K_Y^{\otimes 2}$.
3) $2\mu_P = \mu_{L|H^0(Y, \mathrm{Sym}^2 T_Y)}$.