Sturm-Liouville description of sine-Gordon soliton dynamics

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We analytically demonstrate by use of a double perturbation scheme, and we numerically check that a sine-Gordon soliton trapped in an harmonic potential well $[V(x)=\epsilon x^2]$ exactly oscillates at the Newtonian frequency $\frac{1}{2}\sqrt{\epsilon}+o(\epsilon^{3/2})$. We emphasize the link with the previous Lagrangian collective-coordinate theories. We point out some conceptual problems related to the localization of this oscillating soliton.

I. INTRODUCTION

A static sine-Gordon (SG) soliton:

$$u_s(x)=4\tan^{-1}\exp(-\sigma x) \tag{1}$$

($\sigma=-1$ for a kink, $\sigma=+1$ for an antikink) which is trapped in the bottom of an harmonic potential well according to the following partial differential equation (PDE) dynamical problem[1-3]

$$u_{tt}-u_{xx}+(1+\tfrac{1}{4}\epsilon x^2)\sin u=0 \quad (\epsilon \ll 1) \tag{2a}$$

$$u(x,0)=u_s(x),\quad u_t(x,0)\equiv 0\ , \tag{2b}$$

has the energy

$$H_\epsilon=\int_{-\infty}^{+\infty}h_\epsilon(u_s(x))dx=H_0+\Delta H\ , \tag{3a}$$

where

$$h_\epsilon(u)=\tfrac{1}{2}u_t^2+\tfrac{1}{2}u_x^2+(1+\tfrac{1}{4}\epsilon x^2)(1-\cos u) \tag{3b}$$

is the Hamiltonian density related to the (Hamiltonian) dynamical system (2), H_0 is the energy (or the "mass") on the unperturbed soliton (1):

$$H_0=\int_{-\infty}^{+\infty}h_{\epsilon=0}(u_s(x))dx=8\ , \tag{3c}$$

and ΔH is the energy increase of the system due to the presence of the perturbation:

$$\Delta H=H_\epsilon-H_0=\tfrac{1}{4}\epsilon\int_{-\infty}^{+\infty}x^2[1-\cos(u_s(x))]dx=\frac{\pi^2\epsilon}{12}\ . \tag{3d}$$

This latter value is obtained from the following algebraic identity

$$\cos[u_s(x)]=1-2\,\text{sech}^2x\ . \tag{4}$$

Since the soliton, as an extended particlelike solitary wave, has gained energy from its interaction with the trapping potential well, it oscillates on infinitely discrete degrees of freedom.[2] In this latter reference, these degrees of freedom were defined as the a_n's according to

$$\Psi(x,t)=u(x,t)-u_s(x)=\sum_{n=0}^{\infty}a_n(t)\phi_n(x)\ , \tag{5a}$$

where the ϕ_n's, which satisfy the following well-known Sturm-Liouville (SL) eigenvalue problem:

$$\phi_{n,xx}+\tfrac{1}{4}\epsilon x^2\phi_n=(\omega_n^2-1)\phi_n \tag{5b}$$

$$=(n+\tfrac{1}{2})\sqrt{\epsilon}\phi_n\ , \tag{5c}$$

span an orthonormal and complete basis of the Hilbert space, which the perturbation function Ψ is assumed to belong to.

It was shown in Ref. 2 that the ansatz of $u(x,t)$, defined by (5), in (2a) leads to a discrete infinite-dimension dynamical system described by the following sequence of ordinary differential equations

$$\ddot{a}_n(t)+\omega_n^2a_n+\sum_{p=0}^{\infty}\alpha_{np}a_p=\gamma_n\ , \tag{6a}$$

where

$$\alpha_{np}=\alpha_{pn}=-2\langle\phi_p(x)\,|\,(1+\tfrac{1}{4}\epsilon x^2)\text{sech}^2x\,|\,\phi_n(x)\rangle \tag{6b}$$

$$\gamma_n=-\tfrac{1}{2}\epsilon\sigma\langle x^2\sinh x\ \text{sech}^2x\,|\,\phi_n(x)\rangle\ , \tag{6c}$$

the dot and the bracket respectively mean, as usual, a time derivation and the scalar product in the Hilbert space

$$\langle f(x)\,|\,g(x)=\int_{-\infty}^{+\infty}f(x)\bar{g}(x)dx\ . \tag{6d}$$

Comparing with the direct numerical simulation of the PDE (2a)—starting with the initial conditions (2b)—the dynamical problem (6) leads to quite acceptable results concerning the so-called "odd" spectrum (i.e., the soliton internal-vibration spectrum) and "even" spectrum (i.e., the soliton dynamical-oscillation spectrum), except for the fundamental level of the dynamical system (2).

Actually, when increasing the number of modes n, we obtain (see Fig. 1) the limit value ω_0^2 for this fundamental eigenvalue, which fits the standard perturbation theory applied to the original SL-SG problem[4,5]

$$-f_{xx}^j(x)+f^j(x)\cos u_s(x)=\omega^2f^j(x)\ , \tag{7}$$

namely $[f_b(x)=(1/\sqrt{2}\,\text{sech}x)$ being the Goldstone mode]

$$\omega_0^2=\frac{\epsilon}{4}\langle f_b(x)\,|\,x^2\cos u_s(x)\,|\,f_b(x)\rangle\ . \tag{8}$$

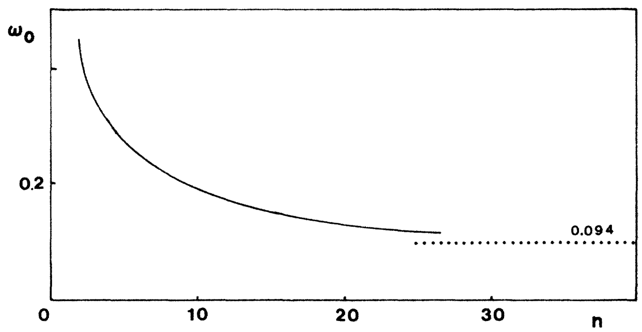

FIG. 1. Evolution of ω_0 when increasing the number of modes (n) in the linear system (6), when $\epsilon=0.09$. ω_0 asymptotically tends towards the value 0.094, the square of which is given by (8) or (24b).

Unfortunately this lowest eigenvalue leads to the frequency Ω_0^a which differs by the factor $\langle f_b(x) \mid x^2\cos u_s(x) \mid f_b(x)\rangle^{1/2} \sim 0.7$ from the observed value

$$\Omega_N = \tfrac{1}{2}\epsilon^{1/2} + O(\epsilon^{3/2}) \ . \tag{9a}$$

This frequency, which is numerically observed with a great accuracy even for very small displacements $Y(t)$ (for instance less than the soliton width), emphasizes a surprisingly simple equation of motion of the particle-like soliton, the mass of which is $H_0=8$ [cf. (3c)], namely:

$$8\ddot{Y} + 2\epsilon Y = O(\epsilon^2) \ . \tag{9b}$$

This "Newtonian" SG soliton equation of motion is recovered either by mean of very simple heuristic arguments based on the Hamiltonian nature of the system (2) (Ref. 3), or by mean of a Lagrangian collective-coordinate theory introducing a high-frequency (compared to the reference plasma frequency equal to unity) parameter in addition to the (low-frequency) soliton position Y (Ref. 1).

The aim of the present paper is to point out the perturbation treatment which must actually be developed in order to demonstrate the Newtonian equation of motion (9b). Moreover, while looking for an acceptable definition of the soliton position, we reach the limit of such intuitive mechanical concepts as particle energy, particle position, particle velocity. This limit is unavoidable, because of the dual wave-particle nature of the SG soliton.

This paper is intended as follows: in Secs. II and III, we show that the right profile $u^*(x)$ which one has to linearize about in order to define the perturbation function $\Psi(x,t)$ is not $u_s(x)$, as was done in Ref. 2, but the homoclinic orbit of the following stationary Hamiltonian system related to (2):

$$-u^*_{xx} + (1+\tfrac{1}{4}\epsilon x^2)\sin u^* = 0 \ . \tag{10}$$

In Sec. II, we calculate $\Delta u = u^*(x) - u_s(x)$ by use of the orthonormal complete basis defined through the SL problem (7). We compare it to the variation $\tilde{u}^*(x) - u_S(x)$ obtained by the numerical integration of (10), as well as to the variation corresponding to: (i) the stationary profile $u_s(\bar{k}x)$, where $\bar{k} = 1+\pi^2\epsilon/32$, is obtained by the collective-coordinate theory,[1] and to (ii) the zero-wave-vector approximation performed in the perturbed phonon spectrum.[6]

In Sec. III, we solve the corresponding SL eigenproblem

$$-f^j_{xx}(x) + [(1+\tfrac{1}{4}\epsilon x^2)\cos u^*(x)]f^j(x) = \Omega_j^2 f^j(x) \ , \tag{11a}$$

for the fundamental eigenvalue Ω_0^2 by use of the standard perturbation theory applied to the SG Goldstone mode, when adding to the unperturbed soliton potential $\cos u_s(x)$ [see (7)] the perturbative potential [cf. (11a)]:

$$W(x) = \frac{\epsilon}{4}x^2\cos u_s(x) - \Delta u \ \sin u_s(x) + O(\epsilon^2) \ . \tag{11b}$$

Therefore we have

$$\Omega_0^2 = \langle f_b(x) \mid W(x) \mid f_b(x)\rangle + O(\epsilon^2) \tag{11c}$$

and we show that

$$\Omega_0 = \Omega_N + O(\epsilon^{3/2}) \ . \tag{12}$$

We compare this value to the square root $\tilde{\Omega}$ of the fundamental obtained by direct numerical resolution of the SL problem

$$-f_{xx} + [(1+\tfrac{1}{4}\epsilon x^2)\cos\tilde{u}^*]f = \tilde{\Omega}^2 f \ , \tag{13}$$

when $\epsilon=0.09$. Having calculated the corresponding homoclinic orbit, we find that $\tilde{\Omega}$ fits the theoretical prediction (12) within the error of order $\epsilon^{3/2}$. Then we analytically solve the SL problem related to the collective-coordinate stationary profile $u_s(\bar{k}x)$ and obtain a fundamental frequency which differs from the theoretical value Ω_N by 10%.

Therefore, although the collective-coordinate theory correctly accounts for the fundamental (9a) of the soliton oscillation spectrum inside of the well ϵx^2, its definition of the corresponding stationary profile $u_s(\bar{k}x)$ is not accurate enough to allow an acceptable link with the related spectral SL treatment. In other words, the collective-coordinate theory[1] is not self-consistent from a spectral SL point of view.

This remark leads us, in Sec. IV, to a rather qualitative discussion of the limits of the particle-like soliton dynamics description.

II. THE SOLITARY WAVE DESCRIBED BY THE PERTURBED STATIONARY SG EQUATION

Define:

$$u^*(x) = u_s(x) + (\Delta u)(x) \tag{14a}$$

and assume that $u^*(x)$ is the (homoclinic) solitary wave solution of (10). The Hamiltonian dynamical system described by (10) always admits such a homoclinic orbit. The soliton solution $u_s(x)$ defined by (1) verifies the unperturbed stationary SG equation. Therefore we have

$$-(\Delta u)_{xx} + (\Delta u)\cos u_s(x) = -\tfrac{1}{4}\epsilon x^2 \sin u_s(x) + O(\epsilon\Delta) \ , \tag{14b}$$

where

$$O(\epsilon\Delta u)=\tfrac{1}{4}\epsilon x^2\cos u_s(x)\Delta u \ . \tag{14c}$$

Expand Δu about the orthogonal basis defined by the unperturbed SL eigenproblem (7):

$$f_b(x)=1/\sqrt{2}\,\mathrm{sech}x,\quad \omega_b^2=0 \ , \tag{15a}$$

$$f_k(x)=\frac{\exp(ikx)}{(2\pi)^{1/2}\omega_k}(k+i\tanh x),\quad \omega_k^2=1+k^2 \ , \tag{15b}$$

$$(\Delta u)(x)=a_b f_b(x)+\int_{-\infty}^{+\infty}a_k f_k(x)dk \ . \tag{15c}$$

Insert (15c) in (14b). We obtain by use of (7):

$$-\tfrac{1}{4}\alpha x^2\sin u_s(x)=\int_{-\infty}^{+\infty}a_k\omega_k^2 f_k(x)dk \ . \tag{16a}$$

Therefore, using the orthogonality of the f_k's:

$$a_k=-\frac{\epsilon}{4\omega_k^2}\int_{-\infty}^{+\infty}\bar{f}_k(x')x'^2\sin u_s(x')dx' \ , \tag{16b}$$

where $\bar{f}_k(x')$ is the complex conjugate of $f_k(x')$.

There is an undetermination of a_b which is overcome by use of (14c). Indeed, we have by inserting (14c) and (15c) into (14b), and then projecting onto $f_b(x)$:

$$a_b\int_{-\infty}^{+\infty}x'^2\cos u_s(x')f_b^2(x')dx' = -\int_{-\infty}^{+\infty}x'^2\sin u_s(x')f_b(x')dx' \ . \tag{16c}$$

Since

$$\sin u_s(x)=(u_s)_{xx}=-2\sqrt{2}\sigma\frac{d}{dx}f_b(x) \ , \tag{17}$$

the rhs of (16c) vanishes and $a_b=0$. Therefore (15c), (16b), and (17) yield:

$$(\Delta u)(x)=\frac{\epsilon\sigma}{\sqrt{2}}\int_{-\infty}^{+\infty}x'^2 I(x,x')\frac{d}{dx'}f_b(x')dx' \ , \tag{18a}$$

where

$$I(x,x')=\int_{-\infty}^{+\infty}\frac{1}{\omega_k^2}f_k(x)\bar{f}_k(x')dk \ . \tag{18b}$$

This integral is calculated by contour integration and yields

$$I_1=\tfrac{1}{4}\exp(x'-x)[(x'-x)(1+\tanh x-\tanh x'-\tanh x\tanh x')+1+\tanh x\tanh x'] \quad \text{for } x>x' \ , \tag{18c}$$

and

$$I_2=\tfrac{1}{4}\exp(x-x')[(x'-x)(-1+\tanh x-\tanh x'+\tanh x\tanh x')+1+\tanh x\tanh x'] \quad \text{for } x<x' \ . \tag{18d}$$

Therefore (18a) gives:

$$(\Delta u)(x)=-\frac{\epsilon\sigma}{2}\left[\int_{-\infty}^{x}x'^2\mathrm{sech}^2x'\sinh x' I_1(x,x')dx'+\int_{x}^{+\infty}x'^2\mathrm{sech}^2x'\sinh x' I_2(x,x')dx'\right] \ . \tag{19}$$

The function $-8(\Delta u)(x)/\sigma\epsilon$ is displayed in Fig. 2. Note that a very crude approximation of I consists in dropping the ω_k^{-2} factor in (18b), thus obtaining by use of the completeness equation

$$I(x,x')\simeq\int_{-\infty}^{+\infty}f_k(x)\bar{f}_k(x')dk = \delta(x-x')-f_b(x)f_b(x') \ . \tag{20a}$$

Then (18a) gives

$$(\Delta u)(x)=\frac{\epsilon\sigma}{\sqrt{2}}x^2\frac{d}{dx}f_b(x)=-\tfrac{1}{2}\sigma\epsilon x^2\sinh x\,\mathrm{sech}^2x \ , \tag{20b}$$

which is shown in Fig. 2.

Finally, the stationary profile obtained by the collective-coordinate (CC) theory[1] applied to (2):

$$u_s(\bar{k}x, Y\equiv 0)=4\tan^{-1}\exp[-\sigma\bar{k}x] \ , \tag{21a}$$

with

$$\bar{k}=1+\frac{\pi^2\epsilon}{32}+O(\epsilon^2) \ , \tag{21b}$$

leads to the function

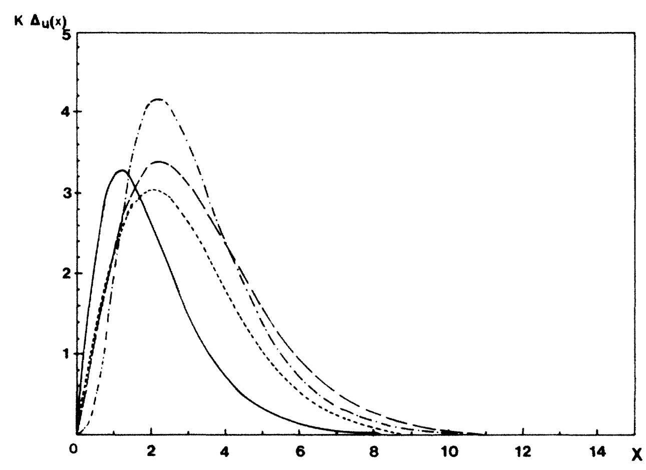

FIG. 2. We show in the case $\epsilon=0.09$ the four expressions of the function $K(\Delta u)(x)$, where $K=-8/\sigma\epsilon$, corresponding to the different approximations (19): — — —, (20b): —·—·—·, (21c): ——, and to the numerical integration of (10)(· · · ·). All functions are odd. Note that, due to factor $K\sim\epsilon^{-1}$, the discrepancy between the dotted [Eq. (10): · · · ·] and the dashed [Eq. (19): — — —] curves is of order ϵ, as it should be, since the analytical treatment leading to formula (19) is a perturbative scheme performed at first order in ϵ.

$$(\Delta u)_{\rm CC}(x)=u_s(\bar{k}x)-u_s(x)=-\frac{\pi^2\epsilon\sigma}{16}x\,\mathrm{sech}x+O(\epsilon^2)\;, \tag{21c}$$

which is also displayed in Fig. 2 for comparison with (19) and (20).

The fact that the function $(\Delta u)(x)$ given by (20) is "not too bad" compared to (19), is related to the predominance of the $k=0$ wave vectors in SG soliton perturbation problems.[6]

III. THE PERTURBED STURM-LIOUVILLE PROBLEM

Consider the $\omega^2=0$ fundamental Goldstone level of the SL-SG problem (7). Note that the linearization of the solution $u(x,t)$ about $u^*(x)$ according to

$$u(x,t)=u^*(x)+\Psi(x,t) \tag{22}$$

leads to the eigenproblem (11a), when assuming, as usual, an $e^{i\omega t}$ time dependence of Ψ. Developing $\cos u^*(x)$ about $\cos u_s(x)$ at first order in Δu and neglecting the $O(\epsilon\Delta u)\sim o(\epsilon^2)$ terms in (11a) leads to the additive (perturbative) potential (11b). Then the standard perturbation theory of the eigenproblem (7) by this perturbative potential (11b) leads to the value (11c) of the perturbation of the $\omega^2=0$ fundamental eigenvalue. Defining $B(x)$ as:

$$(\Delta u)(x)=-\sigma\Omega_N^2B(x)\quad[\text{see (9a) and (19)}]\;, \tag{23}$$

formulae (11b) and (11c) give

$$\Omega_0^2=(\Omega_0^a)^2+(\Omega_0^b)^2+O(\epsilon^2)\;, \tag{24a}$$

where $(\Omega_0^a)^2$ is given by (4) and (8):

$$\begin{aligned}(\Omega_0^a)^2&=\Omega_N^2\langle (f_b(x)\,|\,x^2(1-2\,\mathrm{sech}^2x)\,|\,f_b(x)\rangle\\ &=\Omega_N^2\left[\frac{2}{3}-\frac{\pi^2}{36}\right]\;,\end{aligned} \tag{24b}$$

and $(\Omega_0^b)^2$ yields [cf. (17)]

$$(\Omega_0^b)^2=-\langle f_b(x)\,|\,(\Delta u)\sin u_s(x)\,|\,f_b(x)\rangle \tag{24c}$$

$$=-2\sqrt{2}\Omega_N^2\langle f_b(x)\left|\frac{d}{dx}f_b(x)B(x)\right|f_b(x)\rangle \tag{24d}$$

$$=\Omega_N^2\int_{-\infty}^{+\infty}\mathrm{sech}^4x\,\sinh x B(x)dx \tag{24e}$$

where (6d) has been used for (24b) and (24e).

Therefore

$$\Omega_0=\Omega_N\left[\frac{2}{3}-\frac{\pi^2}{36}+\int_{-\infty}^{+\infty}\mathrm{sech}^4x\,\sinh xB(x)dx\right]^{1/2}+O(\epsilon^{3/2}) \tag{25a}$$

$$=\Omega_N+O(\epsilon^{3/2})\quad(\text{QED})\;. \tag{25b}$$

Indeed, the large parenthesis in (25) is numerically calculated: we first computed $B(x)$ by a Gauss integration techniques, then we calculated the integral in (25a) by a Simpson method [both programs are in Centre d'Etudes et de Recheres Nucleaire (CERN) library]. This large parenthesis yields 1, with an accuracy of 10^{-6}.

Therefore $\Omega_0=\Omega_N$ and (12) is demonstrated. Note that the right choice of the profile $u^*(x)$ defined by (10), which one has to linearize about, leads to the correction of order 30% provided by $(\Omega_0^b)^2$ with respect to $(\Omega_0^a)^2$, and allows Ω_0^2 to exactly equal Ω_N^2. Hence this choice is crucial. For instance, if we adopt for $u^*(x)$ the stationary solitary wave (21a) and (21b) given by the collective-coordinate described in Ref. 1, the corresponding SL eigenproblem reads

$$-f_{xx}^j(x)+(1+\tfrac{1}{4}\epsilon x^2)(1-2\,\mathrm{sech}^2\bar{k}x)f^j(x)=\tilde{\omega}_j^2f^j(x)\;. \tag{26a}$$

We first consider the SL equation

$$-f_{xx}^j(x)+(1-2\,\mathrm{sech}^2\bar{k}x)f^j(x)=(\tilde{\omega}_j)^2f_j(x)\;. \tag{26b}$$

The solution of (26b) is straightforward.[7] Since $\bar{k}>1$ [cf. (21b)], there is a single discrete level, given by

$$(\tilde{\omega}_0^a)^2=\frac{\pi^2\epsilon}{48}=\frac{\pi^2}{12}\Omega_N^2\;. \tag{26c}$$

The corresponding eigenvalue, normalized to $O(\epsilon^2)$, is

$$\tilde{f}_0(x)=\frac{1}{\sqrt{2}}\left\{\mathrm{sech}\left[\left[1+\frac{\pi^2\epsilon}{32}\right]x\right]\right\}^{1-\pi^2\epsilon/24}\;. \tag{26d}$$

We note that we recover, as expected, the SG soliton translation (Goldstone) made in the limit $\epsilon=0$.

Now we consider the additional term $\frac{1}{4}\epsilon x^2(1-2\,\mathrm{sech}^2\bar{k}x)$ in (26a) as a perturbative potential for the SL eigenproblem (26b). Then the original eigenvalue (26c) must be corrected, at order ϵ, by the following additive term:

$$(\tilde{\omega}_0^b)^2=\Omega_N^2\langle\tilde{f}_0(x)\,|\,x^2(1-\mathrm{sech}^2\bar{k}x)\,|\,\tilde{f}_0(x)\rangle \tag{27a}$$

$$=\Omega_N^2\langle f_b(x)\,|\,x^2(1-\mathrm{sech}^2x)\,|\,f_b(x)\rangle+O(\epsilon^2) \tag{27b}$$

$$=(\Omega_0^a)^2=\Omega_N^2\left[\frac{2}{3}-\frac{\pi^2}{36}\right]+O(\epsilon^2)\quad[\text{see (24b)}]\;. \tag{27c}$$

Therefore

$$\tilde{\omega}_0^2=(\tilde{\omega}_0^a)^2+(\tilde{\omega}_0^b)^2=(\Omega_N)^2\left[\frac{2}{3}+\frac{\pi^2}{18}\right]+O(\epsilon^2)\;, \tag{28a}$$

and

$$\tilde{\omega}_0=1.102\Omega_N+O(\epsilon^{3/2})\;. \tag{28b}$$

Hence the approximation of $u^*(x)$ by the stationary solution (21a) and (21b) obtained through the two collective-coordinates theory[1] leads to an error of 10% with respect to the accurate value (25), whereas this very theory provides us with this correct value of the particlelike soliton frequency in the well. Indeed the Eq. (23a) of Ref. 1 is precisely the equation of motion (9b) in the present paper. This point will be discussed in the next part.

Finally, note that the zero-wave-number-phonon approximation (20) leads, through (11b) and (11c) to

$$\tilde{\omega}_0^2=(\Omega_0^a)^2+2\Omega_N^2\int_{-\infty}^{+\infty}x^2(\operatorname{sech}^4x-\operatorname{sech}^6x)dx \quad (29a)$$

$$=\Omega_N^2\left[\frac{2}{3}+\frac{\pi^2}{60}\right], \quad (29b)$$

and

$$\tilde{\omega}=0.912\Omega_N . \quad (29c)$$

Hence the approximation (20) leads to the same error of about 10% as the collective-coordinate description.

The three results (25), (28), and (29) are displayed in Fig. 3, where the fundamental frequency is plotted versus the amplitude of the soliton distortion function, defined as

$$A(\Delta u)=\|\Delta u\|=\langle \Delta u \mid \Delta u\rangle^{1/2} , \quad (30a)$$

with Δu respectively given by (19), (21), and (20). Using the scalar product (6d), we respectively obtain:

$$A_{19}=1.055\epsilon \quad (30b)$$

$$A_{20}=\tfrac{1}{2}\epsilon\left[\int_{-\infty}^{+\infty}x^4\sinh^2x\,\operatorname{sech}^4x\,dx\right]^{1/2}=1.138\epsilon \quad (30c)$$

$$A_{21}=\frac{\pi^2\epsilon}{16}\left[\int_{-\infty}^{+\infty}x^2\operatorname{sech}^2x\,dx\right]^{1/2}=\frac{\pi^3}{16(6)^{1/2}}\epsilon=0.791\epsilon . \quad (30d)$$

Concluding this part, we emphasize that we numerically checked the whole scheme (22)–(25), in the case $\epsilon=0.09$: calculating the homoclinic orbit $u^*(x)$ defined by (10), then directly solving, by use of a DO2KAF code from Numerical Algorithm Group (NAG) library, the SL problem (11a), we obtain $\Omega_0=0.1457$ instead of the expected value $\Omega_N=0.15$.

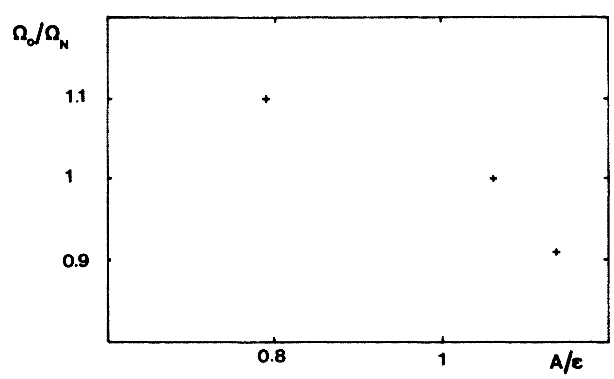

FIG. 3. The fundamental frequency Ω_0/Ω_N, $\tilde{\omega}_0/\Omega_N$, and $\tilde{\omega}_0/\Omega_N$ is plotted vs the corresponding A/ϵ. Where A is the amplitude of the soliton distortion function given by (30). Note that the fundamental level concerning the SG kink, which corresponds to $A=0$ and $\Omega_0/\Omega_N=0.094$, is not displayed.

IV. CONCLUSION: THE DUAL WAVE-PARTICLE SOLITON DESCRIPTION

The physical description of the motion of a SG soliton is not simple. It is related to the right choice of the "collective-coordinate" parameter which is regarded to as describing the soliton position. As long as the soliton dynamics is determined by the unperturbed SG equation, the answer is unambiguous. The (zero-frequency) Goldstone mode related to the simple discrete eigenvalue of the SL problem (7) provides a basis element on which one projects the perturbation function $\Psi(x,t)$ describing the change of state of the soliton between two time values: t_0 and $t\geq t_0$. If we assume at $t=t_0$ a pure (propagating at velocity c) soliton wave, one recovers of course $\Psi\sim -c(t-t_0)u_{s,x}$, i.e., the function Ψ is parallel (in the sense of the scalar product) of the Goldstone mode $\sim u_{s,x}$ and its projection of $u_{s,x}$ is the value of the soliton position.

The situation drastically changes when introducing a perturbation of the SG equation. Then two problems arise:

(i) What is the right profile $u^*(x,t_0)$ which one has to linearize about, in order to define Ψ? We have shown in the present paper that, in the case of a bound soliton dynamics, this profile must be the homoclinic orbit (10) of the perturbed SG equation. Note that, although $\|u^*-u_s\|\sim\epsilon$ (cf. Fig. 3), approximating $u^*(x)$ by $u_s(x)$ leads to a dramatic error of about 30% in the calculation of the fundamental frequency of the system. This is of course because this frequency is in order $\epsilon^{1/2}$.

(ii) Once the corresponding SL eigenproblem (11a) is solved, at least for the fundamental eigenvalue Ω_0^2 of the system, then comes the problem of the physical meaning of the eigenfunction $f_0(x)$ related to Ω_0^2. Clearly $\Omega_0^2\sim\epsilon$. This suggests to consider $f_0(x)$ as the "perturbed Goldstone mode" of the system, and Ω_0 as the low frequency (compared to the plasma frequency equal to unity) describing the particlelike soliton oscillations in the potential well.

In the collective-coordinate approximation (21), we calculated both Ω_0 and $f_0(x)$ [cf. (26d) and (28)], and indeed obtained for $f_0(x)$ a slightly perturbed soliton Goldstone mode. Nevertheless, there remains the problem of the physical meaning of the expression:

$$u(x,t)=u^*(x)+\xi(t)f_0(x) , \quad (31a)$$

where

$$\xi(t)=\langle\Psi(x,t)\mid f_0(x)\rangle . \quad (31b)$$

Indeed, the "soliton position" extrapolated from the unperturbed case now reads, since $u(x,t)=u^*(x)+[\xi(t)f_0(x)/u_x^*(x)]u_x^*(x)$:

$$\Delta=-\xi(t)\frac{f_0(x)}{u_x^*(x)},\quad [u_x^*(x)\neq 0\forall x] . \quad (32)$$

Hence Δ is nonuniform. In the approximation (21c)–(26d), we obtain $\Delta\to\infty$ when $x\to\pm\infty$.

Strictly speaking, the basic result of this paper, namely $\Omega_0=\Omega_N+O(\epsilon^{3/2})$, means that "something" that has neither a definite profile nor a clear position, is oscillating at the Newtonian frequency $\frac{1}{2}\sqrt{\epsilon}$. Of course—the formulae (30) confirm it—it is very much like a SG kink, but at the same time, Fig. 3 shows the dramatic dependence of the fundamental eigenfrequency on the fine structure of this profile.

Qualitatively speaking, this situation reminds the quantum uncertainty principle related to the dual wave-particle nature of the quantum objects. If we project the perturbation function Ψ on u_x^*, we gain a uniform definition of the position of the quasisoliton $u^*(x)$. But we lose the sharpness of its corresponding Fourier spectrum, since u_x^* now has a nonzero component on all eigenvectors related to the (high frequency) eigenvalues Ω_j^2 $(j>1)$ of the SL problem (11a). On the other hand, if we project Ψ on the eigenfunction $f_0(x)$ corresponding to the fundamental level Ω_0^2, we gain of course a monochromatic Fourier spectrum of this projection, but we lose the unambiguous definition of the quasisoliton position, as shown by (32).

In this context, we appreciate the quality of the two-collective-coordinates description of Ref. 1, since the related Lagrangian principle seems to minimize the combination of both above uncertainties: it provides us with the correct frequency Ω_N of the particlelike quasisoliton oscillations in the potential well, although the profile (21a) of this quasisoliton is not the correct one, defined by (10) and given by (19).

Finally concluding, we emphasize the theoretical interest of this work, as it provides us with a fully linear, nonsecular theory of the motion of the SG soliton trapped in a (harmonic) potential well, whereas the Lagrange equations of motion of the two above mentioned collective coordinates are basically nonlinear.

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