On Pryms, rank 2 bundles and nonabelian theta functions

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1 Introduction

Let $\mathcal{M}_0$ (resp. $\mathcal{M}_p$) denote the moduli space parametrizing semistable rank 2 bundles with determinant equal to $\mathcal{O}_C$ (resp. $\mathcal{O}_C(p)$) over a smooth, projective curve $C$ of genus $g \geq 2$; $p$ is a fixed point of $C$. The Picard group of both moduli spaces is isomorphic to $\mathbb{Z}$ and we denote by $\mathcal{L}$ (resp. $\mathcal{L}_p$) their ample generators. Then the Verlinde formula gives the dimension of the vector spaces $H^0(\mathcal{M}_0, \mathcal{L}^k)$ and $H^0(\mathcal{M}_p, \mathcal{L}_p^k)$ which consist of what is called generalized or nonabelian theta functions of level $k$.

Several authors have studied the geometry of the moduli space $\mathcal{M}_0$ in connection with the Jacobian $J$ and the Prym variety $P_x$ of an unramified double cover of the curve $C$ associated to a nonzero 2-torsion point $x$. The Kummers of all these abelian varieties can be mapped naturally to $\mathcal{M}_0$ and the intersection points of two distinct Kummers give the Schottky-Jung and Donagi relations between their theta-nulls [vG-P1]. As a consequence of these identities, van Geemen and Previato [vG-P]

\[ m_4 : S^4 H^0(\mathcal{M}_0, \mathcal{L}) \longrightarrow H^0(\mathcal{M}_0, \mathcal{L}^4) \]

is surjective.

In analogy with $\mathcal{M}_0$, the moduli space $\mathcal{M}_p$ also contains the Prym varieties $P_x$ and (a blown-up of) the Jacobian $\tilde{J}$. We observe that the varieties $\tilde{J}$ and $P_x$ intersect and that two orthogonal Pryms $P_x$ and $P_y$, although they don’t intersect, verify a geometric property, which lead to new relations among theta-constants (see section 4). Finally, we can adapt the method of [vG-P1,2] to prove the main theorem:

**Theorem 1.1** For a generic curve, the multiplication map

\[ m_2 : S^2 H^0(\mathcal{M}_p, \mathcal{L}_p) \longrightarrow H^0(\mathcal{M}_p, \mathcal{L}_p^2) \]

is surjective.

As was shown in [O-P], surjectivity of $m_2$ implies that the natural homomorphism

\[ \bigoplus_{x \in J[2]} H^0(P_x, \mathcal{O}(3\Xi_x))^\vee \longrightarrow H^0(\mathcal{M}_p, \mathcal{L}_p^2) \]
is an isomorphism. Similarly, surjectivity of $m_4$ implies an analogous isomorphism of $H^0(\mathcal{M}_0, L^4)$ with theta spaces. Recently, Ramanan [R] obtained a different (and more general) proof of both isomorphisms.

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2 Theta functions and the geometry of $\mathcal{M}_p$

2.1 Theta functions and the Heisenberg group

Let $A$ be a principally polarized abelian variety of dimension $g$ and let $\Theta$ be a symmetric divisor defining the principal polarization. The line bundle $L := \mathcal{O}_A(2\Theta)$ does not depend on the choice of $\Theta$. We will denote by $V$ the vector space of global sections $H^0(A, L)$ and by $A[2]$ the group of points of order 2. The line bundle $L$ defines a skew-symmetric bilinear pairing $\langle \cdot , \cdot \rangle : A[2] \times A[2] \rightarrow \{\pm 1\}$.

A theta characteristic [B2] of a ppav $A$ is a function $\kappa : A[2] \rightarrow \{\pm 1\}$ such that

$$\kappa(x + y) = \kappa(x)\kappa(y)\langle x, y \rangle$$

The group $A[2]$ acts on the set $\vartheta(A)$ of theta characteristics of $A$ by:

$$(x \cdot \kappa)(y) = \langle x, y \rangle\kappa(y)$$

A theta characteristic $\kappa$ is even ($\varepsilon(\kappa) = 1, \kappa \in \vartheta^+(A)$) if it takes $2^{g-1}(2^g + 1)$ times the value +1 and $2^{g-1}(2^g - 1)$ times the value −1. Otherwise $\kappa$ is odd ($\varepsilon(\kappa) = -1, \kappa \in \vartheta^-(A)$).

We have the following formula:

$$\varepsilon(x \cdot \kappa) = \kappa(x)\varepsilon(\kappa) \quad (1)$$

Suppose that $A$ is the Jacobian $J$ of a smooth projective curve $C$ of genus $g$. A theta characteristic of $C$ is a line bundle $\kappa$ such that $\kappa^{\otimes 2} \cong K_C$. One can associate to $\kappa$ a theta characteristic of $J$ by

$$\kappa(x) = (-1)^{h^0(\kappa \otimes x) + h^0(\kappa)}$$

This correspondence gives a bijection between theta characteristics of $J$ and those of $C$. Furthermore, $x \cdot \kappa$ corresponds to the line bundle $\kappa \otimes x$ and $\varepsilon(\kappa) = (-1)^{h^0(\kappa)}$.

We fix a theta structure for $L$, that is an isomorphism between the group $\mathcal{G}(L) := \{(x, \varphi) \mid \varphi : t_xL \cong L\}$, where $t_x$ denotes translation by $x$, and the Heisenberg group [M1] defined as a set by

$$\text{Heis}(g) = \mathbb{C}^* \times K(g) \times \widehat{K(g)}$$

where $K(g) = (\mathbb{Z}/2)^g$, $\widehat{K(g)} = \text{Hom}((\mathbb{Z}/2)^g, \mathbb{C}^*)$ and multiplication is defined by

$$(s, a, \alpha)(t, b, \beta) = (st\beta(a), a + b, \alpha\beta)$$

2
Via the theta structure \( \text{Heis}(g) \) acts on the vector space \( V \) and, by Mumford’s theta theory, \( V \) is an irreducible representation of \( \text{Heis}(g) \). There exists a unique (up to scalar) basis \( \{X_b\} \), with \( b \in K(g) \), for \( V \) such that

\[
(s, a, \alpha)X_b = s\alpha(b + a)X_{b+a} \tag{2}
\]

The theta structure allows us to define for \( (c, \gamma) \in (\mathbb{Z}/2)^g \times \text{Hom}((\mathbb{Z}/2)^g, \mathbb{C}^*) \) a theta characteristic \( \kappa = \kappa \left[ \begin{smallmatrix} c \\ \gamma \end{smallmatrix} \right] \) by the formula

\[
\kappa(a, \alpha) = \gamma(a)a(a + c)
\]

One has the formula \( \varepsilon(\kappa) = \gamma(c) \).

The Abelian variety \( A \) can be identified to the quotient of the tangent space \( T_0(A) \) by a lattice \( \Gamma \). We choose a symplectic basis \( \{\xi_1, \ldots, \xi_{2g}\} \) of \( \Gamma \) with respect to the non-degenerate skew-symmetric bilinear pairing given by the principal polarization on \( A \). The first \( g \) vectors form a basis of \( T_0(A) \) and there exists a matrix \( \Omega \) of the Siegel upper half-space such that \( \Gamma = \Gamma_\Omega = \mathbb{Z}^g \oplus \Omega\mathbb{Z}^g \). The space \( V \) can be identified to the space of \( \Gamma \)-quasi-periodic second order theta functions. The symplectic basis \( \{\gamma_1, \ldots, \gamma_{2g}\} \) defines an isomorphism between \( K(g) \times \widehat{K(g)} \) and \( A[2] \), which extends to a theta structure for \( L \).

For \( (c, \gamma) \in K(g) \times \widehat{K(g)} \) we can define a theta function

\[
\theta \left[ \begin{smallmatrix} c \\ \gamma \end{smallmatrix} \right](z, \Omega) = \sum_m \exp\pi i l^t(m + \frac{1}{2}\hat{c})\Omega(m + \frac{1}{2}\hat{c}) + 2^t(m + \frac{1}{2}\hat{c})(z + \frac{1}{2}\hat{\gamma}))
\]

where we sum over \( m \in \mathbb{Z}^g \), and \( \hat{c} \) and \( \hat{\gamma} \) are representatives in \( \mathbb{Z}^g \) of \( c \) and \( \gamma \) (via the canonical isomorphism of \( K(g) \) with \( \widehat{K(g)} \)). This theta function will also be denoted by \( \theta_\kappa(z, \Omega) \). A basis, verifying relation (2), is given by the functions

\[
X_b = \theta \left[ \begin{smallmatrix} b \\ 0 \end{smallmatrix} \right](2z, 2\Omega) \quad \forall b \in K(g)
\]

We can associate to a theta characteristic \( \kappa = \kappa \left[ \begin{smallmatrix} c \\ \gamma \end{smallmatrix} \right] \) a character \( \chi_\kappa \) of order 2 of the Heisenberg group \( \text{Heis}(g) \) defined by \( \chi_\kappa(s, a, \alpha) = s^2\gamma(a)a(c) \), and an element

\[
\xi_\kappa = \sum_{b \in K(g)} \gamma(b)X_b \otimes X_{b+c} \in V \otimes V
\]

\( \xi_\kappa \) is an eigenvector under the action of \( \text{Heis}(g) \) associated to the character \( \chi_\kappa \) and the vectors \( \{\xi_\kappa\} \) for \( \kappa \in \vartheta^+(A) \) (resp. \( \kappa \in \vartheta^-(A) \)) form a basis of \( S^2V \) (resp. \( \Lambda^2V \)). We shall identify \( S^2V \) (resp. \( \Lambda^2V \)) with the invariant (resp. anti-invariant) subspace of \( V \otimes V \) under the involution \( \varphi \otimes \psi \leftrightarrow \psi \otimes \varphi \). In the sequel, we will use the addition formula

\[
\theta_\kappa(z + u, \Omega)\theta_\kappa(z - u, \Omega) = \sum_{b \in K(g)} \gamma(b)\theta \left[ \begin{smallmatrix} b \\ 0 \end{smallmatrix} \right](2z, 2\Omega)\theta \left[ \begin{smallmatrix} b + c \\ 0 \end{smallmatrix} \right](2u, 2\Omega) \tag{3}
\]

Finally, recall that the theta functions \( \theta_\kappa(2z, \Omega) \) of order 4 are eigenvectors under the natural action of \( \text{Heis}(g) \) on \( H^0(A, L^{\otimes 2}) \) associated to the character \( \chi_\kappa \) and that they form a basis of \( H^0(A, L^{\otimes 2}) \) for \( \kappa \in \vartheta(A) \).
2.2 The space of sections $H^0(\mathcal{M}_p, \mathcal{L}_p)$

We recall the main results of [B2]. Let $l \in J$, such that $l^{\otimes 2} \not\equiv \mathcal{O}_C$. We can define a stable rank 2 bundle $F_l \in \mathcal{M}_p$ which fits in an exact sequence

$$0 \to l \oplus l^{-1} \to F_l \to \mathcal{C}_p \to 0$$

The map $l \mapsto F_l$ can be extended to a morphism $j_p : \hat{J} \to \mathcal{M}_p$ where $\hat{J}$ denotes the blow-up at the points of order 2 of $J$. Let $F \in \mathcal{M}_p$. For any surjective morphism $u : F \to \mathcal{C}_p$, the vector bundle $\ker(u)$ is semistable. Thus we obtain a morphism from the projective line $\mathbb{P} \mathcal{F}$, which parametrize, up to homothety, nonzero morphisms $F \to \mathcal{C}_p$, to the moduli space $M_0$. In [B1], Beauville gives an identification of $H^0(M_0, \mathcal{L})$ with $V := H^0(J, \mathcal{O}(2\Theta))$. Therefore we obtain a morphism $\varphi : M_0 \to \mathbb{P} \mathcal{V}$. The image of the composite $\mathbb{P} \mathcal{F} \to M_0 \to \mathbb{P} \mathcal{V}$ is a line $\mathbb{A}^1$ by the Plücker embedding. This defines a map $\varphi_p : \mathcal{M}_p \to \mathbb{A}^1 \mathcal{V}$ and the composite $\varphi_p \circ j_p : \hat{J} \to \mathbb{A}^1 \mathcal{V}$ is the Gauss map associated to the vector field $D$, tangent to $C$ at $p$, where $C$ is embedded in $J$ by the map $q \mapsto \mathcal{O}_C(q - p)$. The following diagram is commutative:

$$\begin{array}{ccc}
\Lambda^2 V & \xrightarrow{\varphi_p^*} & H^0(M_p, \mathcal{L}_p) \\
w & & \downarrow j_p^* \\
H_0^0(J, \mathcal{O}(4\Theta)) & & \\
\end{array}$$

where $w_D$ is the Wahl map associated to the vector field $D$ and the subscript $-$ means odd theta functions. Choose a period matrix $\tau$ for the Jacobian $J$. In this paper we will assume that

\[
\begin{cases}
\forall \kappa \in \vartheta^-(J), \ h^0(C, \kappa(-p)) = 0 \iff D\theta_\kappa(0, \tau) \neq 0 \\
\forall \kappa \in \vartheta^+(J), \ h^0(C, \kappa) = 0 \iff \theta_\kappa(0, \tau) \neq 0
\end{cases}
\]

These conditions are verified for a generic curve. Under these assumptions all the linear maps of the diagram above are isomorphisms and there exist a nonzero constant $c$ such that $\forall \kappa \in \vartheta^-(J)$

$$w_D(\xi_\kappa) = c \cdot D\theta_\kappa(0, \tau) \cdot \theta_\kappa(2z, \tau)$$

3 Prym varieties

We can associate to any nonzero 2-torsion point $x \in J[2]$ an unramified double cover $\pi_x : C_x \to C$. We shall denote by $\sigma$ the involution of $C_x$ given by sheet-interchange over $C$. The norm map $\text{Nm}_x : \text{Jac}(C_x) \to J$ maps $D \mapsto \pi_x D$ for any divisor $D$ on $C_x$. The kernel $\ker(\text{Nm}_x)$ has two connected components. We will denote by $P_x$, the Prym variety,
the component containing 0 and by $P_x^-$ the other component. $P_x$ has a natural principal polarization $\Xi_x$. Let’s recall the following facts ([M2] and [vG-P1]):

We have $\ker(\pi_x^*) = \langle x \rangle$ and $\pi_x^*$ induces a symplectic isomorphism

$$\pi_x^*: x^\perp/\langle x \rangle \xrightarrow{\sim} P_x[2]$$

where $x^\perp = \{ y \in J[2] \mid \langle x, y \rangle = 1 \}$. Any $y \in J[2]$ with $\langle x, y \rangle = -1$ gives by tensor product with $\pi_x^*(y)$ an isomorphism $P_x \xrightarrow{\sim} P_x^-$. Fix $z_x \in J$ with $z_x^{\otimes 2} \cong x$, we get a map:

$$\psi_x: \ker(Nm_x) = P_x \cup P_x^- \rightarrow \mathcal{M}_0$$

$$M \mapsto \pi_{x*}(M) \otimes z_x$$

The image of $\psi_x$ does not depend on the choice of $z_x$ (although $\psi_x$ does!) and $\psi_x$ is equivariant for the actions of $J[2]$ on $\ker(Nm_x)$ and $\mathcal{M}_0$. The group $J[2]$ acts by tensorization on $\mathcal{M}_0$ and this action induces a projective representation of $J[2]$ in $\mathbb{P}H^0(\mathcal{M}_0, \mathcal{L})$, which can be identified with $\mathbb{P}\mathcal{V}$. The composite map $\Phi_x = \varphi \circ \psi_x: P_x \cup P_x^- \rightarrow \mathbb{P}\mathcal{V}$ is also $J[2]$-equivariant. The (projective) linear map corresponding to $x$ has two eigenspaces $\mathbb{P}\mathcal{V}_x$ and $\mathbb{P}\mathcal{V}_x^-$ in $\mathbb{P}\mathcal{V}$ and there is one component of $\ker(Nm_x)$ in each eigenspace. By proposition 1 [vG-P1] the (restricted) map $\phi_x: P_x \rightarrow \mathbb{P}\mathcal{V}_x$ is given by the linear series $|2\Xi_x|$ and by an identification $\mathbb{P}\mathcal{V}_x \cong \mathbb{P}H^0(P_x, \mathcal{O}(2\Xi_x))$.

We can summarize these facts in the following commutative diagram:

Now we will describe an “odd degree” version of this diagram. Fix a point $p_x \in C_x$ with $\pi_x(p_x) = p$ and consider the map:

$$\psi_x': P_x \cup P_x^- \rightarrow \mathcal{M}_p$$

$$M \mapsto \pi_{x*}(M(p_x)) \otimes z_x$$

This map is $J[2]$-equivariant and the image of $P_x \cup P_x^-$ in $\mathcal{M}_p$ does not depend on the choice of the point $p_x$. Moreover, the images of both components are the same. Thus we will restrict $\psi_x'$ to the zero component $P_x$.

**Lemma 3.1**

$$(\psi_x' \circ t_{\alpha^{-1}_x})^* \mathcal{L}_p \cong \mathcal{O}_{P_x}(4\Xi_x)$$

where $\alpha_x \in P_x$ is any point satisfying $\alpha_x^{\otimes 4} \cong \mathcal{O}_{C_x}(2p_x - 2\sigma p_x)$, and $t_{\alpha^{-1}_x}$ denotes translation by $\alpha^{-1}_x$. 

5
Lemma 3.3

∀ the eigenspaces = \{x\} in the subspace I P

[2] denotes

the duplication map on P_x and \Xi_N the theta divisor

on P_x with support \{M \in P_x \mid h^0(C_x, M \otimes N) > 0\}. The line bundle N has degree 2g - 2 and can be computed (lemma 1.5 [B2]) N = O_{C_x}(p_x - \sigma p_x) \otimes \alpha_x^{\otimes -2} \otimes \pi_x^* \kappa. A basic fact for Prym varieties is that O_{C_x}(p_x - \sigma p_x) \subseteq P_x. Hence, there exists x' \in J[2] with \langle x, x' \rangle = -1 such that \pi_x^*(x') = \alpha_x^{\otimes 2} \otimes O_{C_x}(\sigma p_x - p_x) and we have N = \pi_x^*(\kappa \otimes x'). Therefore N^{\otimes 2} = \pi_x^*(K_C) = K_{C_x}.

\[ \text{Proof:} \quad \varphi \circ \psi' : P_x \rightarrow M_p \rightarrow IP^2V \text{ is contained in the subspace } IP^2V. \]

\[ \text{Proof:} \quad \text{Fix a theta structure for } O(2\Theta) \text{ on } J. \text{ The point } x \in J[2] \text{ is represented by } x = [a]. \quad \text{and } (1, a, \alpha) \cdot (1, a, \alpha) = (\alpha(a), 0, 1). \text{ The eigenvalues of } (1, a, \alpha) \text{ corresponding to the eigenspaces } V_x \text{ and } V_x^- \text{ are of opposite sign, hence } (1, a, \alpha) \text{ acts on } \Lambda^2 V_x \text{ and } \Lambda^2 V_x^- \text{ as } \alpha(a) Id \text{ and on } W_x \text{ as } -\alpha(a) Id. \text{ Consider } \kappa \in \vartheta^-(J). \text{ The relation } \]

\[ (1, a, \alpha)\xi_\kappa = \chi_\kappa(1, a, \alpha)\xi_\kappa = \alpha(a)\kappa(x)\xi_\kappa \]

implies that

\[ \xi_\kappa \in W_x \iff \kappa(x) = -1 \]

\[ \iff \varepsilon(x \cdot \kappa) = 1 \quad \text{relation(1)} \]

\[ \iff h^0(C, \kappa \otimes x) = 0 \quad \text{assumption(*)} \]

Now we can conclude using lemma 1.5 a) and b) of [B2].

Thus we obtain a commutative diagram:

\[ P_x \cup P_x^- \xrightarrow{h_x} IP^2V_x \]

\[ \psi \downarrow \quad \cap \quad \underline{\phi} \downarrow \]

\[ M_p \xrightarrow{\varphi_p} IP^2V \]

Lemma 3.3 \forall M \in P_x, the line h_x(M) in IPV passes through the points \phi_x(M \otimes \alpha_x^{-1}) \in IPV_x \text{ and } \phi_x(M \otimes \alpha_x^{-1}(p_x - \sigma p_x)) \in IPV_x-.
Proof: By definition of the line $\varphi_p^\gamma(\psi^\gamma_x(M))$, it suffices to prove that $\psi_x(M \otimes \alpha^{-1}_x)$ and $\psi_x(M \otimes \alpha^{-1}_x(p_x - \sigma p_x))$ are subbundles of $\psi^\gamma_x(M)$. Consider the exact sequence on $C_x$

$$0 \rightarrow \mathcal{O}_{C_x} \rightarrow \mathcal{O}_{C_x}(p_x) \rightarrow \mathcal{C}_{p_x} \rightarrow 0$$

Tensorizing with $M \otimes \alpha^{-1}_x \otimes \pi^*_x z_x$, taking direct image by $\pi_x$ and using the projection formula gives

$$0 \rightarrow \pi^*_x(M \otimes \alpha^{-1}_x) \rightarrow \psi^\gamma_x(M) \rightarrow \mathcal{C}_p \rightarrow 0$$

The other point is obtained in the same way from the exact sequence

$$0 \rightarrow \mathcal{O}_{C_x} \rightarrow \mathcal{O}_{C_x}(\sigma p_x) \rightarrow \mathcal{C}_{\sigma p_x} \rightarrow 0$$

after tensorization with $M \otimes \alpha^{-1}_x(p_x - \sigma p_x) \otimes \pi^*_x z_x$ \hspace{1cm} \Box

Using lemma 3.2 and 3.3, we can decompose the composite $h_x : P_x \rightarrow \mathbb{P}W_x$

$$P_x \rightarrow \mathbb{P}V_x \times \mathbb{P}V_x^- \xrightarrow{\mu} \mathbb{P}W_x$$

where the first arrow maps $M \mapsto (\phi_x(M \otimes \alpha^{-1}_x), \phi_x(M \otimes \alpha^{-1}_x(p_x - \sigma p_x))$ and $\mu$ maps the pairs of points $(\nu, \eta)$ to the line in $\mathbb{P}V$ passing through $\nu$ and $\eta$. Recall that the choice of $\alpha_x$ determines a 2-torsion point $x' \in J[2]$ with $\langle x, x' \rangle = -1$ such that $\pi^*_x(x') = \alpha^{\otimes 2}_x \otimes \mathcal{O}_{C_x}(\sigma p_x - p_x)$.

Fix a theta structure on $J$, such that $x = [0, 0]_1$ and $x' = [0, 1]_1$. We can choose e.g. a symplectic basis $(\gamma_1, \ldots, \gamma_2g)$ such that $x = \frac{1}{2} \gamma_1 + \Gamma \tau \in \mathbb{C}^g/\Gamma \tau \cong J$ and $x' = \frac{1}{2} \gamma_2 + \Gamma \tau$, where $\Gamma$ is the lattice associated to the period matrix $\tau$ of $J$. By the canonical isomorphism $\pi^*_x : x^+/\langle x \rangle \rightarrow P_x[2]$, we can identify $P_x[2]$ with the subgroup of $K(g) \times \widehat{K(g)}$ given by $\{ [d, \delta] \} : (d, \delta) \in K(g - 1) \times \widehat{K(g - 1)}$. We have

$$\xi_x \in W_x \iff \kappa = \kappa [d, 1] \delta \begin{array}{c} 0 \\ 1 \end{array}$$

with $\delta(d) = -1$

or $\kappa = \kappa [d, 1] \delta \begin{array}{c} 0 \\ 1 \end{array}$ with $\delta(d) = 1$

Consider $\kappa \in \vartheta^-(J)$, with $\varepsilon(x \cdot \kappa) = 1$ (i.e. $\xi_x \in W_x$). Then $(x' \cdot \kappa)$ induces a theta characteristic on $x^+/\langle x \rangle$, because

$$\forall y \in x^+ (x' \cdot \kappa)(y + x) = (x' \cdot \kappa)(y)(x' \cdot \kappa)(x) y, x)$$

$$= (x' \cdot \kappa)(y) \kappa(x, x')(y, x)$$

$$= (x' \cdot \kappa)(y)$$

This theta characteristic of $P_x$ will be denoted by $\bar{\kappa} = \bar{\kappa} [d, 1]$. This correspondence gives a (noncanonical) bijection
\[ \vartheta_x(J) \sim \vartheta(P_x) \]
\[ \kappa \mapsto \bar{\kappa} \]

where \( \vartheta_x(J) = \{ \kappa \in \vartheta(J) \mid \varepsilon(x \cdot \kappa) = 1 \} \). A basis of \( V_x \) (resp. \( V_x^- \)) is given by \( \{X_{(b,0)}\} \) (resp. \( \{X_{(b,1)}\} \)) for \( b \in K(g - 1) \).

**Lemma 3.4** Choose a period matrix \( \omega_x \) for the Prym \( P_x \). Then the composite \( h_x \) induces by pull-back (up to homothety) the linear map

\[ h_x^*: W_x \rightarrow H^0(P_x, \mathcal{O}(4\Xi_x)) \]
\[ \xi_{\kappa} \mapsto \theta_{\bar{\kappa}}(-2\tilde{\alpha}_x, \omega_x) \theta_{\bar{\kappa}}(2z, \omega_x) \]

**Proof:** An easy computation shows that the pull-back \( \mu^*: W_x \rightarrow V_x \otimes V_x^- \) maps \( \xi_{\kappa} \mapsto 2 \sum_b \delta(b)X_{(b,0)} \otimes X_{(b+d,1)} \), with \( b \) running over \( K(g-1) \). The 2-torsion point \( x' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) gives a linear isomorphism, up to homothety, \( V_x^- \sim \rightarrow V_x \) with \( X_{(b,1)} \mapsto X_{(b,0)} \). Under the natural identifications of \( V_x \) with \( H^0(P_x, \mathcal{O}(2\Xi_x)) \), and \( H^0(P_x, \mathcal{O}(4\Xi_x)) \) with the space of \( \Gamma_{\omega_x} \)-quasi-periodic theta functions of order 4, we obtain a map (up to homothety)

\[ \xi_{\kappa} \mapsto 2 \sum_b \delta(b) \theta \begin{bmatrix} b \\ 0 \end{bmatrix} (2z - 2\tilde{\alpha}_x, 2\omega_x) \theta \begin{bmatrix} b + d \\ 0 \end{bmatrix} (2z + 2\tilde{\alpha}_x, 2\omega_x) \]

\[ = \theta_{\bar{\kappa}}(-2\tilde{\alpha}_x, \omega_x) \theta_{\bar{\kappa}}(2z, \omega_x) \]

where the last equality is obtained by the addition formula (3) and \( \tilde{\alpha}_x \) is a representative in \( \mathbb{C}^g \) of \( \alpha_x \in \mathbb{C}^g/\Gamma_{\omega_x} \cong P_x \). \( \square \)

### 4 Schottky-Jung and Donagi relations

We shall now describe the intersection points of (the Kummers of) the Jacobian and the Pryms mapped to \( \mathcal{M}_0 \) and \( \mathcal{M}_g \). This method was used [vG-P1] to give a vector bundle theoretic proof of the Schottky-Jung and Donagi relations. We denote by \( \varphi_L: J \rightarrow \mathbb{P}V \) the map given by the linear series \( |2\Theta| \). We consider two orthogonal nonzero 2-torsion points \( x, y \in J[2], \langle x, y \rangle = 1 \), which define 2-torsion points of \( P_x \) and \( P_y \), namely \( \bar{x} := x + (y) \in y^+/(y) \cong P_y[2] \) and \( \bar{y} := y + \langle x \rangle = x^+/(x) \cong P_x[2] \).

**Proposition 4.1** (a) the Schottky-Jung relations: \( \exists z_x \in J \) with \( z_x^{\otimes 2} \cong x \) such that

\[ \varphi_L(z_x) = \phi_x(0) \]

(b) the Donagi relations: \( \exists u_y \in P_y \) with \( u_y^{\otimes 2} \cong \bar{x} \) and \( \exists u_x \in P_x \) with \( u_x^{\otimes 2} \cong \bar{y} \) such that

\[ \phi_x(u_x) = \phi_y(u_y) \]
The following proposition is an “odd degree” version of the classical relations.

**Proposition 4.2** With the same notation as above, we have

(a) \( \varphi_p \circ j_p(z_x) = h_x(\alpha_x) \)

\[
\begin{array}{c}
\varphi_p \circ j_p \\
\downarrow \quad \varphi_p \circ j_p \\
\text{IP}^2V \\
\cup \\
P_x \quad h_x \\
\quad \text{IP} W_x
\end{array}
\]

(b) \( p_{x,y}[h_x(u_x \otimes \alpha_x)] = p_{x,y}[h_y(u_y \otimes \alpha_y)] \)

\[
\begin{array}{c}
P_x \quad h_x \\
\quad \text{IP} W_x \\
\quad p_{x,y} \\
\quad \text{IP} W_{x,y} \\
\quad p_{x,y} \\
P_y \quad h_y \\
\quad \text{IP} W_y
\end{array}
\]

where \( p_{x,y} \) is the projection \( \text{IP}^2V \to \text{IP} W_{x,y} := \text{IP} W_x \cap W_y \).

**Remark:** One can show, using proposition 2.4 [R], that the Pryms \( P_x \) and \( P_y \) don’t intersect in \( \mathcal{M}_p \). However, two Pryms associated to nonorthogonal 2-torsion points intersect in \( \mathcal{M}_p \). This intersection property, although natural, is not used in the sequel of this paper.

**Proof:** Let \( z_x \), with \( z_x^\otimes 2 \cong x \), define the map \( \psi_x : P_x \to \mathcal{M}_0 \). Then

\[
\psi_x(0) = \pi_{x*}(\mathcal{O}_{C_x}) \otimes z_x = (\mathcal{O}_C \oplus x) \otimes z_x = z_x \oplus z_x^{-1}
\]

Taking the image by \( \varphi_C \) gives the classical Schottky-Jung relation (prop. 4.1(a)). The two rank 2 bundles \( \psi_x'(\alpha_x) = \pi_{x*}(\mathcal{O}_{C_x}(p_x)) \otimes z_x \) and \( F_{x} = j_p(z_x) \) (see section 2) fit into the exact sequence

\[
0 \longrightarrow z_x \oplus z_x^{-1} \longrightarrow \psi_x'(\alpha_x) \longrightarrow \mathcal{C}_p \longrightarrow 0
\]

As \( \psi_x'(\alpha_x) \) is a stable bundle, it is nonsplit and hence isomorphic to \( j_p(z_x) \). Now apply \( \varphi_p \) to get the first relation.

The main step of the proof given by van Geemen of the Donagi relations is the existence of a stable rank 2 bundle

\[
E := \psi_x(u_x) = \psi_y(u_y) \in \mathcal{M}_0
\]

Consider the projective line \( \text{IP} := \text{IP} \text{Ext}^1(\mathcal{C}_p, E) \), parametrizing isomorphism classes of extensions of \( \mathcal{C}_p \) by \( E \), and the classifying morphism \( \text{IP} \to \mathcal{M}_p \). The proof of the following lemma is similar to the proof of lemma 3.4 of [B2].
Lemma 4.3 The image of the composite $\mathbb{IP} \rightarrow \mathcal{M}_p \xrightarrow{\varphi_p} \mathbb{IP}^2 \mathcal{V}$ is a conic, hence contained in a plane.

Consider the four lines in $\mathbb{IP} \mathcal{V}$ (here we identify bundles in $\mathcal{M}_p$ with their image by $\varphi_p$ in $\mathbb{IP}^2 \mathcal{V}$)

$$l_x := \pi_{xx}(u_x(p_x)) \otimes z_x = h_x(u_x \otimes \alpha_x)$$
$$l'_x := \pi_{xx}(u_x(\sigma p_x)) \otimes z_x$$
$$l_y := \pi_{yy}(u_y(p_y)) \otimes z_y = h_y(u_y \otimes \alpha_y)$$
$$l'_y := \pi_{yy}(u_y(\sigma p_y)) \otimes z_y$$

with $l_x, l'_x \in \mathbb{IP} W_x$ and $l_y, l'_y \in \mathbb{IP} W_y$. They pass through the point $A = \phi_x(u_x) = \phi_y(u_y) \in \mathbb{IP} V_x \cap V_y$.

By lemma 3.3 the line $l_x$ passes through the point

$$B := \phi_x(u_x \otimes \alpha_x^2) \otimes x' = \phi_x(u_x(p_x - \sigma p_x)) \in \mathbb{IP} V_x^-$$
Similarly we define the points (see picture)

\[
B' := \phi_x(u_x(\sigma p_x - p_x)) \in \mathbb{IP}V_x^- \\
C := \phi_y(u_y(p_y - \sigma p_y)) \in \mathbb{IP}V_y^- \\
C' := \phi_y(u_y(\sigma p_y - p_y)) \in \mathbb{IP}V_y^- 
\]

Recall that the eigenspaces \( \mathbb{IP}V_x \) and \( \mathbb{IP}V_x^- \) are stable under the action of \( y \), since \( \langle x, y \rangle = 1 \). We have \( \forall M \in \ker(Nx), \phi_x(M) = \phi_x(M^{-1}) \), which implies that

\[
B' = \phi_x(u_x^{-1}(p_x - \sigma p_x)) = \phi_x(u_x(p_x - \sigma p_x)) \otimes \bar{y} = y.B
\]

where \( y.B \) denotes the image of \( B \) by the projective linear action of \( y \) on \( \mathbb{IP}V_x^- \). Similarly \( C' = x.C \).

Consider the pencil \( \varphi_x \) of lines generated by the lines \( l_x \) and \( l'_x \). Each line of \( \varphi_x \) passes through the point \( A \) and intersects the eigenspace \( \mathbb{IP}V_x^- \) in a point of the line \( (BB') \). Similarly we consider the pencil \( \varphi_y \) generated by \( l_y \) and \( l'_y \) (see picture). The pencils \( \varphi_x \) and \( \varphi_y \) determine a line in \( \mathbb{IP}A^2V \) passing through the points \( l_x \) and \( l'_x \) (for \( \varphi_x \)) and through \( l_y \) and \( l'_y \) (for \( \varphi_y \)). Now the four points \( l_x, l'_x, l_y, l'_y \in \mathbb{IP}A^2V \) lie on a conic (lemma 4.3) and therefore are contained in a plane. Hence the lines \( (l_x l'_x) \) and \( (l_y l'_y) \) intersect in \( \mathbb{IP}A^2V \), i.e. the pencils \( \varphi_x \) and \( \varphi_y \) have a common line \( l \). This line \( l \) intersects \( \mathbb{IP}V_x^- \) in a point on \( (BB') \) and \( \mathbb{IP}V_y^- \) in a point on \( (CC') \). Thus \( l \) intersects the subspace \( \mathbb{IP}V_x^- \cap V_y^- \) in a point \( S = (CC') \cap (BB') \).

Consider the direct sum:

\[
W_y = W_{x,y} \oplus W_y \cap (\Lambda^2V_x \oplus \Lambda^2V_x^-)
\]

where \( W_{x,y} = W_x \cap W_y \). We will show that \( p_{x,y}(l_y) = l \) (up to a scalar) where \( p_{x,y} : W_x \to W_{x,y} \) is the projection given by the direct sum. Combining this result with its analogue \( p_{x,y}(l_x) = l \) allows us to conclude.

Choose a theta structure for \( L = \mathcal{O}(2\Theta) \) on \( J \). The point \( x \in J[2] \) is represented by \( x = (1, a, \alpha) \). As in the proof of lemma 3.2, \( x \) acts on \( W_{x,y} \) as \( -\alpha(a)I_d \) and on \( W_y \cap (\Lambda^2V_x \oplus \Lambda^2V_x^-) \) as \( \alpha(a)I_d \). Choose a representative in \( W_y \), which we also denote by \( l_y \), of the line \( l_y = (AC) \in \mathbb{IP}W_y \). As \( C' = x.C \) and \( A = x.A \), we have \( x.l_y = (AC') = l'_y \) and \( x.l'_y = l_y \). Thus we obtain the direct sum decomposition:

\[
l_y = \frac{1}{2}(l_y - \alpha(a)l'_y) + \frac{1}{2}(l_y + \alpha(a)l'_y)
\]

Now the line \( \frac{1}{2}(l_y - \alpha(a)l'_y) \in \mathbb{IP}W_x \) belongs to the pencil \( \varphi_y \) and intersects \( \mathbb{IP}V_x^- \). Hence \( l = \frac{1}{2}(l_y - \alpha(a)l'_y) = p_{x,y}(l_y) \) (up to a scalar). \( \square \)

We choose a symplectic basis \( (\gamma_1, \ldots, \gamma_2) \) for \( J \) such that \( x = \frac{1}{2} \gamma_g + \Gamma, x' = \frac{1}{2} \gamma_2 + \Gamma, y = \frac{1}{2} \gamma_{g-1} + \Gamma \) and \( y' = \frac{1}{2} \gamma_{2g-1} + \Gamma \), with \( \Gamma \) the period matrix of \( J \). In the corresponding theta structure these points are represented by

\[
x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad x' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad y = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad y' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

(4)
where the zeros in the first column belong to $K(g-2)$. We also choose period matrices $\omega_x$ and $\omega_y$ for the Pryms $P_x$ and $P_y$. We are now able to express the preceding relations in terms of theta-constants.

**Proposition 4.4 (“even version”)** There exist nonzero constants $c_1, c_2$ such that

(a) the Schottky-Jung relations : $\forall b \in K(g-1)$

\[
\theta \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} (0, 2\tau) = c_1 \theta \begin{bmatrix} b \\ 0 \end{bmatrix} (0, 2\omega_x)
\]

(b) the Donagi relations : $\forall d \in K(g-2)$

\[
\theta \begin{bmatrix} d & 0 \\ 0 & 1 \end{bmatrix} (0, 2\omega_x) = c_2 \theta \begin{bmatrix} d \\ 0 \end{bmatrix} (0, 2\omega_y)
\]

**Proof:** We refer to proposition 5 of [vG] for (i) and to lemma 1 of [vG-P1] for (ii).

**Proposition 4.5 (“odd version”)** There exist nonzero constants $d_1, \ldots, d_4$ such that

(a) $\forall \kappa \in \vartheta^{-}_x(J) \cong \vartheta(P_x)$

\[
\varepsilon(\kappa)\theta^2_\kappa(2\tilde{\alpha}_x, \omega_x) = d_1 \theta_{x,\kappa}(0, \tau) D\theta_\kappa(0, \tau)
\]

(a') $\forall b \in K(g-1)$

\[
D\theta \begin{bmatrix} b & 1 \\ 0 & 1 \end{bmatrix} (0, 2\tau) = d_2 \theta \begin{bmatrix} b \\ 0 \end{bmatrix} (4\tilde{\alpha}_x, 2\omega_x)
\]

(b) $\forall d \in K(g-2)$

\[
\theta \begin{bmatrix} d & 1 \\ 0 & 1 \end{bmatrix} (4\tilde{\alpha}_x, 2\omega_x) = d_3 \theta \begin{bmatrix} d \\ 0 \end{bmatrix} (4\tilde{\alpha}_y, 2\omega_y)
\]

(b') $\forall \kappa \in \vartheta^{-}_x(J) \cap \vartheta^{-}_y(J)$

\[
\theta_\kappa(2\tilde{\alpha}_x, \omega_x)\theta_{\tilde{y},\kappa}(2\tilde{\alpha}_x, \omega_x) = d_4 \theta_{\kappa}(2\tilde{\alpha}_y, \omega_y)\theta_{\tilde{x},\kappa}(2\tilde{\alpha}_y, \omega_y)
\]

**Proof:**

(a) We express the coordinates of the point $\varphi_p \circ j_p(z_x) = h_x(\alpha_x)$ in the basis $\{\xi_\kappa\}$ of $W_x$ with $\kappa \in \vartheta^{-}_x(J)$. The map $\varphi_p \circ j_p$ induces by pull-back the Wahl map $w_D$ and $w_D(\xi_\kappa) = cD\theta_\kappa(0, \tau)\theta_\kappa(2z, \tau)$. Therefore the coordinates of $\varphi_p \circ j_p(z_x)$ are (up to scalar)

\[
D\theta_\kappa(0, \tau)\theta_{x,\kappa}(0, \tau)
\]

since $\theta_\kappa(\frac{1}{2} \gamma_g, \tau) = \theta_{x,\kappa}(0, \tau)$. By lemma 3.4 the coordinates of $h_x(\alpha_x)$ are (up to scalar)

\[
\theta_\kappa(-2\tilde{\alpha}_x, \omega_x)\theta_\kappa(2\tilde{\alpha}_x, \omega_x) = \varepsilon(\kappa)\theta^2_\kappa(2\tilde{\alpha}_x, \omega_x)
\]
(a') The line \( t \) in \( IPV \), corresponding to the intersection point in \( IP\Lambda^2V \) of \( J \) and \( P_x \), has two descriptions: first, \( t \) passes through \( \varphi_L(z_x) = \phi_x(0) \in IPV_x \) and its direction is given by a tangent vector \( v \) (to \( J \)), whose coordinates are (up to a scalar)

\[
\left[ \ldots D\theta \begin{bmatrix} 0 & 2\tau \\ 0 & 1 \end{bmatrix} \right]_{V_x} \left( \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} \right) (0,2\tau) \ldots \left[ D\theta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right]_{V_x^-} (0,2\tau) \ldots
\]

Since \( \theta \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} \) (\( z, 2\tau \)) is an even theta function, \( D\theta \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} \) (\( z, 2\tau \)) is odd and thus vanishes at the origin. Hence the tangent vector \( v \) is contained in \( IPV_x^- \).

Secondly, by lemma 3.3 \( t \) passes through the point \( \phi_x(p_x - \sigma p_x) \in IPV_x^- \) and since \( t \) intersects \( IPV_x^- \) in a unique point, the coordinates of this point can be expressed in two different ways.

**Remark:** One should notice that the statements (a) and (a’) are equivalent, namely by applying the differential operator \( D \) to the addition formula (3) -treat one variable as a constant- evaluating at the origin and using the “even” Schottky-Jung relations.

(b) A basis of \( V_y \) (resp. \( V_y^- \)) is given by \( \{X_{(d,0,\epsilon)}\} \) (resp. \( \{X_{(d,1,\epsilon)}\} \)) for \( d \in K(g − 2) \) and \( \epsilon \in \mathbb{Z}/2 \). A basis of the intersection \( V_x^\perp \cap V_y^- \) is given by \( \{X_{(d,1,1)}\} \) for \( d \in K(g − 2) \). We can decompose \( V_y^- = V_x^\perp \cap V_y \oplus V_y^\perp \cap V_y^- \). The point \( D = (BB') \cap (CC') \in IPV_x^\perp \cap V_y^- \) (see proof of proposition 4.2 (b)) is the image of \( B \) by the projection \( IPV_x^\perp \to IPV_x^\perp \cap V_y^- \). Therefore the coordinates of \( S \) (up to a scalar)

\[
\theta \begin{bmatrix} d & 1 \\ 0 & 0 \end{bmatrix} (2\tilde{u}_x + 4\tilde{a}_x, 2\omega_x) = \theta \begin{bmatrix} d & 1 \\ 0 & 1 \end{bmatrix} (4\tilde{a}_x, 2\omega_x)
\]

where \( \tilde{u}_x \) is a representative of \( u_x \in \mathcal{C}^{y-1}/\Gamma_{\omega_x} \cong P_x \) and \( 2\tilde{u}_x = \tilde{y} + \Gamma_{\omega_y} \) with \( \tilde{y} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \).

We obtain the relation (b) doing the same reasoning for \( V_y^- \) and the point \( C \).

(b') We can deduce relation (b') from (b) and the Donagi relations using the addition formula (3) with \( z = 2\tilde{a}_x + \tilde{u}_x \) and \( u = \tilde{u}_x \). Another method is to express the coordinates of the line \( l = (AS) = p_x,y[h_x(u_x \otimes \alpha_x)] = p_x,y[h_y(u_y \otimes \alpha_y)] \) in the basis \( \{\xi_z\} \) of \( W_{x,y} \). We immediately get the result from lemma 3.4.

**Remark:** For a generic curve \( C \), the conditions (*) are verified. Hence by relation (a) \( \theta_\kappa(2\tilde{a}_x, \omega_x) \neq 0, \forall \kappa \in \vartheta(P_x) \), i.e. \( h^* : W_x \to H^0(P_x, \mathcal{O}(4\Xi_x)) \) is an isomorphism.

**Proposition 4.6 ("symmetry")** There exists a nonzero constant \( e \) such that \( \forall d \in K(g − 2) \)

\[
\theta \begin{bmatrix} d & 1 \\ 0 & 0 \end{bmatrix} (4\tilde{a}_x, 2\omega_x) = e\theta \begin{bmatrix} d & 0 \\ 0 & 1 \end{bmatrix} (4\tilde{a}_x, 2\omega_x)
\]

**Proof:** We need more information about the configuration of lines described in the proof of prop. 4.2 (b). First, the intersection point \( A \) (see picture) also lies on the Kummer of the Prym \( P_{x+y} \) (see [D]). Analogously, we can consider the lines \( l_{x+y} \) and \( l'_{x+y} \).
which intersect the eigenspace $\mathbb{P}V_{x+y}$ in two points $D$ and $D'$. Therefore we obtain six points $B, B', C, C', D, D'$ lying on a conic and, by doing the same reasoning as above, we get intersection points

$$(BB') \cap (DD') = S' \in \mathbb{P}V_x \cap V_y \quad \text{and} \quad (CC') \cap (DD') = S'' \in \mathbb{P}V_x \cap V_y$$

Consider the subgroup $G$ of $K(g) \times \widehat{K(g)}$ generated by the elements $x$ and $y$. Since $G$ is totally isotropic, there exists a level subgroup $\tilde{G} \subset \text{Heis}(g)$ over $G$. We can choose $\tilde{x}, \tilde{y} \in \tilde{G}$ such that they act as $-Id$ on $V_\tilde{x}$ and $V_\tilde{y}$. We have a decomposition

$$V_{x+y}^- = V_x^- \cap V_y \oplus V_x \cap V_y^- \quad (5)$$

Consider the involution $\tau \in \text{Aut}(K(g) \times \widehat{K(g)})$ interchanging the points $x$ and $y$ and leaving the subgroup $K(g-2) \times K(g-2)$ invariant. Then $\tau$ lifts to an involution $\tilde{\tau}$ of $\text{Heis}(g)$, which interchanges $\tilde{x}$ and $\tilde{y}$. By prop. 3 [M1], $\text{Heis}(g)$ has a unique irreducible representation $\rho : \text{Heis}(g) \to GL(V)$ on which $\mathbb{C}^*$ acts by its natural character. Hence, there exists a linear involution $i$ of $V$ such that

$$\rho \circ \tilde{\tau}(\alpha) = i \circ \rho(\alpha) \circ i \quad \forall \alpha \in \text{Heis}(g)$$

In particular: $i(V_x) = V_y$, $i(V_x^-) = V_y^-$ and $i$ acts on the eigenspace $V_{x+y}^-$ interchanging the two factors of the decomposition (5). The points $D$ and $D'$ are invariant under $i$, therefore the line $(DD')$ is preserved under $i$ and we can conclude that $i(S') = S''$. Now we express the coordinates of $S'$ and $S''$ in there natural bases and we obtain the relation for the theta-constants on the Prym $P_{x+y}$. The same method gives the symmetric relations for $P_x$ and $P_y$. \hfill $\Box$

5 Quadrics in $\mathbb{IP} \Lambda^2 V$

We can associate to any $x \in J[2]$ a character of $J[2]$, also denoted by $x$, defined by $y \mapsto \langle x, y \rangle$. This correspondence is one-to-one and allows us to identify $J[2]$ with its character group.

The Heisenberg group $\text{Heis}(g)$ acts on $S^2(\Lambda^2 V)$ and its center $\mathbb{C}^*$ acts by $t \mapsto t^4$. This action factors over the abelian group $J[2]$. Consider the decomposition into character spaces

$$S^2(\Lambda^2 V) = \bigoplus_{x \in J[2]} S^2(\Lambda^2 V)_x$$

**Lemma 5.1**

a) The elements $\{\xi_\kappa \otimes \xi_\kappa\}$ for $\kappa \in \vartheta^-(J)$ form a basis of $S^2(\Lambda^2 V)_0$. In particular, $\dim S^2(\Lambda^2 V)_0 = 2^{g-1}(2^g - 1)$.

b) For a nonzero $x \in J[2]$, the distinct elements $\{\xi_\kappa \otimes \xi_{x\kappa}\}$ for $\kappa \in \vartheta^-(J)$ with $x \cdot \kappa \in \vartheta^-(J)$ form a basis of $S^2(\Lambda^2 V)_x$. In particular, $\dim S^2(\Lambda^2 V)_x = 2^{g-2}(2^{g-1} - 1)$.
Proof: Choose a theta structure for $\mathcal{O}(2\Theta)$ on $J$. A point $x \in J[2]$ is represented by $x = (1, a, \alpha)$. Then for any theta characteristic $\kappa = \kappa \left[ \begin{array}{c} e \\ \gamma \end{array} \right]$ and $x \cdot \kappa = \kappa \left[ \begin{array}{c} e + a \\ \alpha \gamma \end{array} \right]$

\begin{align*}
(1, b, \beta)\xi_\kappa &= \gamma(b)\beta(c)\xi_\kappa \\
(1, b, \beta)\xi_{x,\kappa} &= (\alpha\gamma)(b)\beta(c + a)\xi_{x,\kappa}
\end{align*}

Hence $(1, b, \beta)\xi_\kappa \otimes \xi_{x,\kappa} = \alpha(b)\beta(a)\xi_\kappa \otimes \xi_{x,\kappa}$ i.e. $\xi_\kappa \otimes \xi_{x,\kappa} \in S^2(\Lambda^2 V)_x$. Now the elements $\{\xi_\kappa \otimes \xi_\kappa\}$ and $\{\xi_\kappa \otimes \xi_{x,\kappa}\}$ for all $x \neq 0$ generate the space $S^2(\Lambda^2 V)$, since $\{\xi_\kappa\}$ is a basis of $\Lambda^2 V$ with $\kappa \in \vartheta^{-1}(J)$. Identifying $\xi_\kappa \otimes \xi_{x,\kappa}$ and $\xi_{x,\kappa} \otimes \xi_\kappa$, the number of all these elements is equal to $\dim S^2(\Lambda^2 V)$. \hfill \Box

5.1 Invariant quadrics

Fix a theta structure for $\mathcal{O}(2\Theta)$ on $J$. We define a linear map for any nonzero $x = \left[ \begin{array}{c} a \\ a \end{array} \right]$

\begin{align*}
M_x : S^2(\Lambda^2 V)_0 &\longrightarrow (S^2 V \otimes \Lambda^2 V)_x \\
\xi_\kappa \otimes \xi_\kappa &\mapsto \gamma(a)\xi_{x,\kappa} \otimes \xi_\kappa \text{ if } \kappa \in \vartheta_x^{-1}(J) \\
&= 0 \quad \text{otherwise}
\end{align*}

for all theta characteristics $\kappa = \kappa \left[ \begin{array}{c} c \\ \gamma \end{array} \right]$. The space $(S^2 V \otimes \Lambda^2 V)_x$ denotes the subspace of $S^2 V \otimes \Lambda^2 V$ corresponding to the character $x$.

Remark: We defined the map $M_x$ using a theta structure on $J$. However, in the proposition 5.2, we will use the fact that $M_x$ admits an intrinsic definition (up to homothety): consider for $x \in J[2]$ the linear automorphism $U(x)$ of $V$, unique up to homothety, given by the projective representation of $J[2]$ on $\text{IP} V$. Let $v_\kappa \otimes v_\kappa \in V \otimes V$ be an eigenvector (unique up to scalar) w.r.t. the character $\chi_\kappa$ for the action of $G(L)$ on $V \otimes V$ (see section 2). Recall that $v_\kappa$ and $\xi_\kappa$ are proportional via a theta structure on $J$. We consider the automorphism $\tilde{U}(x) = U(x) \otimes \text{Id}$ of $V \otimes V$ and define $M_x : v_\kappa \otimes v_\kappa \mapsto v_\kappa \otimes \tilde{U}(x)v_\kappa$. This defines a linear map, up to homothety, since the definition does not depend on the choice of the eigenvectors $v_\kappa$. Let us check that the two definitions are equivalent. With the same notation as above:

\begin{align*}
(U(x) \otimes \text{Id})\xi_\kappa &= \sum_b \gamma(b)\alpha(b + a)\psi_{b+a} \otimes \psi_{b+c} \\
&= \sum_b \gamma(b + a)\alpha(b)\psi_b \otimes \psi_{b+a+c} \\
&= \gamma(a)\xi_{x,\kappa}
\end{align*}

Consider the product morphism $\tilde{J} \rightarrow \text{IP} V \times \text{IP} \Lambda^2 V$. The elements of $(S^2 V \otimes \Lambda^2 V)_x$ are polynomials on $\text{IP} V \times \text{IP} \Lambda^2 V$ of degree 2 on the first factor and linear on the second one.
Proposition 5.2 An invariant quadric \( F \in S^2(\Lambda^2 V)_0 \) vanishes on the Prym \( P_x \) if and only if \( M_x(F) \) vanishes on \( \hat{J} \).

Proof: Since \( M_x \) is defined intrinsically, we can choose a theta structure such that the conditions preceding proposition 4.4 hold, i.e. \( x = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \) where the zeros of the first column belong to \( K(g-1) \). In particular, \( a = 0 \). By lemma 5.1 a generic polynomial \( F \in S^2(\Lambda^2 V)_0 \) and its image \( M_x(F) \) can be written (with \( a_\kappa \in \mathbb{C} \)):

\[
F = \sum_{\kappa \in \vartheta(J)} a_\kappa \xi_\kappa \otimes \xi_\kappa \\
M_x(F) = \sum_{\kappa \in \vartheta_x(J)} a_\kappa \xi_{x,\kappa} \otimes \xi_\kappa
\]

First, we rephrase the condition that \( F \) vanishes on \( P_x \): if \( \varepsilon(x \cdot \kappa) = -1 \), the polynomial \( \xi_\kappa \otimes \xi_\kappa \in S^2(\Lambda^2 V)_0 \) vanishes on \( P_x \) (lemma 3.2); if \( \kappa \in \vartheta_x(J) \), the polynomial \( \xi_\kappa \otimes \xi_\kappa \) restricts to the theta function \( \theta_\kappa^2(-2\bar{\alpha}_x, \omega_x)\theta_\kappa^2(2z, \omega_x) \). By the addition formula (3) we have:

\[
\theta_\kappa^2(2z, \omega_x) = \sum_b \delta(b) \theta \begin{bmatrix} b \\ 0 \end{bmatrix} (4z, 2\omega_x) \theta \begin{bmatrix} b+d \\ 0 \end{bmatrix} (0, 2\omega_x)
\]

with \( \bar{\kappa} = \kappa \begin{bmatrix} d \\ \delta \end{bmatrix} \) and \( b \) running over \( K(g-1) \). Note that the family of theta functions of order 8 \( \{ \theta \begin{bmatrix} b \\ 0 \end{bmatrix} (4z, 2\omega_x) \} \) is linearly independent. Therefore \( F \) vanishes on \( P_x \)

\[
\Leftrightarrow \sum_\kappa a_\kappa \sum_b \delta(b) \theta_\bar{\kappa}^2(2\bar{\alpha}_x, \omega_x) \theta \begin{bmatrix} b+d \\ 0 \end{bmatrix} (0, 2\omega_x) = 0 \\
\Leftrightarrow \forall b \in K(g-1) \sum_{\kappa \in \vartheta_x(J)} a_\kappa \delta(b) \theta_\bar{\kappa}^2(2\bar{\alpha}_x, \omega_x) \theta \begin{bmatrix} b+d \\ 0 \end{bmatrix} (0, 2\omega_x) = 0 \quad (6)
\]

Now we rephrase the condition that \( M_x(F) \) vanishes on \( \hat{J} \): the polynomial \( \xi_{x,\kappa} \otimes \xi_\kappa \in (S^2 V \otimes \Lambda^2 V)_x \) restricts to the theta function (see section 2)

\[
\theta_{x,\kappa}(0, \tau) \theta_{x,\kappa}(2z, \tau) D\theta_\kappa(0, \tau) \theta_\kappa(2z, \tau)
\]

Again by the addition formula (3), we have:

\[
\theta_\kappa(2z, \tau) \theta_{x,\kappa}(2z, \tau) = \delta(d) \sum_b \delta(b) \theta \begin{bmatrix} b \\ 0 \end{bmatrix} (4z, 2\tau) \theta \begin{bmatrix} b+d \\ 0 \end{bmatrix} (0, 2\tau)
\]

16
Note that the family of theta functions of order 8 \( \{ \theta \left[ \begin{smallmatrix} b & 1 \\ 0 & 1 \end{smallmatrix} \right] (4z, 2\tau) \} \) is linearly independent. Therefore \( M_x(F) \) vanishes on \( \tilde{J} \) if and only if

\[
\sum_{\kappa, b} a_{\kappa} \delta(d) \delta(b) \theta_{x_{\kappa}}(0, \tau) D\theta_{\kappa}(0, \tau) \theta \left[ \begin{smallmatrix} b + d & 0 \\ 0 & 1 \end{smallmatrix} \right] (0, 2\tau) \theta \left[ \begin{smallmatrix} b & 1 \\ 0 & 1 \end{smallmatrix} \right] (4z, 2\tau) = 0
\]

\( \forall b \in K(g - 1) \sum_{\kappa} \varepsilon(\kappa) a_{\kappa} \delta(b) \theta_{x_{\kappa}}(0, \tau) D\theta_{\kappa}(0, \tau) \theta \left[ \begin{smallmatrix} b + d & 0 \\ 0 & 1 \end{smallmatrix} \right] (0, 2\tau) = 0 \) \quad (7)

The Schottky-Jung relations of proposition 4.4 and 4.5 imply that the set of equations (6) and (7) are proportional, hence the two vanishing conditions are equivalent. \( \Box \)

Let us introduce the linear map

\[
N : S^2(\Lambda^2 V)_0 \longrightarrow [V \otimes \Lambda^2 V]_0
\]

\[
\xi_{\kappa} \otimes \xi_{\kappa} \mapsto X_\kappa \otimes \xi_{\kappa}
\]

where \( \kappa = \kappa \left[ \begin{smallmatrix} c \\ \gamma \end{smallmatrix} \right] \) and \([V \otimes \Lambda^2 V]_0 \) is the subspace of \( V \otimes \Lambda^2 V \) invariant under the maximal level subgroup \( \widetilde{K}(g) \subset \text{Heis}(g) \). By abuse of notation we also call \( \widetilde{K}(g) \) its lift in \( \text{Heis}(g) \). It is easy to check that \( N \) is an isomorphism. Consider the multiplication map of theta functions

\[
m : V \otimes \Lambda^2 V \longrightarrow H^0(J, \mathcal{O}(2\Theta)) \otimes H_0^\perp(J, \mathcal{O}(4\Theta)) \longrightarrow H^0(J, \mathcal{O}(6\Theta))
\]

where the first arrow is the Wahl map, which is an isomorphism under the assumption (*). The composite \( m \) is \( \text{Heis}(g) \)-equivariant. By Mumford’s theta theory [M1], \( H^0(J, \mathcal{O}(6\Theta)) \) is a direct sum of irreducible representations of dimension \( 2^g \) (the center \( \mathbb{C}^* \) of \( \text{Heis}(g) \) acts by \( t \mapsto t^3 \)). A result of Kempf [K] asserts that \( m \) is surjective. Take invariants under the level subgroup \( K(g) \), which gives a surjective map:

\[
[m]_0 : [V \otimes \Lambda^2 V]_0 \longrightarrow [H^0_0(J, \mathcal{O}(6\Theta))]_0
\]

Now we can state the main theorem of this subsection

**Theorem 5.3** An invariant quadric \( F \in S^2(\Lambda^2 V)_0 \) vanishes on all Pryms if and only if \( N(F) \in \ker[m]_0 \)

**Proof:** Let \( G_0 = N(F) \) and \( \forall b \in K(g) \quad G_b = (1, b, 0). G_0 \in V \otimes \Lambda^2 V. \) Consider a point \( x = \left[ \begin{smallmatrix} a \\ \alpha \end{smallmatrix} \right] \in J[2] \). Let us prove the following relation, for \( x \neq 0 \):

\[
M_x(F) = \sum_{b \in K(g)} \alpha (b + a) X_{b + a} \otimes G_b
\]

(8)

The right-hand side is considered as an element of \( S^2 V \otimes \Lambda^2 V \) (after the projection \( V \otimes V \otimes \Lambda^2 V \rightarrow S^2 V \otimes \Lambda^2 V \)). By linearity, it suffices to prove this relation for \( F = \xi_{\kappa} \otimes \xi_{\kappa} \).

If \( \kappa \in \mathfrak{g}_+(J) \), \( M_x(F) = \gamma(a) \xi_{x_{\kappa}} \otimes \xi_{\kappa} \) by definition. We compute
\[ G_b = (1, b, 1)X_c \otimes \xi_{\kappa} \]
\[ = X_{b+c} \otimes \chi_{\kappa}(1, b, 1)\xi_{\kappa} \]
\[ = \gamma(b)X_{b+c} \otimes \xi_{\kappa} \]

Hence, the right-hand side is equal to
\[
\left( \sum_b \alpha(b + a) \gamma(b)X_{b+a} \otimes X_{b+c} \right) \otimes \xi_{\kappa}
\]
\[ = \gamma(a) \left( \sum_b \alpha(b) \gamma(b)X_b \otimes X_{b+a+c} \right) \otimes \xi_{\kappa} \]
\[ = \gamma(a)\xi_{x_{\kappa}} \otimes \xi_{\kappa} \]

If \( \varepsilon(x_{\cdot \kappa}) = -1 \), \( M_x(F) = 0 \) and the right-hand side \( \gamma(a)\xi_{x_{\kappa}} \otimes \xi_{\kappa} \) vanishes by projection to \( S^2V \otimes \Lambda^2V \), which proves relation (8).

Assume \( N(F) = G_0 \in \ker[m]_0 \). Since \( m \) is Heis\((g)\)-equivariant, \( G_b \in \ker m \), \( \forall b \in K(g) \). Hence, by relation (8), \( M_x(F) \) vanishes on \( J \) for all nonzero \( x \in J[2] \). By proposition 5.2 we deduce that \( F \) vanishes on \( P_x \).

Conversely, assume that \( F \) vanishes on all Pryms. Then, by proposition 5.2, \( M_x(F) \) vanishes on \( J \) for all nonzero \( x \in J[2] \). Fix a nonzero \( a \in K(g) \). In particular, by relation (8), \( \forall \alpha \in K(g) \sum_b \alpha(b + a)X_{b+a} \otimes G_b \) vanishes on \( \hat{J} \). Taking suitable linear combinations of these polynomials, we deduce that \( \forall b \in K(g) \), \( X_{b+a} \otimes G_b \) vanishes on \( \hat{J} \). Since \( X_{b+a} \)
does not vanish on \( J \), we obtain that \( \forall b \in K(g) \), \( G_b \in \ker m \).

### 5.2 Noninvariant quadrics

In this subsection we fix a nonzero \( x \in J[2] \) and we shall prove a similar result concerning the vanishing on all Pryms of a polynomial \( F \in S^2(\Lambda^2V)_x \).

**Proposition 5.4** Consider \( y \in J[2] \) with \( \langle x, y \rangle = 1 \) and \( x \neq y \). Then a quadric \( F \in S^2(\Lambda^2V)_x \) vanishes on \( P_y \) if and only if \( F \) vanishes on \( P_{y+x} \).

**Proof:** We can choose a theta structure such that \( x + y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \) and \( y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \).

By lemma 5.1 a generic polynomial \( F \in S^2(\Lambda^2V)_x \) can be written \( F = \sum a_{\kappa}\xi_{\kappa} \otimes \xi_{x_{\kappa}} \) where we sum over \( \kappa \in \vartheta^-(J) \) with \( \varepsilon(x \cdot \kappa) = -1 \). For any such \( \kappa \), we have \( 1 = \kappa(x) = \kappa(x + y)\kappa(y) \langle x + y, y \rangle = \varepsilon(x \cdot \kappa) \).

If \( \varepsilon(y \cdot \kappa) = -1 \), the polynomial \( \xi_{\kappa} \otimes \xi_{x_{\kappa}} \) vanishes on \( P_y \) (lemma 3.2); if \( \varepsilon(y \cdot \kappa) = 1 \), the polynomial \( \xi_{\kappa} \otimes \xi_{x+ y_{\kappa}} \) restricts to the theta function

\[ \theta_{\kappa}(-2\tilde{\alpha}_y, \omega_y)\theta_{\kappa}(2z, \omega_y)\theta_{\tilde{\kappa}}(-2\tilde{\alpha}_y, \omega_y)\theta_{\tilde{\kappa}}(2z, \omega_y) \]

By the addition formula, we have
\[ \theta_\kappa(2z, \omega_y) \theta_{\bar{x}\kappa}(2z, \omega_y) = \epsilon(e) \sum_d \epsilon(d) \theta \begin{bmatrix} d \ 1 \\ 0 \ 1 \end{bmatrix} (4z, 2\omega_y) \theta \begin{bmatrix} d \ e \ 0 \\ 0 \ 0 \ 1 \end{bmatrix} (0, 2\omega_y) \]

with \( \bar{\kappa} = \kappa \left[ \begin{array}{c} e^1 \\ 0 \end{array} \right] \) and \( \bar{x}\bar{\kappa} = \kappa \left[ \begin{array}{c} e^1 \\ 1 \end{array} \right] \). We have \( \epsilon(\bar{\kappa}) = \epsilon(e) \) and \( \epsilon(\bar{\kappa}) \cdot \epsilon(\bar{x}\bar{\kappa}) = -1 \). The family \( \{ \theta \begin{bmatrix} d \ 1 \\ 0 \ 1 \end{bmatrix} (4z, 2\omega_y) \} \) is linearly independent. Hence \( F \) vanishes on \( P_y \) if and only if \( \forall d \in K(g-2) \)

\[ \sum_{\kappa} \epsilon(\bar{\kappa}) a_\kappa \epsilon(d) \theta_\kappa(2\bar{\alpha}_y, \omega_y) \theta_{\bar{x}\kappa}(2\bar{\alpha}_y, \omega_y) \theta \begin{bmatrix} d \ e \ 0 \\ 0 \ 0 \ 1 \end{bmatrix} (0, 2\omega_y) = 0 \]  \( \text{(9)} \)

where we sum over \( \kappa \in \vartheta_y^-(J) \cap \vartheta_{x+y}^-(J) \). The vanishing condition for the Prym \( P_{x+y} \) is obtained from equation (9) by replacing \( \bar{x} \) by \( \bar{y} \) and the subscript \( y \) by \( x + y \). The Donagi relations and proposition 4.5 (b') -written for the orthogonal pair \((y, x+y)\) - imply that the two conditions are equivalent. \( \square \)

From now on, we fix a theta structure for \( \mathcal{O}(2\Theta) \) such that \( x = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \) and \( x' = \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] \).

We define a linear map \( M_y^{(x)} \) for any nonzero \( y = \left[ \begin{array}{c} a \ 0 \\ \alpha \ 0 \end{array} \right] \)

\[ M_y^{(x)} : S^2(\Lambda^2 V)_x \rightarrow (W^+_x \otimes W^-_x)_{x+y} \]

\[ \xi_\kappa \otimes \xi_{x,\kappa} \rightarrow \delta(a)\xi_{(x+y)-\kappa'} \otimes \xi_{\kappa'} \text{ if } \kappa \in \vartheta_y^-(J) \]
\[ \text{0 otherwise} \]

here \( W^\pm \) is the subspace of \( W_x \) isomorphic (via \( h^*_x \)) to \( H^0_x(P_x, \mathcal{O}(4\Xi_x)) \). A basis of \( S^2(\Lambda^2 V)_x \) is given by the tensors \( \xi_\kappa \otimes \xi_{x,\kappa} \) for \( \kappa = \kappa \left[ \begin{array}{c} d \ 0 \\ \delta \ 0 \end{array} \right] \) such that \( \delta(d) = -1 \). We note \( \kappa' = x' \cdot \kappa \). The linear map \( M_y^{(x)} \) also admits an intrinsic definition (up to homothety) - see remark in subsection 5.1.

**Proposition 5.5** A quadric \( F \in S^2(\Lambda^2 V)_x \) vanishes on \( P_y \) if and only if \( M_y^{(x)}(F) \) vanishes on \( P_x \)

**Proof:** Choose a theta structure on \( J \) such that conditions (4) hold. The theta characteristics \( \kappa = \kappa \left[ \begin{array}{c} d \ 0 \\ \delta \ 0 \end{array} \right] \in \vartheta_y^-(J) \) are of the form

\[ \kappa = \kappa \left[ \begin{array}{c} e \ 1 \\ 0 \ e \ 0 \end{array} \right] \epsilon(e) = -1 \text{ or } \kappa = \kappa \left[ \begin{array}{c} e \ 1 \ 0 \\ 1 \ e \ 0 \end{array} \right] \epsilon(e) = 1 \]  \( \text{(10)} \)

They restrict to the theta characteristics \( \bar{\kappa} = \bar{\kappa} \left[ \begin{array}{c} e_0 \\ \epsilon_0 \end{array} \right] \) on \( P_y \). A generic polynomial \( F = \sum a_\kappa \xi_\kappa \otimes \xi_{x,\kappa} \in S^2(\Lambda^2 V)_x \) restricts to the theta function on \( P_y \) (lemma 3.4)

\[ \sum_{\kappa} a_\kappa \theta_\kappa(-2\bar{\alpha}_y, \omega_y) \theta_{\bar{x}\kappa}(2z, \omega_y) \theta_{\bar{x}\kappa}(-2\bar{\alpha}_y, \omega_y) \theta_{\bar{x}\kappa}(2z, \omega_y) \]

\[ = \sum_{\kappa} a_\kappa \epsilon(d) \theta_\kappa(2\bar{\alpha}_y, \omega_y) \theta_{\bar{x}\kappa}(2\bar{\alpha}_y, \omega_y) \theta \begin{bmatrix} d \ 0 \\ 0 \ 1 \end{bmatrix} (4z, 2\omega_y) \theta \begin{bmatrix} d \ e \ 0 \\ 0 \ 0 \ 1 \end{bmatrix} (0, 2\omega_y) \]  \( \text{(11)} \)
where we sum over \( \kappa \) of the form (10) and \( d \in K(g-2) \). The last equation is obtained using the addition formula (3). The polynomial \( M_g^{(x)}(F) = \sum \alpha_n \xi(y + \kappa) \otimes \xi_{\kappa'} \) restricts to the theta function on \( P_x \)

\[
\sum \alpha_n \theta_{\kappa'}(-2\tilde{\alpha}_x, \omega_x) \theta_{\kappa'}(2z, \omega_x) \theta_{\bar{\kappa}^\prime}(2\tilde{\alpha}_x, \omega_x) \theta_{\bar{\kappa}^\prime}(2z, \omega_x)
= \sum \alpha_n d(d) \theta_{\kappa'}(2\tilde{\alpha}_x, \omega_x) \theta_{\bar{\kappa}^\prime}(2\tilde{\alpha}_x, \omega_x) \theta \left[ \begin{array}{c} d \ 0 \\ 0 \ 1 \end{array} \right] \theta \left[ \begin{array}{c} d + e \ 0 \\ 0 \ 1 \end{array} \right] (0, 2\omega_x)
\]  

(12)

with the same summation as in (11). We apply the addition formula again: \( \forall (e, \epsilon) \in K(g-2) \times K(\bar{g}-2) \)

\[
\theta_{\kappa}(2\tilde{\alpha}_y, \omega_y) \theta_{\bar{\kappa}}(2\tilde{\alpha}_y, \omega_y) = \sum d \epsilon(d) \theta \left[ \begin{array}{c} d \ 0 \\ 0 \ 1 \end{array} \right] \theta \left[ \begin{array}{c} d + e \ 0 \\ 0 \ 1 \end{array} \right] (0, 2\omega_y)
\]

(12)

Thus the two vanishing conditions (11) and (12) are proportional, hence equivalent.

\( \square \)

We introduce the linear isomorphism

\[
N^{(x)} : S^2(\Lambda^2 V_\kappa) \rightarrow [V^-_x \otimes W^-_x]_0
\]

\[
\xi_\kappa \otimes \xi_{\kappa'} \rightarrow X_{(d1)} \otimes \xi_{\kappa'}
\]

As above, \([V^-_x \otimes W^-_x]_0 \) denotes the invariant subspace of \( V^-_x \otimes W^-_x \) under the level subgroup \( K(g-1) \) of \( Heis(g-1) \) acting on \( V^-_x \otimes W^-_x \). Doing the same reasoning as in subsection 5.1, we get a surjective map

\[
[m^{(x)}]_0 : [V^-_x \otimes W^-_x]_0 \rightarrow [H^0_\kappa(P_x, 6\Xi)]_0
\]

The following theorem is the analogue of theorem 5.3

**Theorem 5.6** A quadric \( F \in S^2(\Lambda^2 V_\kappa) \) vanishes on all Pryms if and only if \( N^{(x)}(F) \in \ker[m^{(x)}]_0 \).

**Proof:** First, by prop. 5.4, it is sufficient to look at the vanishing of \( F \) on all the Pryms \( P_y \) with \( y = \begin{bmatrix} \alpha & 0 \\ \alpha & 0 \end{bmatrix} \). The rest follows, as in the proof of theorem 5.3, from relation (8) which we can write

\[
M_y^{(x)}(F) = \sum_{b \in K(g)} \hat{\alpha}(b + \hat{\alpha}) X_{b+\hat{\alpha}} \otimes G_b
\]

where \( G_b = (1, b, 0)N^{(x)}(F) \in V \otimes W_x \) and \( \hat{\alpha} = (\alpha \ 1), \hat{\alpha} = (a \ 0) \).
6 Proof of the main theorem

Now we are able to prove theorem 1.1: consider a curve $C$ verifying condition (*). The following diagram is commutative

\[
\begin{array}{ccc}
\bigoplus_{x \in J[2]} S^2(\Lambda^2 V)_x & \xrightarrow{\oplus (m_2)_x} & H^0(\mathcal{M}_p, \mathcal{L}_p^2) \\
\oplus n_x & \xrightarrow{\text{res}} & \bigoplus_{x \in J[2], x \neq 0} H^0(P_x, 8\Xi_x)
\end{array}
\]

The vertical arrow is the restriction map to all the Pryms lying in $\mathcal{M}_p$. We identify $H^0(\mathcal{M}_p, \mathcal{L}_p)$ with $\Lambda^2 V$ via $\varphi^*_p$. The horizontal arrow is the multiplication map $m_2 = \oplus (m_2)_x$, which is decomposed under the action of $Heis(g)$. We deduce from theorem 5.3 and 5.6 that

\[
\begin{align*}
\dim \ker n_0 &= 2^{g-1}(2^g - 1) - \frac{3^g - 1}{2} \\
\dim \ker n_x &= 2^{g-2}(2^{g-1} - 1) - \frac{3^{g-1} - 1}{2} \quad \text{for } x \neq 0
\end{align*}
\]

Commutativity of the diagram above implies that $\dim \ker (m_2)_x \leq \dim \ker n_x$. Hence:

\[
\dim H^0(\mathcal{M}_p, \mathcal{L}_p^2) \geq \sum_{x \in J[2]} \dim S^2(\Lambda^2 V)_x - \dim \ker (m_2)_x
\]

\[
\geq \sum_{x \in J[2]} \dim S^2(\Lambda^2 V)_x - \dim \ker n_x
\]

\[
= \frac{3^g - 1}{2} + (2^{2g} - 1) \frac{3^{g-1} - 1}{2}
\]

Now the Verlinde formula (see e.g. [vG-P2]) tells us that we have equality, hence equality at each level and in particular $m_2$ is surjective.

\[\square\]

**Corollary 6.1** A quadric on $\mathcal{I} \Lambda^2 V$ vanishes on $\mathcal{M}_p$ if and only if it vanishes on all Pryms.

This corollary is the analogue of theorem 4.2(ii) [vG-P2]:

A quartic on $\mathcal{I} V$ vanishes on $\mathcal{M}_0$ if and only if it vanishes on the Kummers of the Jacobian and of all the Pryms.
References


