A SMOOTH COUNTEREXAMPLE TO NORI'S CONJECTURE ON THE FUNDAMENTAL GROUP SCHEME

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Abstract. We show that Nori's fundamental group scheme \( \pi(X, x) \) does not base change correctly under extension of the base field for certain smooth projective ordinary curves \( X \) of genus 2 defined over a field of characteristic 2.

1. Introduction

In the paper [N] Madhav Nori introduced the fundamental group scheme \( \pi(X, x) \) for a reduced and connected scheme \( X \) defined over an algebraically closed field \( k \) as the Tannaka dual group of the Tannakian category of essentially finite vector bundles over \( X \). In characteristic zero \( \pi(X, x) \) coincides with the \( \acute{e} \)tale fundamental group, but in positive characteristic it does not (see e.g. [MS]). By analogy with the \( \acute{e} \)tale fundamental group, Nori conjectured that \( \pi(X, x) \) base changes correctly under extension of the base field. More precisely:

Nori's conjecture (see [MS] page 144 or [N] page 89) If \( K \) is an algebraically closed extension of \( k \), then the canonical homomorphism

\[
h_{X, K} : \pi(X_K, x) \rightarrow \pi(X, x) \times_k \text{Spec}(K)
\]

is an isomorphism.

In [MS] V.B. Mehta and S. Subramanian show that Nori’s conjecture is false for a projective curve with a cuspidal singularity. In this note (Corollary 4.2) we show that certain smooth projective ordinary curves of genus 2 defined over a field of characteristic 2 also provide counterexamples to Nori’s conjecture.

The proof has two ingredients: the first is an equivalent statement of Nori’s conjecture in terms of \( F \)-trivial bundles due to V.B. Mehta and S. Subramanian (see section 2) and the second is the description of the action of the Frobenius map on rank-2 vector bundles over a smooth ordinary curve \( X \) of genus 2 defined over a field of characteristic 2 (see section 3). In section 4 we explicitly determine the set of \( F \)-trivial bundles over \( X \).

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2. Nori’s conjecture and \( F \)-trivial bundles

Let \( X \) be a smooth projective curve defined over an algebraically closed field \( k \) of characteristic \( p > 0 \). Let \( F : X \rightarrow X \) denote the absolute Frobenius of \( X \) and \( F^n \) its \( n \)-th iterate for some positive integer \( n \).

2.1. Definition. A rank-\( r \) vector bundle \( E \) over \( X \) is said to be \( F^n \)-trivial if

\[
E \text{ stable and } F^n E \cong \mathcal{O}_X^r.
\]

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2. **Proposition** ([MS] Proposition 3.1). *If the canonical morphism \( h_{X,K} \) (1.1) is an isomorphism, then any \( F^n \)-trivial vector bundle \( E_K \) over \( X_K := X \times_k \text{Spec}(K) \) is isomorphic to \( E_k \otimes_k K \) for some \( F^n \)-trivial vector bundle \( E_k \) over \( X \).*

3. The action of the Frobenius map on rank-2 vector bundles

We briefly recall some results from [LP1] and [LP2].

Let \( X \) be a smooth projective ordinary curve of genus 2 defined over an algebraically closed field \( k \) of characteristic 2. By [LP2] section 2.3 the curve \( X \) equipped with a level-2 structure can be uniquely represented by an affine equation of the form

\[
y^2 + x(x + 1)y = x(x + 1)(ax^3 + (a + b)x^2 + cx + c),
\]

for some scalars \( a, b, c \in k \). Let \( M_X \) denote the moduli space of \( S \)-equivalence classes of semistable rank-2 vector bundles with trivial determinant over \( X \)—see e.g. [LeP]. We identify \( M_X \) with the projective space \( \mathbb{P}^3 \) (see [LP1] Proposition 5.1). We denote by \( V : \mathbb{P}^3 \to \mathbb{P}^3 \) the rational map induced by pull-back under the absolute Frobenius \( F : X \to X \). There are homogeneous coordinates \((x_0 : x_1 : x_10 : x_{11})\) on \( \mathbb{P}^3 \) such that the equations of \( V \) are given as follows (see [LP2] section 5)

\[
V(x_0 : x_1 : x_{10} : x_{11}) = (\sqrt{abc}P_{00}^2(x) : \sqrt{b}P_{01}^2(x) : \sqrt{c}P_{10}^2(x) : \sqrt{a}P_{11}^2(x)),
\]

with

\[
P_{00}(x) = x_0^2 + x_{10}^2 + x_{11}^2,
\]

\[
P_{01}(x) = x_0x_{10} + x_{10}x_{11},
\]

\[
P_{11}(x) = x_0x_{11} + x_{11}x_{10}.
\]

Given a semistable rank-2 vector bundle \( E \) with trivial determinant, we denote by \([E] \in M_X = \mathbb{P}^3\) its \( S \)-equivalence class. The semistable boundary of \( M_X \) equals the Kummer surface \( \text{Kum}_X \) of \( X \). Given a degree 0 line bundle \( N \) on \( X \), we also denote the point \([N \oplus N^{-1}] \in \mathbb{P}^3\) by \( N \).

3.1. **Proposition** ([LP1] Proposition 6.1 (4)). *The preimage \( V^{-1}(N) \) of the point \( N \in \text{Kum}_X \subset M_X = \mathbb{P}^3 \) with coordinates \((x_0 : x_1 : x_{10} : x_{11})\)*

- is a projective line, if \( x_{00} = 0 \).
- consists of the 4 square-roots of \( N \), if \( x_{00} \neq 0 \).

4. Computations

In this section we prove the following

4.1. **Proposition.** *Let \( X = X_{a,b,c} \) be the smooth projective ordinary curve of genus 2 given by the affine model (3.1). Suppose that*

\[
a^2 + b^2 + c^2 + a + c = 0.
\]

*Then there exists a nontrivial family \( \mathcal{E} \to X \times S \) parametrized by a 1-dimensional variety \( S \) (defined over \( k \)) of \( F^4 \)-trivial rank-2 vector bundles with trivial determinant over \( X \). Moreover any \( F^4 \)-trivial rank-2 vector bundle \( E \) with trivial determinant appears in the family \( \mathcal{E} \), i.e., is of the form \((\text{id}_X \times s)^*\mathcal{E}\) for some \( k \)-valued point \( s : \text{Spec}(k) \to S \).*

We therefore obtain a counterexample to Nori’s conjecture.

4.2. **Corollary.** *Let \( X = X_{a,b,c} \) be a curve satisfying (4.1). Then for any algebraically closed extension \( K \), the morphism \( h_{X,K} \) is not an isomorphism.*

*Proof.* Since \( S \) is 1-dimensional, there exists a \( K \)-valued point \( s : \text{Spec}(K) \to S \), which is not a \( k \)-valued point. Then the bundle \( E_K = (\text{id}_X \times s)^*\mathcal{E} \) over \( X_K \) is not of the form \( E_k \otimes_k K \). Now apply Proposition 2.2. \( \square \)
Proof of Proposition 4.1. The method of the proof is to determine explicitly all $F^n$-trivial rank-2 vector bundles $E$ over $X$ for $n = 1, 2, 3, 4$. Taking tensor product of $E$ with $2^{n+1}$-torsion line bundles allows us to restrict attention to $F^n$-trivial vector bundles with trivial determinant.

We first compute the preimage under iterates of $V$ of the point $A_0 \in \mathbb{P}^3$ determined by the trivial rank-2 vector bundle over $X$. We recall (see e.g. [LP1] Lemma 2.11 (i)) that the coordinates of $A_0 \in \mathbb{P}^3$ in the coordinate system $(x_{00} : x_{01} : x_{10} : x_{11})$ are $(1 : 0 : 0 : 0)$. It follows from Proposition 3.1 and equations (3.2) that $V^{-1}(A_0)$ consists of the 4 points
\[(4.2) \quad (1 : 0 : 0 : 0), \quad (0 : 1 : 0 : 0), \quad (0 : 0 : 1 : 0) \quad \text{and} \quad (0 : 0 : 0 : 1),\]
which correspond to the 2-torsion points of the Jacobian of $X$. Abusing notation we denote by $A_1$ both the 2-torsion line bundle on $X$ and the point $(0 : 1 : 0 : 0) \in \mathbb{P}^3$.

Both points $A_0$ and $A_1$ correspond to $S$-equivalence classes of semistable rank-2 vector bundles. The set of isomorphism classes represented by the two $S$-equivalence classes $A_0$ and $A_1$ equal $\mathbb{P} \text{Ext}^1(A_1, A_1) \cup \{0\}$ and $\mathbb{P} \text{Ext}^1(O_X, O_X) \cup \{0\}$ respectively, where $0$ denotes the trivial extensions $A_1 \oplus A_1$ and $O_X \oplus O_X$. Note that the two cohomology spaces $\text{Ext}^1(A_1, A_1)$ and $\text{Ext}^1(O_X, O_X)$ are canonically isomorphic to $H^1(O_X)$. The pull-back by the absolute Frobenius $F$ of $X$ induces a rational map
\[F^* : \mathbb{P} \text{Ext}^1(A_1, A_1) \longrightarrow \mathbb{P} \text{Ext}^1(O_X, O_X),\]
which coincides with the projectivized $p$-linear map on the cohomology $H^1(O_X) \rightarrow H^1(O_X)$ induced by the Frobenius map $F$. Since we have assumed $X$ ordinary, this $p$-linear map is bijective. Hence we obtain that there is only one (strictly) semistable bundle $E$ such that $[E] = A_1$ and $F^*E \cong O_X^c$, namely $E = A_1 \oplus A_1$. In particular there are no $F^1$-trivial rank-2 vector bundles over $X$.

By Proposition 3.1 and using the equations (3.2), we easily obtain that the preimage $V^{-1}(A_1)$ is a projective line $\mathbb{L} \cong \mathbb{P}^1$, which passes through the two points
\[(1 : 1 : 1 : 1) \quad \text{and} \quad (0 : 0 : 1 : 1).\]
We now determine the bundles $E$ satisfying $F^*E \cong A_1 \oplus A_1$. Given $E$ with $[F^*E] = A_1 \in \mathbb{P}^3$ we easily establish the equivalence
\[F^*E \cong A_1 \oplus A_1 \iff \dim \text{Hom}(F^*E, A_1) = \dim \text{Hom}(E, F_*A_1) = 2.\]
Suppose that $E$ is stable and $F^*E \cong A_1 \oplus A_1$. The quadratic map
\[\text{det} : \text{Hom}(E, F_*A_1) \longrightarrow \text{Hom}(\text{det } E, \text{det } F_*A_1) = H^0(O_X(w))\]
has nontrivial fibre over $0$, since $\dim \text{Hom}(E, F_*A_1) = 2$. Hence there exists a nonzero $f \in \text{Hom}(E, F_*A_1)$ not of maximal rank. We consider the line bundle $N = \text{im } f \subset F_*A_1$. Since $F_*A_1$ is stable (see [LaP] Proposition 1.2), we obtain the inequalities
\[0 = \mu(E) < \deg N < \frac{1}{2} = \mu(F_*A_1),\]
a contradiction. Therefore $E$ is strictly semistable and $[E] = [A_2 \oplus A_2^{-1}]$ for some 4-torsion line bundle $A_2$ with $A_2^{\otimes 2} = A_1$. The $S$-equivalence class $[A_2 \oplus A_2^{-1}]$ contains three isomorphism classes and a standard computation shows that only the decomposable bundle $A_2 \oplus A_2^{-1}$ is mapped by $F^*$ to $A_1 \oplus A_1$. In particular there are no $F^2$-trivial rank-2 bundles.

We now determine the coordinates of $A_2$ by intersecting the line $\mathbb{L}$, which can be parametrized by $(r : r : s : s)$ with $r, s \in k$, with the Kummer surface, whose equation is (see [LP2] Proposition 3.1)
\[c(x_{00}^2 x_{10}^2 + x_{01}^2 x_{11}^2) + b(x_{00}^2 x_{01}^2 + x_{10}^2 x_{11}^2) + a(x_{00}^2 x_{11}^2 + x_{10}^2 x_{01}^2) + x_{00} x_{01} x_{10} x_{11} = 0.\]
The computations are straightforward and will be omitted. Let $u \in k$ be a root of the equation
\[(4.3) \quad u^2 + u = b.\]
Then $u + 1$ is the other root. The coordinates of the two 4-torsion line bundles (modulo the canonical involution of the Jacobian of $X$) $A_2$ such that $A_2^{\otimes 2} = A_1$ are
\[
(u : u : \sqrt{b} : \sqrt{b}) \quad \text{and} \quad (u + 1 : u + 1 : \sqrt{b} : \sqrt{b}).
\]
Now the equation $u = 0$ (resp. $u + 1 = 0$) implies by (4.3) $b = 0$, which is excluded because we have assumed $X$ smooth. So by Proposition 3.1 the preimage $V^{-1}(A_2)$ consists of the 4 line bundles $A_3$ such that $A_3^{\otimes 2} = A_2$. In particular there are no $F^3$-trivial rank-2 bundles.

One easily verifies that the image under the rational map $V$ given by (3.2) of the hyperplane $x_{00} = 0$ is the quartic surface given by the equation
\[(4.4) \quad bx_{11}^2x_{10}^2 + cx_{11}^2x_{01}^2 + ax_{10}^2x_{01}^2 + x_{00}x_{10}x_{01}x_{11} = 0.
\]
When we replace $(x_{00} : x_{01} : x_{10} : x_{11})$ with $(u : u : \sqrt{b} : \sqrt{b})$ in (4.4) we obtain the equation
\[(4.5) \quad b^2 + a^2(1 + a + c) = 0.
\]
Similarly replacing $(x_{00} : x_{01} : x_{10} : x_{11})$ with $(u + 1 : u + 1 : \sqrt{b} : \sqrt{b})$ in (4.4) we obtain the equation
\[(4.6) \quad b^2 + (a^2 + 1)(1 + a + c) = 0.
\]
Finally the product of (4.5) with (4.6) equals (here one uses (4.3)) equation (4.1) up to a factor $b^4$, which we can drop since $b \neq 0$ — note that we have assumed $X$ smooth, hence $b \neq 0$ by [LP2] Lemma 2.1. To summarize we have shown that if (4.1) holds, then by Proposition 3.1 there exists an 8-torsion line bundle $A_3$ with $A_3^{\otimes 2} = A_1$ and such that the preimage $V^{-1}(A_3)$ is a projective line $\Delta \subset \mathbb{P}^3$.

Consider a point $[E] \in \Delta$ away from the Kummer surface — note that $\Delta$ is not contained in the Kummer surface $\text{Kum}_X$ because its intersection is contained in the set of 16-torsion points. Then $E$ is stable and $[F^*E] = [A_3 \oplus A_3^{-1}]$. There are three isomorphism classes represented by the $S$-equivalence class $[A_3 \oplus A_3^{-1}]$, namely the trivial extension $A_3 \oplus A_3^{-1}$ and two nontrivial extensions (for the details see [LP1] Remark 6.2). Since $E$ is invariant under the hyperelliptic involution we obtain $F^*E = A_3 \oplus A_3^{-1}$ and finally that $E$ is $F^4$-trivial. Hence any stable point on $\Delta$ is $F^4$-trivial.

Therefore, assuming (4.1), there exists a 1-dimensional subvariety $\Delta_0 \subset \mathcal{M}_X \setminus \text{Kum}_X$ parametrizing all $F^4$-trivial rank-2 bundles. Passing to an étale cover $S \rightarrow \Delta_0$ ensures existence of a “universal” family $\mathcal{E} \rightarrow X \times S$ and we are done.

\[\square\]

**Remark.** Note that equation (4.1) depends on the choice of a nontrivial 2-torsion line bundle $A_1$. If one chooses the 2-torsion line bundle $(0 : 0 : 1 : 0)$ or $(0 : 0 : 0 : 1)$ — see (4.2) — the corresponding equations are
\[a^2 + b^2 + c^2 + a + b = 0 \quad \text{or} \quad a^2 + b^2 + c^2 + b + c = 0.
\]

**References**


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