On cubics and quartics through a canonical curve

Christian Pauly

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Abstract

We construct families of quartic and cubic hypersurfaces through a canonical curve, which are parametrized by an open subset in a Grassmannian and a Flag variety respectively. Using G. Kempf’s cohomological obstruction theory, we show that these families cut out the canonical curve and that the quartics are birational (via a blowing-up of a linear subspace) to quadric bundles over the projective plane, whose Steinerian curve equals the canonical curve.

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1 Introduction

Let \( C \) be a smooth nonhyperelliptic curve of genus \( g \geq 4 \) defined over the complex numbers, which we consider as an embedded curve \( \iota : C \hookrightarrow \mathbb{P}^{g-1} \) by its canonical linear series \( |\omega| \). Let \( I = \bigoplus_{n \geq 2} I(n) \) be the graded ideal of the canonical curve. It was classically known (Noether-Enriques-Petri theorem, see e.g. [ACGH] p. 124) that the ideal \( I \) is generated by its elements of degree 2, unless \( C \) is trigonal or a plane quintic.

It was also classically known how to construct some distinguished quadrics in \( I(2) \). We consider a double point of the theta divisor \( \Theta \subset \text{Pic}^{g-1}(C) \), which corresponds by Riemann’s singularity theorem to a degree \( g-1 \) line bundle \( L \) satisfying \( \dim |L| = \dim |\omega L^{-1}| = 1 \) and we observe that the morphism \( \iota_L \times \iota_{\omega L^{-1}} : C \longrightarrow C' \subset |L|^* \times |\omega L^{-1}|^* = \mathbb{P}^1 \times \mathbb{P}^1 \) (here \( C' \) denotes the image curve) followed by the Segre embedding into \( \mathbb{P}^3 \) factorizes through the canonical space \( |\omega|^* \), i.e.,

\[
\begin{align*}
C & \quad \hookrightarrow |\omega|^* \\
\mathbb{P}^1 \times \mathbb{P}^1 & \quad \hookrightarrow \mathbb{P}^3,
\end{align*}
\]

where \( \pi \) is projection from a \((g-5)\)-dimensional vertex \( \mathbb{P}V^\perp \) in \( |\omega|^* \). We then define the quadric \( Q_L := \pi^{-1}(\mathbb{P}^1 \times \mathbb{P}^1) \), which is a rank \( \leq 4 \) quadric in \( I(2) \) and coincides with the projectivized tangent cone at the double point \([L] \in \Theta\) under the identification of \( H^0(C, \omega)^* \) with the tangent space \( T_{[L]}\text{Pic}^{g-1}(C) \). The main result, due to M. Green [Gr], asserts that the set of quadrics \( \{Q_L\} \), when \( L \) varies over the double points of \( \Theta \), linearly spans \( I(2) \). From this result one infers a constructive Torelli theorem by intersecting all quadrics \( Q_L \) — at least for \( C \) general enough.

The geometry of the theta divisor \( \Theta \) at a double point \([L]\) can also be exploited to produce higher degree elements in the ideal \( I \) as follows: we expand in a suitable set of coordinates a local equation \( \theta \) of \( \Theta \) near \([L]\) as \( \theta = \theta_2 + \theta_3 + \ldots \), where \( \theta_i \) are homogeneous forms of degree \( i \). Having seen that \( Q_L = \text{Zeros}(\theta_2) \), we denote by \( S_L \) the cubic \( \text{Zeros}(\theta_3) \subset |\omega|^* \), the osculating cone of
θ at [L]. The cubic S_L has many nice geometric properties: under the blowing-up of the vertex \( \mathbb{P}^L \subset S_L \), the cubic S_L is transformed into a quadric bundle \( \tilde{S}_L \) over \( \mathbb{P}^1 \times \mathbb{P}^1 \) and it was shown by G. Kempf and F.-O. Schreyer [KS] that the Hessian and Steinerian curves of \( \tilde{S}_L \) are \( C' \subset \mathbb{P}^1 \times \mathbb{P}^1 \) and \( C \subset |\omega|^* \) respectively, which gives another proof of Torelli’s theorem.

In this paper we construct and study distinguished cubics and quartics in the ideal I by adapting the methods of [KS] to rank-2 vector bundles over C. Our construction basically goes as follows (section 2): we consider a general 3-plane adapting the methods of [KS] to rank-2 vector bundles over \( \mathbb{P}^3 \) and bundle \( E_W \) as follows (section 2): we consider a general 3-plane adapting the methods of [KS] to rank-2 vector bundles over \( \mathbb{P}^3 \).

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2 Some constructions for rank-2 vector bundles with canonical determinant

In this section we briefly recall some known results from [BV], [vGI] and [PP] on rank-2 vector bundles over $C$.

2.1 Bundles $E$ with $\dim H^0(C, E) \geq 3$

Let $W \subset H^0(C, \omega)$ be a 3-plane. We denote by $[W] \in \text{Gr}(3, H^0(\omega))$ the corresponding point in the Grassmannian and by $B \subset \text{Gr}(3, H^0(\omega))$ the codimension 2 subvariety consisting of $[W]$ such that the net $\mathbb{P}W \subset |\omega|$ has a base point. For $[W] \not\in B$ we consider (see [vGI] section 4) the rank-2 vector bundle $E_W$ defined by the exact sequence

$$0 \longrightarrow E_W^* \longrightarrow \mathcal{O}_C \otimes W \xrightarrow{ev} \omega \longrightarrow 0. \quad (2.1)$$

Here $E_W^*$ denotes the dual bundle of $E_W$. We have $\text{det}E_W = \omega$ and $W^* \subset H^0(C, E_W)$. We denote by $D$ the effective divisor in $|\mathcal{O}_{Gr}(g-2)|$ defined by the condition

$$[W] \in D \iff \dim H^0(C, E_W) \geq 4.$$

We have the inclusion $B \subset D$. If $[W] \not\in D$, then $E_W$ is stable ([vGI] Lemma 4.2).

Let $W^\perp \subset H^0(\omega)^* = H^1(\mathcal{O})$ denote the annihilator of $W \subset H^0(\omega)$. We call the projective subspace $\mathbb{P}W^\perp \subset |\omega|^*$ the vertex and denote by

$$\pi : |\omega|^* \dashrightarrow \mathbb{P}W^*, \quad \pi : C \to \Gamma \subset \mathbb{P}W^*,$$

denote the projection with center $\mathbb{P}W^\perp$. Abusing notation we also denote by $\pi$ a linear lift $\pi : H^0(\omega)^* \to W^*$. If $[W] \not\in B$, then $C \cap \mathbb{P}W^\perp = \emptyset$ and $\pi$ restricts to a morphism $C \to \mathbb{P}W^*$. Its image is a plane curve $\Gamma$ of degree $2g - 2$. We note that $E_W = \pi^*(T(-1))$, where $T$ is the tangent bundle of $\mathbb{P}W^* = \mathbb{P}^2$.

Conversely any globally generated bundle $E$ with $\text{det} E = \omega$ is of the form $E_W$.

2.2 Bundles $E$ with $\dim H^0(C, E) \geq 4$

Following [BV] (see also [PP] section 5.2) we associate to a bundle $E$ with $\dim H^0(C, E) = 4$ a rank $\leq 6$ quadric $Q_E \in |I(2)|$, which is defined as the inverse image of the Klein quadric under the dual $\mu^*$ of the exterior product map

$$\mu^* : |\omega|^* \longrightarrow \mathbb{P}(\Lambda^2 H^0(E)^*) \supset \text{Gr}(2, H^0(E)^*), \quad Q_E := (\mu^*)^{-1}(\text{Gr}).$$

Composing with the previous construction, we obtain a rational map

$$\alpha : D \dashrightarrow |I(2)|, \quad \alpha([W]) = Q_{E_W}.$$

Moreover given a $Q \in |I(2)|$ with $\text{rk} Q \leq 6$ and $\text{Sing} Q \cap C = \emptyset$, it is easily shown that

$$\alpha^{-1}(Q) = \{[W] \in D \mid \mathbb{P}W^\perp \subset Q\}.$$

If $\text{rk} Q = 6$, then $\alpha^{-1}(Q)$ has two connected components, which are isomorphic to $\mathbb{P}^3$. 
2.1 **Lemma.** We have \([W] \notin \mathcal{D}\) if and only if the linear map induced by restricting quadrics to the vertex \(\mathbb{P}W^\perp\)
\[
\text{res} : I(2) \longrightarrow H^0(\mathbb{P}W^\perp, \mathcal{O}(2))
\]
is an isomorphism.

**Proof.** It is enough to observe that the two spaces have the same dimension and that a nonzero element in \(\ker \text{res}\) corresponds to a \(Q \in |I(2)|\) with \(\text{rk} Q \leq 6\). \(\square\)

2.3 **Definition of the quartic** \(F_W\)

We will now define the main object of this paper. Given \([W] \notin \mathcal{B}\), we consider the \(2\theta\)-divisor \(D(E_W) \subset JC\) (see e.g. [BV],[vGI],[PP]), whose set-theoretical support equals
\[
D(E_W) = \{ \xi \in JC \mid \dim H^0(C, \xi \otimes E_W) > 0 \}.
\]
Since \(\text{mult}_\mathcal{O}D(E_W) \geq \dim H^0(C, E_W) \geq 3\) and since any \(2\theta\)-divisor is symmetric, the first nonzero term of the Taylor expansion of a local equation of \(D(E_W)\) at the origin \(\mathcal{O}\) is a homogeneous polynomial \(F_W\) of degree 4. The hypersurface in \(|\omega^* = \mathbb{P}T_OJC|\) associated to \(F_W\) is also denoted by \(F_W\). Here we restrict attention to the case \(\dim H^0(C, E_W) = 3\) or 4. We have
\[
F_W := \text{Cone}_\mathcal{O}(D(E_W)) \subset |\omega|^*.
\]
The study of the quartics \(F_W\) for \([W] \in \text{Gr}(3, H^0(\omega)) \setminus \mathcal{D}\) is the main purpose of this paper. If \([W] \in \mathcal{D}\), the quartics \(F_W\) have already been described in [PP] Proposition 5.12.

2.2 **Proposition.** If \(\dim H^0(C, E_W) = 4\), then \(F_W\) is a double quadric
\[
F_W = Q^2_{E_W}.
\]

Since \(|I(2)|\) is linearly spanned by rank \(\leq 6\) quadrics (see [PP] section 5), we obtain the following fact, which will be used in section 6.

2.3 **Proposition.** The linear subsystem \(|F_4|\) contains all squares of quadrics in \(|I(2)|\).

Although we will not use that fact, we mention that the rational map (1.1) is given by a linear subsystem \(\Pi \subset |\mathcal{J}_B(g-1)|\), where \(\mathcal{J}_B\) is the ideal sheaf of the subvariety \(B\). If \(g = 4\), the inclusion is an equality (see [OPP] section 6). If \(g > 4\), a description of \(\Pi\) is not known.

3 **Kempf’s cohomological obstruction theory**

In this section we outline Kempf’s deformation theory [K1] and apply it to the study of the tangent cones \(F_W\) of the divisors \(D(E_W)\).

3.1 **Variation of cohomology**

Let \(\mathcal{E}\) be a vector bundle over the product \(C \times S\), where \(S = \text{Spec}(A)\) is an affine neighbourhood of the origin of \(JC\). We restrict attention to the case
\[
\mathcal{E} = \pi^*_C E_W \otimes \mathcal{L},
\]
for some 3-plane \(W\), and recall that Kempf’s deformation theory was applied [K1], [K2], [KS] to the case \(\mathcal{E} = \pi^*_C M \otimes \mathcal{L}\), for a line bundle \(M\) over \(C\). The line bundle \(\mathcal{L}\) denotes the restriction of
a Poincaré line bundle over $C \times JC$ to the neighbourhood $C \times S$. The fundamental idea to study the variation of cohomology, i.e., the two upper-semicontinuous functions on $S$

$$s \mapsto h^0(C \times \{s\}, \mathcal{E} \otimes_A \mathcal{C}_s), \quad s \mapsto h^1(C \times \{s\}, \mathcal{E} \otimes_A \mathcal{C}_s),$$

where $\mathcal{C}_s = A/\mathfrak{m}_s$ and $\mathfrak{m}_s$ is the maximal ideal of $s \in S$, is based on the existence of an approximating homomorphism.

**3.1 Theorem (Grothendieck, [K1] section 7).** Given a family $\mathcal{E}$ of vector bundles over $C \times S$, there exist two flat $A$-modules $F$ and $G$ of finite type and an $A$-homomorphism $\alpha : F \to G$ such that for all $A$-modules $M$, we have isomorphisms

$$H^0(C \times S, \mathcal{E} \otimes_A M) \cong \ker (\alpha \otimes_A \text{id}_M), \quad H^1(C \times S, \mathcal{E} \otimes_A M) \cong \coker (\alpha \otimes_A \text{id}_M).$$

By considering a smaller neighbourhood of the origin, we may assume the $A$-modules $F$ and $G$ to be locally free (Nakayama’s lemma). Moreover ([K1] Lemma 10.2) by restricting further the neighbourhood, we may find an approximating homomorphism $\alpha : F \to G$ such that $\alpha \otimes \mathcal{C}_0 : F \otimes_A A/\mathfrak{m}_0 \to G \otimes_A A/\mathfrak{m}_0$ is the zero homomorphism.

We apply this theorem to the family $\mathcal{E} = \pi_C^* E_W \otimes \mathcal{L}$, for $[W] \notin \mathcal{D}$. Since by Riemann-Roch $\chi(\mathcal{E} \otimes \mathcal{C}_s) = \chi(E_W \otimes \mathcal{L}_s) = 0$, $\forall s \in S$, and since $h^0(C, E_W) = 3$, the local equation $f$ of the divisor

$$D(E_W)|_S = \{s \in S | h^0(C \times \{s\}, E_W \otimes \mathcal{L}_s) > 0\}$$

is given at the origin $\mathcal{O}$ by the determinant of a $3 \times 3$ matrix of regular functions $f_{ij}$ on $S$, with $1 \leq i, j \leq 3$, which vanish at $\mathcal{O}$, i.e., the $A$-modules $F$ and $G$ are free and of rank 3. Hence

$$f = \det (f_{ij}).$$

The linear part of the regular functions $f_{ij}$ is related to the cup-product as follows ([K1] Lemma 10.3 and Lemma 10.6): let $\mathfrak{m} = \mathfrak{m}_0$ be the maximal ideal of the origin $\mathcal{O} \in S$ and consider the exact sequence of $A$-modules

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to A/\mathfrak{m}^2 \to A/\mathfrak{m} \to 0.$$

After tensoring with $\mathcal{E}$ over $C \times S$ and taking cohomology, we obtain a coboundary map

$$H^0(C, E_W) = H^0(C \times \{s\}, \mathcal{E} \otimes_A A/\mathfrak{m}) \xrightarrow{\delta} H^1(C \times \{s\}, \mathcal{E} \otimes_A m/\mathfrak{m}^2) = H^1(C, E_W) \otimes m/\mathfrak{m}^2,$$

where $m/\mathfrak{m}^2$ is the Zariski cotangent space at $\mathcal{O}$ to $JC$. Note that we have a canonical isomorphism $(m/\mathfrak{m}^2)^* \cong H^1(\mathcal{O})$ and that a tangent vector $b \in H^1(\mathcal{O})$ gives, by composing with the linear form $l_b : m/\mathfrak{m}^2 \to \mathbb{C}$, a linear map $\delta_b : H^0(E_W) \to H^1(E_W)$. As in the line bundle case [K1], one proves

**3.2 Lemma.** For any nonzero $b \in H^1(\mathcal{O}) = T_{\mathcal{O}}JC$, we have

1. The linear map $\delta_b : H^0(E_W) \to H^1(E_W)$ coincides with the cup-product $(\cup b)$ with the class $b$, and is skew-symmetric after identifying $H^1(E_W)$ with $H^0(E_W)^*$ (Serre duality).

2. The coboundary map $\delta : H^0(E_W) \to H^1(E_W) \otimes m/\mathfrak{m}^2$ is described by a skew-symmetric $3 \times 3$ matrix $(x_{ij})$, with $x_{ij} \in H^1(\mathcal{O})^*$. Moreover the linear form $x_{ij}$ coincides with the differential $(df_{ij})_0$ of $f_{ij}$ at the origin $\mathcal{O}$.
The coboundary map $\delta$ induces a linear map
\[ \Delta : H^1(\mathcal{O}) \rightarrow \Lambda^2 H^0(E_W)^*, \quad b \mapsto \delta b, \]
which coincides with the dual of the multiplication map of global sections of $E_W$. Moreover
\[ \ker \Delta = W^\perp = \{ x_{12} = x_{13} = x_{23} = 0 \}. \]

Using a flat structure [K2] we can write the power series expansion of the regular functions $f_{ij}$ around $O$
\[ f_{ij} = x_{ij} + q_{ij} + \cdots, \]
where $x_{ij}$ and $q_{ij}$ are linear and quadratic polynomials respectively. We easily calculate the expansion of $f$: by skew-symmetry its cubic term is zero, and its quartic term equals
\[ F_W : q_{11} x_{23}^2 + q_{22} x_{13}^2 + q_{33} x_{12}^2 + x_{12} x_{23} (q_{13} + q_{31}) - x_{12} x_{13} (q_{13} + q_{31}) - x_{12} x_{13} (q_{23} + q_{32}). \]

We straightforwardly deduce from this equation the following properties of $F_W$.

3.3 Proposition. 1. The quartic $F_W$ is singular along the vertex $P^\perp W$.

2. For any $x \in W^\perp$, the cubic polar $P_x(F_W)$ is singular along the vertex $P^\perp W$.

3.2 Infinitesimal deformations of global sections of $E_W$

We first recall some elementary facts on principal parts. Let $V$ be an arbitrary vector bundle over $C$ and let $\text{Rat}(V)$ be the space of rational sections of $V$ and $p$ be a point of $C$. The space of principal parts of $V$ at $p$ is the quotient
\[ \text{Prin}_p(V) = \text{Rat}(V) / \text{Rat}_p(V), \]
where $\text{Rat}_p(V)$ denotes the space of rational sections of $V$ which are regular at $p$. Since a rational section of $V$ has only finitely many poles, we have a natural mapping
\[ \text{pp} : \text{Rat}(V) \rightarrow \text{Prin}(V) := \bigoplus_{p \in C} \text{Prin}_p(V), \quad s \mapsto (s \text{ mod } \text{Rat}_p(V))_{p \in C}. \quad (3.1) \]

Exactly as in the line bundle case ([K1] Lemma 3.3), one proves

3.4 Lemma. There are isomorphisms
\[ \ker \text{pp} \cong H^0(C, V), \quad \text{coker } \text{pp} \cong H^1(C, V). \]

In the particular case $V = \mathcal{O}$, we see that a tangent vector $b \in H^1(\mathcal{O}) = T_OJC$ can be represented by a collection $\beta = (\beta_p)_{p \in I}$ of rational functions $\beta_p \in \text{Rat}(\mathcal{O})$, where $p$ varies over a finite set of points $I \subset C$. We then define $\text{pp}(\beta) = (\omega_p)_{p \in I} \in \text{Prin}(\mathcal{O})$, where $\omega_p$ is the principal part of $\beta_p$ at $p$. We denote by $[\beta] = b$ its cohomology class in $H^1(\mathcal{O})$. Note that we can define powers of $\beta$ by $\beta^k = (\beta_p^k)_{p \in I}$.

For $i \geq 1$, let $D_i$ be the infinitesimal scheme $\text{Spec}(A_i)$, where $A_i$ is the Artinian ring $\mathbb{C}[\epsilon]/\epsilon^{i+1}$. As explained in [K2] section 2, a tangent vector $b \in H^1(\mathcal{O})$ determines a morphism
\[ \exp_{i,b} : D_i \rightarrow JC, \]
with \( \exp_{i,b}(x_0) = \mathcal{O} \), where \( x_0 \) is the closed point of \( D_i \). Let \( \mathbb{L}_{i+1}(b) \) denote the pull-back of the Poincaré sheaf \( \mathcal{L} \) under the morphism \( \exp_{i,b} \times id_C \). Note that we have the following exact sequences

\[
D_1 \times C : \quad 0 \longrightarrow \epsilon \mathcal{O} \longrightarrow \mathbb{L}_2(b) \longrightarrow \mathcal{O} \longrightarrow 0, \tag{3.2}
\]

\[
D_2 \times C : \quad 0 \longrightarrow \epsilon^2 \mathcal{O} \longrightarrow \mathbb{L}_3(b) \longrightarrow \mathbb{L}_2(b) \longrightarrow 0. \tag{3.3}
\]

The second arrows in each sequence correspond to the restriction to the subschemes \( \{x_0\} \times C \subset D_1 \times C \) and \( D_1 \times C \subset D_2 \times C \) respectively. As above we choose a representative \( \beta \) of \( b \). Following [K2] section 2, one shows that the space of global sections \( H^0(C \times D_1, \mathbb{L}_{i+1}(b) \otimes E) \), with \( E = E_W \) and \([W] \not\in \mathcal{D} \), is isomorphic to the \( A_i \)-module

\[
V_i(\beta) = \{ f = f_0 + \cdots + f_i \epsilon^i \in \text{Rat}(E) \otimes A_i \text{ such that } f \exp(\epsilon) \text{ is regular } \forall p \in C \}. \tag{3.4}
\]

An element \( f \in V_i(\beta) \) is called an \( i \)-th order deformation of the global section \( f_0 \in H^0(E) \). In the case \( i = 2 \), the condition \( f \in V_i(\beta) \) is equivalent to the following three elements,

\[
f_0, \quad f_1 + f_0 \beta, \quad f_2 + f_1 \beta + f_0 \frac{\beta^2}{2}, \tag{3.5}
\]

being regular at all points \( p \in C \) — for \( i = 1 \), we consider the first two elements. Alternatively this means that their classes in \( \text{Prin}(E) \) are zero. We note that, given two representatives \( \beta = \langle \beta_p \rangle_{p \in I} \) and \( \beta' = \langle \beta'_p \rangle_{p \in I'} \), with \( [\beta] = [\beta'] \), the two subspaces \( V_i(\beta) \) and \( V_i(\beta') \) of \( \text{Rat}(E) \otimes A_i \) are different and that any rational function \( \varphi \in \text{Rat}(\mathcal{O}) \) satisfying \( \text{pp}(\varphi) = \text{pp}(\beta' - \beta) \) induces an isomorphism \( V_i(\beta) \cong V_i(\beta') \).

We consider a class \( b \in H^1(\mathcal{O}) \setminus W^\perp \) and a representative \( \beta \) such that \( [\beta] = b \). By taking cohomology of (3.2) tensored with \( E \), we observe that a first order deformation of \( f_0 \), i.e., a global section \( f = f_0 + f_1 \epsilon \in V_1(\beta) \cong H^0(C \times D_1, \mathbb{L}_2(b) \otimes E) \) always exists. Since \( \text{rk}(\cup b) = 2 \), the global section \( f_0 \) is uniquely determined up to a scalar

\[
f_0 \cdot \mathbb{C} = \ker \left( \cup b : H^0(E) \longrightarrow H^1(E) \right).
\]

Moreover any two first order deformations of \( f_0 \) differ by an element in \( \epsilon H^0(E) \).

We now state a criterion for a tangent vector \( b = [\beta] \) to lie on the quartic tangent cone \( F_W \) in terms of a second order deformation of \( f_0 \in H^0(E) \).

**3.5 Lemma.** A cohomology class \( b = [\beta] \in H^1(\mathcal{O}) \setminus W^\perp \) is contained in the cone over the quartic \( F_W \) if and only if there exists a global section

\[
f = f_0 + f_1 \epsilon + f_2 \epsilon^2 \in V_2(\beta) \cong H^0(C \times D_2, \mathbb{L}_3(b) \otimes E).
\]

**Proof.** The proof is similar to [KS] Lemma 4. We work over the Artinian ring \( A_4 \), i.e., \( \epsilon^5 = 0 \). By Theorem 3.1 applied to the family \( \mathbb{L}_3(b) \otimes E \) over \( C \times D_4 \), there exists an approximating homomorphism of \( A_4 \)-modules

\[
A_4^{\mathbb{R}^3} \xrightarrow{\varphi} A_4^{\mathbb{R}^3}, \tag{3.6}
\]

such that \( \ker \varphi|_{D_2} \cong H^0(C \times D_2, \mathbb{L}_3(b) \otimes E) \), \( \text{coker} \varphi|_{D_2} \cong H^1(C \times D_2, \mathbb{L}_3(b) \otimes E) \), and \( \varphi \otimes \mathbb{C}_0 = 0 \). We denote by \( \varphi|_{D_2} \) the homomorphism obtained from (3.6) by projecting to \( A_2 \). Note that any \( A_4 \)-module is free. The matrix \( \varphi \) is equivalent to a matrix

\[
M := \begin{pmatrix} \epsilon^u & 0 & 0 \\ 0 & \epsilon^v & 0 \\ 0 & 0 & \epsilon^w \end{pmatrix}.
\]
Since $\varphi \otimes \mathbb{C}_0 = 0$, we have $u, v, w \geq 1$. Moreover we can order the exponents so that $1 \leq u \leq v \leq w$. It follows from the definition of $D(E_W)$ as a determinant divisor that the pull-back of $D(E_W)$ by $\exp_1 : D_4 \longrightarrow JC$ is given by the equation (in $A_4$)

$$\det M = e^{v+w}.$$

We immediately see that $b \in F_W$ if and only if $u + v + w \geq 5$. Let us now restrict $\varphi$ to $D_1$, i.e., we project (3.6) to $A_1$. Since we assume $b \notin W^\perp = \ker \Delta$, the restriction $\varphi|_{D_1}$ is nonzero and by skew-symmetry of rank 2, i.e., $u = v = 1$ and $w \geq 2$. Hence $b \in F_W$ if and only if $w \geq 3$.

On the other hand the $A_2$-module $\ker \varphi|_{D_2} \cong H^0(C \times D_2, \mathcal{L}_3(b) \otimes E)$ has length $2 + w$. Let $\mu$ be the multiplication by $e^2$ on this $A_2$-module. Then by (3.4) the $A_2$-module $\ker \mu$ is isomorphic to the $A_1$-module $H^0(C \times D_1, \mathcal{L}_2(b) \otimes E)$, which is of length 4, provided $b \notin W^\perp$. Hence we obtain that $w \geq 3$ if and only if there exists an $f \in H^0(C \times D_2, \mathcal{L}_3(b) \otimes E)$ such that $\mu(f) = e^2f_0$. This proves the lemma.

\[ \Box \]

4 Study of the quartic $F_W$

In this section we prove geometric properties of the quartic $F_W$.

4.1 Criteria for $b \in F_W$

We now show that the criterion of Lemma 3.5 simplifies to a criterion involving only a first order deformation $f = f_0 + f_1 \epsilon \in V_1(\beta)$ of $f_0$. As above we assume $b \notin W^\perp$.

First we observe that the rational differential form $f_1 \wedge f_0$ is independent of the choice of the representative $\beta$, i.e., $f_1 \wedge f_0$ only depends on the cohomology class $b = |\beta|$: suppose we take $\beta' = (\beta_p \cdot \varphi)_{p \in I}$, where $\varphi \in \text{Rat}(\omega)$. Then $f_0$ and $f_1$ transform into $f_0' = f_0$ and $f_1' = f_1 + \varphi f_0$, from which it is clear that $f_1' \wedge f_0' = f_1 \wedge f_0$.

Secondly one easily sees that $f_0 = \pi(b)$ (section 2.1) and that, under the canonical identification $\Lambda^2W^* = \Lambda^2H^0(E) = W$, the 2-plane $H^0(E) \wedge f_0$ coincides with the intersection $V_0 := H_b \cap W$, where $H_b$ denotes the hyperplane determined by $b \in H^1(\mathcal{O})$.

It follows from these two remarks that, given $b$ and $W$, the form $f_1 \wedge f_0$ is well-defined up to a regular differential form in $V_0 \subset W$.

4.1 Proposition. We have the following equivalence

$$b \in F_W \quad \iff \quad f_1 \wedge f_0 \in H_b.$$

Proof. Since $f_1 \wedge f_0$ does not depend on $\beta$, we may choose a $\beta$ with simple poles at the points $p \in I$. By Lemma 3.5 and relation (3.5) we see that $b \in F_W$ if and only if the cohomology class $[f_1\beta + f_0 \beta^2]_\mathbb{C}$ is zero in $H^1(E)/\text{im} (\cup b)$ — we recall that $f_1$ is defined up to $H^0(E)$.

First we will prove that $[f_0 \beta^2 ]_\mathbb{C} \in \text{im} (\cup b)$. The commutativity of the upper right triangle of the
As already observed, we have diagram (see e.g. [K1])

\[ \begin{array}{cccc}
H^0(E) & \rightarrow & H^0(E(2I)) & \rightarrow & E(2I)_2I & \rightarrow & H^1(E) \\
\downarrow \beta^2/2 & \nearrow \cup [\beta^2/2] & & & & & \\
\cap & \cap & \cap & & & & \\
\end{array} \]

\[ \text{Rat}(E) \xrightarrow{pp} \text{Prin}(E) \]

implies that \([f_0\beta^2/2] = f_0 \cup [\beta^2/2]\). Moreover the skew-symmetric cup-product map \(\cup b\)

\[ \cup b = \land b : H^0(E) = W^* \rightarrow H^1(E) = W = \Lambda^2W^* \]

identifies with the exterior product \(\land b\), where \(b = \pi(b) \in W^*\). It is clear that \(\text{im}(\cup b) = \text{im}(\land b) = \ker(\land b)\), where \(\land b\) also denotes the linear form

\[ \land b : \Lambda^2W^* \rightarrow \Lambda^3W^* \cong \mathbb{C}. \quad (4.1) \]

As already observed, we have \(f_0 = \land b\). Denoting by \(c \in W^*\) the class \(\pi([\beta^2/2])\), we see that the relation \((f_0 \land c) \land b = \land b \land c \land b = 0\) implies that \(f_0 \cup [\beta^2/2] \in \ker(\land b) = \text{im}(\cup b)\).

Therefore the previous condition simplifies to \([f_1\beta] \in \text{im}(\cup b)\). We next observe that the linear form \(\land b\) on \(H^1(E)\) \((4.1)\) identifies with the exterior product map

\[ H^1(E) \xrightarrow{\land f_0} H^1(\omega) \cong \mathbb{C}. \]

Since we have a commutative diagram

\[ \begin{array}{cccc}
f_1 \in H^0(E(I)) & \rightarrow & \text{Prin}(E) & \rightarrow & H^1(E) \\
\downarrow \land f_0 & & \downarrow \land f_0 & & \\
f_1 \land f_0 \in H^0(\omega) & \rightarrow & \text{Prin}(\omega) & \rightarrow & H^1(\omega), \\
\end{array} \]

and since \(f_1 \land f_0 \in H^0(\omega) \subset \text{Rat}(\omega)\), we easily see that the condition \([f_1\beta] \in \text{im}(\cup b)\) is equivalent to \(f_1 \land f_0 \in H_b = \ker(\cup b : H^0(\omega) \rightarrow H^1(\omega))\).

\[ \square \]

In the following proposition we give more details on the element \(f_1 \land f_0 \in H^0(\omega)\). We additionally assume that \(\pi(b) \notin \Gamma\), which implies that the global section \(f_0 \in H^0(E)\) does not vanish at any point and hence determines an exact sequence

\[ 0 \rightarrow O \xrightarrow{f_0} E \xrightarrow{\land f_0} \omega \rightarrow 0. \quad (4.2) \]

The coboundary map of the associated long exact sequence

\[ \ldots \rightarrow H^0(\omega) \xrightarrow{\cup e} H^1(O) \rightarrow \ldots \]

is symmetric and coincides (e.g. [K1] Corollary 6.8) with cup-product \(\cup e\) with the extension class \(e \in \mathbb{P}H^1(\omega^{-1}) = [\omega^2]^*\). Moreover \(\cup e\) is the image of \(e\) under the dual of the multiplication map

\[ H^1(\omega^{-1}) = H^0(\omega^2)^* \hookrightarrow \text{Sym}^2H^0(\omega)^*, \quad e \mapsto \cup e. \quad (4.4) \]

We note that \(\text{corank}(\cup e) = 2\) and that \(\ker(\cup e) = V_b\). Hence \((f_1 \land f_0) \cup e\) is well-defined.
4.2 Proposition. If \( \pi(b) \notin \Gamma \), then \( f_1 \land f_0 \notin \ker(\cup e) \) and we have (up to a nonzero scalar)

\[
(f_1 \land f_0) \cup e = b \in H^1(\mathcal{O}).
\]

Proof. We keep the notation of the previous proof. The condition \( f_1 \land f_0 \in V_b \) implies that \( f_1 \) is a regular section and, by (3.5), that \( f_0 \) vanishes at the support of \( b \), i.e., \( \pi(b) \in \Gamma \). As for the equality of the proposition, we introduce the rank-2 vector bundle \( \hat{E} \) which is obtained from \( E \) by (positive) elementary transformations at the points \( p \in I \) and with respect to the line in \( E_p \) spanned by the nonzero vector \( f_0(p) \). Then we have \( E \subset \hat{E} \subset E(I) \) and \( \hat{E} \) fits into the exact sequence

\[
0 \rightarrow E \rightarrow \hat{E} \rightarrow \mathcal{O}_I \rightarrow 0.
\]

Moreover \( f_1 \in H^0(\hat{E}) \), which follows from condition (3.5). We also have the following exact sequences

\[
0 \rightarrow \mathcal{O}(I) \rightarrow \hat{E} \underset{f_0}{\rightarrow} \omega \rightarrow 0 \quad (\hat{e})
\]

\[
0 \rightarrow \mathcal{O} \underset{f_0}{\rightarrow} E \underset{f_0}{\rightarrow} \omega \rightarrow 0 \quad (e),
\]

and the extension class \( \hat{e} \in H^1(\omega^{-1}(D)) \) is obtained from \( e \) by the canonical projection \( H^1(\omega^{-1}) \rightarrow H^1(\omega^{-1}(I)) \). Taking the associated long exact sequences, we obtain

\[
f_1 \in H^0(\hat{E}) \underset{f_0}{\rightarrow} H^0(\omega) \underset{\cup e}{\rightarrow} H^1(\mathcal{O}(I))
\]

\[
H^0(E) \underset{f_0}{\rightarrow} H^0(\omega) \underset{\cup e}{\rightarrow} H^1(\mathcal{O}),
\]

where the two squares commute. This means that

\[
\pi_I ((f_1 \land f_0) \cup e) = (f_1 \land f_0) \cup \hat{e} = 0.
\]

Since \( f_1 \land f_0 \) does not depend on \( \beta \) (nor on \( I \)), the latter relation holds for any \( I \) with \( I = \text{supp } \beta \). Hence, denoting by \( (I) \) the linear span in \( |\omega|^* \) of the support \( I \) of \( \beta \), we obtain

\[
(f_1 \land f_0) \cup e \in \bigcap_{I=\text{supp } \beta} \ker \pi_I = \bigcap_{b \in (I)} \langle I \rangle = b.
\]

\( \square \)

4.2 Geometric properties of \( F_W \)

4.3 Proposition. For any \([W] \notin \mathcal{D}\) we have the following

1. The quartic \( F_W \) contains the canonical curve \( C \), i.e., \( F_W \in |I(4)| \).

2. The quartic \( F_W \) contains the secant line \( \overline{pq} \), with \( p \neq q \), if and only if \( \overline{pq} \cap PW^\perp \neq \emptyset \) or \( \dim W \cap H^0(\omega(-2p - 2q)) > 0 \).

3. Let \( \Sigma \) be the set of points \( p \) at which the tangent line \( T_p(C) \) intersects the vertex \( PW^\perp \). Then \( \Sigma \) is empty for general \([W]\) and finite for any \([W]\). Moreover any point \( p \in C \setminus \Sigma \) is smooth on \( F_W \) and the embedded tangent space \( T_p(F_W) \) is the linear span of \( T_p(C) \) and \( PW^\perp \).
Proof. All statements are easily deduced from Proposition 4.1. Given a point \( p \in C \) we denote by \( p_\mu \in \text{Prin}_p(O) \) the principal part supported at \( p \) of a rational function with a simple pole at \( p \). Then the class \([p_\mu] \in H^1(O)\) is proportional to \( i_\omega(p) \in |\omega|^* = \mathbb{P}H^1(O) \) and the section \( f_0 \) vanishes at \( p \). Hence \( f_0|_{p_\mu} \in \text{Prin}(E) \) is everywhere regular and we may choose \( f_1 = 0 \). This proves part 1. See also [PP].

As for part 2, we introduce \( \beta_{\lambda,\mu} = \lambda p_\mu + \mu q_\mu \in \text{Prin}(O) \) for \( \lambda, \mu \in \mathbb{C} \) and denote by \( s_p \) and \( s_q \) the global sections \( \pi([p_\mu]) \) and \( \pi([q_\mu]) \), which vanish at \( p \) and \( q \) respectively. Then one checks that \( f_0 = \lambda s_p + \mu s_q \in \ker(\cup[\beta_{\lambda,\mu}]) \) and \( \text{pp}(f_1) = \lambda\mu s_p s_q + s_p q_\mu \in \text{Prin}(E) \). With this notation the condition of Proposition 4.1 transforms into

\[
0 = l_{\lambda,\mu}(f_0 \wedge f_1) = \lambda\mu(\lambda^2 \gamma_p + \mu^2 \gamma_q),
\]

where \( l_{\lambda,\mu} \) is the linear form defined by \([\beta_{\lambda,\mu}] \in H^1(O)\). The scalars \( \gamma_p \) and \( \gamma_q \) are the values of the section \( s_p \wedge s_q \in W \cap H^0(\omega(-p - q)) \) at \( p \) and \( q \) respectively. We now conclude noting that \( s_p \wedge s_q = 0 \) if and only if \( \mathbb{P}q \cap \mathbb{P}W^\perp \neq \emptyset \).

As for part 3, we first observe that the assumption \( \Sigma = C \) implies that the restriction \( \pi|_C : C \rightarrow \mathbb{P}W^* \) contracts \( C \) to a point, which is impossible. Next we consider the tangent vector \( t_q \) at \( p \) given by the direction \( q \). By putting \( \lambda = 1 \) and \( \mu = \epsilon \), with \( \epsilon^2 = 0 \), into equation (4.5) we obtain that \( t_q \in T_p(F_W) \) if and only if \( \epsilon \gamma_p = 0 \), i.e., \( \pi(q) \in T_{\pi(p)}(\Gamma) \). Hence \( T_p(F_W) = \pi^{-1}(T_{\pi(p)}(\Gamma)) \), which proves part 3.

4.3 The cubic polar \( P_x(F_W) \)

Firstly we deduce from Propositions 4.1 and 4.2 a criterion for \( b \in P_x(F_W) \), with \( x \in W^\perp \). Let \( H_x \) be the hyperplane determined by \( x \in H^1(O) \). As above we assume \( b \notin W^\perp \) and \( \pi(b) \notin \Gamma \), i.e., the pencil \( V = V_b \) is base-point-free.

4.4 Proposition. We have the following equivalence

\[
b \in P_x(F_W) \iff f_1 \wedge f_0 \in H_x.
\]

Proof. We recall from section 4.1 that \( \cup e \) induces a symmetric isomorphism \( \cup e : (V^\perp)^* \sim \rightarrow V^\perp \) and we denote by \( Q^* \subset \mathbb{P}(V^\perp)^* \) and \( Q \subset \mathbb{P}V^\perp \) the two associated smooth quadrics. Note that \( Q \) and \( Q^* \) are dual to each other. Combining Propositions 4.1, 4.2 and 3.3 (1) we see that the restriction of the quartic \( F_W \) to the linear subspace \( \mathbb{P}V^\perp \subset |\omega|^* \) splits into a sum of divisors

\[
(F_W)|_{\mathbb{P}V^\perp} = 2\mathbb{P}W^\perp + Q.
\]

We also observe that \( Q \) only depends on \( V \) (and on \( W \)) and not on \( b \). Taking the polar with respect to \( x \in W^\perp \), we obtain

\[
(P_x(F_W))|_{\mathbb{P}V^\perp} = 2\mathbb{P}W^\perp + P_x(Q).
\]

Finally we see that the condition \( b \in P_x(Q) \) is equivalent to \( f_0 \wedge f_1 = (\cup e)^{-1}(b) \in H_x \).

We easily deduce from this criterion some properties of \( P_x(F_W) \).

4.5 Proposition. The cubic \( P_x(F_W) \) contains the canonical curve \( C \), i.e., \( P_x(F_W) \in |I(3)| \).
Proof. We first observe that the two closed conditions of Proposition 4.4 are equivalent outside \( \pi^{-1}(\Gamma) \). Hence they coincide as well on \( \pi^{-1}(\Gamma) \) and we can drop the assumption \( \pi(b) \notin \Gamma \). Now, as in the proof of Proposition 4.3(1), we may choose \( f_1 = 0 \).

4.6 Proposition. We have the following properties

\[
\bigcap_{x \in W^\perp} P_x(F_W) = S_W \cup \mathbb{P}W^\perp \cup \bigcup_{n \geq 2} \Lambda_n,
\]

where \( S_W \) is an irreducible surface. For \( n \geq 0 \), we denote by \( \Lambda_n \) the union of \((n + 1)\)-secant \( \mathbb{P}^n \)'s to the canonical curve \( C \), which intersect the vertex \( \mathbb{P}W^\perp \) along a \( \mathbb{P}^{n-1} \). If \( W \) is general, then \( \Lambda_n = \emptyset \) for \( n \geq 2 \) and \( \Lambda_1 \) is the union of \( 2(g - 1)(g - 3) \) secant lines.

Proof. We consider \( b \) in the intersection of all \( P_x(F_W) \) and we first suppose that \( \pi(b) \notin \Gamma \). Then by Propositions 4.1 and 4.4 we have

\[
f_0 \land f_1 = \bigcap_{x \in W^\perp} H_x = W.
\]

Hence we obtain that \( \mathbb{P}V^\perp \cap \bigcap_{x \in W^\perp} P_x(F_W) \) is reduced to the point \( (\cup c)(W) \in \mathbb{P}V^\perp \). On the other hand a standard computation shows that \( S_W \) is the image of \( \mathbb{P}^2 \) under the linear system of the adjoint curves of \( \Gamma \). Hence \( S_W \) is irreducible.

If \( \pi(b) \in \Gamma \), we denote by \( p_1, \ldots, p_{n+1} \in C \) the points such that \( \pi(p_i) = \pi(b) \). Then \( f_0 \) vanishes at \( p_1, \ldots, p_{n+1} \). Since \( f_1 \land f_0 \) does not depend on the support of \( b \), we can choose \( \text{supp } b \) such that \( p_i \notin \text{supp } b \). Then \( f_1 \) is regular at \( p_1 \) and we deduce that \( f_1 \land f_0 \in H^0(\omega(- \sum p_i)) \cap W = V_b \). Now any rational \( f_1 \) satisfying \( f_1 \land f_0 \in V_b = \text{im } (\land f_0) \) is regular everywhere, which can only happen when \( f_0 \) vanishes at the support of \( b \). By uniqueness we have \( \text{supp } b \subset \{ p_1, \ldots, p_{n+1} \} \) and \( b \in \Lambda_n \). Note that \( \Lambda_0 = C \). This proves the first equality.

If \( b \in F_W \cap S_W \), we have \( f_1 \land f_0 \in W \cap H_b = V_b \) and we conclude as above. Note that \( \Lambda_1 \) is contained in \( S_W \) and is mapped by \( \pi \) to the set of ordinary double points of \( \Gamma \).

For any \([W] \in \text{Gr}(3, H^0(\omega)) \setminus \mathcal{D}\) we introduce the subspace of \( I(3) \)

\[
L_W = \{ R \in I(3) \mid R \text{ is singular along the vertex } \mathbb{P}W^\perp \}.
\]

Then Propositions 4.5 and 3.3(2) imply that \( P_x(F_W) \in L_W \). More precisely, we have

4.7 Proposition. The restriction of the polar map of the quartic \( F_W \) to its vertex \( \mathbb{P}W^\perp \)

\[
P : W^\perp \longrightarrow L_W, \quad x \mapsto P_x(F_W),
\]

is an isomorphism.

Proof. First we show that \( \dim L_W = g - 3 \). We choose a complementary subspace \( A \) to \( W^\perp \), i.e., \( H^0(\omega)^* = W^\perp \oplus A \), and a set of coordinates \( x_1, \ldots, x_{g-3} \) on \( W^\perp \) and \( a_1, a_2, a_3 \) on \( A \). This enables us to expand a cubic \( F \in S^3 H^0(\omega) \)

\[
F = F_3(x) + F_2(x)G_1(a) + F_1(x)G_2(a) + G_3(a), \quad F_i \in \mathbb{C}[x_1, \ldots, x_{g-3}], \ G_i \in \mathbb{C}[a_1, a_2, a_3],
\]

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with \( \deg F_i = \deg G_i = i \). Let \( S_A \) denote the subspace of cubics singular along \( PA \), i.e. \( G_2 = G_3 = 0 \). We consider the linear map
\[
\alpha : I(3) \longrightarrow S_A, \quad F \longmapsto F_3(x) + F_2(x)G_1(a).
\]
Since by Lemma 2.1 any monomial \( x_ix_j \in H^0(\mathbb{P}W^\perp, O(2)) \) lifts to a quadric \( Q_{ij} \in I(2) \), we observe that the monomials \( x_ix_jx_k \) and \( x_ix_aj_l \) which generate \( S_A \), also lift e.g. to \( Q_{ij}x_k \) and \( Q_{ij}a_l \) in \( I(3) \). Hence \( \alpha \) is surjective and \( \dim L_W = \dim \ker \alpha \) is easily calculated. One also checks that this computation does not depend on \( A \).

In order to conclude, it will be enough to show that \( \mathcal{P} \) is injective. Suppose that the contrary holds, i.e., there exists a point \( x \in W^\perp \) with \( P_x(F_W) = 0 \). Given any base-point-free pencil \( V \subset W \) and any \( b \in V^\perp \), we obtain by Proposition 4.4 that \( f_0 \wedge f_1 \in H_x \). Since \( \cup e : (V^\perp)^* \longrightarrow V^\perp \) is an isomorphism, we see that for \( b \notin (\cup e)^{-1}(H_x) \) the element \( f_0 \wedge f_1 \) must be zero. This implies that \( b \in \Lambda \) and since \( b \) varies in an open subset of \( |\omega|^* \), we obtain \( \Lambda = |\omega|^* \), a contradiction.

\[ \square \]

4.4 The quadric bundle associated to \( F_W \)

Let \( \mathbb{P}^g_W^{g-1} \rightarrow |\omega|^* \) denote the blowing-up of \( |\omega|^* \) along the vertex \( \mathbb{P}W^\perp \subset |\omega|^* \). The rational projection \( \pi : |\omega|^* \longrightarrow \mathbb{P}^2 = \mathbb{P}W^* \) resolves into a morphism \( \hat{\pi} : \mathbb{P}^g_W^{g-1} \rightarrow \mathbb{P}^2 \). Since \( F_W \) is singular along \( \mathbb{P}W^\perp \) (Proposition 3.3 (2)), the proper transform \( \tilde{F}_W \subset \mathbb{P}^g_W^{g-1} \) admits a structure of a quadric bundle \( \hat{\pi} : \tilde{F}_W \rightarrow \mathbb{P}^2 \).

The contents of Propositions 4.3 and 4.5 can be reformulated in a more geometrical way.

4.8 Theorem. For any \( [W] \in \text{Gr}(3, H^0(\omega)) \setminus \mathcal{D} \), the quadric bundle \( \hat{\pi} : \tilde{F}_W \rightarrow \mathbb{P}^2 \) has the following properties

1. Its Hessian curve is \( \Gamma \subset \mathbb{P}^2 \).

2. Its Steinerian curve is the (proper transform of the) canonical curve \( C \subset |\omega|^* \).

3. The rational Steinerian map \( \text{St} : \Gamma \longrightarrow C \), which associates to a singular quadric its singular point, coincides with the adjoint map \( \text{ad} \) of the plane curve \( \Gamma \). Moreover the closure of the image \( \text{ad}(\mathbb{P}^2) \) equals \( S_W \).

4.9 Remark. We note that Theorem 4.8 is analogous to the main result of [KS] (replace \( \mathbb{P}^2 \) with \( \mathbb{P}^1 \times \mathbb{P}^1 \)). In spite of this striking similarity and the relation between the two parameter spaces \( \text{Sing} \Theta \) and \( \text{Gr}(3, H^0(\omega)) \) (see [PP]), we were unable to find a common frame for both constructions.

5 The cubic hypersurface \( \Psi_V \subset \mathbb{P}^{g-3} \) associated to a base-point-free pencil \( \mathbb{P}V \subset |\omega| \)

In this section we show that the symmetric cup-product maps \( \cup e \in \text{Sym}^2 H^0(\omega)^* \) (see (4.3)) arise as polar quadrics of a cubic hypersurface \( \Psi_V \), which will be used in the proof of Theorem 6.1.

Let \( V \) denote a base-point-free pencil of \( H^0(\omega) \). We consider the exact sequence given by evaluation of sections of \( V \)
\[
0 \longrightarrow \omega^{-1} \longrightarrow O_C \otimes V \longrightarrow \omega \longrightarrow 0.
\]
Its extension class \( v \in \text{Ext}^1(\omega, \omega^{-1}) \cong H^1(\omega^{-2}) \cong H^0(\omega^3)^* \) corresponds to the hyperplane in \( H^0(\omega^3) \), which is the image of the multiplication map

\[
\text{im} \left( V \otimes H^0(\omega^2) \to H^0(\omega^3) \right).
\]

We consider the cubic form \( \Psi_V \) defined by

\[
\Psi_V : \text{Sym}^3 H^0(\omega) \xrightarrow{\mu} H^0(\omega^3) \xrightarrow{\bar{v}} \mathbb{C},
\]

where \( \mu \) is the multiplication map and \( \bar{v} \) the linear form defined by the extension class \( v \). It follows from the description (5.2) that \( \Psi_V \) factorizes through the quotient

\[
\Psi_V : \text{Sym}^3 \mathcal{V} \to \mathbb{C},
\]

where \( \mathcal{V} := H^0(\omega)/V \). We also denote by \( \Psi_V \subset \mathbb{P} \mathcal{V} \) its associated cubic hypersurface.

A 3-plane \( W \supset V \) determines a nonzero vector \( w \) in the quotient \( \mathcal{V} = H^0(\omega)/V \) and a general \( w \) determines an extension (4.2) — recall that \( W^* \cong H^0(E) \). Hence we obtain an injective linear map \( \mathcal{V} \hookrightarrow H^1(\omega^{-1}), w \mapsto e \), which we compose with (4.4)

\[
\Phi : \mathcal{V} \hookrightarrow H^1(\omega^{-1}) = H^0(\omega^2)^* \hookrightarrow \text{Sym}^2 H^0(\omega)^*, \quad w \mapsto e \mapsto \cup e.
\]

Since \( V \subset \ker(\cup e) \), we note that \( \text{im} \Phi \subset \text{Sym}^2 \mathcal{V}^* \).

We now can state the main result of this section.

**5.1 Proposition.** The linear map \( \Phi : \mathcal{V} \to \text{Sym}^2 \mathcal{V}^* \) coincides with the polar map of the cubic form \( \Psi_V \), i.e.,

\[
\forall w \in \mathcal{V}, \quad \Phi(w) = P_w(\Psi_V).
\]

**Proof.** This is straightforwardly read from the diagram obtained by relating the exact sequences (5.1) and (2.1) via the inclusion \( V \subset W \). We leave the details to the reader. \( \square \)

We also observe that, by definition of the Hessian hypersurface (see e.g. [DK] section 3), we have an equality among degree \( g - 2 \) hypersurfaces of \( \mathbb{P} \mathcal{V} = \mathbb{P}^{g-3} \)

\[
\text{Hess}(\Psi_V) = \mathcal{D} \cap \mathbb{P} \mathcal{V},
\]

where we use the inclusion \( \mathbb{P} \mathcal{V} \subset \text{Gr}(3, H^0(\omega)) \).

**5.2 Remark.** We recall (see [DK] (5.2.1)) that the Hessian and Steinerian of a cubic hypersurface coincide and that the Steinerian map is a rational involution \( i \). In the case of the cubic \( \Psi_V \), the involution

\[
i : \text{Hess}(\Psi_V) \to \text{Hess}(\Psi_V)
\]

corresponds to the involution of [BV] Propositions 1.18 and 1.19, i.e., \( \forall w \in \mathcal{D} \cap \mathbb{P} \mathcal{V} \), the bundles \( E_w \) and \( E_{i(w)} \) are related by the exact sequence

\[
0 \to E_{i(w)}^* \to \mathcal{O}_C \otimes H^0(E_w) \xrightarrow{ev} E_w \to 0.
\]

Since we will not use that result, we leave its proof to the reader.

**5.3 Remark.** The construction which associates to a base-point-free pencil \( V \subset H^0(\omega) \) the extension class \( v \in |\omega^3|^* \) induces a rational map

\[
\text{Gr}(2, H^0(\omega)) \to |\omega^3|^*, \quad V \mapsto v.
\]

It is worthwhile to investigate the possible relations between that map and the Wahl map

\[
\text{Gr}(2, H^0(\omega)) \to |\omega^3|, \quad V = \langle s, t \rangle \mapsto t^{\otimes 2}d(s/t).
\]
Let us denote by $|F_3| \subset |I(3)|$ and $|F_4| \subset |I(4)|$ the linear subsystems spanned by the image of the rational maps $F_3$ and $F_4$ respectively. Then we have the following

**6.1 Theorem.** The base loci of $|F_3|$ and $|F_4|$ coincides with the canonical curve $C \subset |\omega|^*$. 

**Proof.** Let $b \in \text{Bs}|F_3|$ and let us suppose that $b \notin C$. We consider a base-point-free pencil $V \subset H_b$. With the notation of section 5, we introduce the rational map $r_b : \mathbb{P}V \dashrightarrow \mathbb{P}V$, $w \mapsto r_b(w) = w'$, with $\Psi_V(w, w', \cdot) = b$, where $\Psi_V$ is the symmetric trilinear form of $\Psi_V$. We note (Proposition 4.2) that, for $w \in \mathbb{P}(H_b/V)$, the element $r_b(w)$ is collinear with the nonzero element $f_0 \land f_1 \mod V$ and that $r_b$ is defined away from the hypersurface $\text{Hess}(\Psi_V)$, which we assume to be nonzero. Since $b \in \text{Bs}|F_3|$ we obtain by Proposition 4.4 that $r_b(w) = \left( \bigcap_{x \in W^\perp} H_x \right) \mod V = W \mod V = w$. Hence $r_b$ is the identity map (away from $\text{Hess}(\Psi_V)$). This implies that $\Psi_V(w, w, \cdot) = b$ for any $w \in \mathbb{P}V$, hence $\Psi_V = x_0^3$, where $x_0$ is the equation of the hyperplane $\mathbb{P}(H_b/V) \subset \mathbb{P}V$. This in turn implies that $\text{Hess}(\Psi_V) = 0$, i.e., $\mathbb{P}V \subset D$. Since for a general $[W] \in \text{Gr}(3, H^0(\omega))$ the pencil $V = W \cap H_b$ is base-point-free, we obtain that a general $[W]$ lies on the divisor $D$, which is a contradiction.

As for $|F_4|$, we recall that the fact $\text{Bs}|F_4| = C$ follows from [We]. Alternatively, it can also be deduced by noticing (see Proposition 2.3) that $\text{Bs}|F_4| \subset \text{Bs}|I(2)|$. Hence, if $C$ is not trigonal nor a plane quintic, we are done. In the other cases, the result can be deduced from Proposition 4.3 — we leave the details to the reader. 

### 7 Open questions

#### 7.1 Dimensions

The projective dimensions of the linear systems $|F_3|$ and $|F_4|$ are not known for general $g$. The known values of $\text{dim } |F_4|$ for a general curve $C$ are given as follows (see [PP]).

<table>
<thead>
<tr>
<th>$g$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim $</td>
<td>F_4</td>
<td>$</td>
<td>4</td>
<td>15</td>
</tr>
</tbody>
</table>

The examples of [PP] section 6 show that $\text{dim } |F_4|$ depends on the gonality of $C$. Moreover it can be shown that $|F_4| \neq |I(4)|$.

#### 7.2 Prym-canonical spaces and symplectic bundles

The construction of the quartic hypersurfaces $F_W$ admits various analogues and generalizations, which we briefly outline.

1. Let $P_\alpha := \text{Prym}(C_\alpha/C)$ denote the Prym variety of the étale double cover $C_\alpha \to C$ associated to the nonzero 2-torsion point $\alpha \in J_C$. Given a general 3-plane $Z \subset H^0(C, \omega_\alpha)$, we associate the rank-2 vector bundle $E_Z$ defined by

$$0 \to E_Z^* \to O_C \otimes Z \xrightarrow{\text{ev}} \omega_\alpha \to 0.$$
By [IP] Proposition 4.1 we can associate to $E_Z$ the divisor $\Delta(E_Z) \in |2\Xi|$, where $\Xi$ is a symmetric principal polarization on $P_\alpha$. Its projectivized tangent cone at the origin $0 \in P_\alpha$ is a quartic hypersurface $F_Z$ in the Prym-canonical space $\mathbb{P}T_0P_\alpha \cong |\omega_\alpha|^*$. Kempf’s obstruction theory equally applies to the quartics $F_Z$. We note that $F_Z$ contains the Prym-canonical curve $i_{\omega_\alpha}(C) \subset |\omega_\alpha|^*$.

(2) Let $W$ be a vector space of dimension $2n + 1$, for $n \geq 1$. We consider a general linear map

$$\Phi : \Lambda^2 W^* \to H^0(C, \omega).$$

By taking the $n$-th symmetric power $\text{Sym}^n \Phi$ and using the canonical maps $\text{Sym}^n(\Lambda^2 W^*) \to \Lambda^{2n} W^* \cong W$ and $\text{Sym}^n H^0(\omega) \to H^0(\omega^{\otimes n})$, we obtain a linear map

$$\alpha : W \to H^0(\omega^{\otimes n}),$$

which we assume to be injective. We then define the rank $2n$ vector bundle $E_\Phi$ by

$$0 \to E_\Phi^* \to O_C \otimes W \xrightarrow{\text{ev}} \omega^{\otimes n} \to 0.$$  

The bundle $E_\Phi$ carries an $\omega$-valued symplectic form and the projectivized tangent cone at $O \in JC$ to the divisor $D(E_\Phi)$ is a hypersurface $F_\Phi$ in $|\omega|^*$ of degree $2n + 2$. Moreover $F_\Phi \in |I(2n + 2)|$.

References


[BV] S. Brivio, A. Verra: The theta divisor of $\text{SU}_C(2, 2d)$ is very ample if $C$ is not hyperelliptic, Duke Math. J. 82 (1996), no. 3, 503-552


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Christian Pauly
Laboratoire J.-A. Dieudonné
Université de Nice-Sophia-Antipolis
Parc Valrose
06108 Nice Cedex 2
France
e-mail: pauly@math.unice.fr