Statistical estimation of the division rate of a size-structured population

M. Doumic, M. Hoffmann, P. Reynaud-Bouret, V. Rivoirard

INRIA Rocquencourt, ENSAE-CREST, Nice, Dauphine

1 The problem

The informal problem and the PDE translation

- A cell grows.
- Depending on its size x, the cell has a certain chance to divide itself in 2 offsprings, ie 2 cells of size $x/2$.
- We are interesting by the evolution of the whole population of cells, each of them having this behavior.

The informal problem and the PDE translation

- A cell grows.
- Depending on its size x, the cell has a certain chance to divide itself in 2 offsprings, ie 2 cells of size $x/2$.
- We are interesting by the evolution of the whole population of cells, each of them having this behavior.

Size-Structured Population Equation (finite time)

$$\begin{cases} \frac{\partial}{\partial t}(n(t,x)) + \kappa \frac{\partial}{\partial x}(g(x)n(t,x)) + B(x)n(t,x) = 4B(2x)n(t,2x), \\ n(t,x=0) = 0, \quad t > 0 \\ n(0,x) = n_0(x), \quad x \geq 0. \end{cases}$$

- $n(t,x)$ the "amount" of cells with size x ($\neq$ density),
- g the "qualitative" growth rate of one cell: linear is $g = 1$ ...
- B is the division rate, which depends on the size

Asymptotics of the PDE

It can be shown (Perthame Ryzhik 2005 for instance) that

- $n(t,.)$ grows exponentially fast ie $I_t = \int n(t,x)dx$ asymptotically proportional to $e^{\lambda t}$,
- the renormalized $n(t,x)/I_t$ tends to a density N, which satisfies

Size-Structured Population Equation (asymptotics)

$$\begin{cases} \kappa \dfrac{\partial}{\partial x}\big(g(x)N(x)\big) + \lambda N(x) = \mathcal{L}\big(BN\big)(x), \\ B(0)N(0) = 0, \qquad \int N(x)dx = 1, \end{cases}$$

where

- for any real-valued function $x \rightsquigarrow \varphi(x)$, $\mathcal{L}\big(\varphi\big)(x) := 4\varphi(2x) - \varphi(x)$.
- $\kappa = \lambda \dfrac{\int_{\mathbb{R}_+} xN(x)dx}{\int_{\mathbb{R}_+} g(x)N(x)dx}$.

The inverse problem

Under the previous differential equation, we consider the inverse problem of finding B given a "noisy" version of N.

The inverse problem

Under the previous differential equation, we consider the inverse problem of finding B given a "noisy" version of N.

- Practical: biologists take a sample of, say, plankton in a lake, and they look at the respective size of the cells.

The inverse problem

Under the previous differential equation, we consider the inverse problem of finding B given a "noisy" version of N.

- Practical: biologists take a sample of, say, plankton in a lake, and they look at the respective size of the cells. Then they perform a preprocessing, by, say a kernel estimator. This is N_ϵ.

The inverse problem

Under the previous differential equation, we consider the inverse problem of finding B given a "noisy" version of N.

- Practical: biologists take a sample of, say, plankton in a lake, and they look at the respective size of the cells. Then they perform a preprocessing, by, say a kernel estimator. This is N_ϵ. (probably more approximation than that).

The inverse problem

Under the previous differential equation, we consider the inverse problem of finding B given a "noisy" version of N.

- Practical: biologists take a sample of, say, plankton in a lake, and they look at the respective size of the cells. Then they perform a preprocessing, by, say a kernel estimator. This is N_ϵ. (probably more approximation than that).
- Analytical point of view: N_ϵ is a noisy version of N, less regular than N (it is likely that no derivative exists) and $\|N - N_\epsilon\|_2 \leq \epsilon$. (see Perthame, Zubelli, etc)

The inverse problem

Under the previous differential equation, we consider the inverse problem of finding B given a "noisy" version of N.

- Practical: biologists take a sample of, say, plankton in a lake, and they look at the respective size of the cells. Then they perform a preprocessing, by, say a kernel estimator. This is N_ϵ. (probably more approximation than that).
- Analytical point of view: N_ϵ is a noisy version of N, less regular than N (it is likely that no derivative exists) and $\|N - N_\epsilon\|_2 \leq \epsilon$. (see Perthame, Zubelli, etc)
- Statistical point of view: we observe a n-sample $X_1, ..., X_n$ of iid variables with density N.

Pro and Con

Analytical point of view

Pro: taking into account maybe more approximations (but not all), results true for any N_ϵ.

Pro and Con

Analytical point of view

Pro: taking into account maybe more approximations (but not all), results true for any N_ϵ.
Con: N_ϵ is probably differentiable. If there are numerical methods which adapt to the regularity of N (discrepancy principle), they need to know ϵ.

Pro and Con

Analytical point of view

Pro: taking into account maybe more approximations (but not all), results true for any N_ϵ.
Con: N_ϵ is probably differentiable. If there are numerical methods which adapt to the regularity of N (discrepancy principle), they need to know ϵ.

Statistical point of view

Pro: Framework close to what biologists do, true inverse problem. We can adapt to the regularity, noise is given by the sample size.

Pro and Con

Analytical point of view

Pro: taking into account maybe more approximations (but not all), results true for any N_ϵ.
Con: N_ϵ is probably differentiable. If there are numerical methods which adapt to the regularity of N (discrepancy principle), they need to know ϵ.

Statistical point of view

Pro: Framework close to what biologists do, true inverse problem. We can adapt to the regularity, noise is given by the sample size.
Con: We only take one approximation into account and assume that we have access to the sample. Results true in expectation.

Pro and Con

Analytical point of view

Pro: taking into account maybe more approximations (but not all), results true for any N_ϵ.
Con: N_ϵ is probably differentiable. If there are numerical methods which adapt to the regularity of N (discrepancy principle), they need to know ϵ.

Statistical point of view

Pro: Framework close to what biologists do, true inverse problem. We can adapt to the regularity, noise is given by the sample size.
Con: We only take one approximation into account and assume that we have access to the sample. Results true in expectation.

Estimation of N

Let $K : \mathbb{R} \rightarrow \mathbb{R}_+$ continuous function / $\int K = 1$ and $\int K^2 < \infty$.

Estimation of N

Let $K : \mathbb{R} \to \mathbb{R}_+$ continuous function / $\int K = 1$ and $\int K^2 < \infty$.

$$\hat{N}_h(x) := \frac{1}{n} \sum_{i=1}^{n} K_h(x - X_i),$$

$K_h = \frac{1}{h} K(./h)$.

Bias-Variance decomposition

$$\mathbb{E}\left(\left\| N - \hat{N}_h \right\|_2 \right) \leq \| N - K_h \star N \|_2 + \frac{1}{\sqrt{nh}} \| K \|_2,$$

where $K_h \star N = \mathbb{E}(\hat{N}_h)$

How to adaptively select h ?

Estimation of N

Let $K : \mathbb{R} \to \mathbb{R}_+$ continuous function / $\int K = 1$ and $\int K^2 < \infty$.

$$\hat{N}_h(x) := \frac{1}{n} \sum_{i=1}^{n} K_h(x - X_i),$$

$K_h = \frac{1}{h} K(./h)$.

Bias-Variance decomposition

$$\mathbb{E}\left(\left\|N - \hat{N}_h\right\|_2\right) \leq \|N - K_h \star N\|_2 + \frac{1}{\sqrt{nh}} \|K\|_2,$$

where $K_h \star N = \mathbb{E}(\hat{N}_h)$

How to adaptively select h ? Recent work of Goldenshluger and Lepski (2009, 2010)

Estimation of N

Let $K : \mathbb{R} \to \mathbb{R}_+$ continuous function / $\int K = 1$ and $\int K^2 < \infty$.

$$\hat{N}_h(x) := \frac{1}{n} \sum_{i=1}^{n} K_h(x - X_i),$$

$K_h = \frac{1}{h} K(./h)$.

Bias-Variance decomposition

$$\mathbb{E}\left(\left\|N - \hat{N}_h\right\|_2\right) \leq \|N - K_h \star N\|_2 + \frac{1}{\sqrt{nh}} \|K\|_2,$$

where $K_h \star N = \mathbb{E}(\hat{N}_h)$

How to adaptively select h ? Recent work of Goldenshluger and Lepski (2009, 2010) Here just a "toy" version, but that's exactly what we needed.

Selection of bandwidth

Set for any x and any $h, h' > 0$,
$\hat{N}_{h,h'}(x) := \frac{1}{n}\sum_{i=1}^{n}(K_h \star K_{h'})(x - X_i) = (K_h \star \hat{N}_{h'})(x)$,

Selection of bandwidth

Set for any x and any $h, h' > 0$,
$\hat{N}_{h,h'}(x) := \frac{1}{n}\sum_{i=1}^{n}(K_h \star K_{h'})(x - X_i) = (K_h \star \hat{N}_{h'})(x)$,

"estimator" of the bias term

$$A(h) := \sup_{h' \in \mathcal{H}} \left\{ \|\hat{N}_{h,h'} - \hat{N}_{h'}\|_2 - \frac{\chi}{\sqrt{nh'}}\|K\|_2 \right\}_+$$

where, given $\varepsilon > 0$, $\chi := (1 + \varepsilon)(1 + \|K\|_1)$.

Selection of bandwidth

Set for any x and any $h, h' > 0$,
$\hat{N}_{h,h'}(x) := \frac{1}{n}\sum_{i=1}^{n}(K_h \star K_{h'})(x - X_i) = (K_h \star \hat{N}_{h'})(x)$,

"estimator" of the bias term

$$A(h) := \sup_{h' \in \mathcal{H}} \left\{ \|\hat{N}_{h,h'} - \hat{N}_{h'}\|_2 - \frac{\chi}{\sqrt{nh'}}\|K\|_2 \right\}_+$$

where, given $\varepsilon > 0$, $\chi := (1+\varepsilon)(1+\|K\|_1)$.

$$\hat{h} := \arg\min_{h \in \mathcal{H}} \left\{ A(h) + \frac{\chi}{\sqrt{nh}}\|K\|_2 \right\} \quad \text{and} \quad \hat{N} := \hat{N}_{\hat{h}}.$$

First result

Oracle inequality

If $\mathcal{H} = \{1/\ell \ / \ \ell = 1, ..., \ell_{max}\}$ and if $\ell_{max} = \delta n$, if moreover $\|N\|_\infty < \infty$,
then for any $q \geq 1$,

$$\mathbb{E}\left(\|\hat{N} - N\|_2^{2q}\right) \leq \square_q \chi^{2q} \inf_{h \in \mathcal{H}} \left\{\|K_h \star N - N\|_2^{2q} + \frac{\|K\|_2^{2q}}{(hn)^q}\right\} + \square_{q,\varepsilon,\delta,\|K\|_2,\|K\|_1,\|N\|_\infty} \frac{1}{n^q}.$$

Estimation of $D = \frac{\partial}{\partial x}\big(g(x)N(x)\big)$

If K is differentiable, $\int K = 1$ and $\int |K'|^2 < \infty$.

$$\hat{D}_h(x) := \frac{1}{n}\sum_{i=1}^{n} g(X_i) K_h'(x - X_i)$$

Estimation of $D = \frac{\partial}{\partial x}\big(g(x)N(x)\big)$

If K is differentiable, $\int K = 1$ and $\int |K'|^2 < \infty$.

$$\hat{D}_h(x) := \frac{1}{n}\sum_{i=1}^{n} g(X_i)K_h'(x - X_i)$$

Estimation of $D = \frac{\partial}{\partial x}(g(x)N(x))$

If K is differentiable, $\int K = 1$ and $\int |K'|^2 < \infty$.

$$\hat{D}_h(x) := \frac{1}{n}\sum_{i=1}^{n} g(X_i)K_h'(x - X_i)$$

Bias-Variance de composition:

$$\mathbb{E}\left(\left\|D - \hat{D}_h\right\|_2\right) \leq \|D - K_h \star D\|_2 + \frac{1}{\sqrt{nh^3}}\|g\|_\infty \|K'\|_2.$$

GL's trick

$\hat{D}_{h,h'}(x) := \frac{1}{n}\sum_{i=1}^{n} g(X_i)(K_h \star K_{h'})'(x - X_i)$,

$$\tilde{A}(h) := \sup_{h' \in \tilde{\mathcal{H}}} \left\{ \|\hat{D}_{h,h'} - \hat{D}_{h'}\|_2 - \frac{\tilde{\chi}}{\sqrt{nh'^3}}\|g\|_\infty \|K'\|_2 \right\}_+ ,$$

where, given $\tilde{\varepsilon} > 0$, $\tilde{\chi} := (1 + \tilde{\varepsilon})(1 + \|K\|_1)$.

Estimation of $D = \frac{\partial}{\partial x}(g(x)N(x))$

If K is differentiable, $\int K = 1$ and $\int |K'|^2 < \infty$.

$$\hat{D}_h(x) := \frac{1}{n}\sum_{i=1}^{n} g(X_i)K_h'(x - X_i)$$

GL's trick

$\hat{D}_{h,h'}(x) := \frac{1}{n}\sum_{i=1}^{n} g(X_i)(K_h \star K_{h'})'(x - X_i)$,

$$\tilde{A}(h) := \sup_{h' \in \tilde{\mathcal{H}}} \left\{ \|\hat{D}_{h,h'} - \hat{D}_{h'}\|_2 - \frac{\tilde{\chi}}{\sqrt{nh'^3}}\|g\|_\infty \|K'\|_2 \right\}_+ ,$$

where, given $\tilde{\varepsilon} > 0$, $\tilde{\chi} := (1 + \tilde{\varepsilon})(1 + \|K\|_1)$.

Finally, we estimate D by using $\hat{D} := \hat{D}_{\tilde{h}}$ with

$$\tilde{h} := \mathrm{argmin}_{h \in \tilde{\mathcal{H}}} \left\{ \tilde{A}(h) + \frac{\tilde{\chi}}{\sqrt{nh^3}}\|g\|_\infty \|K'\|_2 \right\}.$$

Result for the derivative D

Oracle inequality for D

If $\tilde{\mathcal{H}} = \{1/\ell \ / \ \ell = 1, \dots, \ell_{max}\}$ and if $\ell_{max} = \sqrt{\delta' n}$, if moreover $\|N\|_\infty$ and $\|g\|_\infty < \infty$, then for any $q \geq 1$,

$$\mathbb{E}\left(\|\hat{D} - D\|_2^{2q}\right) \leq \square_q \tilde{\chi}^{2q} \inf_{h \in \tilde{\mathcal{H}}} \left\{ \|K_h \star D - D\|_2^{2q} + \left[\frac{\|g\|_\infty \|K'\|_2}{\sqrt{nh^3}}\right]^{2q} \right\}$$

$$+ \square_{q, \tilde{\varepsilon}, \delta', \|K'\|_2, \|K\|_1, \|K'\|_1, \|N\|_\infty, \|g\|_\infty} \frac{1}{n^q}.$$

Estimation of λ and κ

λ is estimated via another (or simultaneous experiment).

Estimation of λ and κ

λ is estimated via another (or simultaneous experiment).

Assumption on $\hat{\lambda}$

There exist some $q > 1$ such that

- $\varepsilon_\lambda = \mathbb{E}(|\hat{\lambda} - \lambda|^q) < \infty$,
- $R_\lambda = \mathbb{E}(\hat{\lambda}^{2q}) < \infty$.

Estimation of λ and κ

λ is estimated via another (or simultaneous experiment).

Assumption on $\hat{\lambda}$

There exist some $q > 1$ such that

- $\varepsilon_\lambda = \mathbb{E}(|\hat{\lambda} - \lambda|^q) < \infty$,
- $R_\lambda = \mathbb{E}(\hat{\lambda}^{2q}) < \infty$.

Let $c > 0$,

$$\hat{\kappa} = \hat{\lambda} \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n g(X_i) + c}.$$

The inversion of $\mathcal{L}$

It remains to (approximately) invert $\mathcal{L}$. (see Perthame, Zubelli, Doumic (2009))

The inversion of $\mathcal{L}$

It remains to (approximately) invert $\mathcal{L}$. (see Perthame, Zubelli, Doumic (2009))
Define $T > 0$, an integer $k \geq 1$ and the regular grid on $[0, T]$ with mesh $k^{-1}T$ defined by
$0 = x_{0,k} < x_{1,k} < \cdots < x_{i,k} := \frac{i}{k}T < \ldots < x_{k,k} = T$.
Set $\varphi_{i,k} =: \frac{k}{T}\int_{x_{i,k}}^{x_{i+1,k}} \varphi(x)dx$ for $i = 0, \ldots, k-1$, and define by induction the sequence

$$H_{i,k}(\varphi) := \frac{1}{4}(H_{i/2,k}(\varphi) + \varphi_{i/2,k}) \text{ with } \begin{cases} H_0(\varphi) := & \frac{1}{3}\varphi_{1,k}, \\ H_1(\varphi) := & \frac{4}{21}\varphi_{0,k} + \frac{1}{7}\varphi_{1,k} \end{cases}$$

The inversion of $\mathcal{L}$

It remains to (approximately) invert $\mathcal{L}$. (see Perthame, Zubelli, Doumic (2009))
Define $T > 0$, an integer $k \geq 1$ and the regular grid on $[0, T]$ with mesh $k^{-1}T$ defined by
$0 = x_{0,k} < x_{1,k} < \cdots < x_{i,k} := \frac{i}{k}T < \ldots < x_{k,k} = T$.
Set $\varphi_{i,k} =: \frac{k}{T}\int_{x_{i,k}}^{x_{i+1,k}} \varphi(x)dx$ for $i = 0, \ldots, k-1$, and define by induction the sequence

$$H_{i,k}(\varphi) := \frac{1}{4}(H_{i/2,k}(\varphi) + \varphi_{i/2,k}) \text{ with } \begin{cases} H_0(\varphi) := & \frac{1}{3}\varphi_{1,k}, \\ H_1(\varphi) := & \frac{4}{21}\varphi_{0,k} + \frac{1}{7}\varphi_{1,k} \end{cases}$$

for any sequence $u_i, i = 1, 2, \ldots,$

$$u_{i/2} := \begin{cases} u_{i/2} & \text{if } i \text{ is even} \\ \frac{1}{2}(u_{(i-1)/2} + u_{(i+1)/2}) & \text{otherwise.} \end{cases}$$

The inversion of $\mathcal{L}$

It remains to (approximately) invert $\mathcal{L}$. (see Perthame, Zubelli, Doumic (2009))

Define $T > 0$, an integer $k \geq 1$ and the regular grid on $[0, T]$ with mesh $k^{-1}T$ defined by

$0 = x_{0,k} < x_{1,k} < \cdots < x_{i,k} := \frac{i}{k}T < \ldots < x_{k,k} = T.$

Set $\varphi_{i,k} =: \frac{k}{T}\int_{x_{i,k}}^{x_{i+1,k}} \varphi(x)dx$ for $i = 0, \ldots, k-1$, and define by induction the sequence

$$H_{i,k}(\varphi) := \frac{1}{4}(H_{i/2,k}(\varphi)+\varphi_{i/2,k}) \text{ with } \begin{cases} H_0(\varphi) := & \frac{1}{3}\varphi_{1,k}, \\ H_1(\varphi) := & \frac{4}{21}\varphi_{0,k} + \frac{1}{7}\varphi_{1,k} \end{cases}$$

Finally, we define

$$\mathcal{L}_k^{-1}(\varphi)(x) := \sum_{i=0}^{k-1} H_{i,k}(\varphi)1_{[x_{i,k},x_{i+1,k})}(x).$$

The inversion of $\mathcal{L}$

It remains to (approximately) invert $\mathcal{L}$. (see Perthame, Zubelli, Doumic (2009))
Define $T>0$, an integer $k \geq 1$ and the regular grid on $[0,T]$ with mesh $k^{-1}T$ defined by
$0 = x_{0,k} < x_{1,k} < \cdots < x_{i,k} := \frac{i}{k}T < \ldots < x_{k,k} = T.$
Set $\varphi_{i,k} =: \frac{k}{T}\int_{x_{i,k}}^{x_{i+1,k}} \varphi(x)dx$ for $i = 0, \ldots, k-1$, and define by induction the sequence

$$H_{i,k}(\varphi) := \frac{1}{4}(H_{i/2,k}(\varphi) + \varphi_{i/2,k}) \text{ with } \begin{cases} H_0(\varphi) := & \frac{1}{3}\varphi_{1,k}, \\ H_1(\varphi) := & \frac{4}{21}\varphi_{0,k} + \frac{1}{7}\varphi_{1,k} \end{cases}$$

Finally, we define

$$\mathcal{L}_k^{-1}(\varphi)(x) := \sum_{i=0}^{k-1} H_{i,k}(\varphi) 1_{[x_{i,k}, x_{i+1,k})}(x).$$

Hence we are able to estimate $H = BN$ by

$$\hat{H} = \mathcal{L}_k^{-1}(\hat{\kappa}\hat{D} + \hat{\lambda}\hat{N}).$$

Oracle inequality for the estimation of $H = BN$

here

Theorem

Under suitable assumptions, $\mathbb{E}(\left\|\hat{H} - H\right\|_{2,T}^{q}) \leq$

Oracle inequality for the estimation of $H = BN$

here

Theorem

Under suitable assumptions, $\mathbb{E}(\left\| \hat{H} - H \right\|_{2,T}^{q}) \leq$

- $\square_{\dots} \left\{ \sqrt{R_\lambda} \inf_{h \in \tilde{\mathcal{H}}} \left[\|K_h \star D - D\|_2^q + \left(\frac{\|g\|_\infty \|K'\|_2}{\sqrt{nh^3}} \right)^q \right] \right.$

Oracle inequality for the estimation of $H = BN$

here

Theorem

Under suitable assumptions, $\mathbb{E}(\left\|\hat{H} - H\right\|_{2,T}^q) \leq$

- $\square_{\dots} \left\{ \sqrt{R_\lambda} \inf_{h \in \tilde{\mathcal{H}}} \left[\|K_h \star D - D\|_2^q + \left(\frac{\|g\|_\infty \|K'\|_2}{\sqrt{nh^3}} \right)^q \right] \right.$
- $+ \inf_{h \in \mathcal{H}} \left[\|K_h \star N - N\|_2^q + \left(\frac{\|K\|_2}{\sqrt{nh}} \right)^q \right] +$

Oracle inequality for the estimation of $H = BN$

here

Theorem

Under suitable assumptions, $\mathbb{E}(\left\|\hat{H} - H\right\|_{2,T}^{q}) \leq$

- $\square_{\ldots} \left\{\sqrt{R_\lambda} \inf_{h \in \tilde{\mathcal{H}}} \left[\|K_h \star D - D\|_2^q + \left(\frac{\|g\|_\infty \|K'\|_2}{\sqrt{nh^3}}\right)^q\right]\right.$
- $+ \inf_{h \in \mathcal{H}} \left[\|K_h \star N - N\|_2^q + \left(\frac{\|K\|_2}{\sqrt{nh}}\right)^q\right] +$
- $+\varepsilon_\lambda +$

Oracle inequality for the estimation of $H = BN$

here

Theorem

Under suitable assumptions, $\mathbb{E}(\left\|\hat{H} - H\right\|_{2,T}^q) \leq$

- $\square_{\dots} \left\{ \sqrt{R_\lambda} \inf_{h \in \tilde{\mathcal{H}}} \left[\|K_h \star D - D\|_2^q + \left(\frac{\|g\|_\infty \|K'\|_2}{\sqrt{nh^3}} \right)^q \right] \right.$
- $+ \inf_{h \in \mathcal{H}} \left[\|K_h \star N - N\|_2^q + \left(\frac{\|K\|_2}{\sqrt{nh}} \right)^q \right] +$
- $+ \varepsilon_\lambda +$
- $\left. + \left(\square_\alpha |\mathcal{L}(BN)|_{\mathcal{C}^\alpha(T)} T^{\alpha+1/2} k^{-\alpha} \right)^q \right\} +$

Oracle inequality for the estimation of $H = BN$

here

Theorem

Under suitable assumptions, $\mathbb{E}\left(\left\|\hat{H} - H\right\|_{2,T}^{q}\right) \leq$

- $\square_{\dots} \left\{ \sqrt{R_\lambda} \inf_{h \in \tilde{\mathcal{H}}} \left[\|K_h \star D - D\|_2^q + \left(\frac{\|g\|_\infty \|K'\|_2}{\sqrt{nh^3}} \right)^q \right] \right.$
- $+ \inf_{h \in \mathcal{H}} \left[\|K_h \star N - N\|_2^q + \left(\frac{\|K\|_2}{\sqrt{nh}} \right)^q \right] +$
- $+ \varepsilon_\lambda +$
- $\left. + \left(\square_\alpha |\mathcal{L}(BN)|_{\mathcal{C}^\alpha(T)} T^{\alpha + 1/2} k^{-\alpha} \right)^q \right\} +$
- $\square_{\dots} \frac{1}{n^{q/2}}$.

Rate of convergence for the estimation of B

We finally set $\hat{B} = \hat{H}/\hat{N}$ and $\tilde{B} = \max(\min(\hat{B}, \sqrt{n}), -\sqrt{n})$.

Rate of convergence for the estimation of B

We finally set $\hat{B} = \hat{H}/\hat{N}$ and $\tilde{B} = \max(\min(\hat{B}, \sqrt{n}), -\sqrt{n})$.
If $B \in \mathcal{W}_s$ $(s > 1/2)$ and $g \in \mathcal{W}_{s+1}$, then (under suitable assumptions) $N \in \mathcal{W}_{s+1}$.

Rate of convergence for the estimation of B

We finally set $\hat{B} = \hat{H}/\hat{N}$ and $\tilde{B} = \max(\min(\hat{B}, \sqrt{n}), -\sqrt{n})$.
If $B \in \mathcal{W}_s$ ($s > 1/2$) and $g \in \mathcal{W}_{s+1}$, then (under suitable assumptions) $N \in \mathcal{W}_{s+1}$.

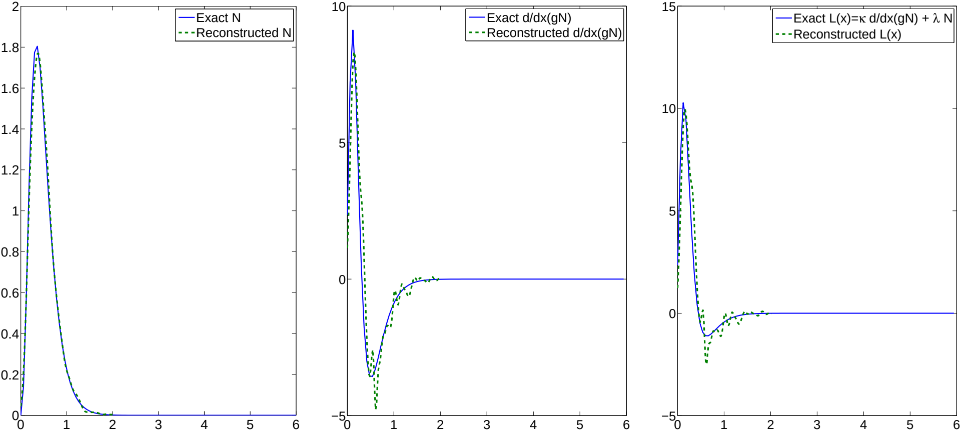
Theorem

If one knows a bound $\alpha \geq s$, one can choose a kernel K and a family of $\mathcal{H}$ and $\mathcal{H}'$ independent of s such that for any compact $[a, b]$ of $[0, T]$ (under technical assumptions),

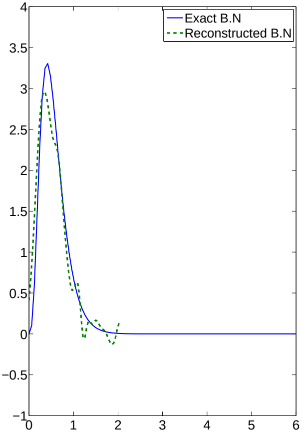
$$\mathbb{E}\left[\left\|(\tilde{B} - B)1_{[a,b]}\right\|_2^q\right] = O\left(n^{-\frac{qs}{2s+3}}\right).$$

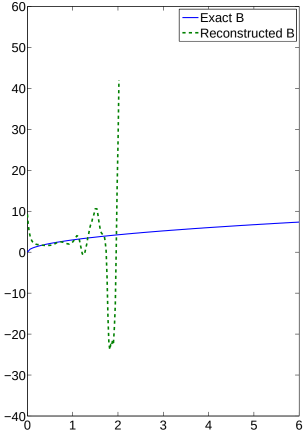
Simulations

n=5000, Gaussian kernel, $B = 3\sqrt{x}$, $g = 1$.



Simulations





Concluding remarks

- work still in progress: simulations and comparison to analytical methods

Concluding remarks

- work still in progress: simulations and comparison to analytical methods
- Probabilistic interpretation not used: evolution of one cell look like TCP window size, but the whole population (?) $\rightsquigarrow$ chaos and not necessarily independence (work in progress of Hoffmann, Krell, Lepoutre ...)

Concluding remarks

- work still in progress: simulations and comparison to analytical methods
- Probabilistic interpretation not used: evolution of one cell look like TCP window size, but the whole population (?) $\rightsquigarrow$ chaos and not necessarily independence (work in progress of Hoffmann, Krell, Lepoutre ...)
- Calibration of GL's method not done, comparison with the L-curve method in analysis (χ N step ?)