Existence and stability of a solution blowing up on a sphere
for a $L^2$ supercritical non linear Schrödinger equation

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Abstract

We consider the quintic two dimensional nonlinear Schrödinger equation $iu_t = -\Delta u - |u|^4 u$ which is $L^2$ supercritical. Even though the existence of finite time blow up solutions in the energy space $H^1$ is known, very little is understood concerning the singularity formation. Numerics suggest the existence of a stable blow up dynamic corresponding to a self similar blow up at one point in space. We prove the existence of a different type of dynamics and exhibit an open set among the $H^1$ radial distributions of initial data for which the corresponding solution blows up in finite time on a sphere. This is the first result of description of a blow up dynamic in the $L^2$ supercritical setting.

1 Introduction

1.1 Setting of the problem

We consider in this paper the quintic two dimensional focusing nonlinear Schrödinger equation

$$(NLS) \begin{cases} 
  i u_t = -\Delta u - |u|^4 u, & (t, x) \in [0, T) \times \mathbb{R}^2, \\
  u(0, x) = u_0(x), & u_0 : \mathbb{R}^2 \to \mathbb{C} 
\end{cases}$$

with $u_0 \in H^1(\mathbb{R}^2)$. It is a special case of the more general system:

$$iu_t = -\Delta u - |u|^{p-1} u \quad (1)$$

with $1 < p < +\infty$ for $N = 1, 2$, $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$, and initial condition $u_0 \in H^1(\mathbb{R}^N)$. From a result of Ginibre Velo [10], (1) is locally well-posed in $H^1$ and thus, for $u_0 \in H^1$, there exists $0 < T \leq +\infty$ such that $u(t) \in C([0, T), H^1)$ and either $T = +\infty$, we say the solution is global, or $T < +\infty$ and then $\limsup_{t \to T} |\nabla u(t)|_{L^2} = +\infty$, we say the solution blows up in finite time.
Let us recall the main known facts about (1). It admits a number of symmetries in the energy space $H^1$: if $u(t, x)$ solves (1), then $\forall (\lambda_0, t_0, \beta_0, \gamma_0) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, so does
$$v(t, x) = \lambda_0^{\frac{2}{p-1}} u(t + t_0, \lambda_0 x + x_0 - \beta_0 t) e^{i \frac{\beta_0}{2} (x - \frac{\beta_0}{2} t)} e^{i \gamma_0}.$$ 
(2)
From Ehrenfest law or direct computation, these symmetries induce invariances in the energy space, namely:
- $L^2$ norm: $\int |u(t, x)|^2 = \int |u_0(x)|^2$;
- Energy: $E(u(t, x)) = \frac{1}{2} \int |\nabla u(t, x)|^2 - \frac{1}{p+1} \int |u(t, x)|^{p+1} = E(u_0)$;
- Momentum: $Im (\int \nabla u \bar{u}(t, x)) = Im (\int \nabla u_0 \bar{u}_0(x))$.

The conservation of the energy expresses the Hamiltonian structure of (1) in $H^1$.

In the subcritical range $p < 1 + \frac{4}{N}$, the conservation of the energy and the $L^2$ norm and a Gagliardo-Nirenberg inequality imply
$$|\nabla u(t)|_{L^2}^2 \leq C(u_0) \left( |\nabla u(t)|_{L^2}^{2\beta(p)} + 1 \right)$$
for some $\beta(p) < 1$,
so that (1) is globally well posed in $H^1$:
$$\forall t \in [0, T), \ |\nabla u(t)|_{L^2} \leq C(u_0) \quad \text{and} \quad T = +\infty.$$

The situation is quite different for $p \geq 1 + \frac{4}{N}$. Let an initial condition $u_0 \in \Sigma = H^1 \cap \{xu \in L^2\}$ and assume $E(u_0) < 0$, then $T < +\infty$ follows from the so called Virial Identity, see [29] and [12]. Indeed, the quantity $y(t) = \int |x|^2 |u|^2(t, x)$ is well defined for $t \in [0, T)$ and satisfies
$$y''(t) \leq C(p)E(u_0) \quad \text{with} \quad C(p) > 0.$$ 
(3)
The positivity of $y(t)$ yields the conclusion.

Special solutions play a fundamental role for the description of the dynamics of (1). They are the so called solitary waves of the form $u(t, x) = e^{i \omega t} W_\omega(x)$, $\omega > 0$, where $W_\omega$ solves
$$\Delta W_\omega + W_\omega |W_\omega|^{p-1} = \omega W_\omega.$$ 
(4)
Equation (4) is a standard nonlinear elliptic equation, and from [1] and [14], there is a unique positive solution up to translation $Q_\omega(x)$. $Q_\omega$ is in addition radially symmetric. Letting $Q = Q_1$, then $Q_\omega(x) = \omega^{\frac{1}{p-1}} Q(\omega^\frac{1}{2} x)$ from scaling property. In the subcritical range $p < 1 + \frac{4}{N}$, the ground state solution is orbitally stable, see [6], while it is unstable for $p \geq 1 + \frac{4}{N}$. 

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1.2 Previous results in the critical case $p = 1 + \frac{4}{N}$

Let us now recall the main known results concerning the critical case $p = 1 + \frac{4}{N}$, that is

$$iu_t = -\Delta u - |u|^{\frac{4}{N}}u$$ \hspace{1cm} (5)

For $u_0 \in H^1$ with $|u_0|_{L^2} < |Q|_{L^2}$, the solution is global in $H^1$ from the conservation of the energy, the $L^2$ norm and Gagliardo-Nirenberg inequality as exhibited by Weinstein in [27]:

$$\forall u \in H^1, \quad E(u) \geq \frac{1}{2} \left( \int |\nabla u|^2 \right) \left( 1 - \left( \frac{\int |u|^2}{\int Q^2} \right)^{\frac{2}{N}} \right).$$ \hspace{1cm} (6)

In addition, this condition is sharp: for $|u_0|_{L^2} \geq |Q|_{L^2}$, blow up may occur. Indeed, (1) admits in the critical case only an additional explicit symmetry known as the pseudo conformal transformation: if $u(t, x)$ solves (5), then so does

$$v(t, x) = \frac{1}{|t|^\frac{N}{2}} \pi \left( \frac{1}{t}, \frac{x}{t} \right) e^{i\frac{|x|^2}{4t}}.$$

This symmetry applied to the stationary solution $e^{it}Q(x)$ yields an explicit blow up solution

$$S(t, x) = \frac{1}{|t|^\frac{N}{2}} Q \left( \frac{x}{t} \right) e^{-i\frac{|x|^2}{4t} + \frac{t}{4}}$$

which blows up at $T = 0$ with $|S(t)|_{L^2} = |Q|_{L^2}$. Note that blow up speed for $S(t)$ is $|\nabla S(t)|_{L^2} \sim \frac{1}{T - t}$. Moreover, from [17], $S(t)$ is the unique minimal mass blow up solution up to the symmetries.

Most results concerning blow up dynamics of (5) now concern the perturbative situation when

$$u_0 \in B_{\alpha^*} = \{u_0 \in H^1 \text{ with } \int Q^2 \leq \int |u_0|^2 < \int Q^2 + \alpha^*\}$$

for some small universal constant $\alpha^* > 0$. At least two different blow up behaviors are known to possibly occur. On the one hand, in dimension $N = 1, 2$, Bourgain and Wang, [4], have constructed $S(t)$ type blow up solutions with blow up speed $|\nabla u(t)|_{L^2} \sim \frac{1}{T - t}$ near blow up time. On the other hand, numerical simulations, [15], and formal arguments, [26], predicted a stable blow up regime $|\nabla u(t)|_{L^2} \sim \left( \frac{\log \log (T - t)}{T - t} \right)^{\frac{1}{4}}$ in dimension $N = 2$, and such a solution has been constructed by Perelman, [23], in dimension $N = 1$.

The situation has been clarified by Merle and Raphaël in the sequel of papers [18], [19], [20], [21], [22], [24]. More precisely, let us consider the following property:

**Spectral Property** Let $N \geq 1$. Consider the two real Schrödinger operators

$$\mathcal{L}_1 = -\Delta + \frac{2}{N} \left( \frac{4}{N} + 1 \right) Q^{\frac{N}{2} - 1} y \cdot \nabla Q, \quad \mathcal{L}_2 = -\Delta + \frac{2}{N} Q^{\frac{N}{2} - 1} y \cdot \nabla Q,$$ \hspace{1cm} (7)
and the real valued quadratic form for $\varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^1$

$$H(\varepsilon, \varepsilon) = (L_1\varepsilon_1, \varepsilon_1) + (L_2\varepsilon_2, \varepsilon_2).$$  \hfill (8)

Then there exists a universal constant $\tilde{\delta}_1 > 0$ such that $\forall \varepsilon \in H^1$, if $(\varepsilon_1, Q) = (\varepsilon_2, Q_1) = (\varepsilon_2, \nabla Q) = 0$, then $H(\varepsilon, \varepsilon) \geq \tilde{\delta}_1 (\int |\nabla \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|})$ where $Q_1 = \frac{N}{2} Q + y \cdot \nabla Q$, $Q_2 = \frac{N}{2} Q_1 + y \cdot \nabla Q_1$.

This property has been proved in [18] for dimension $N = 1$, and there is numerical evidence that it holds true in dimensions $N = 2, 3, 4$.

We have the following result of description of the singularity formation for small supercritical mass solutions, ie $u_0 \in B_{\alpha^*}$.

**Theorem 1 (Dynamics of (5), [18], [19], [20], [21], [22], [24])** Let $N = 1$ or $N \geq 2$ assuming Spectral Property holds true. There exist universal constants $\alpha^* > 0$, $C^* > 0$ such that the following holds true. For $u_0 \in B_{\alpha^*}$, let $u(t)$ the corresponding solution to (14) with $[0, T)$ its maximum time interval existence on the right in $H^1$.

(i) Description of the singularity: Assume $u(t)$ blows up in finite time ie $0 < T < +\infty$, then there exist parameters $(\lambda(t), x(t), \gamma(t)) \in \mathbb{R}_+^* \times \mathbb{R}^N \times \mathbb{R}$ and an asymptotic profile $u^* \in L^2$ such that

$$u(t) - \frac{1}{\lambda(t)} N Q \left( \frac{x - x(t)}{\lambda(t)} \right) e^{i\gamma(t)} \to u^* \text{ in } L^2 \text{ as } t \to T.$$  \hfill (9)

Moreover, blow up point is finite in the sense that

$$x(t) \to x(T) \in \mathbb{R}^N \text{ as } t \to T.$$  

(ii) Estimates on the blow up speed: The scaling parameter $\lambda(t)$ is related to the blow up speed according to

$$\lambda(t)|\nabla u(t)|_{L^2} \to |\nabla Q|_{L^2} \text{ as } t \to T$$

and we have either

$$\lim_{t \to T} \frac{|\nabla u(t)|_{L^2}}{|\nabla Q|_{L^2}} \left( \frac{T - t}{\log \log (T - t)} \right) \frac{1}{2} = \frac{1}{\sqrt{2\pi}}$$  \hfill (10)

or

$$|\nabla u(t)|_{L^2} \geq C(u_0) \frac{T}{T - t}.$$  \hfill (11)

for $t$ close enough to $T$.

(iii) Sufficient condition for log-log blow up: If $E(u_0) < 0$, then $u(t)$ blows up in finite time with the log-log speed (10). More generally, the set of initial data $u_0 \in B_{\alpha^*}$ such that the corresponding solution $u(t)$ blows up in finite time $0 < T < +\infty$ with the log-log speed (10) is open in $H^1$.  

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(iv) Asymptotic of \( u^* \) on the singularity: Assume \( T < +\infty \). If \( u(t) \) satisfies (10), then for all \( R > 0 \) small enough,

\[
\frac{1}{C^*(\log|\log(R)|)^2} \leq \int_{|x-x(T)| \leq R} |u^*(x)|^2 dx \leq \frac{C^*}{(\log|\log(R)|)^2},
\]

and in particular

\[ u^* \notin H^1 \quad \text{and} \quad u^* \notin L^p \quad \text{for} \quad p > 2. \]

If \( u(t) \) satisfies (11), then

\[
\int_{|x-x(T)| \leq R} |u^*|^2 \leq C(u_0) R^2 \quad \text{and} \quad u^* \in H^1.
\]

In other words, we have the two following facts describing the blow up dynamics for \( u_0 \in \mathcal{B}_{\alpha^*} \):

(i) The singular part of the solution has according to (9) a universal space time structure. In particular, the blow up profile is exactly the ground state \( Q \). The singularity formation is moreover a somewhat localized in space property in the sense that the blow up point converges and the profile \( u^* \) remains globally in \( L^2 \), while the singular part forms a Dirac mass in \( L^2 \)

\[ |u(t)|^2 \to \left( \int |Q|^2 \right) \delta_{x=x(T)} + |u^*|^2 \]

in the weak sense of measures.

(ii) There are at least two different blow up dynamics with two different speeds, one of which given by the log-log law (10) is stable in the energy space \( H^1 \).

1.3 Known results and open problems for \( p > 1 + \frac{4}{N} \)

In the super critical case \( p > 1 + \frac{4}{N} \), the existence of finite time blow up solutions is a consequence of virial law (3). Nevertheless, very little is known in this case on the singularity formation.

Let us first recall from [7] that the Cauchy problem for (1) is well posed in \( H^s \) with

\[ s = \frac{N}{2} - \frac{2}{p-1} > 0. \]

Note that the homogeneous \( \dot{H}^s \) norm is left invariant by the scaling (2). Let now \( u_0 \in H^1 \) and assume that the corresponding solution to (1) blows up in finite time \( 0 < T < +\infty \), then the well posedness in \( H^s \) automatically implies the scaling lower bound on blow up rate

\[
|\nabla u(t)|_{L^2} \geq \frac{C(u_0)}{(T-t)^{\frac{1}{p-1} - \frac{N}{4}}} \quad (12)
\]
for $t$ close enough to $T$, see [7]. In the critical case $p = 1 + \frac{4}{N}$, the lower bound becomes

$$|\nabla u(t)|_{L^2} \geq \frac{C(u_0)}{(T-t)^\frac{3}{2}}$$

and is conjectured never to be sharp in the energy space, see [3]. This is indeed a consequence of Theorem 1 for initial data in $B_{\alpha^*}$.

On the contrary, in the super critical range $p > 1 + \frac{4}{N}$, numerical simulations suggest the existence and the stability in $H^1$ of self similar solutions blowing up at one point in space, that is solutions of the form

$$u(t) \sim \frac{1}{\lambda(t)^{\frac{2}{p-1}}} P \left( \frac{x - x(T)}{\lambda(t)} \right) e^{i\gamma(t)} \text{ with } \lambda(t) \sim C\sqrt{T-t}$$

near blow up time. The stationary profile $P$ should then satisfy some non linear elliptic PDE of the form

$$\Delta P - P + ib \left( \frac{2}{p-1} P + y \cdot \nabla P \right) + |P|^{p-1} = 0$$

for some parameter $b > 0$. There nevertheless is no existence result of solutions to this non linear elliptic equation in the energy space. We refer to [26] and references therein for numerical results and formal discussions on this problem.

More generally, there is no construction result of finite time blow up solution with a prescribed blow up dynamic in the super critical range. Note in particular that the constructive techniques developed for the critical case in [4] and [23] both rely on the fact that the linear operator close to the ground state $Q$ is only algebraically unstable, while it can be proved to be exponentially unstable for $p > 1 + \frac{4}{N}$, see [28], and thus these approaches fail.

Another widely open problem concerns the description of the blow up set. In the critical case $p = 1 + \frac{4}{N}$ and due precisely to the criticality of the problem, it has been conjectured in [22] that the blow up set consists of a finite number of points where a Dirac mass singularity is formed in $L^2$ while the rest of the solution remains smooth in some sense outside these points. In the super critical case, one point blow up self similar solutions are believed to exist in $H^1$ from numerics, but they may not be the only possibility. In other instances like for the non linear heat equation, one may exhibit solutions which blow up in finite time on a sphere, see Giga and Kohn [9].
1.4 Statement of the result

From now on and for the rest of this paper, we focus onto the quintic two dimensional nonlinear Schrödinger equation

\[
(NLS) \left\{ \begin{array}{ll}
  iu_t = -\Delta u - |u|^4 u, & (t, x) \in [0, T) \times \mathbb{R}^2; \\
  u(0, x) = u_0(x), & u_0 : \mathbb{R}^2 \to \mathbb{C},
\end{array} \right.
\]

with \( u_0 \in H^1_r(\mathbb{R}^2) = H^1_r \) where \( H^1_r \) denotes the set of \( H^1 \) distributions of \( \mathbb{R}^2 \) with radial symmetry. The critical exponent in dimension \( N = 2 \) is \( p = 1 + \frac{4}{N} = 3 \) and thus (14) is \( L^2 \) supercritical in the previous terminology. We now claim that there exists an open subset of \( H^1_r \) such that for an initial data \( u_0 \) in this set, the corresponding solution to (14) blows up in finite time on a sphere with a blow up speed given by the log-log law (10).

**Theorem 2 (Existence and stability of a solution blowing up on a sphere)** Let \( Q \) be the one dimensional ground state solution to (4) with \( p = 5 \), explicitly

\[
Q(x) = \left( \frac{3}{\text{ch}^2(x)} \right)^{\frac{1}{4}}.
\]

There exists an open subset \( \mathcal{P} \subset H^1_r \) and a universal constant \( C^* > 0 \) such that the following holds true. Let \( u_0 \in \mathcal{P} \), then the corresponding solution \( u(t) \) to (14) blows up in finite time \( 0 < T < +\infty \) according to the following dynamics:

(i) **Description of the singularity formation**: there exist \( \lambda(t) > 0, r(t) > 0 \) and \( \gamma(t) \in \mathbb{R} \) such that

\[
u(t, r) - \frac{1}{\lambda(t)^{\frac{1}{2}}} Q \left( \frac{r - r(t)}{\lambda(t)} \right) e^{i\gamma(t)} \to u^*(r) \quad \text{in} \quad L^2 \quad \text{as} \quad t \to T,
\]

and the radius of the singular circle converges

\[
r(t) \to r(T) > 0 \quad \text{as} \quad t \to T.
\]

(ii) **Estimate on the blow up speed**:

\[
\forall 0 < s \leq 1, \quad \lambda^s(t) |u(t)|_{H^s} \to |Q|_{H^s} \quad \text{as} \quad t \to T
\]

and

\[
\lambda(t) \left( \frac{\log \log (T - t)}{T - t} \right)^{\frac{1}{2}} \to \frac{\sqrt{2\pi}}{|Q|_{L^2}} \quad \text{as} \quad t \to T.
\]

(iii) **Asymptotic of the profile \( u^* \) on the singularity**: \( \forall R > 0 \) small enough,

\[
\frac{1}{C^*(\log \log (R))^2} \leq \int_{|r - r(T)| \leq R} |u^*(r)|^2 dr \leq \frac{C^*}{(\log \log (R))^2},
\]

and in particular

\[
u^* \notin H^1 \quad \text{and} \quad u^* \notin L^p \quad \text{for} \quad p > 2.
\]
(iv) $H^{\frac{1}{2}}$ gain of regularity outside the singular circle: $\forall R > 0$ small enough,
\[ u^* \in H^{\frac{1}{2}} (|r - r(T)| > R). \]  

Comments on the result

1. **On the behavior of the critical norm:** The scaling invariant norm for (14) is the $\dot{H}^{\frac{1}{2}}$ norm. From the point of view of the local well posedness theory in $H^{\frac{1}{2}}$ of [7], it was unclear whether this norm would remain bounded or not under an assumption of finite time blow up. Note that a self similar blow up solution behaving like (13) would indeed have a finite $H^{\frac{1}{2}}$ norm up to blow up time. On the contrary, the only Sobolev norm which remains bounded up to blow up time for the solutions given by Theorem 2 is according to (17) the one conserved by the flow, i.e. the $L^2$ norm. Note also that (15), (20) and (21) provide a precise insight into what happens in the $H^{\frac{1}{2}}$ critical space. The singular part of the solution leaves $H^{\frac{1}{2}}$ by leaving $L^2$ and forming a Dirac mass supported on the circle; the rest of the solution also leaves $H^{\frac{1}{2}}$ on the singular circle because of asymptotic behavior (19) which is the trace of the radiative regime responsible for the log-log correction to the self-similar regime in the $L^2$ critical case. But this singularity formation is a very well localized in space phenomenon in the sense that essentially, nothing happens strictly outside the singular circle where the solution remains $H^{\frac{1}{2}}$ bounded up to blow up time according to (21).

2. **On the blow up speed:** Note that the log-log blow up speed given by (18) is far above the scaling estimate (12) for $N = 2$ and $p = 5$, and thus the scaling lower bound is not optimal. The proof will proceed by proving that in rescaled radial variables centered around the singular circle, the leading order dynamic is given by the one dimensional quintic (NLS) which is $L^2$ critical in one dimension. Note in passing that we will need the one dimensional version only of the Spectral Property stated above which proof has been completed in [18].

3. **On the description of the set $\mathcal{P}$:** An explicit description of the set $\mathcal{P}$ of admissible data is given by Definition 1. Note that from direct verification, the sign of the energy is not prescribed in $\mathcal{P}$, and thus finite time blow up on a circle may happen with either sign of the energy.

4. **On the stability of the singularity formation:** The singularity formation described by Theorem 2 is stable with respect to small $H^1$ radial perturbations of the initial data as $\mathcal{P}$ is open. Whether or not these solutions are stable with respect to a small non radial perturbation is open.

5. **On the non linearity:** The choice of the two dimensional quintic non linearity corresponds to the intersection of two conditions: first we want the problem in radial
coordinates far from zero to reduce to the one dimensional $L^2$ critical (NLS) which is the quintic one; second we want the Sobolev invariant norm to be the $H^2$ norm, see the strategy of the proof below, and this requires being in dimension $N = 2$. Now observe that similar constructive results could be derived for more general non linearities $uf(|u|^2)$ as long as $f$ is smooth enough and $|f(x) - x^2| < C|x|^β$ is strictly subcritical ie $β < 2$.

1.5 Strategy of the proof

Let us explain the main steps of the proof of Theorem 2.

The basic heuristic is the following. Rewrite (14) in radial coordinates

$$iu_t = -\partial_r^2 u - \frac{\partial_r u}{r} - |u|^4 u.$$ 

In general, the two terms $\partial_r^2 u$ and $\frac{\partial_r u}{r}$ forming the Laplacian certainly scale the same way, but if we now assume that the singularity formation a priori takes place exclusively around the circle $r \sim 1$, then on this circle, the term $\frac{\partial_r u}{r}$ scales below $\partial_r^2 u$ and thus the singular part of the equation is governed by the one dimensional quintic (NLS) for which a stable log-log dynamic is described by Theorem 1.

**step 1**: Setting of the bootstrap.

We start with an initial data which corresponds to a well focused initial data around $r = 1$ in rescaled variables for the quintic one dimensional (NLS). For a short time, this configuration remains and the solution can be written explicitly in the form

$$u(t, r) = \frac{1}{\lambda(t)^2} \tilde{Q}_{b(t)} \left( \frac{r - r(t)}{\lambda(t)} \right) e^{i\gamma(t)} + \tilde{u}(t, r), \quad (22)$$

$$\varepsilon(t, y) = \lambda(t)^\frac{1}{2} \tilde{u}(t, \lambda(t)y + r(t)) e^{-i\gamma(t)}, \quad y \in \left[ -\frac{r(t)}{\lambda(t)}, +\infty \right). \quad (23)$$

The profiles $\tilde{Q}_b$ are suitable well localized deformations of the ground state $Q$ corresponding to an extra degeneracy in the $L^2$ problem related to the pseudo-conformal invariance and the parameter $0 < b(t) << 1$. The excess of mass $\varepsilon(t, y)$ is $H^1$ small in a weighted norm corresponding to the condition that $\tilde{u} \in H^1(\mathbb{R}^2)$, and the radius of concentration is $r(t) \sim 1$. The condition that the solution (22) is in the log-log regime for the one dimensional quintic (NLS) corresponds to assuming specific relations between the parameters $b, \lambda$ of the form

$$\lambda(t) \sim e^{-e^{\frac{\pi}{\sqrt{r(t)}}}} \quad \text{and} \quad -\lambda \frac{d\lambda}{dt} \sim b(t) > 0. \quad (24)$$

Integration of this differential inequation yields log-log law (18). Using a bootstrap type argument, we aim at proving that such a configuration is stable and leads to the log-log
blow up solutions of Theorem 2. What we need to prove is that the singularity formation remains well localized in space and in particular does not propagate into the zone $r \sim 0$ where the two dimensional problem is “truly” $L^2$ super critical. This is a problem of propagation of singularity.

**step 2:** Dynamics on the singular circle.

We study the singularity formation on the circle using the tools developed for the $L^2$ critical case in [18], [19], [20], [21], [22], [24]. In particular, we use a very specific feature of the log-log regime described in [21] and which is that the space in rescaled variable may decomposed into three specific regions:

(i) The soliton core $|y| \leq \frac{2}{b(t)}$ where the excess of mass $\varepsilon$ is essentially negligible and the solution looks very much like the non linear ground state $Q$. This is where focusing takes place.

(ii) The radiative zone $\frac{2}{b(t)} \leq |y| \leq e^{C_b(t)}$ where the solution looks like the tale of a self similar solution. This corresponds to a fine refinement of the profile $Q$ necessary to take into account the log-log type corrections.

(iii) The outgoing radiation takes $L^2$ mass out of the soliton core and sends it into the dispersive zone $|y| \geq e^C$ where a purely linear dynamic takes place. The key here is the global $L^2$ constraint on the solution.

Now the first fundamental remark in the log-log regime is that the first two zones which somehow describe the singularity formation are for $|y| \leq e^C << 1$ from (24), or in other words in original variable, they never escape the zone $\frac{1}{2} \leq r \leq \frac{5}{4}$. Second, the nature of the non linear estimates we have provide us with a global control in space of the $\dot{H}^1$ norm of $\tilde{u}$ provided an a priori control of the non local $L^6$ norm of the form

$$\int |\tilde{u}|^6 \ll \int |\partial_r \tilde{u}|^2.$$ \hspace{1cm} (25)

Once this control is established, the reinjection of the control of the $\dot{H}^1$ norm of $\tilde{u}$ into the finite dimensional dynamic driving the geometrical parameters $\lambda(t), r(t), b(t)$ allows us to derive (24) which leads to (18) and the convergence of the radius of concentration (16). Now to prove (25), we use the radial symmetry to divide the space in two regions. In the zone $r \geq \frac{1}{2}$, we use the one dimensional Gagliardo-Nirenberg inequality to derive an estimate of the form

$$\int_{r \geq \frac{1}{2}} |\tilde{u}|^6 \leq C \left( \int |\tilde{u}|^2 \right)^2 \int |\partial_r \tilde{u}|^2$$

and (25) follows from the smallness of the $L^2$ norm which is a consequence of the conservation of the $L^2$ norm. In the zone $r \leq \frac{1}{2}$, we roughly have from the two dimensional Gagliardo-Nirenberg inequality

$$\int_{r \leq \frac{1}{2}} |\tilde{u}|^6 \leq C |\tilde{u}|_{H^\frac{1}{2}(r \leq \frac{1}{2})}^4 \int |\partial_r \tilde{u}|^2.$$
We thus need to prove that the $H^{\frac{1}{2}}$ norm remains small in the ball $r \leq \frac{1}{2}$.

**step 3**: Control of the $H^{\frac{1}{2}}$ norm.

The main difficulty to control the $H^{\frac{1}{2}}$ norm is that $\tilde{u}$ cannot be proved to remain globally $H^{\frac{1}{2}}$ small according to anomalous behavior (19) on the singular circle which implies (20). We thus need to introduce a cut off in space $\chi(r) = 1$ for $r \leq \frac{1}{2}$ and zero otherwise and prove the $H^{\frac{1}{2}}$ boundedness of $w = \chi u$. Now the main terms in the equation satisfied by $w$ are roughly of the form

$$iw_t + \Delta w \sim -\partial_r \chi \partial_r u - |w|^4.$$ 

Writing down Duhamel’s formula, we estimate the $H^{\frac{1}{2}}$ norm of $w$ and two key facts will allow us to conclude:

(i) First, the term corresponding to the quintic non linearity can be estimated using Strichartz estimates as for the proof of the $H^{\frac{1}{2}}$ critical well posedness theory of [7]. In other words, a fixed point type of estimate ensures that if we start with a small $H^{\frac{1}{2}}$ norm, this term cannot grow.

(ii) Second, we have the dangerous linear term $\partial_r \chi \partial_r u$. Now from the well localization in space of this term, the smoothing effect for the linear Schrödinger flow allows us to gain half a derivative $L^\infty_\infty$ in time ie the control of the $H^{\frac{1}{2}}$ norm provided the control

$$\int_0^T |\partial_r \tilde{u}(\tau)|^2_{L^2} d\tau < +\infty$$

where $T > 0$ is the expected blow up time. Observe from (18) that

$$\int_0^T |\partial_r u(\tau)|^2_{L^2} d\tau = +\infty.$$ 

Now (26) holds true as a consequence of the dispersive estimates on the solution obtained for the proof of the log-log lower bound on blow up rate in the $L^2$ critical case in [21]. This a priori estimate is for example at the heart of the proof of $L^2$ convergence (9), and indeed the gain from (27) to (26) is precisely what allows one to separate the singular and the regular parts of the solution in $L^2$, see [22].

The outcome is that we may bootstrap the log-log regime and control the $H^{\frac{1}{2}}$ norm in the ball $r \leq \frac{1}{2}$, what concludes the proof of Theorem 2.

This paper is organized as follows. In section 2, we recall the main objects involve in the geometrical decomposition (22) introduced in [18], [19] which are at the heart of the description of the admissible set $P$ of Theorem 2. In section 3, we adapt the analysis in [20], [21] and derive Lyapounov type of controls of the singular dynamics provided the a priori control of the $H^{\frac{1}{2}}$ norm near the origin which is proved in section 4.
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2 Geometrical description of the set \( \mathcal{P} \)

Our aim in this section is to introduce the geometrical decomposition of the solution \( u(t) \) at the heart of the description of the set \( \mathcal{P} \) of initial data of Theorem 2. This decomposition was first introduced in the critical case in [18], [19] and allows a splitting of the solution into a finite dimensional part governed by an explicit ODE which will yield log-log behavior (18), and an infinite dimensional part controlled by suitable dispersive estimates. On the contrary to the \( L^2 \) critical setting, this last part cannot be controlled directly due to the high power of the non linearity, and requires a bootstrap argument to prove that the solution cannot escape a specific regime. We explicitly describe the bootstrap argument at the end of this section, see Proposition 2, while the last two sections will be devoted to its proof.

We will use the following notation. For a given well localized function \( f \), we let

\[
\begin{align*}
    f_1 &= \frac{1}{2}f + y \cdot \nabla f, \\
    f_2 &= \frac{1}{2}f_1 + y \cdot \nabla f_1.
\end{align*}
\]

2.1 Localized self similar profiles

We recall from [19] the existence of a one parameter family of localized self similar profiles in the vicinity of ground state solution \( Q \). Let a parameter \( 0 < \eta < \eta^* \) small enough to be fixed later. For \( b \neq 0 \), set \( R_b = \frac{2}{|b|} \sqrt{1-\eta} \) and \( R_b^- = \sqrt{1-\eta} R_b \) and let a regular radially symmetric cut-off function \( \phi_b(x) = 0 \) for \( |x| \geq R_b \), \( \phi_b(x) = 1 \) for \( |x| \leq R_b^- \), \( 0 \leq \phi_b(x) \leq 1 \), such that \( |\phi_b'_{L^\infty}| + |\Delta \phi_b|_{L^\infty} \to 0 \) as \( |b| \to 0 \).

**Proposition 1 (Localized self similar profiles)** See Propositions 8 and 9 of [20]. There exist universal constants \( C > 0, \eta^* > 0 \) such that the following holds true. For all \( 0 < \eta < \eta^* \), there exist constants \( \varepsilon^*(\eta) > 0, b^*(\eta) > 0 \) going to zero as \( \eta \to 0 \) such that for all \( |b| < b^*(\eta) \), there exists a unique even solution \( Q_b \) to

\[
\begin{align*}
    \partial_y^2 Q_b - Q_b + i b \left( \frac{1}{2} Q_b + y \partial_y Q_b \right) + Q_b |Q_b|^4 &= 0, \\
    P_b &= Q_b e^{i \frac{|b|^2 y}{4}} > 0 \text{ for } y \in [0, R_b), \\
    Q_b(0) &\in (Q(0) - \varepsilon^*(\eta), Q(0) + \varepsilon^*(\eta)), \quad Q_b(R_b) = 0.
\end{align*}
\]

Moreover, let

\[
Q_b(y) = \frac{\tilde{Q}_b(y)}{\phi_b(y)},
\]
then:

(i) Uniform closeness to the ground state:

\[ \| e^{Cy}(\tilde{Q}_b - Q) \|_{C^3} \to 0 \text{ as } b \to 0. \]

(ii) Degeneracy of the energy and the momentum:

\[ |E(\tilde{Q}_b)| \leq e^{-C_b} \text{ and } \Im \left( \int \partial_y \tilde{Q}_b \overline{\tilde{Q}_b} \right) = 0. \]  

(iii) \( \tilde{Q}_b \) has super-critical mass, and more precisely

\[ 0 < \frac{d}{d(b^2)} \left( \int |\tilde{Q}_b|^2 \right) \bigg|_{b^2=0} = d_0 < +\infty. \]

Profiles \( \tilde{Q}_b \) are not exact self similar solutions and we define the error term \( \tilde{\Psi}_b \) by

\[ \partial_y^2 \tilde{Q}_b - \tilde{Q}_b + ib(\tilde{Q}_b)1 + \tilde{Q}_b|\tilde{Q}_b|^4 = -\tilde{\Psi}_b. \]  

We next introduce the outgoing radiation escaping the soliton core.

**Lemma 1 (Linear outgoing radiation)** See Lemma 15 in [20]. There exist universal constants \( C > 0 \) and \( \eta^* > 0 \) such that \( \forall 0 < \eta < \eta^* \), there exists \( b^*(\eta) > 0 \) such that \( \forall 0 < b < b^*(\eta) \), the following holds true: let \( \Psi_b \) given by (30), there exists a unique even solution \( \zeta_b \) to

\[ \begin{cases} 
\partial_y^2 \zeta_b - \zeta_b + ib(\frac{1}{2} \zeta_b + y \partial_y \zeta_b) = \Psi_b \\
\int |\partial_y \zeta_b|^2 < +\infty.
\end{cases} \]  

Moreover, let

\[ \Gamma_b = \lim_{|y| \to +\infty} |y||\zeta_b(y)|^2, \]

then

\[ \int |\partial_y \zeta_b|^2 \leq \Gamma_b^{1-C\eta}, \]

\[ \forall |y| \geq R_b^2, \quad e^{-(1-C\eta)\frac{y}{2}} \geq |y||\zeta_b(y)|^2 \geq \frac{4}{5} \Gamma_b \geq e^{-(1+C\eta)\frac{y}{2}}. \]

### 2.2 Geometrical characterization of the set \( \mathcal{P} \)

We introduce in this subsection a geometrical decomposition of radial \( H^1 \) functions which are closed to a focused ground state on a circle. This decomposition corresponds to a splitting of the solution into a finite dimensional part governed by an explicit ODE, and the infinite dimensional part controlled by suitable dispersive estimates.
We let $\alpha^* > 0$ be a small enough universal constant to be chosen later. Moreover, given two parameters $\lambda, r > 0$, we denote 

$$\mu_{\lambda, r}(y) = (\lambda y + r)1_{\lambda y + r \geq 0}$$

which will be the weight of the Lebesgue measure in rescaled variables.

**Lemma 2 (Existence of a geometrical decomposition)** Let $u \in H^1_r$ of the form

$$u(r) = \frac{1}{\lambda_u^2} (\tilde{Q}_b + \varepsilon_u) \left( \frac{r - r_u}{\lambda_u} \right) e^{i\gamma_u}$$

for some parameters $\gamma_u \in \mathbb{R}$, $b_u \in \mathbb{R}$, $\lambda_u > 0$, $r_u > 0$ with the following estimates

$$\frac{3}{4} < r_u < \frac{4}{3} \text{ and } 10\lambda_u < b_u < \alpha^*,$$  \hspace{1cm} (35)

$$\int |\partial_y \varepsilon_u(y)|^2 \mu_{\lambda_u, r_u}(y) dy + \int_{|y| \leq \frac{10}{\lambda_u}} |\varepsilon_u(y)|^2 e^{-|y|} dy < \Gamma_{b_u}^2.$$  \hspace{1cm} (36)

Then there exists a universal constant $C > 0$ and parameters $\gamma_0 \in \mathbb{R}$, $\lambda_0 > 0$, $r_0 > 0$ with

$$|b_0 - b_u| + \left| \frac{\lambda_0}{\lambda_u} - 1 \right| + \left| \frac{r_0 - r_u}{\lambda_u} \right| \leq \Gamma_{b_0}^\frac{1}{2}$$

such that

$$\varepsilon(y) = \frac{1}{\lambda_0^2} u(\lambda_0 y + r_0) e^{-i\gamma_0} - \tilde{Q}_{b_0}$$

satisfies the following one dimensional orthogonality conditions:

$$ (\varepsilon_1, |y|^2 \Sigma_{b_0}) + (\varepsilon_2, |y|^2 \Theta_{b_0}) = 0, \hspace{1cm} (37)$$

$$ (\varepsilon_1, y \Sigma_{b_0}) + (\varepsilon_2, y \Theta_{b_0}) = 0, \hspace{1cm} (38)$$

$$ - (\varepsilon_1, (\Theta_{b_0})_2) + (\varepsilon_2, (\Sigma_{b_0})_2) = 0, \hspace{1cm} (39)$$

$$ - (\varepsilon_1, (\Theta_{b_0})_1) + (\varepsilon_2, (\Sigma_{b_0})_1) = 0, \hspace{1cm} (40)$$

where $\varepsilon = \varepsilon_1 + i\varepsilon_2$, $\tilde{Q}_b = \Sigma_b + i\Theta_b$ in terms of real and imaginary parts.

The proof of this lemma is a simple application of the implicit function theorem and similar to the one of Lemma 2 in [19] or Lemma 4 in [21] an we skip it. Let us simply point out that condition (35) ensures from $\text{Supp}(\tilde{Q}_b) \subset B(0, \frac{2}{\lambda})$ that the rescaled ground state part of $u$ is supported in an annulus away from zero, while smallness assumption (36) ensures that this property remains satisfied during the modulation, and thus the one dimensional inner products (37), (38), (39), (40) are well defined.
We now are in position to describe the set $\mathcal{P}$ of initial data for which the corresponding solution to (14) satisfies the conclusions of Theorem 2. We let two smooth radial cut-off functions
\[
\chi_0(r) = \begin{cases} 
1 & \text{for } 0 \leq r \leq \frac{13}{16} \text{ and } r \geq \frac{14}{13}, \\
0 & \text{for } \frac{15}{16} \leq r \leq \frac{16}{15}, 
\end{cases}
\]
and
\[
\psi(r) = \begin{cases} 
1 & \text{for } 0 \leq r \leq \frac{1}{2}, \\
0 & \text{for } 0 \leq r \leq \frac{1}{4}, \text{ and } r \geq 3. 
\end{cases}
\]

**Definition 1 (Geometrical description of the set $\mathcal{P}$)** We let $\mathcal{P}$ be the set of radial distributions $u_0 \in H^1_r$ of the form
\[
u_0(r) = \frac{1}{\lambda_0^2}(\tilde{Q}_{b_0} + \varepsilon_0) \left(\frac{r - r_0}{\lambda_0}\right)e^{i\gamma_0}
\]
with the following controls:

(i) Localization of the singular circle:
\[|r_0 - 1| < \alpha^*; \tag{42}\]

(ii) Closeness to $Q$ on the singular circle:
\[0 < b_0 < \alpha^*; \tag{43}\]

and $\varepsilon(y) = \lambda_0^2 \tilde{u}_0(\lambda_0 y + r_0)e^{-i\gamma_0}$ satisfies orthogonality conditions (37), (38), (39), (40) of Lemma 2 and smallness estimate
\[
\int |\partial_y \varepsilon_0(y)|^2 \mu_{\lambda_0, r_0}(y)dy + \int_{|y| \leq \frac{10}{b_0}} |\varepsilon_0(y)|^2 e^{-|y|}dy < \Gamma_{b_0}^6; \tag{44}\]

(iii) Normalization of the energy and the localized momentum:
\[
\lambda_0^2 |E_0| + \lambda_0 \left| \text{Im} \left( \int \nabla \psi \cdot \nabla u(0) \bar{\pi}(0) \right) \right| < \Gamma_{b_0}^{10}; \tag{45}\]

(iv) $u_0$ is in the log-log regime:
\[0 < \lambda_0 < e^{-e^{-\frac{4\pi}{b_0}}}; \tag{46}\]

(v) Global $L^2$ smallness: Let
\[
\tilde{\nu}_0(r) = \frac{1}{\lambda_0^2} \varepsilon_0 \left(\frac{r - r_0}{\lambda_0}\right)e^{i\gamma_0},
\]
then
\[|\tilde{\nu}_0|_{L^2} < \alpha^*; \tag{47}\]

(vi) $H^{\frac{1}{2}}$ smallness outside the singular circle:
\[|\chi_0 \tilde{\nu}_0|_{H^{\frac{1}{2}}} < \alpha^*. \tag{48}\]
Let us observe that Lemma 2 ensures that $\mathcal{P}$ is an open subset of $H^1$. 

**Remark 1** The non emptyness of $\mathcal{P}$ is also straightforward. Indeed, let $b_0 > 0$ small enough so that (43) holds, $r_0 = 1$ and $\lambda_0 > 0$ small enough so that (46) holds. Let then $f(y)$ be a smooth real valued even function supported in $|y| \leq 2$ with $|f|_{H^1(\mathbb{R})} = 1$ and such that $\varepsilon_0(y) = \nu f(y)$ satisfies orthogonality conditions of Lemma 2 for any parameter $\nu \in \mathbb{R}$. Let

$$u_0(r) = \frac{1}{\lambda_0^2}(\hat{Q}_{b_0} + \nu f) \left( \frac{r - r_0}{\lambda_0} \right),$$

we cannot take $\nu = 0$ because of (45) and (28). Nevertheless, first observe from (46) that for $y \in \text{Supp}(\hat{Q}_{b_0})$, $\frac{3}{4} \leq \lambda_0 y + r_0 \leq \frac{1}{4}$ and thus $\partial_y \psi(\lambda_0 y + r_0) = 1$ from (41). This implies using (28):

$$\lambda_0 \text{Im} \left( \int \nabla \psi \cdot \nabla u(0) \pi(0) \right) = \text{Im} \left( \int \partial_y \psi(\lambda_0 y + r_0) \partial_y (\hat{Q}_{b_0} + \nu f)(\hat{Q}_{b_0} + \nu f)(\lambda_0 y + r_0) dy \right)$$

$$= \lambda_0 \left| \text{Im} \left( \int \partial_y (\hat{Q}_{b_0} + \nu f)(\hat{Q}_{b_0} + \nu f) y dy \right) \right| \leq C \lambda_0 < C_{b_0}^{10}.$$

Next,

$$\left| \lambda_0^2 E(u_0) \right| \leq \left| \frac{1}{2} \int \partial_y (\hat{Q}_{b_0} + \nu f)^2 \mu_{\lambda_0,1}(y) dy - \frac{1}{6} \int |\hat{Q}_{b_0} + \nu f|^6 \mu_{\lambda_0,1}(y) dy \right|$$

$$\leq \left| \frac{1}{2} \int \partial_y (\hat{Q}_{b_0} + \nu f)^2 dy - \frac{1}{6} \int |\hat{Q}_{b_0} + \nu f|^6 \right| + C \lambda_0$$

from the support properties of $\hat{Q}_{b}$ and $f$. Now we may assume $(f, Q) = 1$, and then from $\frac{d}{dt} E_1(Q + \nu f)_{\nu=0} = -(f, Q) = -1$ where $E_1$ denotes the one dimensional energy $E_1(\nu) = \frac{1}{2} \int |\partial_y \nu|^2 dy - \frac{1}{2} \int |\nu|^6 dy$, we may find $\nu = \nu(b_0)$ such that $E(\hat{Q}_{b_0} + \nu f) = 0$ with $|\nu| \leq C_{b_0}^{1-C_0}$ from (28) and (46), and thus (45) and (44) hold. Moreover,

$$|\tilde{u}_0|_{L^2} = |\nu| \left( \int |f|^2(\mu_{\lambda_0,1}(y) dy \right)^{\frac{1}{2}} < \alpha^*$$

and $\chi_0 \tilde{u}_0 = 0$ from support property of $\hat{Q}_{b}$ and $f$, and thus (47) and (48) hold.

### 2.3 The bootstrap argument

Let now an initial data $u_0 \in \mathcal{P}$ and $u(t)$ the corresponding solution to (14) with $[0, T)$, $0 < T \leq +\infty$, its maximum time interval existence in $H^1$. Then from $u \in C([0, T), H^1)$, there exists some time $t_1 \in [0, T]$ such that for all $t \in [0, t_1)$, $u(t) \in \mathcal{P}$ and admits a geometrical decomposition as in Lemma 2 which we denote

$$u(t, r) = \frac{1}{\lambda(t)^2} \hat{Q}_{b(t)} \left( \frac{r - r(t)}{\lambda(t)} \right) e^{i\gamma(t)} + \tilde{u}(t, r), \quad (49)$$
\[ \varepsilon(t, y) = \lambda(t)^{\frac{1}{2}} \tilde{u}(t, \lambda(t)y + r(t))e^{-r(t)}, \]  

(50)

where \( \lambda(t), \gamma(t), r(t) \) are continuous functions of time. We may thus assume: \( \forall t \in [0, t_1), \)

\[ |r(t) - 1| < \sqrt{\alpha^*}, \]  

(51)

\[ 0 < b(t) < (\alpha^*)^{\frac{1}{10}}, \]  

(52)

\[ \int |\partial_y \varepsilon(t, y)|^2 \mu_{\lambda(t), r(t)}(y)dy + \int_{|y| \leq \frac{1}{b(t)}} |\varepsilon(t, y)|^2 e^{-|y|}dy < \Gamma^{\frac{3}{2}}_{b(t)}, \]  

(53)

\[ \lambda^2(t)|E_0| < \Gamma^2_{b(t)}, \]  

(54)

\[ \lambda(t) \left| Im \left( \int \nabla \psi \cdot \nabla u(t) \pi(t) \right) \right| < \Gamma^2_{b(t)}, \]  

(55)

\[ 0 < \lambda(t) < e^{-\frac{\pi}{100(t+1)}}. \]  

(56)

We also assume a global \( L^2 \) bound

\[ |	ilde{u}(t)|_{L^2} < (\alpha^*)^{\frac{1}{10}}, \]  

(57)

and a smallness estimate of \( \tilde{u} \) in \( H^\frac{1}{2} \) outside the singular circle

\[ |\chi_1 \tilde{u}(t)|_{H^\frac{1}{2}} < (\alpha^*)^\frac{1}{2} \]  

(58)

where \( \chi_1 \) is a smooth radial cut off function with

\[ \chi_1(r) = \begin{cases} 
1 & \text{for } 0 \leq r \leq \frac{1}{4} \text{ and } r \geq 4 \\
0 & \text{for } \frac{1}{2} \leq r \leq 2. 
\end{cases} \]

In fact, we need to prove a slightly more refined estimate than (58) related to the well posedness of the Cauchy problem for (14) in the critical space \( H^\frac{1}{2} \). Let us introduce a Besov generalization of \( \dot{H}^s \) for \( s > 0 \) by defining the homogeneous Besov semi-norm

\[ |u|_{\dot{B}^s_{r,2}} = \left( \int_0^\infty (t^{-s} \sup_{|y| \leq t} |u(\cdot - y) - u(\cdot)|_{L^r})^2 \frac{dt}{t} \right)^{\frac{1}{2}}, \quad 1 < r < +\infty, \]  

(59)

and the norm

\[ |u|_{\dot{B}^s_{r,2}} = |u|_{\dot{B}^s_{r,2}} + |u|_{L^r}, \]

see for example [5] p.15. Let us recall that

\[ \dot{B}^s_{2,2} = \dot{H}^s. \]
Moreover, from the two dimensional Sobolev embeddings, \( H^1 \hookrightarrow \dot{H}^{\frac{5}{2}} \hookrightarrow \dot{B}^{\frac{1}{2}}_{3,2} \), \( u \in C([0, T), H^1) \) implies
\[
\forall t \in [0, T), \quad u \in L^6([0, t], \dot{B}^{\frac{1}{2}}_{3,2}).
\]

We then bootstrap the estimate
\[
|\chi_1 \tilde{u}|_{L^6([0, t_1], \dot{B}^{\frac{1}{2}}_{3,2})} < (\alpha^*)^{\frac{3}{8}}.
\]  
(60)

The main part of this paper is devoted to proving that these estimates may be bootstrapped in time as we claim:

**Proposition 2 (Bootstrap on the log-log regime)** For all \( t \in [0, t_1) \), we have:

\[
|r(t) - 1| < (\alpha^*)^{\frac{3}{8}},
\]  
(61)

\[
0 < b(t) < (\alpha^*)^{\frac{3}{8}},
\]  
(62)

\[
\int |\partial_y \varepsilon(t, y)|^2 \mu_{\lambda(t), r(t)}(y) dy + \int_{|y| \leq \frac{1}{\lambda(t)}} |\varepsilon(t, y)|^2 e^{-|y|} dy < \Gamma_{b(t)}^{\frac{3}{4}},
\]  
(63)

\[
\lambda^2(t)|E_0| < \Gamma_{b(t)}^{\frac{3}{4}},
\]  
(64)

\[
\lambda(t) \left| \text{Im} \left( \int \nabla \psi \cdot \nabla u(t) \pi(t) \right) \right| < \Gamma_{b(t)}^{\frac{3}{4}},
\]  
(65)

\[
0 < \lambda(t) < e^{-\frac{8}{9}(\alpha^*)^{\frac{3}{8}}},
\]  
(66)

\[
|\tilde{u}(t)|_{L^2} < (\alpha^*)^{\frac{3}{8}},
\]  
(67)

\[
|\chi_1 \tilde{u}(t)|_{H^{\frac{1}{2}}} < (\alpha^*)^{\frac{3}{8}},
\]  
(68)

\[
|\chi_1 \tilde{u}|_{L^6([0, t_1], \dot{B}^{\frac{1}{2}}_{3,2})} < \sqrt{\alpha^*}.
\]  
(69)

and thus
\[
t_1 = T.
\]

**Remark 2** The need to bootstrap (65) on a localized momentum corresponds to somehow a loss of information in the radial case. Indeed, the conservation of the momentum \( \text{Im} \left( \int \nabla u \pi(t, x) \right) \) does not provide us with any information because \( u \) radial implies that this vector is identically zero. On the other hand, the local dynamic around \( Q \) on the singular circle has a degeneracy related to the translation invariance and following the analysis in [18], the corresponding nonlinear direction is controlled using this conservation law. In our case, we then mimic it by bootstrapping in the log-log regime the control of a local momentum, see the proof of degeneracy estimate (84) below.
3 Lyapounov control of the dynamics on the singular circle

We recall and adapt in this section the techniques involved in the control of the log-log regime as first exhibited for the $L^2$ critical case in [18], [19], [20], [21], [22], [24]. We shall sometimes refer directly to these papers for detailed proofs and explanations. Obtained properties are the starting point of our analysis and rely on the geometrical decomposition of the solution (49), (50) in the time interval $[0, t_1)$. We in particular derive dispersive type estimates for the excess of mass $\tilde{u}$ which allow us to understand part of the interactions between the finite dimensional dynamic of the geometrical parameters $\lambda(t)$, $r(t)$, $\gamma(t)$, and the infinite dimensional part of the solution. The heart of the analysis is to derive Lyapounov type of controls which yield rigidity of the blow up dynamics and allow us to precisely estimate the part of the solution which escapes the singular circle of focusing.

We work in the whole section with the notations of the previous section and in the setting of the bootstrap argument of Proposition 2.

3.1 First estimates on the decomposition

Let us introduce the rescaled time

$$s = \int_0^t \frac{d t'}{\lambda^2(t')} + s_0,$$

where we chose explicitly the origin of rescaled times according to

$$s_0 = e^{\frac{3\pi}{4} b_0}.$$  

We let $s_1 = s(t_1) \in [s_0, +\infty]$. Consider geometrical decomposition (49), (50) and let

$$Q_{sing}(t, r) = \frac{1}{\lambda(t)^{\frac{3}{2}}} \tilde{Q}_{b(t)} \left( r - r(t) \right) e^{i\gamma(t)}.$$  

First observe from bootstrap estimates (51), (56) and $\text{Supp}(\tilde{Q}_b) \subset B(0, \frac{3}{\pi})$ that

$$\forall t \in [0, t_1), \quad \text{Supp}(Q_{sing}(t)) \subset \left\{ \frac{2}{3} \leq r \leq \frac{3}{2} \right\},$$

and thus $\tilde{u} \in H^1$. We now view $\varepsilon(t, y)$ given by (50) as a one dimensional distribution for $y \in [-\frac{r(t)}{\pi(t)}, +\infty)$ with the suitable boundary condition at $y = -\frac{r(t)}{\pi(t)}$ corresponding to $\tilde{u} \in H^1$. $\varepsilon(t, y)$ lives in some weighted Sobolev space and we note to simplify notations

$$\mu(y) = \mu_{\lambda(t), r(t)}(y) = (\lambda(t)y + r(t)) 1_{\lambda(t)y + r(t) \geq 0}.$$
We may in particular rewrite (53)
\[
\int |\partial_y \varepsilon(y)|^2 \mu(y) dy + \int_{|y| \leq \frac{1}{\varepsilon}} |\varepsilon(y)|^2 e^{-|y|} dy \leq \Gamma \varepsilon^3.
\]

To derive the \( \varepsilon \) equation, first recall from a standard argument that \( \{ \lambda(s), \gamma(s), x(s), b(s) \} \) are \( C^1 \) functions of \( s \) on \([s_0, s_1]\), see [18]. Let \( \Psi_b \) given by (30), we define
\[
\tilde{\Psi}_b(t, y) = \Psi_b - \frac{\lambda(t)}{\mu(y)} \partial_y Q_b(y)
\]
so that
\[
\partial_y^2 \tilde{Q}_b + \frac{\lambda(t)}{\mu(y)} \partial_y Q_b - \tilde{Q}_b + ib(\tilde{Q}_b)_1 + \tilde{Q}_b |\tilde{Q}_b|^4 = -\tilde{\Psi}_b.
\]

To simplify notations, we note from now on
\[
\tilde{Q}_b = \Sigma + i\Theta, \quad \tilde{\Psi}_b = Re(\tilde{\Psi}) + iIm(\tilde{\Psi})
\]
in terms of real and imaginary parts. \( \varepsilon \) satisfies the following equation for \( s \in [s_0, s_1], y \in [-r(t)/\lambda(t), +\infty) \) with the suitable boundary condition at \( y = -r(t)/\lambda(t) \):
\[
b_s \frac{\partial \Sigma}{\partial b} + \partial_s \varepsilon_1 - M_-(\varepsilon) + b \left( \frac{1}{2} \varepsilon_1 + y \partial_y \varepsilon_1 \right) = \left( \frac{\lambda_s}{\lambda} + b \right) \Sigma_1 + \tilde{\gamma}_s \Theta + r_s \partial_y \Sigma
\]
\[
+ \left( \frac{\lambda_s}{\lambda} + b \right) \left( \frac{1}{2} \varepsilon_1 + y \partial_y \varepsilon_1 \right) + \tilde{\gamma}_s \varepsilon_2 + r_s \partial_y \varepsilon_1
\]
\[
+ Im(\tilde{\Psi}) - R_2(\varepsilon)
\]
\[
b_s \frac{\partial \Theta}{\partial b} + \partial_s \varepsilon_2 + M_+(\varepsilon) + b \left( \frac{1}{2} \varepsilon_2 + y \partial_y \varepsilon_2 \right) = \left( \frac{\lambda_s}{\lambda} + b \right) \Theta_1 - \tilde{\gamma}_s \Sigma + r_s \partial_y \Theta
\]
\[
+ \left( \frac{\lambda_s}{\lambda} + b \right) \left( \frac{1}{2} \varepsilon_2 + y \partial_y \varepsilon_2 \right) - \tilde{\gamma}_s \varepsilon_1 + r_s \partial_y \varepsilon_2
\]
\[
- Re(\tilde{\Psi}) + R_1(\varepsilon),
\]
with \( \tilde{\gamma}(s) = -s - \gamma(s) \). The linear operator close to \( \tilde{Q}_b \) is \( M = (M_+, M_-) \) with
\[
M_+(\varepsilon) = -\partial_y^2 \varepsilon_1 - \frac{\lambda}{\mu} \partial_y \varepsilon_1 + \varepsilon_1 - \left( \frac{4\Theta^2}{|\tilde{Q}_b|^2} + 1 \right) |\tilde{Q}_b|^4 \varepsilon_1 - 4 \Sigma \Theta |\tilde{Q}_b|^2 \varepsilon_2,
\]
\[
M_-(\varepsilon) = -\partial_y^2 \varepsilon_2 - \frac{\lambda}{\mu} \partial_y \varepsilon_2 + \varepsilon_2 - \left( \frac{4\Theta^2}{|\tilde{Q}_b|^2} + 1 \right) |\tilde{Q}_b|^4 \varepsilon_2 - 4 \Sigma \Theta |\tilde{Q}_b|^2 \varepsilon_1.
\]

Non linear interaction terms are explicitly:
\[
R_1(\varepsilon) = (\varepsilon_1 + \Sigma) \varepsilon - \tilde{Q}_b |\tilde{Q}_b|^4 - \Sigma |\tilde{Q}_b|^4 - \left( \frac{4\Theta^2}{|\tilde{Q}_b|^2} + 1 \right) |\tilde{Q}_b|^4 \varepsilon_1 - 4 \Sigma \Theta |\tilde{Q}_b|^2 \varepsilon_2,
\]
\[
R_2(\varepsilon) = \left( \frac{4 \Theta^2}{|\tilde{Q}_b|^2} + 1 \right) \tilde{Q}_b |\tilde{Q}_b|^4 - 4 \Sigma \Theta |\tilde{Q}_b|^2 \varepsilon_2,
\]
\[ R_2(\varepsilon) = (\varepsilon_2 + \Theta)|\varepsilon + \hat{Q}_b|^4 - \Theta|\hat{Q}_b|^4 - \left( \frac{4\Theta^2}{|\hat{Q}_b|^2} + 1 \right)|\hat{Q}_b|^4 \varepsilon_2 - 4\Sigma \Theta |\hat{Q}_b|^2 \varepsilon_1. \]  

(81)

We now claim the following preliminary estimates for this decomposition:

**Lemma 3** For all \( s \in [s_0, s_1) \), there holds:

(i) Estimate induced by the conservation of the \( L^2 \) norm:

\[ b^2 + \int |\tilde{u}|^2 < C \sqrt{\alpha^*}. \]  

(82)

(ii) Estimates induced by the conservation of the energy and the momentum:

\[
\begin{align*}
&\left| 2(\varepsilon_1, \Sigma) + 2(\varepsilon_1, \Theta) - \int |\partial_y \varepsilon|^2 \mu(y) dy - 5 \int_{|y| \leq \frac{10}{7}} Q_1^4 \varepsilon_1^2 - \int_{|y| \leq \frac{10}{7}} Q_1^4 \varepsilon_2^2 \right| \leq \Gamma_b^{1-C\eta} \\
&\quad + \delta(\alpha^*)(\int |\partial_y \varepsilon|^2 \mu(y) dy + \int_{|y| \leq \frac{10}{7}} |\varepsilon|^2 e^{-|y|} dy), \quad (83)
\end{align*}
\]

\[
\begin{align*}
&\left| (\varepsilon_2, \partial_y \Sigma) \right| \leq C \delta(\alpha^*)(\int |\partial_y \varepsilon|^2 \mu(y) dy + \int_{|y| \leq \frac{10}{7}} |\varepsilon|^2 e^{-|y|} dy)^{\frac{1}{2}} + \Gamma_b^{2} \quad (84)
\end{align*}
\]

where \( \delta(\alpha^*) \to 0 \) as \( \alpha^* \to 0 \).

(iii) Estimates on the modulation parameters:

\[
\begin{align*}
&\left| \frac{\lambda_s}{\lambda} + b + |b_s| \right| \leq C(\int |\partial_y \varepsilon|^2 \mu(y) dy + \int_{|y| \leq \frac{10}{7}} |\varepsilon|^2 e^{-|y|} dy) + \Gamma_b^{1-C\eta}, \quad (85)
\end{align*}
\]

\[
\begin{align*}
&\left| \tilde{\gamma}_s - \frac{1}{|Q_1|^2}(\varepsilon_1, L_+ Q_2) \right| + \frac{r_s}{\lambda} \leq \delta(\alpha^*)(\int |\partial_y \varepsilon|^2 \mu(y) dy + \int_{|y| \leq \frac{10}{7}} |\varepsilon|^2 e^{-|y|} dy)^{\frac{1}{2}} \\
&\quad + \Gamma_b^{1-C\eta} \quad (86)
\end{align*}
\]

where \( L_+ = -\Delta + 1 - 5Q^4 \) is the real part of the linearized operator close to \( Q \).

**Proof of Lemma 3.**

This Lemma is very similar to Lemma 3 in [19] and we shall briefly sketch the proof. It relies on the conservation laws rewritten in the \( \varepsilon \) variable and the computation of the geometrical parameters induced by the choice of orthogonality conditions (37), (38), (39), (40).

(i) \( L^2 \) conservation: We have \( \int |u(t)|^2 = \int |u_0|^2 \). First observe from (29) and (47) that

\[
\left| \int |u_0|^2 - \int Q^2 \right| < C \alpha^*. \]  

(87)
We now rewrite the conservation of the $L^2$ norm in the variables of decomposition (49), (50):

\[
\int |\tilde{Q}_b|^2 \mu(y) dy + 2\text{Re}(\int \varepsilon \tilde{Q}_b \mu(y) dy) + \int |\tilde{u}|^2 = \int |u_0|^2. \tag{88}
\]

A first estimate we will keep using and which follows from bootstrap estimates (52) and (56) is

\[
\lambda(t) < e^{-\varepsilon \frac{\pi}{10} b(t)} < \Gamma_{b(t)}^{10}. \tag{89}
\]

We thus have using (29) and (51):

\[
\int |\tilde{Q}_b|^2 \mu(y) dy - \int |Q|^2 = \lambda(t) \int y |\tilde{Q}_b|^2 dy + (r(t) - 1) \int |\tilde{Q}_b|^2 + \int |\tilde{Q}_b|^2 - \int |Q|^2 \geq Cb^2 - C\sqrt{\alpha^*} \tag{90}
\]

To estimate the crossed term in (88), we first observe from (51) and (56) that

\[
\forall |y| \leq \frac{10}{b}, \quad \frac{2}{3} \leq \mu(y) \leq \frac{3}{2}, \tag{91}
\]

and thus from (53), (87) and (90), we have

\[
\forall t \in [0, t_1), \quad b^2(t) + \int |\tilde{u}(t)|^2 \leq C\sqrt{\alpha^*}.
\]

This is (82) which proves (62) and the control of the $L^2$ norm (67).

(ii) Conservation of the energy: We write it from (49), (50),

\[
2\lambda^2 E_0 = \int |\partial_y (\tilde{Q}_b + \varepsilon)|^2 \mu(y) dy - \frac{1}{3} \int |\tilde{Q}_b + \varepsilon|^6 \mu(y) dy. \tag{92}
\]

We now expand this formula. Note that the support property of $\tilde{Q}_b$ allows us to split the measure $\mu$ into a term with $r(t)$ only for which we perform integration by parts as if we were in the one dimensional situation, and the term with $\lambda(t) y$ for which we estimate

\[
\lambda \int |\varepsilon| |y||\tilde{Q}_b| dy \leq \lambda < \Gamma_b^2
\]

from (53) and (89). In other words, all interaction terms involving $\tilde{Q}_b$ may be estimated as in the one dimensional case to the cost of a $\lambda$ term inherited by the measure $\mu$ which is estimate from (89). Regarding interaction terms of order 3 to 5 in $\varepsilon$ which appear when expanding $|\tilde{Q}_b + \varepsilon|^6$, a useful estimate to treat them is

\[
\forall t \in [0, t_1), \quad \sup_{|y| \leq \frac{10}{b}} |\varepsilon(t,y)| \leq (\alpha^*)^{\frac{1}{4}}. \tag{93}
\]
Indeed, let a smooth cut off function

\[ \phi_1(t, y) = \begin{cases} 
0 & \text{for } |y| \geq \frac{1}{4} \\
1 & \text{for } |y| \leq \frac{1}{4} 
\end{cases} \]

with \( |\partial_y \phi_1|_{L^\infty} \leq C\lambda \), we estimate using the one dimensional Sobolev embeddings

\[
\sup_{|y| \leq \frac{10}{16}} |\varepsilon(t, y)|^2 \leq |\phi_1 \varepsilon|_{L^\infty}^2 \leq |\partial_y (\phi_1 \varepsilon)|_{L^2} |\phi_1 \varepsilon|_{L^2}
\]

\[
\leq \left( \int_{|y| \leq \frac{10}{16}} |\partial_y \varepsilon|^2 + \lambda |\varepsilon|^2 \right)^{\frac{1}{2}} \left( \int_{|y| \leq \frac{1}{4}} |\varepsilon|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \sqrt{\alpha^*}
\]

where we used \( \mu(y) \geq \frac{1}{3} \) for \( |y| \leq \frac{1}{4} \) from (51), and smallness estimates (53), (56) and (57). This proves (93).

We now integrate by parts the first order term in \( \varepsilon \) of formula (92) as in the one dimensional situation. The term \( \lambda^2 |E_0| \) is controlled using (54). The degeneracy of the energy of \( \tilde{Q}_b \) (28) and (89) yield the control of the \( E(\tilde{Q}_b) \) type of term.

We now turn to the proof of the crucial term \( \int |\varepsilon|^6 \mu(y) dy \) in (92) which is not a local term in space. In order to use Sobolev injections, we come back to the \( \tilde{u} \) formulation

\[
\int_{\mathbb{R}} |\varepsilon|^6 \mu(y) dy = \lambda^2 \int_{\mathbb{R}^2} |\tilde{u}|^6.
\]

We will many times use the following splitting in space: on the singularity in the region \( \frac{1}{8} \leq r \leq 8 \), we may use the one dimensional Sobolev embeddings because \( \tilde{u} \) is radial; on the other hand, in the region \( 0 \leq r \leq \frac{1}{4} \) and \( r \geq 4 \), we use the two dimensional Sobolev embeddings which are much worse, except that we now bootstrap a priori informations (58) in this region. More precisely, let two smooth radial cut off functions

\[
\chi_2(r) = \begin{cases} 
1 & \text{for } 0 \leq r \leq \frac{1}{16} \text{ and } r \geq 6 \\
0 & \text{for } \frac{1}{8} \leq r \leq 5.
\end{cases}
\]

\[
\chi_3(r) = \begin{cases} 
0 & \text{for } 0 \leq r \leq \frac{1}{64} \text{ and } r \geq 8 \\
1 & \text{for } \frac{1}{16} \leq r \leq 7.
\end{cases}
\]

First observe that

\[
\int |\tilde{u}|^6 \leq \int |\chi_2 \tilde{u}|^6 + \int |\chi_3 \tilde{u}|^6.
\]

The first term is estimated with the two dimensional Sobolev embedding

\[
\int |\chi_2 \tilde{u}|^6 \leq |\partial_r (\chi_2 \tilde{u})|_{L^2}^2 |\chi_2 \tilde{u}|_4^4 \leq C\alpha^* \left( 1 + \int |\partial_r \tilde{u}|^2 \right)
\]

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where we used $\text{Supp}(\chi_2) \subset \{x, \chi_1(x) = 1\}$, $H^{1/2}$ control (58) and global $L^2$ control (67). For the second term, we use the one dimensional Sobolev embedding according to the following Lemma which proof is standard and recalled for the sake of completeness in Appendix A.

**Lemma 4 (One dimensional type Sobolev embeddings)** Let $0 < s_1 \leq 1$, $1 \leq p_1 < \frac{1}{s_1}$. Let $0 \leq s_2 < s_1$ and
\[-s_2 + \frac{1}{p_2} = -s_1 + \frac{1}{p_1}.
\]
Let $r_0 > 0$, then there exists $C > 0$ such that for all $f \in W^{s_1,p_1}(\mathbb{R}^2)$ with radial symmetry and $\text{Supp}(f) \subset \{x, |x| > r_0\}$, we have
\[f \in W^{s_2,p_2}(\mathbb{R}^2) \quad \text{with} \quad |f|_{W^{s_2,p_2}(\mathbb{R}^2)} \leq C|f|_{W^{s_1,p_1}(\mathbb{R}^2)}.
\]
From (67) and the one dimensional Sobolev embedding $H^{\frac{1}{2}} \hookrightarrow L^6$, we estimate
\[\int |\chi_3 \tilde{u}|^6 \leq |\chi_3 \tilde{u}|_{H^\frac{1}{2}}^2 |\chi_3 \tilde{u}|_{L^2}^4 \leq (\alpha^*)^\frac{\delta}{2} \left(1 + \int |\partial_r \tilde{u}|^2\right)
\]
We thus conclude
\[\int |\tilde{u}|^6 \leq \delta(\alpha^*) \left(1 + \int |\partial_r \tilde{u}|^2\right)
\] or equivalently
\[\int |\tilde{e}|^6 \mu(y)dy \leq \delta(\alpha^*) \left(\lambda^2 + \int |\partial_y \tilde{e}|^2 \mu(y)dy\right) \leq \Gamma_b^2 + \delta(\alpha^*) \int |\partial_y \tilde{e}|^2 \mu(y)dy
\] (94)
where we used (89). This concludes the proof of (83).

Degeneracy induced by the control of the localized momentum: We now turn to the proof of (84) which is a consequence of bootstrap estimate (55). Indeed, we first compute:
\[\text{Im}\left(\int \partial_y \psi(\lambda(t)y + r(t)) \partial_y (\tilde{Q}_b + \tilde{e})(\tilde{Q}_b + \tilde{e}) \mu(y)dy\right) = \lambda(t) \text{Im}\left(\int \nabla \psi \cdot \nabla u(t) \Pi(t)\right).
\]
We expand this formula and observe from the definition of $\psi$ (41) and (89) that $\partial_y \psi(\lambda(t)y + r(t)) = 1$ for $y \in \text{Supp}(\tilde{Q}_b)$. We thus get using (28):
\[2r(t)(\tilde{e}_2, \partial_y \Sigma) = 2r(t)(\tilde{e}_1, \partial_y \Theta) + \lambda(t) \text{Im}\left(\int y|\partial_y \tilde{e}| \tilde{Q}_b + \partial_y \tilde{Q}_b \tilde{e} + \partial_y \tilde{Q}_b \tilde{e} |\partial_y \tilde{Q}_b \tilde{e}|dy\right)
\] + $\lambda(t) \text{Im}\left(\int \nabla \psi \cdot \nabla u(t) \Pi(t)\right)$.

(84) now follows from (55), localization estimate (51), smallness on $b$ (52), $L^2$ smallness estimate (67) and the control of $\lambda$ (89).
Estimates (85) and (86) are a consequence of orthogonality conditions (37), (38), (39), (40). Indeed, we take the one dimensional inner product of the \( \varepsilon \) equation with the corresponding directions, then integrate by parts and estimate the interaction terms. Similarly like when expanding the conservation the energy or the localized momentum, the key here is the support and decay properties of \( \tilde{Q}_b \) which allows treating each term in the measure \( \mu \) separately: the \( r(t) \) term yields one dimensional terms, \( \lambda(t)y \) terms yield a \( \lambda \) smallness estimated with (89). Higher order interaction terms are estimated using the one dimensional Sobolev embedding (93). The same algebra like for the proof of Lemma 3 in [19] now yields the result.

This concludes the proof of Lemma 3.

### 3.2 Control from above of the blow up speed

Our aim in this subsection is to prove bootstrap estimates (61), (62), (64), (65), (66), (67). These estimates are a consequence of the analysis in [18], [19] which led to the proof of the log-log upper bound on blow up rate in the \( L^2 \) critical case. Note that pointwise bootstrap estimate (53) will allow us to simplify part of the argument as it induces a strong decoupling between the finite dimensional dynamic of the geometrical parameters and the infinite dimensional dynamic of the excess of mass \( \varepsilon \). The bootstrap of (63) and the fundamental gain of half a derivative outside the singular circle (68) and (69) will be proved in the next subsection using techniques introduced for the proof of lower estimates on the blow up rate in [20], [21].

**step 1** \( L^2 \) smallness estimate and control of \( b \).

Recall that the conservation of the \( L^2 \) norm implies (82) which yields (62), (67).

**step 2** Proof of (64).

Let us rewrite (85) using (54) and (56)

\[
\frac{\lambda_s}{\lambda} + b < 1, \quad \left| b_s \right| < \Gamma_b^\frac{1}{2}.
\]

This yields on \([0, t_1]\)

\[
\frac{d}{ds}\{\lambda^2 e^{\frac{5\pi}{b}}\} = 2\lambda^2 e^{\frac{5\pi}{b}} \left( \frac{\lambda_s}{\lambda} + b - b - \frac{5\pi b_s}{2b^2} \right) \leq -\lambda^2 b e^{\frac{5\pi}{b}} < 0.
\]

Integrating in time, we conclude from (34) and (45): \( \forall t \in [0, t_1], \)

\[
\frac{\lambda^2(t)|E_0|}{\Gamma_b^\frac{1}{2}(t)} < \lambda^2(t)|E_0|e^{\frac{5\pi}{b(t)}} \leq \lambda^2(0)|E_0|e^{\frac{5\pi}{b(0)}} < 1,
\]

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what proves (64).

**step 3** The local virial inequality.

We now derive the local virial inequality which will provide us with the first Lyapounov type of control of the dynamics on the singular circle. The corresponding dispersive estimate will allow us to derive an upper bound on the blow up speed.

**Lemma 5 (Local virial inequality)** For all $s \in [s_0, s_1)$, there holds

$$b_s \geq \delta_0(\int |\partial_y \psi|^2 \mu(y)dy + \int_{|y| \leq \frac{10}{\Gamma_1}} |\psi| e^{-|y|} dy) - \Gamma_1^{1-C_\eta}$$

(97)

for some universal constant $\delta_0 > 0$.

**Proof of Lemma 5**

The proof is similar to the one of Lemma 8 in [20] or Proposition 3 in [19]. We first compute $b_s$ using orthogonality condition (40). This amounts taking the inner product of (76) with $\Theta_1$ and (77) with $\Sigma_1$, summing the obtain identities and then integrate by parts. This is done in full detail in Appendix C in [20]. The only difference is first for the correction term $-\lambda \mu \partial_y \psi$ in formulas (78), (79) for which we have using (91)

$$\int |y| \leq \frac{10}{\Gamma_1} \mu(y) |\psi| e^{-|y|} dy \leq C \lambda < \Gamma_1^2,$$

and corrections terms induced by the term $\lambda \frac{\eta}{\mu(y)} \partial_y \bar{Q}$ in (74) which similarly are of order $\lambda$. Second order terms in $\epsilon$ involving the geometrical parameters are estimated using (85), (86). Higher order terms all involve local interaction terms with $Q$ and are treated as in the one dimensional case up to the $\int \int \mu(y)dy$ term which is estimated from (94). The outcome is the preliminary estimate

$$\frac{|yQ|^2}{4} b_s \geq \int |\partial_y \psi|^2 \mu(y)dy + 10 \int_{|y| \leq \frac{10}{\Gamma_1}} yQ^3 \partial_y Q \psi^2 + 2 \int_{|y| \leq \frac{10}{\Gamma_1}} yQ^3 \partial_y Q \psi^2$$

$$- \frac{1}{|Q_1|^2} (\epsilon_1, L + Q_2)(\epsilon_1, Q_1) - \delta(\alpha^{*}) (\int |\partial_y \psi|^2 \mu(y)dy + \int_{|y| \leq \frac{10}{\Gamma_1}} |\psi| e^{-|y|} dy) - \Gamma_1^{1-C_\eta}.$$

(98)

Let $\phi_2$ be a cut off

$$\phi_2(t, y) = \begin{cases} 1 & \text{for } |y| \leq \Gamma_1^{5}, \\ 0 & \text{for } |y| \geq 2\Gamma_1^{5}, \end{cases}$$

with $|\partial_y \phi_2|_{L^\infty} \leq C \Gamma_1^{5}$. We may assume $\phi_2 = \tilde{\phi}_2^2$ with $\tilde{\phi}_2$ smooth and $|\partial_y \tilde{\phi}_2|_{L^\infty} \leq C \Gamma_1^{5}$. We split

$$\int |\partial_y \psi|^2 \mu(y)dy = \int \phi_2 |\partial_y \psi|^2 \mu(y)dy + \int (1 - \phi_2) |\partial_y \psi|^2 \mu(y)dy.$$
For the first term, we claim
\[ \left| \int \phi_2 |\partial_y \varepsilon|^2 \mu(y) dy - \int |\partial_y (\tilde{\phi}_2 \varepsilon)|^2 dy \right| \leq \delta(\alpha^*) \int |\partial_y \varepsilon|^2 \mu(y) dy + \Gamma_b^2. \tag{99} \]
Indeed, we first have
\[ \left| \int \phi_2 |\partial_y \varepsilon|^2 \mu(y) dy - \int \phi_2 |\partial_y \varepsilon|^2 dy \right| = \left| \int \phi_2 |\partial_y \varepsilon|^2 (\lambda(t)y + r(t) - 1) dy \right| \leq \delta(\alpha^*) \int |\partial_y \varepsilon|^2 \mu(y) dy \]
where we used (51) and (89) in the last step. Next, from (53) and (67),
\[ \left| \int \phi_2 |\partial_y \varepsilon|^2 dy - \int |\partial_y (\tilde{\phi}_2 \varepsilon)|^2 dy \right| \leq C|\partial_y \tilde{\phi}_2|L^\infty < \Gamma_b^2, \]
and (99) is proved. We thus rewrite (98)
\[ b_s \geq \tilde{H}(\tilde{\phi}_2 \varepsilon, \tilde{\phi}_2 \varepsilon) + \int (1 - \phi_2) |\partial_y \varepsilon|^2 \mu(y) dy \tag{100} \]
where the one dimensional quadratic form \( \tilde{H} \) is
\[ \tilde{H}(\varepsilon, \tilde{\varepsilon}) = \int |\partial_y \varepsilon|^2 + 10 \int yQ^3 |\partial_y Q \varepsilon|^2 + 2 \int yQ^3 |\partial_y Q \varepsilon|^2 - \frac{1}{|Q|^2}(\varepsilon_1, L + Q_2)(\varepsilon_1, Q_1). \]
The following coercivity property has been proved in Lemma 8 in [20] as a consequence of the Spectral Property stated in introduction in dimension \( N = 1 \): \( \forall \varepsilon \in H^1(\mathbb{R}) \),
\[ \tilde{H}(\varepsilon, \tilde{\varepsilon}) \geq \tilde{\delta}_0 \left( \int |\partial_y \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) - \frac{1}{\delta_0} \left\{ (\varepsilon_1, Q)^2 + (\varepsilon_1, |Q|^2) + (\varepsilon_1, yQ)^2 \right\} + (\varepsilon_2, Q_1)^2 + (\tilde{\varepsilon}_2, Q_2)^2 + (\tilde{\varepsilon}_2, \partial_y Q)^2 \]
for some universal constant \( \tilde{\delta}_0 > 0 \). Injecting orthogonality conditions (37), (38), (39), (40) and degeneracy estimates (83) and (84) into (100) now yields
\[ b_s \geq \delta(\alpha^*) \int |\partial_y (\tilde{\phi}_2 \varepsilon)|^2 dy + \int (1 - \phi_2) |\partial_y \varepsilon|^2 \mu(y) dy + \int_{|y| \leq 10^b} |\varepsilon|^2 e^{-|y|} dy \]
\[ - \delta(\alpha^*) \left( \int |\partial_y \varepsilon|^2 \mu(y) dy + \int_{|y| \leq 10^b} |\varepsilon|^2 e^{-|y|} dy \right) - \Gamma_b^1. \]
Now (99) yields (97) for \( \alpha^* > 0 \) small enough. This concludes the proof of Lemma 5.

**step 4** Estimates on the scaling parameter.

We now adapt the analysis in [19] to derive from (95) and (97) a lower bound on \( b \) corresponding to an upper bound on the blow up rate.
Lemma 6 (Upper bound on the blow up rate) Let $s_0$ given by (71). Then $\forall s \in [s_0, s_1)$,

$$b(s) \geq \frac{3\pi}{4 \log s},$$  \hspace{1cm} (101)

$$\lambda(s) \leq \sqrt{\lambda_0 e^{-\frac{3\pi}{8b}}}.$$ \hspace{1cm} (102)

and (66) holds.

Proof of Lemma 6

This is a consequence of a careful integration in time of differential inequation (97) coupled with the law for the scaling parameter (95). Indeed, from $b > 0$ on $[s_0, s_1)$ from (52), we may rewrite (97):

$$\left\{ e^{\frac{3\pi b}{4}} \right\} \leq 1 \text{ ie } e^{\frac{3\pi b}{4}s} \leq s + e^{\frac{3\pi b}{4}s_0} - s_0 = s$$

where we used (71) in the last step, and (101) follows. We now integrate the scaling law using (95) which implies

$$\frac{-\lambda_s}{\lambda} \geq \frac{2b}{3} \text{ ie } -\log \lambda(s) \geq -\log \lambda(s_0) + \int_{s_0}^{s} \frac{\pi}{2 \log \sigma} d\sigma$$

where we used (101). Note that for all $s_0 > 0$ large enough -what we may assume from (71) and (43) for $\alpha > 0$ small enough-, there holds from integration by parts

$$\int_{s_0}^{s} \frac{d\sigma}{\log \sigma} \geq \frac{2}{3} \left( \frac{s}{\log s} - \frac{s_0}{\log s_0} \right)$$

and thus

$$-\log \lambda(s) \geq -\log \lambda(s_0) + \frac{\pi}{3} \left( \frac{s}{\log s} - \frac{s_0}{\log s_0} \right).$$

Now from (46) and (71)

$$-\log \lambda(s_0) \geq e^{\frac{8\pi}{9\alpha}} = s_0^{\frac{32}{27}}$$

and thus

$$-\log \lambda(s) \geq -\frac{1}{2} \log \lambda(s_0) + \frac{\pi}{3} \frac{s}{\log s}.$$ \hspace{1cm} (103)

This first yields uniform upper bound on $\lambda$ (102). This also implies for $s_0$ large enough:

$$-\log(s \lambda(s)) \geq \frac{\pi}{3} \frac{s}{\log s} - \log s \geq \frac{\pi}{3.1 \log s},$$

and taking the log and using (101) yields

$$\log |-\log(s \lambda(s))| \geq \log \left( \frac{s}{\log s} \right) \geq \frac{\log s}{2} \geq \frac{3\pi}{8b} \text{ ie } s \lambda(s) \leq e^{-\frac{3\pi}{8b}}.$$ \hspace{1cm} (104)
which in particular implies (66). This concludes the proof of Lemma 6.

**step 5** Control of the radius of concentration.

We now turn to the proof of (61). We first have from (86) the rough estimate
\[
\left| r_s \lambda \right| \leq 1.
\]
We thus get from (102):

\[
\forall s \in [s_0, s_1),
\left| r(s) - r_0 \right| \leq \int_{s_0}^{s} |r_s| d\sigma \leq \int_{s_0}^{s} \lambda(\sigma) d\sigma \leq \sqrt{\lambda_0} \int_{2}^{+\infty} e^{-\frac{7}{2} \pi \sigma^2} d\sigma \leq \alpha^*.
\]
where the last step follows from (43) and (46). (61) now follows from (42).

**step 6** Control of the localized momentum.

We now are in position to prove (65). We first recall a classical identity obtained by multiplying (14) by \( \frac{\Delta \psi}{2} + \nabla \psi \cdot \nabla \bar{u} \) and taking the real part:
\[
\frac{1}{2} \frac{d}{dt} \text{Im} \left( \int \nabla \psi \cdot \nabla u \bar{u} \right) = \int \psi'' |\partial_t u|^2 - \frac{1}{4} \int \Delta^2 |u|^2 - \frac{1}{3} \int \Delta \psi |u|^6.
\]
From the support localization of \( \psi \) given by (41), we may use the one dimensional Sobolev embeddings of Lemma 4 and estimate:
\[
\left| \frac{d}{dt} \text{Im} \left( \int \nabla \psi \cdot \nabla u \bar{u} \right) \right| \leq C |u(t)|^2_{H^1} \leq C \lambda^2,
\]
where we used (53) and (57) in the last step. Integrating this in time, we remark from (70) that
\[
\int_{0}^{t} \frac{d\sigma}{\lambda^2(\tau)} = \int_{s_0}^{s} d\sigma \leq s
\]
and thus: \( \forall t \in [0, t_1) \),
\[
\lambda(t) \left| \text{Im} \left( \int \nabla \psi \cdot \nabla u(t) \bar{u} \right) \right| \leq \lambda(t) \left| \text{Im} \left( \int \nabla \psi \cdot \nabla u_0 \bar{u_0} \right) \right| + C\lambda(t)s(t).
\]
We now recall (104) which implies:
\[
\forall s \in [s_0, s_1), \ s\lambda(s) \leq \Gamma_{b(s)}^{10}.
\]
Next, we have from (95):
\[
\frac{d}{ds} \{ \lambda e^{\frac{6\pi}{\lambda}} \} = \lambda e^{\frac{6\pi}{\lambda}} \left( \frac{\lambda s}{\lambda} + b - b - \frac{6\pi b_s}{b^2} \right) \leq -\frac{1}{2} \lambda be^{\frac{6\pi}{\lambda}} < 0
\]
29
and thus
\[
\frac{\lambda(t)|Im(\int \nabla \psi \cdot \nabla u_0 \, d\sigma)}{\Gamma_0^{\psi}} < \lambda(t)|Im(\int \nabla \psi \cdot \nabla u_0 \, e^{\frac{it}{\Delta}})|e^{\frac{6\pi}{\lambda}} \leq \lambda_0|Im(\int \nabla \psi \cdot \nabla u_0 \, e^{\frac{it}{\Delta}})|e^{\frac{6\pi}{\lambda_0}} < 1
\]
where we used (45) in the last step. Injecting this together with (107) into (106) yields (65).

### 3.3 \(L^2\) type estimates and control from below of the blow up speed

We recall and adapt in this subsection the techniques first introduced in [20] and refined in [21] which are at the heart of the proof of the log-log lower bound on blow up rate and the key dispersive control on the reminder term \(\tilde{u}\):
\[
\int_0^T \int_{\mathbb{R}^2} |\partial_r \tilde{u}(t)|^2 \, dt < +\infty.
\]
We shall mainly recall the main steps and explain how to adapt them to our situation.

**Step 1** The radiative virial estimate.

We introduce as in [21] the radiative tale \(\zeta_b\) escaping the soliton core given by Lemma 1 expecting that the profile \(\tilde{Q}_b + \zeta_b\) will be a better approximation of the solution. Nevertheless, due to a radiative tale at infinity which induces slow decay (32), \(\zeta_b\) is never in \(L^2\), and thus the approximation that the new profile is indeed more accurate holds true only in a specific region in space, this is the radiative regime, while at infinity in space, a third regime takes place which corresponds to pure linear dispersion at infinity. To measure this, we introduce as in [22] an even cut off parameter
\[
\phi_3(y) = \begin{cases} 
1 & \text{for } 0 \leq y \leq 1 \\
0 & \text{for } y \geq 2.
\end{cases}
\]
\[
A(t) = e^{\frac{2a}{t}},
\]
for some small parameter \(0 < a << 1\) and let
\[
\tilde{\zeta} = \phi_3\left(\frac{y}{A}\right)\zeta_b.
\]
Observe that (89) and (108) ensure
\[
A << \frac{1}{\lambda}.
\]
The equation satisfied by \(\tilde{\zeta}\) is
\[
\Delta \tilde{\zeta}_b - \tilde{\zeta}_b + i b(\tilde{\zeta}_b)_1 = \Psi_b + F.
\]

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with
\[ F = \frac{1}{A^2} \Delta \phi_3 \left( \frac{y}{A} \right) \zeta_b + \frac{2}{A} \nabla \phi_3 \left( \frac{y}{A} \right) \cdot \nabla \zeta_b + i b \frac{y}{A} \cdot \nabla \phi_3 \left( \frac{y}{A} \right) \zeta_b. \]  
(109)

We then let
\[ \tilde{\varepsilon} = \varepsilon - \tilde{\zeta}. \]

Note that at the linear level, $F$ plays for $\tilde{\varepsilon}$ the role of $\Psi_b$ for $\varepsilon$.

The first step of the analysis in [21] is to derive a local virial estimate for $\tilde{\varepsilon}$ similar to (97).

**Lemma 7 (Virial dispersion in the radiative regime)** There holds for some universal constants $\delta_1, c > 0$: \[ \forall s \in [s_0, s_1), \]
\[ \{ f_1(s) \} \geq \delta_1 \left( \int \left| \partial_y \tilde{\varepsilon}(y) \right|^2 \mu(y) dy + \int_{|y| \leq \frac{b}{10}} |\tilde{\varepsilon}|^2 e^{-|y|} dy \right) + c \Gamma_b - \frac{1}{\delta_1} \int_{A \leq |y| \leq 2A} |\varepsilon|^2 dy, \]
(110)

with
\[ f_1(s) = \frac{b}{4} |y\tilde{Q}_b|^2 + \frac{1}{2} \text{Im} \left( \int y \cdot \nabla \tilde{\zeta} \right) + (\varepsilon_2, (\tilde{\zeta}_{re})_1) - (\varepsilon_1, (\tilde{\zeta}_{im})_1). \]
(111)

The proof is very similar to the one of Lemma 6 in [21] and we skip it. Let us simply say that as for the proof of Lemma 5, it relies on taking the inner product of the $\varepsilon$ equation by suitable directions build on $\tilde{Q}_b + \tilde{\zeta}$. Now as this function is supported in $B(0, 2A)$ far away from the $\frac{1}{A}$ zone in space, all corresponding interaction estimates are one dimensional estimates, up to possible $\lambda$ type of corrections induced by the metric $\mu$ which are controlled thanks to $\lambda < \Gamma_b^{10}$ from (89). Note also that the key term $+\Gamma_b$ appearing in the right hand side in (110) is obtained from a $L^2$ flux type of computation relying on exact decay rate (32), while the $L^2$ localized term $\int_A^{2A} |\varepsilon|^2 dy$ corresponds to the estimate of the linear term induced by the presence of the term $F$ given by (109) on the right hand side of the $\tilde{\varepsilon}$ equation. We refer again to [21] for the details.

**step 2** Control of the $L^2$ flux at infinity.

We now need to control the $L^2$ type of term $\int_{A \leq |y| \leq 2A} |\varepsilon|^2 dy$ in (110). This is achieved by computing the flux of $L^2$ norm escaping the radiative zone. **On the contrary to the proof of Lemma 7, this estimate requires seeing the whole space and not only the zone $r \sim 1$.** Thus one should be extra careful here and we sketch the proof in details. Let a smooth non negative even cut off function
\[ \phi_4(y) = \begin{cases} 0 & \text{for } 0 \leq y \leq \frac{1}{2}, \\ 1 & \text{for } y \geq 3, \end{cases} \]
and with $\frac{1}{4} \leq \phi_4'(y) \leq \frac{1}{2}$ for $1 \leq y \leq 2$, $\phi_4'(y) \geq 0$ for $y \geq 0$. We claim:
Lemma 8 \((L^2 \text{ dispersion at infinity in space})\) There holds for some universal constant \(C > 0\) and \(s \in [s_0, s_1]\):

\[
\left\{ \frac{1}{r_s} \int \phi_4 \left( \frac{y}{A} \right) |\varepsilon|^2 \mu(y) dy \right\}_s \geq \frac{b}{400} \int_{A \leq |y| \leq 2A} |\varepsilon|^2 - \frac{\gamma^2}{b^2} \int |\partial_y \varepsilon|^2 \mu(y) dy.
\] (112)

Proof of Lemma 8

We compute from (14):

\[
\frac{1}{2} \frac{d}{dt} \left( \frac{\phi_4 \left( \frac{y}{A} \right)}{\lambda A} \right) |u|^2 = \frac{1}{\lambda A} Im \left( \int \phi_4 \left( \frac{y}{A} \right) \partial_y \varepsilon \mu(y) dy \right) + b \int \frac{\phi_4 \left( \frac{y}{A} \right)}{A} |\varepsilon|^2 \mu(y) dy
\]

We rewrite this identity with the \(\varepsilon\) variable. Note from \(\text{Supp}(\hat{Q}_b) \subset B(0, \frac{10}{b})\) and choice of \(A\) (108) that \(\hat{Q}_b = 0\) on \(\text{Supp}(\phi_4 \left( \frac{y}{A} \right))\). We thus get the purely linear identity:

\[
\frac{1}{2} \frac{d}{ds} \int \phi_4 \left( \frac{y}{A} \right) |\varepsilon|^2 \mu(y) dy = \frac{1}{A} Im \left( \int \phi_4 \left( \frac{y}{A} \right) \partial_y \varepsilon \mu(y) dy \right) + b \int \frac{\phi_4 \left( \frac{y}{A} \right)}{A} |\varepsilon|^2 \mu(y) dy
\]

First observe from (85), (86), (108) and (53) that

\[
\left| \frac{1}{2A} \int \left[ \left( \frac{\lambda_s}{\lambda} + b + \frac{A^s}{A} \right) y + \frac{r_s}{\lambda} \right] \phi_4 \left( \frac{y}{A} \right) |\varepsilon|^2 \mu(y) dy \right| \leq \frac{b}{40} \int \phi_4 \left( \frac{y}{A} \right) |\varepsilon|^2 \mu(y) dy.
\]

Next, we estimate:

\[
\left| \frac{1}{A} Im \left( \int \phi_4 \left( \frac{y}{A} \right) \nabla \varepsilon \mu(y) dy \right) \right| \leq \frac{1}{A} \int |\nabla \varepsilon|^2 \mu(y) dy \right| \left( \int \phi_4 \left( \frac{y}{A} \right) |\varepsilon|^2 \mu(y) dy \right) \right| \leq \frac{b}{40} \int \phi_4 \left( \frac{y}{A} \right) |\varepsilon|^2 \mu(y) dy + \frac{\gamma^2}{b^2} \int |\partial_y \varepsilon|^2 \mu(y) dy.
\]

We now inject these estimates into (113) and use from the choice of \(\phi_3\) and (51) that

\[
\int \frac{\phi_4 \left( \frac{y}{A} \right)}{A} |\varepsilon|^2 \mu(y) dy \geq \frac{1}{5} \int \phi_3 \left( \frac{y}{A} \right) |\varepsilon|^2 \mu(y) dy \geq \frac{1}{30} \int_{A \leq |y| \leq 2A} |\varepsilon|^2 dy
\]

to derive so far

\[
\left\{ \int \phi_4 \left( \frac{y}{A} \right) |\varepsilon|^2 \mu(y) dy \right\}_s \geq \frac{b}{300} \int_{A \leq |y| \leq 2A} |\varepsilon|^2 - \frac{\gamma^2}{b^2} \int |\partial_y \varepsilon|^2 \mu(y) dy.
\] (114)

We now estimate

\[
\frac{r_s}{r^2} \int \phi_4 \left( \frac{y}{A} \right) |\varepsilon|^2 \mu(y) dy \leq C \lambda \int |\tilde{u}|^2 \leq \Gamma_b^2
\]

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from (67), (89) and (105). This together with (114) yields (112) and concludes the proof of Lemma 8.

**step 3** Exhibition of the Lyapounov function.

Combining estimates of Lemma 7 and Lemma 8, we recover the existence of a Lyapounov function in time which is the core of the proof of the log-log lower bound on blow up rate and the existence of the asymptotic profile \( u^* \in L^2 \).

**Proposition 3 (Lyapounov functional in \( H^1 \))** There holds for some universal constant \( C > 0 \) and for \( s \in [s_0, s_1] \):

\[
\mathcal{J}(s) \leq -Cb \left( \Gamma_b + \int |\partial_y \varepsilon|^2 \mu(y) dy + \int_{|y| \leq \frac{10}{a}} |\varepsilon|^2 e^{-|y|} dy + \int_{A \leq |y| \leq 2A} |\varepsilon|^2 \right),
\]

(115)

with

\[
\mathcal{J}(s) = \left( \int |\tilde{Q}_b|^2 - \int Q^2 + 2(\varepsilon_1, \Sigma) + 2(\varepsilon_2, \Theta) \right) + \frac{1}{r(s)} \int (1 - \phi_4 \left( \frac{y}{A} \right)) |\varepsilon|^2 \mu(y) dy
\]

\[
- \frac{\delta_1}{800} \left( b \tilde{f}_1(b) - \int_0^b \tilde{f}_1(v) dv + b \{(\varepsilon_2, (\zeta_{re})_1) - (\varepsilon_1, (\zeta_{im})_1)\} \right),
\]

(116)

where

\[
\tilde{f}_1(b) = \frac{b}{4} |y \tilde{Q}_b|^2 + \frac{1}{2} Im \left( \int y \cdot \nabla \zeta \right).
\]

(117)

**Proof of Proposition 3**

We follow the lines of the proof of Proposition 4 in [21].

Multiply (110) by \( \frac{\delta_1 b}{800} \) and sum with (112). We get:

\[
\left\{ \frac{1}{r(s)} \right\} \int \phi_4 \left( \frac{y}{A} \right) |\varepsilon|^2 \mu(y) dy + \frac{\delta_1 b}{800} \left\{ f_1 \right\}_s \geq \frac{\delta_4 b}{800} \left( \int |\partial_y \varepsilon|^2 \mu(y) dy + \int_{|y| \leq \frac{10}{a}} |\varepsilon|^2 e^{-|y|} dy \right)
\]

\[
+ \frac{b}{800} \int_{A \leq |y| \leq 2A} |\varepsilon|^2 + \frac{c_0 \delta_1 b}{1000} \Gamma_b - \Gamma_b^2 \int |\partial_y \varepsilon|^2 \mu(y) dy,
\]

(118)

\( f_1 \) given by (111). First estimate from (33)

\[
\Gamma_b^2 \int |\partial_y \varepsilon|^2 \mu(y) dy \leq \Gamma_b^2 \left( \Gamma_b^{1-C\eta} + \int |\partial_y \varepsilon|^2 \mu(y) dy \right) \leq \Gamma_b^{1+C\eta} + \Gamma_b^2 \int |\partial_y \varepsilon|^2 \mu(y) dy
\]

where the last step follows from additional constraint \( a > C\eta \)-see Remark 8 in [21]. Next, we integrate the left hand side of (118) by parts in time

\[
b \left\{ f_1 \right\}_s = \left\{ b \tilde{f}_1(b) - \int_0^b \tilde{f}_1(v) dv + b \{(\varepsilon_2, (\zeta_{re})_1) - (\varepsilon_1, (\zeta_{im})_1)\} \right\}_s
\]

\[
- b_s \left\{ (\varepsilon_2, (\zeta_{re})_1) - (\varepsilon_1, (\zeta_{im})_1) \right\},
\]

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\( \tilde{f}_1 \) given by (117). (118) now becomes using (85)

\[
\begin{aligned}
\left\{ \frac{1}{r(s)} \int \phi_4 \left( \frac{y}{A} \right) |\varepsilon|^2 \mu(y) dy + \frac{\delta_1}{800} \left[ b \tilde{f}_1(b) - \int_0^b \tilde{f}_1(v) dv + b\{(\varepsilon_2, (\tilde{\zeta}_{re})_1) - (\varepsilon_1, (\tilde{\zeta}_{im})_1)\} \right] \right\}_s \\
\geq \frac{\delta_2 b}{800} \left( \int |\partial_y \varepsilon|^2 \mu(y) dy + \int_{|y| \leq \frac{b}{|\varepsilon|}} |\varepsilon|^2 e^{-|y|/|\varepsilon|} dy + \int_{A} |\varepsilon|^2 \right) + \frac{c \delta_1 b}{1000} \Gamma_b.
\end{aligned}
\]  

(119)

Next, we inject the conservation of the \( L^2 \) norm:

\[
\int |\varepsilon|^2 \mu(y) dy + \int |\tilde{Q}_b|^2 \mu(y) dy + 2 \text{Re} \left( \int \varepsilon \tilde{Q}_b \mu(y) dy \right) = \int |u_0|^2
\]

which we rewrite

\[
\int |\varepsilon|^2 \mu(y) dy + r(t) \left( \int |\tilde{Q}_b|^2 - \int Q^2 + 2 \text{Re}(\varepsilon \tilde{Q}_b) \right) = \int |u_0|^2 - r(t) \int Q^2 - 2 \lambda \text{Re}(\varepsilon, y \tilde{Q}_b).
\]

We then write \( \int \phi_4 \left( \frac{y}{A} \right) |\varepsilon|^2 \mu(y) dy = \int |\varepsilon|^2 \mu(y) dy - \int (1 - \phi_4 \left( \frac{y}{A} \right)) |\varepsilon|^2 \mu(y) dy |\varepsilon|^2 \) and obtain

\[
\begin{aligned}
\left\{ \frac{1}{r(s)} \phi_4 \left( \frac{y}{A} \right) |\varepsilon|^2 \mu(y) dy \right\}_s &= -\left\{ \left( \int |\tilde{Q}_b|^2 - \int Q^2 + 2(\varepsilon_1, \Sigma) + 2(\varepsilon_2, \Theta) \right) \right. \\
+ \frac{1}{r(s)} \left( \int (1 - \phi_4 \left( \frac{y}{A} \right)) |\varepsilon|^2 \mu(y) dy \right)_s - 2 \left\{ \lambda \text{Re}(\varepsilon, y \tilde{Q}_b) \right\}_s - \frac{r_s}{r^2} \int |u_0|^2.
\end{aligned}
\]

We easily estimate using the \( \varepsilon \) equation, (85) and (89)

\[
\left| \left\{ \lambda \text{Re}(\varepsilon, y \tilde{Q}_b) \right\}_s \right| \leq C \lambda \leq \Gamma_b^2.
\]

Moreover, from (51), (105) and (89),

\[
\left| \frac{r_s}{r^2} \int |u_0|^2 < C \lambda \int |u_0|^2 < \Gamma_b^2.
\]

(115) now follows from (119). This concludes the proof of Proposition 3.

**step 4** Log-log lower bound on the blow up rate.

We now claim the following dispersive controls which imply the log-log lower bound on blow up rate.

**Lemma 9 (Lower bound on the blow up rate)** There holds for some universal constant \( C > 0 \): \( \forall s \in [s_0, s_1) \),

\[
b(s) \leq \frac{4\pi}{3\log s},
\]

(120)
\[
\int |\partial_y \varepsilon(s, y)|^2 \mu(y) dy + \int_{|y| \leq \frac{10}{10\text{th}}} |\varepsilon(s, y)|^2 e^{-|y|} dy \leq \Gamma_{b(s)}^{\frac{1}{2}}
\]  \hspace{1cm} (121)

and

\[
\int_{s_0}^{s} \left( \Gamma_b(\sigma) + \int |\partial_y \varepsilon(\sigma, y)|^2 \mu(y) dy + \int_{|y| \leq \frac{10}{10\text{th}}} |\varepsilon(\sigma, y)|^2 e^{-|y|} dy \right) d\sigma \leq C\alpha^*.
\]  \hspace{1cm} (122)

**Remark 3** Note that (121) is equivalent to bootstrap estimate (63).

**Proof of Lemma 9**

This lemma is a consequence of Lyapunov control (115) together with a pointwise control of the Lyapunov function $\mathcal{J}$ which is a consequence of the conservation of the energy. We adapt here the proof of Proposition 5 in [21].

We first claim:

\[
\forall s \in [s_0, s_1), \quad |\mathcal{J} - d_0 b^2| < \delta_1 b^2
\]  \hspace{1cm} (123)

for some universal constant $0 < \delta_1 << 1$. Indeed, we estimate each term in (116). Local terms are estimated from (53). The $L^2$ term is estimated from support property of $\phi_4$ and (108):

\[
\int (1-\phi_4(y/A))|\varepsilon|^2 \mu(y) dy \leq C \int_{-2A}^{2A} |\varepsilon|^2 \leq CA^2 \left( \int_{|y| \leq 3A} |\partial_y \varepsilon|^2 + \int_{|y| \leq \frac{10}{10\text{th}}} |\varepsilon|^2 e^{-|y|} dy \right) \leq \Gamma_b^{1-C\eta}.
\]

(123) follows by remarking that the universal constant $\delta_1$ which comes from estimate (110) may be chosen small with respect to $d_0$ -provided $\alpha^* > 0$ small enough-. We now prove (122). From (123), we may divide (115) by $\sqrt{\mathcal{J}}$ and integrate in time to get

\[
\int_{s_0}^{s} \left( \Gamma_b(\sigma) + \int |\partial_y \varepsilon(\sigma, y)|^2 \mu(y) dy + \int_{|y| \leq \frac{10}{10\text{th}}} |\varepsilon(\sigma, y)|^2 e^{-|y|} dy \right) d\sigma
\]

\[
\leq C \left( \sqrt{\mathcal{J}(s_0)} - \sqrt{\mathcal{J}(s)} \right) \leq Cb(s_0)
\]

and (122) follows from (43).

We now prove (120) which is a consequence of Lyapunov control (115) together with (123) which yields a differential inequation for $b$. Indeed, they imply: $\forall s \in [s_0, s_1)$,

\[
\left\{ e^\frac{\sigma}{T} \sqrt{\frac{\sigma}{\mathcal{J}(\sigma)}} \right\}_s \geq 1 \quad \text{i.e.} \quad e^\frac{\sigma}{T} \sqrt{\frac{\sigma}{\mathcal{J}(\sigma)}} \geq e^\frac{\sigma}{T} \sqrt{\frac{\sigma_0}{\mathcal{J}(\sigma_0)}} + s - s_0.
\]
We then observe from (71) and (123) that
\[
\sqrt{\frac{2^\pi}{e^r}} \sqrt{\frac{d_0}{\mathcal{J}(s_0)}} > e^{\frac{2^\pi}{e^r}} > s_0,
\]
and thus using (123) again
\[
e^{\frac{4^\pi}{e^r}} \geq s
\]
and (120) follows.

It remains to prove (121). The proof of this estimate requires a more refined estimate on the Lyapounov function \(\mathcal{J}\) than (123): \(\forall s \geq s_1,\)
\[
\mathcal{J}(s) - f_2(b(s)) \begin{cases} 
\geq -\Gamma_{b}^{1-C_a} + \frac{1}{C} \left( \int |\partial_y \varepsilon|^2 \mu(y) dy + \int_{|y| \leq \frac{10}{\Gamma} \varphi(s,y)} |\varepsilon|^2 e^{-|y|} dy \right), \\
\leq CA^2 \left( \int |\partial_y \varepsilon|^2 \mu(y) dy + \int_{|y| \leq \frac{10}{\Gamma} \varphi(s,y)} |\varepsilon|^2 e^{-|y|} dy \right) + \Gamma_{b}^{1-C_a},
\end{cases}
\tag{124}
\]
where \(f_2\) given by
\[
f_2(b) = \left( \int |\bar{Q}_b|^2 - \int Q^2 \right) - \frac{\delta_1}{800} \left( b\bar{f}_1(b) - \int_0^b \bar{f}_1(v)dv \right)
\]
satisfies
\[
0 < \frac{df_2}{db^2} |b^2=0 < +\infty.
\tag{125}
\]
Indeed, (125) is a consequence of the fact that the constant \(\delta_1 > 0\) in (110) may be chosen small with respect to \(d_0\) given by (29); (124) is a consequence of the conservation of the energy and follows as in step 2 of the proof of Proposition 5 in [21] to which we refer for the details.

Let now \(s \in [s_0, s_1]\). If \(b_s \leq 0\), then (121) follows from local virial inequality (97). If \(b_s > 0\), then let \(s_2 \in [s_0, s)\) the smallest time such that \(\forall \sigma \in (s_2, s), b_s(\sigma) > 0\). Then either \(s_2 = s_0\) or \(b_s(s_2) = 0\). In both cases, using either (44) or (97) at \(s_2\), we have for \(a > 0\) small enough
\[
b(s_2) \leq b(s) \quad \text{and} \quad \int |\partial_y \varepsilon(s_2, y)|^2 \mu(y) dy + \int_{|y| \leq \frac{10}{\Gamma} \varphi(s_2)} |\varepsilon(s_2, y)|^2 e^{-|y|} dy \leq \Gamma_{b(s_2)}^{b(s_2)}.
\tag{126}
\]
In particular, we have from (124) at \(s_2\)
\[
\mathcal{J}(s_2) - f_2(b(s_2)) \leq \Gamma_{b(s_2)}^{b(s_2)}.
\tag{127}
\]
We now use that \(\mathcal{J}\) is non increasing from (115) and sharp control (124) to derive from (127)
\[
f_2(b(s)) + \frac{1}{C} \left( \int |\partial_y \varepsilon(s, y)|^2 \mu(y) dy + \int_{|y| \leq \frac{10}{\Gamma} \varphi(s)} |\varepsilon(s, y)|^2 e^{-|y|} dy \right)(s) \leq \mathcal{J}(s) + \Gamma_{b(s)}^{1-C_a}
\leq \mathcal{J}(s) + \Gamma_{b(s)}^{1-C_a} \leq f_2(b(s_2)) + \Gamma_{b(s_2)}^{b(s)} + \Gamma_{b(s_2)}^{\frac{5}{8}},
\]

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Now from (125) and (126),
\[ f_2(b(s_2)) \leq f_2(b(s)) \quad \text{and} \quad \Gamma_{b(s_2)} \leq \Gamma_{b(s)} \]
and (121) follows for \( a > 0 \) small enough.
This concludes the proof of Lemma 9.

4 \( H^{\frac{1}{2}} \) control of the solution outside the singular circle

We prove in this section bootstrap estimates (68) and (69) what will conclude the proof of Proposition 2. We then finish the proof of Theorem 2 in the last subsection.
Our main task is to bootstrap \( H^{\frac{1}{2}} \) smallness estimate (68). Let us recall that this estimate has been the key to control the non linear term \( \int |\varepsilon|^6 \mu(y)dy \) according to (94) which was a crucial step in the proof of local virial estimate (97).

Estimate (68) is a smoothness estimate of the solution outside the singularity and corresponds to a gain of half a derivative with respect to (15). Recall that the global regularity of \( u^* \) is no better than \( L^2 \) from behavior (19) near the singularity. The proof relies on dispersive estimate (122) which will imply a uniform control:

\[ \int_0^T \int_{0 \leq r \leq \frac{3}{4}, \ r \geq \frac{4}{3}} |\partial_r u(t)|^2 dt << 1. \]

Estimate (68) will then follow from the smoothing effect for the linear Schrödinger flow which precisely implies a gain of half a derivative, together with the fact that the non linear Cauchy problem for (14) is locally well posed in \( H^{\frac{1}{2}} \) from [7].

4.1 \( H^{\frac{1}{2}} \) smallness outside zero and the singular circle

This subsection is devoted to the proof of \( H^{\frac{1}{2}} \) smallness estimate (68) outside zero. The key here is that we may use from the radial assumption and Lemma 4 the one dimensional Sobolev embeddings. Let a smooth non negative cut off function

\[ \chi_4(r) = \begin{cases} 
0 & \text{for} \quad 0 \leq r \leq \frac{1}{10} \quad \text{and} \quad \frac{3}{4} \leq r \leq \frac{4}{5} \\
1 & \text{for} \quad \frac{1}{5} \leq r \leq \frac{2}{3} \quad \text{and} \quad r \geq \frac{3}{2}.
\end{cases} \]

**Lemma 10 (\( H^{\frac{1}{2}} \) regularity outside zero)** There holds:

\[ \forall t \in [0, t_1), \quad |\chi_4 u(t)|_{H^{\frac{1}{2}}} < \sqrt{\alpha^*}. \quad (128) \]

**Proof of Lemma 10**
**step 1** Time averaged $H^1$ regularity outside the singularity.

Let a smooth non-negative cut off function

$$\chi_5(r) = \begin{cases} 1 & \text{for } 0 \leq r \leq \frac{10}{11} \text{ and } r \geq \frac{11}{11} \\ 0 & \text{for } \frac{12}{13} \leq r \leq \frac{13}{12} \end{cases}.$$  

We first claim the fundamental estimate

$$\int_0^{t_1} |\chi_5 u(t)|_{H^1}^2 dt < C\alpha^* \tag{129}$$

for some universal constant $C > 0$. Indeed, from (56) and (108),

$$u(t, r) = \tilde{u}(t, r) = \frac{1}{\lambda_2(t)} \tilde{r}(t, r - r(t)) e^{i\gamma(t)} \text{ on Supp}(\chi_5).$$

Using now (50) and $\frac{ds}{dt} = \frac{1}{\lambda_2}$, we have

$$\int_0^t |\chi_5 u(\tau)|_{H^1}^2 d\tau = \int_0^s |\chi_5 \tilde{u}(\tau)|_{H^1}^2 d\tau \leq C \int_{s_0}^s \int |\partial_y \tilde{z}(s)|^2 \mu(y) dy ds + C \int_0^t |\tilde{u}(\tau)|_{L^2}^2 d\tau \leq C(\alpha^* + t_1(\alpha^*)^\frac{2}{3})$$

where we used (67) and (122) in the last step. Observe now from (102) that

$$t_1 = \int_{s_0}^{s_1} \lambda^2(s) ds \leq C\lambda_0 \int_2^{+\infty} e^{-\frac{3}{2} \mu^*} ds < \alpha^*, \tag{130}$$

from (43) and (46), and (129) follows.

**step 2** $H^\nu$ regularity outside zero and the singularity for $0 < \nu < \frac{1}{2}$.

Let us enlarge a little the support of $\chi_4$ and consider

$$\tilde{\chi}_4(r) = \begin{cases} 0 & \text{for } 0 \leq r \leq \frac{1}{12} \text{ and } \frac{4}{5} \leq r \leq \frac{5}{4} \\ 1 & \text{for } \frac{1}{11} \leq r \leq \frac{3}{4} \text{ and } r \geq \frac{4}{3}. \end{cases}$$

and set

$$v = \tilde{\chi}_4 u,$$

then $v$ satisfies

$$i\partial_t v + \Delta v = 2\partial_r \tilde{\chi}_4 \partial_r u + u \Delta \tilde{\chi}_4 - v|u|^4. \tag{131}$$

Let now $0 < \nu < \frac{1}{2}$, we claim a uniform smallness estimate for the $H^\nu$ norm

$$\forall t \in [0, t_1), \quad |v|_{H^\nu} \leq C_\nu \sqrt{\alpha^*}. \tag{132}$$
Proof of (132): Let $D^\nu$ denote the Fourier multiplier $\hat{D^\nu v}(\xi) = |\xi|^\nu \hat{v}(\xi)$. We compute from (131):

\[ \frac{1}{2} \left| \frac{d}{dt} |D^\nu v|^2_{L^2} \right| = |Re(\partial_t v, \overline{D^\nu v})| \leq C \left| Im(\partial_t \chi_4 \partial_t u + u \Delta \chi_4, \overline{D^\nu v}) \right| + C |D^\nu v|_{L^2} |D^\nu (|v|^4)|_{L^2} \]

\[ \leq C |\chi_5 u|^2_{H^1} + C |D^\nu v|_{L^2} |D^\nu (|v|^4)|_{L^2}. \quad (133) \]

To estimate the non linear term, let a cut off

\[ \chi_6(r) = \begin{cases} 0 & \text{for } 0 \leq r \leq \frac{1}{2} \text{ and } \frac{8}{5} \leq r \leq \frac{9}{5} \\ 1 & \text{for } \frac{1}{18} \leq r \leq \frac{7}{8} \text{ and } r \geq \frac{8}{9}. \end{cases} \]

First observe that $\chi_6 = 1$ on $\operatorname{Supp}(\chi_4)$ and $|\chi_6 u|_{H^1} \leq C |\chi_5 u|_{H^1}$. Let us now recall the following standard commutation estimate, see [13].

**Lemma 11 (Commutation estimates)** Let $0 < \nu < 1$ and $1 < p < +\infty$, then

\[ |D^\nu (fg)|_{L^p} \leq C \left( |f|_{L^p} |D^\nu g|_{L^q} + |D^\nu f|_{L^p} |g|_{L^q} \right) \quad (134) \]

with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$, $1 < p_2, p_3 < +\infty$.

We now let

\[ q = \frac{1}{\nu} > 2 \text{ and } \frac{1}{p} = \frac{1}{2} - \frac{1}{q} \]

and estimate from (134)

\[ |D^\nu (|v|^4)|_{L^2} = |D^\nu (|\chi_6 u|^4)|_{L^2} \leq C \nu \left( |D^\nu v|_{L^2} |\chi_6 u|^4_{L^\infty} + |v|_{L^p} |D^\nu (|\chi_6 u|^4)|_{L^q} \right). \quad (135) \]

For the first term, we have Strauss’ radial interpolation inequality

\[ |\chi_6 u|_{L^\infty} \leq C |\partial_t (\chi_6 u)|^\frac{1}{2}_{L^2} |\chi_6 u|^\frac{1}{2}_{L^2} \]

and thus using global $L^2$ bound (67),

\[ |D^\nu v|_{L^2} |\chi_6 u|^4_{L^\infty} \leq C |v|_{H^\nu} |\chi_5 u|^2_{H^1}. \quad (137) \]

For the second term, we first use iteratively (134) to derive

\[ |D^\nu (|\chi_6 u|^4)|_{L^p} \leq C |\chi_6 u|^3_{L^\infty} |\chi_6 u|_{W^{\nu, q}}. \quad (138) \]

We now use Lemma 4 with the one dimensional Sobolev embeddings $H^\nu \hookrightarrow L^p$ from $\nu < \frac{1}{2}$, $H^\frac{1}{2} \hookrightarrow W^{\nu, q}$, (136) and (138) to estimate the second term in (135) as

\[ |v|_{L^p} |D^\nu (|\chi_6 u|^4)|_{L^q} \leq C_{\nu} |v|_{H^\nu} |\chi_6 u|^3_{L^\infty} |\chi_6 u|_{W^{\nu, q}} \leq C_{\nu} |v|_{H^\nu} |\chi_6 u|^3_{L^\infty} |\chi_6 u|_{H^\frac{1}{2}} \leq C_{\nu} |v|_{H^\nu} |\chi_5 u|^2_{H^1} \]

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where we used $L^2$ bound (67). We thus conclude together with (135) and (137)
\[
|D^\nu (v|u|^4)|_{L^2} \leq C_\nu |v|_{H^\nu} |\chi_5 u|^2_{H^1}.
\]
Injecting this into (133) together with global $L^2$ bound (67) yields
\[
\frac{d}{dt} |D^\nu v(t)|_{L^2}^2 \leq C_\nu \left(1 + |D^\nu v(t)|_{L^2}^2\right) |\chi_5 u(t)|_{H^1}^2.
\]
From Gronwall’s lemma and (129), we get
\[
|D^\nu v(t)|_{L^2}^2 \leq C_\nu \left(|D^\nu v(0)|_{L^2}^2 + \int_0^t |\chi_5 u|^2_{H^1} dt\right) \leq C_\nu \alpha^s,
\]
from (48) and (129). This concludes the proof of (132).

**step 3** Proof of (128).

Let
\[
w = \chi_4 u.
\]
We argue similarly as for the previous step with $\nu = \frac{1}{2}$ and get as for (133)
\[
\frac{d}{dt} |D^{\frac{1}{2}} w(t)|_{L^2}^2 \leq C \left(|\chi_5 u|^2_{H^1} + |D^{\frac{1}{2}} w|_{L^2} |D^{\frac{1}{2}} (w|u|^4)|_{L^2}\right).
\]
To estimate the last term, we may no longer use the one dimensional Sobolev estimate $H^\nu \hookrightarrow L^p$ for $\frac{1}{p} = \frac{1}{2} - \nu$ which holds only for $\nu < \frac{1}{2}$. But we now may use (132). Estimate from (134):
\[
|D^{\frac{1}{2}} (w|u|^4)|_{L^2} \leq C \left(|w|_{H^{\frac{3}{2}}} |\chi_6 u|^4_{L^\infty} + |w|_{L^6} |D^{\frac{1}{2}} (|\chi_4 u|^4)|_{L^3}\right) \\
\leq C \left(|w|_{H^{\frac{3}{2}}} |\chi_6 u|^4_{L^\infty} + |w|_{L^6} |\tilde{\chi}_4 u|^3_{L^\infty} |\chi_4 u|_{W^{\frac{1}{2}, 3}}\right).
\]
We now use Strauss’ radial interpolation inequality (136), $L^2$ bound (67) and Lemma 4 with the one dimensional Sobolev embeddings $H^{\frac{5}{4}} \hookrightarrow L^{\frac{8}{5}}, H^{\frac{4}{5}} \hookrightarrow W^{\frac{1}{2}, 3}$, to get
\[
|w|_{L^6} |\tilde{\chi}_4 u|^3_{L^\infty} |\chi_4 u|_{H^{\frac{1}{2}}} \leq C |w|_{H^{\frac{5}{4}}} |\chi_5 u|^2_{H^1} |\tilde{\chi}_4 u|^\frac{1}{2}_{H^{\frac{5}{4}}} \leq C |w|_{H^{\frac{5}{4}}} |\chi_5 u|^2_{H^1} |\tilde{\chi}_4 u|^\frac{1}{2}_{H^{\frac{5}{4}}}.
\]
Using (132) with $\nu = \frac{1}{4}$ and (67), we conclude
\[
\frac{d}{dt} |D^{\frac{1}{2}} w(t)|_{L^2}^2 \leq C \left(1 + |D^{\frac{1}{2}} w(t)|_{L^2}^2\right) |\chi_5 u|^2_{H^1} + C |w|_{H^{\frac{5}{4}}} |\chi_5 u|^2_{H^1} |\tilde{\chi}_4 u|^\frac{1}{2}_{H^{\frac{5}{4}}} \\
\leq C \left(1 + |D^{\frac{1}{2}} w(t)|_{L^2}^2\right) |\chi_5 u|^2_{H^1},
\]
and (128) now follows from Gronwall lemma, (48) and (129). This concludes the proof of Lemma 10.
4.2 \( H^{\frac{1}{2}} \) smallness on the origin

We now turn to the proof of the \( H^{\frac{1}{2}} \) smallness of \( u \) at the origin which together with (128) concludes the proof of bootstrap estimate (68). We will also prove (69) and thus finish the proof of Proposition 2. Note that in term of Sobolev embedding, the region near the origin clearly is the place where the problem is truly \( L^2 \) super critical. We will use two main ingredients: a priori estimate (129) together with the smoothing effect for the linear Schrödinger flow will allow us to gain half a derivative on the linear type of terms induced by the cut off in space -which is required to avoid the singularity of \( u^* \) on the circle according to (20)-; the quintic non linear term will then be treated using a \( H^{\frac{1}{2}} \) critical estimate at the heart of the critical local wellposedness theory of [7].

Let
\[ f = \chi_1 u, \]
and consider the norm:
\[
|f|_X = |f(t)|_{L^\infty([0,t_1],H^{\frac{1}{2}})} + |f(t)|_{L^6([0,t_1],B^{\frac{1}{2}}_{2,2})}. \tag{140}
\]
We claim
\[
|f|_X < (\alpha^*)^{\frac{3}{8}}, \tag{141}
\]
which together with (128) concludes the proof of bootstrap estimates (68) and (69).

**Proof of (141)**

\( f \) satisfies the following equation:
\[
i\partial_t f + \Delta f = 2\partial_r \chi_1 \partial_r u + u \Delta \chi_1 + (\chi_5^1 - \chi_1)u|u|^4 - f|f|^4
\]
which we rewrite using Duhamel’s formula:
\[
f(t) = e^{it\Delta} f(0) + \int_0^t e^{i(t-\tau)\Delta} u \Delta \chi_1 + \int_0^t e^{i(t-\tau)\Delta} (\chi_5^1 - \chi_1)u|u|^4
\]
\[
+ 2 \int_0^t e^{i(t-\tau)\Delta} \partial_r \chi_1 \partial_r u - \int_0^t e^{i(t-\tau)\Delta} f|f|^4. \tag{142}
\]
Let us recall the following well known properties of the linear Schrödinger flow.

**Proposition 4 (Strichartz and smoothing for \( e^{it\Delta} \))** We say \((q,r)\) is an admissible pair iff
\[
\frac{2}{q} = 1 - \frac{2}{r}, \quad 2 \leq r < \infty.
\]
Then for all admissible pairs \((q,r), (\gamma, \rho)\), there exists a constant \(C > 0\) such that for any time interval \(I\) containing zero, we have the following:

(i) **Homogeneous Strichartz estimates:**

\[
\left| e^{it\Delta}u_0 \right|_{L^q(I, \dot{B}^{\frac{1}{2}}_{r,2})} \leq C|u_0|_{H^{\frac{1}{2}}}. \tag{143}
\]

(ii) **Inhomogeneous Strichartz estimate and smoothing effect:** Let \(u\) be the solution to the inhomogeneous linear Schrödinger equation

\[
\begin{aligned}
  iu_t + \Delta u &= F, \\
  u_{t=0} &= 0,
\end{aligned}
\]

then

\[
|u(t)|_{L^q(I, \dot{B}^{\frac{1}{2}}_{r,2})} \leq C|F|_{L^{\gamma'}(I, B^{\frac{1}{2\rho'}}_{\rho',2})} \tag{144}
\]

with \(\gamma' = \frac{\gamma}{\gamma - 1}\), \(\rho' = \frac{\rho}{\rho - 1}\). Assume moreover that

\[
\forall t \in I, \quad \text{Supp}(F(t)) \subset B(0, 10),
\]

then we have the smoothing estimate

\[
|u|_{L^q(I, \dot{B}^{\frac{1}{2}}_{r,2})} \leq C|F|_{L^2(I, L^2)}. \tag{145}
\]

(143), (144) are the standard Strichartz estimates, see [25] and [11]. This version with Besov spaces can be found in [5], Corollary 2.3.9 p39. For the sake of completeness, we briefly recall the proof of smoothing effect (145) in Appendix B.

We now estimate each term in Duhamel’s formula (142) in the \(X\) norm defined by (140). Let us indeed observe that (6, 3) is an admissible pair.

The first term is estimated using (47) and (143):

\[
|e^{it\Delta}f(0)|_X \leq |f(0)|_{H^{\frac{1}{2}}} \leq \alpha^*. \tag{146}
\]

For the second term, first remark that \(\text{Supp}\Delta \chi_1 \subset \{x, \chi_4(x) = 1\}\), and we estimate from uniform \(H^{\frac{3}{2}}\) bound (128) and (144)

\[
\left| \int_0^t e^{i(t-\tau)\Delta} u\Delta \chi_1 \right|_X \leq C|u\Delta \chi_1|_{L^1([0,t_1), H^{\frac{3}{2}})} \leq Ct_1|u\Delta \chi_1|_{L^\infty([0,t_1), H^{\frac{1}{2}})} \leq C\alpha^* \tag{147}
\]

where we also used (130) in the last step. For the third term, we similarly have

\[
\left| \int_0^t e^{i(t-\tau)\Delta}(\chi_4^5 - \chi_1)u^4 \right|_X \leq \left| (\chi_4^5 - \chi_1)u^4 \right|_{L^1([0,t_1), H^{\frac{1}{2}})}.
\]
Now from $\text{Supp}(\chi_5^5 - \chi_1) \subset \{ x, \chi_4(x) = 1 \}$, we use Lemma 11 and Strauss’ radial interpolation inequality (136) to estimate
\[
|(\chi_5^5 - \chi_1)u||u|^4|_{H^{\frac{1}{2}}} \leq C|\chi_5 u|^4|_{L^\infty} |\chi_4 u|_{H^{\frac{1}{2}}} \leq C\sqrt{\alpha^*} |\chi_5 u|^2_{H^1}
\]
where we used uniform $L^2$ bound (67) and $H^{\frac{1}{2}}$ control (128). We thus have
\[
\left| \int_0^t e^{i(t-\tau)\Delta} (\chi_5^5 - \chi_1)u|u|^4 \right|_X \leq C\sqrt{\alpha^*} \int_0^t |\chi_5 u(\tau)|^2_{H^1} d\tau < \alpha^* \tag{148}
\]
where we used (129) in the last step.

From $\text{Supp}(\partial_r \chi_1) \subset \{ x, \chi_5(x) = 1 \}$, the fourth term may be estimated using smoothing effect (145) which gives
\[
\left| \int_0^t e^{i(t-\tau)\Delta} \partial_r \chi_1 \partial_r u \right|_X \leq C|\partial_r \chi_1 \partial_r u|_{L^2([0,t_1],L^2)} \leq C \left( \int_0^{t_1} |\chi_5 u|^2_{H^1} \right)^{\frac{1}{2}} < C \sqrt{\alpha^*} \tag{149}
\]
where we used (129) in the last step.

For the last term, we argue as in [7]. First use (144) to estimate
\[
\left| \int_0^t e^{i(t-\tau)\Delta} f|f|^4 \right|_X \leq |f| |f|^4|_{L^5([0,t_1],B^{\frac{1}{2},3}_{2,2})}.
\]

Next, we recall the non linear estimate
\[
|f| |f|^4|_{B^{\frac{1}{2},2}_{2,2}} \leq C|f|^4|_{L^{12}} |f|^\frac{1}{2}_{B^{\frac{1}{2},2}_{2,2}}.
\]

which is a simple consequence of (59), see [5], Prop. 4.9.4. p126. From the two dimensional Sobolev injection $B^{\frac{1}{2},2}_{2,2} \hookrightarrow L^{12}$, this yields
\[
|f| |f|^4|_{B^{\frac{1}{2},2}_{2,2}} \leq C|f|^5|_{B^{\frac{3}{2},2}_{2,2}}.
\]

We thus conclude using bootstrap estimate (60):
\[
\left| \int_0^t e^{i(t-\tau)\Delta} f|f|^4 \right|_X \leq C|f|^5|_{L^6([0,t_1],B^{\frac{1}{2},2}_{2,2})} \leq \frac{1}{2} |f|_X. \tag{150}
\]

Putting together (146), (147), (148), (149) and (150) yields $|f|_X \leq C\sqrt{\alpha^*}$ and concludes the proof of (141).

This concludes the proof of Proposition 2.
4.3 Conclusion of the proof of Theorem 2

We now derive blow up result of Theorem 2 as a consequence of the estimates of Proposition 2.

**Proof of Theorem 2**

**step 1** Finite time blow up and log-log speed.

First observe from Proposition 2 that \( t_1 = T \), and thus from (130),

\[
T < +\infty
\]

and \( u(t) \) blows up in finite time. From the local wellposedness theory in \( H^1 \) and the uniform weighted \( H^1 \) bound on \( \varepsilon \) (63) and (67), this implies

\[
\lambda(t) \to 0 \text{ as } t \to T. \tag{151}
\]

Observe also from (62) and (95) that \( \left| \frac{\Delta x}{\alpha} \right| < 1 \) on \([s_0, s_1]\). This implies

\[
\forall s \in [s_0, s_1), \quad |\log \lambda(s)| \leq C(1 + s).
\]

Now from (151), \( \lambda(s) \to 0 \text{ as } s \to s_1 = s(t_1) \), and thus

\[
s_1 = +\infty.
\]

We now claim that \( \lambda \) satisfies for \( t \) close enough to \( T \) the pointwise differential inequation

\[
\frac{1}{C} \leq -\left( \lambda^2 \log |\log \lambda| \right)_t \leq C \tag{152}
\]

for some universal constant \( C > 0 \).

Proof of (152): First observe from (95) that

\[
\frac{b}{2} \leq -\frac{\lambda s}{\lambda} \leq 2b. \tag{153}
\]

Next, we have for \( t \) close to \( T \),

\[
\left( \lambda^2 \log |\log (\lambda)| \right)_t = -\lambda \lambda_t \log |\log (\lambda)| \left( 2 + \frac{1}{|\log (\lambda)| \log |\log (\lambda)|} \right) = -\frac{\lambda s}{\lambda} \log |\log (\lambda)| \left( 2 + \frac{1}{|\log (\lambda)| \log |\log (\lambda)|} \right)
\]

and thus (152) follows from

\[
\frac{1}{C} \leq b \log |\log \lambda| \leq C.
\]
The lower bound follows from (66). For the upper bound, we first have from (153) and (120) for \( s \) large enough
\[
-\log \lambda(s) \leq C + 2 \int_{s_0}^{s} b \leq C + \frac{8\pi}{3} \int_{s_0}^{s} \frac{d\sigma}{\log \sigma} \leq \frac{16\pi}{3} \frac{s}{\log s}.
\]
Taking the log of this inequality and reinjecting (120) yields
\[
\log |\log \lambda(s)| \leq 2\log s \leq \frac{8\pi}{3b}
\]
and the upper bound in (152) follows.

Integrating differential inequation (152) together with (151) yields
\[
\frac{T - t}{C} \leq \lambda^2(t) \log |\log(\lambda(t))| \leq C(T - t)
\]
and an estimate
\[
\frac{1}{C} \left( \frac{T - t}{\log |\log(T - t)|} \right)^\frac{1}{2} \leq \lambda(t) \leq C \left( \frac{T - t}{\log |\log(T - t)|} \right)^\frac{1}{2} \tag{154}
\]
for \( t \) close enough to \( T \) follows. From \( \frac{ds}{dt} = \frac{1}{\lambda^2} \), this implies
\[
\frac{1}{C} |\log(T - t)| \leq s \leq C |\log(T - t)|
\]
and thus from (101) and (120), there holds for \( t \) close enough to \( T \)
\[
\frac{1}{C \log |\log(T - t)|} \leq b \leq \frac{C}{\log |\log(T - t)|}. \tag{155}
\]
The proof of exact convergence (18) requires further simple technical refinements like for the proof of the same estimate in [21] and is left to the reader -see Proposition 6 in [21]-.

**step 2** Convergence of the radius of concentration.

The convergence of the radius of concentration (16) is a consequence of (105) which implies
\[
\int_{0}^{T} |r_t| dt = \int_{0}^{T} \frac{dr}{\lambda} \leq \int_{0}^{T} \frac{dt}{\lambda} < +\infty
\]
where the last step follows from (154).

**step 3** Uniform \( H^{\frac{1}{2}} \) bound on \( \tilde{u} \) outside the singular circle.
Let $0 < R < \frac{1}{10}$. We claim that there exists $C(R) > 0$ such that

$$
\forall t \in [0, T), \quad |\tilde{u}(t)|_{H^\frac{1}{4}(|r-r(T)| \geq \frac{R}{4})} < C(R). \tag{156}
$$

Proof of (156): First observe that a uniform $H^\frac{1}{4}$ bound on $\tilde{u}$ holds in the region $R^2/\{ \frac{1}{10} \leq r \leq 4 \}$ from (68). Next, recall that we now have proved finite time blow up and the convergence of the radius of concentration (16). Thus, from (66) and (108), there exists $t(R) \in [0, T)$ such that for all $t \in [t(R), T)$,

$$
\int_0^T \int_{|r-r(T)| \geq \frac{R}{4}} |\partial_r u(\tau)|^2 d\tau = \int_0^T \int_{|r-r(T)| \geq \frac{R}{4}} |\partial_r \tilde{u}(\tau)|^2 d\tau \leq \int_0^{\infty} \int |\partial_y \tilde{z}(s)|^2 \mu(y) dy \leq C \alpha^s
$$

where the last step follows from (122). We conclude:

$$
\forall R \in (0, \frac{1}{10}), \quad \int_0^T \int_{|r-r(T)| \geq \frac{R}{4}} |\partial_r u(t)|^2 < C(R). \tag{157}
$$

Let now a radial cut off function

$$
\phi_R(r) = \phi\left(\frac{r-r(T)}{R}\right) \quad \text{where} \quad \phi(x) = \begin{cases} 
0 & \text{for } |x| \leq \frac{1}{2} \\
1 & \text{for } |x| \geq 1.
\end{cases} \tag{158}
$$

Let $w_R = \phi_R u$, it is an easy task to adapt the proof of estimate (128) to derive similarly as (139): $\forall t \in [t(R), T)$,

$$
\left| \frac{d}{dt} |D^\frac{1}{2} w_R(t)|^2_{L^2} \right| \leq C(R)(1 + |D^\frac{1}{2} w_R(t)|^2_{L^2}) |u|^2_{H^1(|r-r(T)| \geq \frac{R}{4})}.
$$

(156) now follows from Gronwall lemma and (157).

**step 4** $L^2$ convergence of $\tilde{u}$ outside the singular circle.

Let $R > 0$. Let us prove $L^2$ convergence (15) outside the singular circle, ie

$$
\tilde{u}(t) \rightarrow u^* \quad \text{in} \quad L^2(|r-r(T)| \geq R). \tag{159}
$$

We follow the proof of Proposition 9 in [22] and claim as a consequence of (156) that $\tilde{u}$ is a Cauchy sequence in $L^2(|r-r(T)| \geq R)$. Indeed, pick a small $\varepsilon_0 > 0$. We may assume from (157) that $t(R)$ is close enough to $T$ so that

$$
\int_{t(R)}^T \left( \int_{|r-r(T)| \geq \frac{R}{4}} |\partial_r u(t)|^2 \right) dt < \varepsilon_0. \tag{160}
$$

Next, given a fixed parameter $\tau > 0$, let $v^\tau(t, x) = u(t+\tau, x) - u(t, x)$. $u(t)$ is strongly continuous in $L^2$ at time $t(R)$, so there exists $\tau_0 > 0$ such that

$$
\forall \tau \in [0, \tau_0], \quad \int |v^\tau(t(R))|^2 < \varepsilon_0. \tag{161}
$$
We now claim:

\[
\forall \tau \in [0, \tau_0], \ \forall t \in [t(R), T - \tau), \ \int_{|r - r(T)| \geq \frac{\tau}{2}} |v^\tau(t)|^2 < C\varepsilon_0,
\]

(162)

for some universal constant \( C > 0 \) which yields the claim.

Proof of (162): \( v^\tau(t, x) \) satisfies:

\[
iv^\tau_t + \Delta v^\tau = - \left( u|u|^4(t + \tau) - u|u|^4(t) \right).
\]

Let the cut off function \( \phi_R \) given by (158), we compute:

\[
\frac{1}{2} \left( \int \phi_R |v^\tau|^2 \right)_t = Im \left( \int \partial_r \phi_R \partial_r v^\tau \overline{v^\tau} \right).
\]

From Cauchy-Schwarz and the conservation of the \( L^2 \) norm:

\[
\left| \int \phi_R \partial_r v^\tau \overline{v^\tau} \right| \leq C(R) \left( 1 + \int_{|r - r(T)| \geq \frac{\tau}{2}} |\partial_r u(t)|^2 + |\partial_r u(t + \tau)|^2 \right).
\]

For the second term, we first have by homogeneity:

\[
|v^\tau| \left( |u(t)|^5 + |u(t + \tau)|^5 \right) \leq C(|u(t)|^6 + |u(t + \tau)|^6).
\]

Then, we may assume cut off function \( \phi \) satisfies \( \phi_R = \tilde{\phi}_R^6 \) with \( \tilde{\phi}_R \) regular, and thus from Gagliardo-Nirenberg inequality and the uniform control of the \( H^{1/2} \) norm (156):

\[
\int \phi_R |u|^6 = \int |\tilde{\phi}_R u|^6 \leq C|\partial_r (\tilde{\phi}_R u)|^2_{L^2} |\tilde{\phi}_R u|^4_{H^{1/2}} \leq C(R) \left( 1 + \int_{|r - r(T)| \geq \frac{\tau}{2}} |\partial_r u(t)|^2 + |\partial_r u(t + \tau)|^2 \right).
\]

We thus conclude: \( \forall \tau \in [0, \tau_0), \ \forall t \) with \( [t, t + \tau] \in [t(R), T) \),

\[
\left( \int \phi_R |v^\tau|^2 \right)_t \leq C(R) \left( 1 + \int_{|r - r(T)| \geq \frac{\tau}{2}} |\partial_r u(t)|^2 + |\partial_r u(t + \tau)|^2 \right).
\]

Integrating this in time with controls (160) and (161) now yields (162).

**step 5** Non concentration of \( \tilde{u} \) on the singular circle.

We follow the proof of section 4.3 in [22]. Let

\[
R(t) = A(t)\lambda(t)
\]

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with $A(t)$ given by (108) so that $R(t) \to 0$ as $t \to T$ from (154) and (155). Note from (108) and (155) that there holds for some constant $C > 0$

$$A(t) \leq \frac{1}{|\log(T-t)|^C}. \quad (163)$$

Let $\phi$ be given by (158). We compute:

$$\begin{align*}
\frac{1}{2} \left| \int_\Gamma \phi \left( \frac{r - r(t)}{R(t)} \right) |u|^2 \right|_\tau & = \frac{1}{R(t)} \left| \int_\Gamma \phi \left( \frac{r - r(t)}{R(t)} \right) \cdot \partial_r u \right|_\tau \\
- \frac{r_T}{2} \int_\Gamma \phi \left( \frac{r - r(T)}{R(t)} \right) |u(t)|^2 & \leq \frac{C}{A(t)} \left( \int_\Gamma |\partial_r u|^2 \right)^{\frac{1}{2}},
\end{align*}$$

where we used (105) in the last step. We integrate this in time and use (154) and (159) to derive

$$\begin{align*}
\left| \int_\Gamma \phi \left( \frac{r - r(t)}{R(t)} \right) |u|^2 \right| - \int_\Gamma \phi \left( \frac{r - r(t)}{R(t)} \right) |u(t)|^2 & \leq \frac{C}{A(t)} \int_\Gamma |\partial_r u|^2 |d\tau |
\leq \frac{C}{|\log(T-t)|^C} \int_t^T |\partial_r u|^2 |d\tau |.
\end{align*}$$

Letting $t \to T$, we conclude

$$\int |u|^2 = \lim_{t \to T} \int_\Gamma \phi \left( \frac{r - r(t)}{R(t)} \right) |u(t)|^2. \quad (164)$$

We now claim

$$\lim_{t \to T} \int \phi \left( \frac{r - r(t)}{R(t)} \right) |u(t)|^2 = \int_{\Gamma} \int_\Gamma \phi \left( \frac{r - r(t)}{R(t)} \right) |u(t)|^2 = \lim_{t \to T} \int |\tilde{u}(t)|^2 dt, \quad (165)$$

which concludes the proof of $L^2$ convergence (15).

Proof of (165): We first have from (49)

$$\int_{|r-r(t)| \leq 2R(t)} |\tilde{u}(t)|^2 = \int_{|y| \leq \frac{2}{\pi(t)}} |\varepsilon(t, y)|^2 \mu(y) dy \leq 2 \int_{|y| \leq \frac{2}{\pi(t)}} |\varepsilon(t, y)|^2 dy$$

where we used (66) and (108) in the last step. Moreover, we have the following Sobolev type estimate as observed in [20],

$$\forall D \geq 1, \int_{|y| \leq D} |\varepsilon(y)|^2 dy \leq C D^2 \left( \int_{|y| \leq 2D} |\partial_y \varepsilon(y)|^2 dy + \int_{|y| \leq 1} |\varepsilon(y)|^2 e^{-|y|} \right)$$

for some universal constant $C > 0$. We thus conclude from pointwise estimate (121)

$$\int_{|r-r(t)| \leq 2R(t)} |\tilde{u}(t)|^2 \leq C A^2(t) \left( \int |\partial_y \varepsilon(t, y)|^2 \mu(y) dy + \int_{|y| \leq \frac{2}{\pi(t)}} |\varepsilon| e^{-|y|} dy \right) \leq \Gamma^2 \delta(t),$$

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Now $\Gamma_{b(t)} \to 0$ as $t \to T$ from (155) what concludes the proof of (165).

Last, estimate (17) is a straightforward consequence of the uniform weighted $H^1$ bound on $\varepsilon$ from (63) and (67).

This concludes the proof of Theorem 2.

**Appendix A: Proof of Lemma 4**

The proof follows similarly as in [16] and we briefly sketch it for the sake of completeness. Let $0 < s_1 \leq 1$ and $1 \leq p_1 < \frac{1}{s_1}$, $\frac{1}{q_1} = -s_1 + \frac{1}{p_1}$. First observe that the spaces $W^{s_2,p_2}$, $0 \leq s_2 < s_1$, $-s_2 + \frac{1}{p_2} = -s_1 + \frac{1}{p_1}$ can be obtained by complex interpolation between $L^{q_1}$ and $W^{s_1,p_1}$, see [2]. It thus suffices to prove the Sobolev injection with $s_2 = 0$, $p_2 = q_1$. Let $\phi \in C_0^\infty(\mathbb{R}^2)$ with radial symmetry and $\text{Supp}(\phi) \subset \mathbb{R}^2 / B(0,1)$, we need to prove

$$|\phi|_{L^{q_1}(\mathbb{R}^2)} \leq C|\phi|_{W^{s_1,p_1}(\mathbb{R}^2)}.$$ (166)

We first claim

$$|\phi|_{W^{s_1,p_1}(\mathbb{R}^2)} \sim |r^{\frac{1}{p_1}} \phi|_{W^{s_1,p_1}(1, +\infty)}.$$ (166)

Assume (166), then observe that

$$|\phi|_{L^{q_1}(\mathbb{R}^2)} = \int_{r \geq 1} |r|^{-s_1} |\phi(r)|^{q_1} dr \leq \int_{r \geq 1} r^{-s_1 q_1} |\phi(r)|^{q_1} dr \leq r^{\frac{1}{p_1}} |\phi|_{L^{q_1}(1, +\infty)}^{q_1},$$

and thus from (166) and the one dimensional Sobolev injection $W^{s_1,p_1} \hookrightarrow L^{q_1}$,

$$|\phi|_{L^{q_1}(\mathbb{R}^2)} \leq |r^{\frac{1}{p_1}} \phi|_{L^{q_1}(1, +\infty)} \leq C|\phi|_{W^{s_1,p_1}(1, +\infty)} \leq C|\phi|_{W^{s_1,p_1}(\mathbb{R}^2)}.$$ (166)

It thus remains to prove (166). The case $s_1 = 0$ is trivial. Now by complex interpolation, $W^{s_1,p_1} = [L^{p_1}, W^{1,p_1}]_{s_1}$, see [2], and thus it suffices to prove (166) for $s_1 = 1$. Let $\psi = r^{\frac{1}{p_1}} \phi$, then $\psi' = -\frac{1}{p_1 r} r^{\frac{1}{p_1}} \phi + r^{\frac{1}{p_1}} \phi'$, and thus

$$|\psi'|_{L^{1, +\infty}(\mathbb{R}^2)} \leq C \left( |\phi|_{L^{p_1}(\mathbb{R}^2)} + |\phi'|_{L^{p_1}(\mathbb{R}^2)} \right).$$

Similarly, $r^{\frac{1}{p_1}} \phi' = \psi' + \frac{\psi}{p_1 r}$ and thus

$$|\phi'|_{L^{p_1}(\mathbb{R}^2)} \leq C \left( |\psi'|_{L^{p_1}(1, +\infty)} + |\psi|_{L^{p_1}(1, +\infty)} \right).$$

This concludes the proof of (166) and of Lemma 4.
Appendix B: Proof of smoothing effect (145)

The proof of smoothing effect (145) uses a standard $TT^*$ argument. Let $T$ the operator $T\phi = e^{it\Delta}\phi$, then from Strichartz estimates (144), $T$ sends $\dot{H}^{\frac{1}{2}}$ continuously into $L^q(I, \dot{B}^{\frac{1}{2}}_{r,2})$ for any admissible pair $(q, r)$. Consider now the weighted $L^2$ space $L^2_{loc}$ equipped with the norm

$$|\phi|_{L^2_{loc}} = \sup_{j \geq 0} 2^j |\phi|_{L^2(|x| \sim 2^j)} + |\phi|_{L^2(B(0,1))}$$

where the physical space has been partitioned into annuli of radius $2^j$. Kato’s standard smoothing effect for $T$ is

$$|T\phi|_{L^2(I, \dot{H}^{\frac{1}{2}}_{loc})} \leq C|\phi|_{L^2}, \quad (167)$$

with a constant $C > 0$ independent of the time interval $I$. Let the adjoint norm of the local $L^2$ norm

$$|\psi|_{L^2_{comp}} = \sum_{j \geq 0} 2^{-j} |\psi|_{L^2(|x| \sim 2^j)} + |\psi|_{L^2(B(0,1))},$$

than the adjoint version of (167) implies

$$|T^*\psi|_{\dot{H}^{\frac{1}{2}}} = \left| \int_{\mathbb{R}} e^{-is\Delta}\psi(s)ds \right|_{\dot{H}^{\frac{1}{2}}} \leq C|\psi|_{L^2(I, L^2_{comp})}. \quad (168)$$

Using now (143) yields for any admissible pair $(q, r)$

$$|TT^*F|_{L^q(I, \dot{B}^{\frac{1}{2}}_{r,2})} \leq C|T^*F|_{\dot{H}^{\frac{1}{2}}} \leq C|F|_{L^2(I, L^2_{comp})}. \quad (169)$$

Let now $u$ be the solution to the inhomogeneous linear Schrödinger equation

$$\begin{cases}
    iu_t + \Delta u = F; \\
    u_{t=0} = 0,
\end{cases} \quad I = [0, t] \text{ with } \forall s \in I, \text{ Supp}(F(s)) \subset B(0, 10).$$

Then

$$u(t, x) = \int_0^t e^{i(t-s)\Delta} F(s, x)ds \quad \text{while} \quad TT^*F(t, x) = \int_{\mathbb{R}} e^{i(t-s)\Delta} F(s, x).$$

Nevertheless, the Christ and Kiselev Lemma, [8], applies from $q > 2$ to deal with the boundary of the time integration and implies from (168)

$$|u|_{L^q(I, \dot{B}^{\frac{1}{2}}_{r,2})} \leq C|F|_{L^2(I, L^2_{comp})}. \quad (170)$$

(145) now follows from the localization of the support of $F$. 

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References


