Sub-Riemannian curvature: a variational approach

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References:

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1 Motivation

2 Affine control systems and geodesic cost

3 Geodesic growth vector, ample geodesics and curvature

4 Applications: Laplacian of the sub-Riemannian distance

5 Applications: SR geodesic dimension and MCP
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1. Motivation
2. Affine control systems and geodesic cost
3. Geodesic growth vector, ample geodesics and curvature
4. Applications: Laplacian of the sub-Riemannian distance
5. Applications: SR geodesic dimension and MCP
In Riemannian geometry the curvature tensor plays a major role in many aspects:

- Ricci bounds
- comparison theorems (volume, distance, Laplacian, Hessian etc.)
- asymptotic and estimates of the heat kernel

**Obstruction 1.** In sub-Riemannian geometry no canonical affine connection

Recently, many efforts to generalize the notion of Ricci/curvature bounds

- via optimal transport techniques
  - Lott, Villani - Sturm
  - Ambrosio, Gigli, Savaré (and coauthors)
- via gen. curvature dimension inequalities / heat equation techniques
  - Garofalo, Baudoin (and coauthors)
  - Thalmaier, Grong
- via measure contraction properties / other geometric techniques
  - Julliet - Rifford - Vassilev
  - Agrachev, Lee - Lee, Li, Zelenko
In this talk I will be interested in looking for a definition/generalization of the notion of (sectional) curvature in sub-Riemannian geometry.

This curvature is based on two main tools

- distance (→ or more generally a cost)
- geodesics (→ or more generally minimizers for the cost)

From these two ingredients we define a “directional curvature”.

Naive idea:

→ The curvature can be found in the asymptotic of the distance along geodesics

Obstruction 2. Non-smoothness of the distance in sub-Riemannian geometry
Let \((M, d)\) be a Riemannian manifold and \(x \in M\).

- Let \(\gamma_v, \gamma_w\) two geodesics with unit tangent vector \(v, w \in T_x M\).

\[
C(t, s) := \frac{1}{2} d^2(\gamma_v(t), \gamma_w(s))
\]

is smooth in \((0, 0)\)

We expect to recover the scalar product and the curvature from the asymptotic expansion of \(C(t, s)\).
Asymptotic expansion

\[ C(t, s) \approx \frac{1}{2}(t^2 + s^2) - \langle v, w \rangle ts + \frac{1}{4} \partial_{tts} C(0, 0) t^2 s^2 + \ldots \]

Euclidean

Theorem (Loeper, Villani)

\[-\frac{3}{2} \partial_{tts} C(0, 0) = \langle R^\nabla (v, w)v, w \rangle \quad (= \text{Sec}(v, w))\]

Remarks

→ In SR geometry the geodesic is not uniquely associated with its tangent vector.

→ The function \( C(t, s) \) is not smooth at \( (0,0) \).
Lemma: differential of $d^2(x_0, \cdot)$

Let $x_0 \in M$ and define

$$f_{x_0}(\cdot) = \frac{1}{2} d^2(x_0, \cdot)$$

- $f_{x_0}$ smooth at $x \iff \exists$! minimal non conjugate geodesic $\gamma$ joining $x_0$ and $x$.
- $\nabla f_{x_0}(x) = T \gamma'_v(T)$

Remark. If we define $c_t(x) := -\frac{1}{2t} d^2(x, \gamma_v(t))$ then $\nabla c_t(x_0) = v$ (indep. on $t$).

$$\nabla_{x_0} \left( -\frac{1}{2t} d^2(\cdot, \gamma_v(t)) \right) = -\frac{1}{t} \nabla f_{\gamma_v(t)}(x_0) = -\frac{1}{t} (-tv) = v$$
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Affine optimal control problems

**[Dynamic]** Let us consider a smooth affine control system on a manifold $M$

\[
\dot{x} = X_0(x) + \sum_{i=1}^{k} u_i X_i(x), \quad x \in M, \, u \in \mathbb{R}^k.
\]

- we call $\mathcal{D}_x = \text{span}_x \{X_1, \ldots, X_k\}$ the **distribution**.
- $u \in L^\infty([0, T], \mathbb{R}^k) \to \gamma_{x_0, u}(\cdot)$ the solution with $x(0) = x_0$ (**admissible curve**)

**[Cost]** Given a smooth function $L : M \times \mathbb{R}^k \to \mathbb{R}$ we define the **cost at time** $T$ as the functional

\[
J_T(u) := \int_0^T L(\gamma_u(t), u(t)) \, dt,
\]

**Definition**

For two given points $x, y \in M$ and $T > 0$, we define the **value function**

\[
S_T(x, y) = \inf \{ J_T(u) \mid u \text{ admissible, } \gamma_u(0) = x, \gamma_u(T) = y \},
\]
Assumptions

**(A1)** The weak bracket generating condition is satisfied

\[ \text{Lie}_x \left\{ (\text{ad} \ X_0)^i X_j, i \in \mathbb{N}, j = 1, \ldots, k \right\} = T_x M, \]

[the vector field $X_0$ is not included in the generators of the Lie algebra]

**(A2)** The Lagrangian $L : M \times \mathbb{R}^k \to \mathbb{R}$ is of Tonelli-type, i.e.

- (A2.a) $L$ has strictly positive Hessian in $u$, for all $x \in M$.
- (A2.b) $L$ has superlinear growth, i.e. $L(x, u)/|u| \to +\infty$ when $|u| \to +\infty$.

(A1) does not ask that the linear part is bracket-generating.

→ when $X_0 = 0$ it is the classical Hormander condition.

(A2) ensures that the Hamiltonian is well defined and smooth
Geometric framework

This setting includes control problems associated with many different geometric structures such as

- Riemannian structures
- sub-Riemannian structures
- smooth Finsler structures (or smooth sub-Finsler).

The (sub-)Riemannian case corresponds to the case when

- the system is driftless \( X_0 \equiv 0 \) and bracket generating \((A1)\)
- \( k < n \) \((k = n \) corresponds to Riemannian\)
- the cost is quadratic \( L(x, u) = \frac{1}{2}|u|^2 \) \((A2)\)

→ The cost is induced by a scalar product such that \( X_1, \ldots, X_k \) are orthonormal.

\[
S_T(x, y) = \frac{1}{2T} d^2(x, y)
\]

In particular

\[
c_t(x) = -\frac{1}{2t} d^2(x, \gamma(t)) = -S_t(x, \gamma(t))
\]
Main idea: microlocal approach

Fix $x_0 \in M$ and a minimizing trajectory $\gamma = \gamma_{x_0,u}(t)$.

$$c_t : M \rightarrow \mathbb{R}, \quad t > 0$$

$$c_t(x) = -S_t(x, \gamma(t))$$

The curvature appears in the asymptotic expansion of the second derivative of the cost along a minimizing trajectory.

We need some assumptions on $\gamma$ to guarantee
- the smoothness of $c_t$ ($\rightarrow$ strongly normal)
- the existence of the asymptotic. ($\rightarrow$ ample)
End point map

Definition

Fix a point \( x_0 \in M \) and \( T > 0 \). The end-point map at time \( T \) is the map

\[
E_{x_0, T} : \mathcal{U} \rightarrow M, \quad u \mapsto \gamma_{x_0, u(T)},
\]

For every \( x_0, x_1 \in M \) one has

\[
S_T(x_0, x_1) = \inf_{E_{x_0, T}^{-1}(x_1)} J_T = \inf \{ J_T(u) \mid E_{x_0, T}(u) = x_1 \}
\]

In general \( E_{x_0, T} \) is smooth but

- \( D_u E_{x_0, T} \) is not surjective
- the set \( E_{x_0, T}^{-1}(x_1) \) is not a smooth manifold.
Lagrange multipliers rule

A necessary condition for constrained critical point of $\inf_{E^{-1}_{x_0,T}(x_1)} J_T$.

**Theorem**

Assume $u \in \mathcal{U}$ is a constr. crit. point, with $x = E_{x_0,T}(u)$. Then (at least) one of the two following statements hold true

(i) $\exists \lambda_T \in T^*_x M$ s.t. $\lambda_T \cdot D_u E_{x_0,T} = D_u J_T$, $\lambda_T \cdot D_u E_{x_0,T} = D_u J_T$,

(ii) $\exists \lambda_T \in T^*_x M$ s.t. $\lambda_T \cdot D_u E_{x_0,T} = 0$ (here $\lambda_T \neq 0$).

→ A control $u$ (reps. the associated trajectory $\gamma_u$) is called

- **normal** in case (i),
- **abnormal** in case (ii).

A priori a single control can be associated with two covectors such that both (i) and (ii) are satisfied.

- In what follows we will call **geodesic** any normal extremal.

A geodesic $\gamma : [0, T] \rightarrow M$ is called **strongly normal** if for all $s \in [0, T]$ the restriction $\gamma_{|[0,s]}$ is not abnormal.
Hamiltonian and PMP for normal geodesics

\[ H(\lambda) = \max_{u \in \mathbb{R}^k} \left( \langle \lambda, X_u(x) \rangle - L(x, u) \right), \quad \lambda \in T^*M, \ x = \pi(\lambda). \]

where \( X_u(x) = X_0(x) + \sum u_i X_i(x) \)

**Theorem (PMP, normal case)**

*Let \((u(t), \gamma_u(t))\) be a normal geodesic. Then there exists a Lipschitz curve \(\lambda(t) \in T^\ast_{\gamma_u(t)}M\) such that*

\[ \dot{\lambda}(t) = \overrightarrow{H}(\lambda(t), u(t)), \]

*Any normal geodesic starting at \(x_0\) is characterized by its initial covector \(\lambda_0 \in T^*_{x_0}M\) and we write*

\[ \gamma(t) = \text{Exp}_{x_0}(t, \lambda_0) \]
Distance from a minimizer

Fix \( x_0 \in M \) and a strongly normal geodesic \( \gamma(t) = \text{Exp}_{x_0}(t, \lambda_0) \) and define the geodesic cost

\[
c_t : M \to \mathbb{R}, \quad t > 0
\]

\[
c_t(x) = -S_t(x, \gamma(t))
\]

**Theorem**

There exist \( \varepsilon > 0 \) and an open set \( U \subset (0, \varepsilon) \times M \) such that \( (t, x_0) \in U \)

(i) The geodesic cost function \( (t, x) \mapsto c_t(x) \) is smooth on \( U \).

(ii) For every \( (t, x) \in U \) there exists a unique minimizer connecting \( x \) and \( \gamma(t) \)

Moreover \( d_{x_0} c_t = \lambda_0 \) for any \( t \in (0, \varepsilon) \).

Then \( d_{x_0} \dot{c}_t = 0 \rightarrow d_{x_0}^2 \dot{c}_t : T_{x_0} M \to \mathbb{R} \) is a well defined family of quadratic forms.
Fix any Riemannian geodesic $\gamma \Rightarrow$ well defined quadratic form $d^2_{x_0} \dot{c}_t : T_{x_0} M \rightarrow \mathbb{R}$

Define the family of operators $Q_\gamma(t) : T_{x_0} M \rightarrow T_{x_0} M$ by

$$d^2_{x_0} \dot{c}_t(v) = \langle Q_\gamma(t)v, v \rangle, \quad v \in T_{x_0} M, \quad t \in (0, \epsilon)$$

- smooth family of symmetric operators for $t \in (0, \epsilon)$
- singular at $t = 0$

Regularity theorem (Riemannian case)

The family of symmetric operators has a second order pole at $t = 0$ and

$$Q_\gamma(t) = \frac{1}{t^2} \mathbb{I} + \frac{1}{3} R^\nabla (\dot{\gamma}, \cdot) \dot{\gamma} + O(t)$$

$R^\nabla (\dot{\gamma}, \cdot) \dot{\gamma}$ can be though as “directional curvature operator” in the direction $\dot{\gamma}$
Interpretation for $\dot{c}_t(x)$ in Riemannian geometry

Recall that, for a Riemannian structure

$$c_t(x) := -S_t(x, \gamma(t)) = -\frac{1}{2t} d^2(x, \gamma(t))$$

**Lemma**

For a Riemannian structure, the derivative of the geodesic cost function is

$$\dot{c}_t(x) = \frac{1}{2} \| W_{t}^{x_0, \gamma(t)} - W_{t}^{x, \gamma(t)} \|^2 - \frac{1}{2}$$

where $W_{t}^{x, y} \in T_y M$ is the tangent vector at time $t$ of the unique minimizer connecting $x$ with $y$ in time $t$

- It encodes the curvature without using parallel transport.
Figure 1. A geometrical interpretation for the function $\dot{c}_t$. 

Motivation
Affine control systems and geodesic cost
Geodesic growth vector, ample geodesics and curvature
Applications: Laplacian...
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Geodesic growth vector

Let $\gamma$ be smooth admissible curve. Let $T \in \text{Vec}(M)$ an admissible extension of $\dot{\gamma}$

Geodesic flag

$$\mathcal{F}_\gamma^i(t) = \text{span}\{[T, \ldots, [T, X]]|_{\gamma(t)} \mid X \in \Gamma(D), \ j = 0, \ldots, i - 1\}$$

These subspaces define a natural flag

$$\mathcal{F}_\gamma^1(t) \subset \mathcal{F}_\gamma^2(t) \subset \ldots \subset T_{\gamma(t)}M$$

- $\mathcal{F}_\gamma^1(t)$ is the distribution $D_{\gamma(t)}$.
- Does not depend on the choice of $T$
- Depends on the germ of $\gamma$ and of the distribution at $\gamma(t)$ (→ microlocal)

Geodesic growth vector

$$\mathcal{G}_\gamma(t) = \{k_1(t), k_2(t), \ldots\}, \quad k_i(t) = \dim \mathcal{F}_\gamma^i(t)$$
Ample geodesic

**Definition (Ample geodesic)**

A geodesic $\gamma$ is **ample at $t$** if $\exists m = m(t) > 0$ s.t.

$$\mathcal{F}^1(t) \subset \ldots \subset \mathcal{F}^m(t) = T_{\gamma(t)}M$$

An ample geodesic is **equiregular** if its growth vector does not depend on $t$.

- is a “microlocal Hörmander condition”.
- ample at $t = 0 \Rightarrow$ strongly normal

**Theorem (The generic normal SR geodesic is ample at $t = 0$)**

*Let $M$ be a sub-Riemannian manifold. For every $x_0 \in M$ there exists a non-empty open Zariski $A_{x_0} \subset T^*_{x_0}M$ such that, for all $\lambda \in A_{x_0}$, the geodesic $\gamma(t) = \exp(t\lambda)$ is ample at $t = 0$.***
Hamiltonian scalar product on $\mathcal{D}_x$

- In the Riemannian case we used the scalar product to transform quadratic forms in operators.
- In the sub-Riemannian case we have a scalar product on $\langle \cdot, \cdot \rangle$ on $\mathcal{D}_x$
  $\rightarrow$ restriction of our quadratic forms $d^2_{x_0} \dot{c}_t$ to $\mathcal{D}_{x_0}$

This is a scalar product induced by the Hamiltonian $H$. In the general case
- $L$ is Tonelli (i.e. $d^2_u L$ non degenerate)
- the second derivative of $H|_{T^*_x M}$ at $\lambda$ defines a scalar product on $\mathcal{D}_x$
  - $d^2_\lambda H_x : T^*_x M \rightarrow T_x M$ self-adjoint linear map
  - $\text{Im}(d^2_\lambda H_x) = \mathcal{D}_x$ for every $\lambda \in T^*_x M$.

$$\langle v_1 | v_2 \rangle_\lambda := \langle \xi_1, d^2_\lambda H_x(\xi_2) \rangle, \quad d^2_\lambda H_x(\xi_i) = v_i.$$ 

- Scalar product $\langle \cdot, \cdot \rangle_\lambda$ on $\mathcal{D}_x$, depending on $\lambda$.
- if $H$ is (quadratic) sub-Riemannian, then it is the SR scalar product (not dep on $\lambda$!).
Main result

Consider the restriction $d^2_{x_0} \dot{c}_t \big|_{\mathcal{D}_{x_0}} : \mathcal{D}_{x_0} \to \mathbb{R}$ to the distribution.

→ the scalar product $\langle \cdot, \cdot \rangle_\lambda$ on $\mathcal{D}_{x_0}$ let us to define a family of symmetric operators

$$Q_\lambda(t) : \mathcal{D}_{x_0} \to \mathcal{D}_{x_0}, \quad d^2_{x_0} \dot{c}_t(v) = \langle Q_\lambda(t)v, v \rangle_\lambda, \quad v \in \mathcal{D}_{x_0}$$

Theorem

Assume the geodesic is ample. Then $Q_\lambda(t)$ has a second order pole at $t = 0$ and

$$Q_\lambda(t) \simeq \frac{1}{t^2} \mathcal{I}_\lambda + \frac{1}{3} \mathcal{R}_\lambda + O(t), \quad \text{for } t \to 0$$

where

- $\mathcal{I}_\lambda \geq \mathbb{I} > 0$
- $\mathcal{R}_\lambda$ is the curvature operator. We def $\text{Ric}(\lambda) := \text{trace}(\mathcal{R}_\lambda)$.

→ $\mathcal{I}_\lambda$ and $\mathcal{R}_\lambda$ are operators defined on $\mathcal{D}_{x_0}$ (symmetric wrt $\langle \cdot, \cdot \rangle_\lambda$).
<table>
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<th>Riemannian</th>
<th>Finsler</th>
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<tbody>
<tr>
<td>• any geodesic is ample with trivial growth vector $G_\gamma = {n}$</td>
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<tr>
<td>• $\mathcal{R}_\lambda = R^\nabla(\dot{\gamma}, \cdot)\dot{\gamma}$</td>
<td>• $\mathcal{R}<em>\lambda = R^F</em>\gamma$ - Flag curvature operator of Finsler manifolds</td>
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The Heisenberg case

The Heisenberg group $\mathbb{R}^3 = \{(x, y, z)\}$ with standard left-invariant structure

\[ X = \partial_x - \frac{y}{2} \partial_z, \quad Y = \partial_y + \frac{x}{2} \partial_z \]

Every (non trivial) geodesic is

- ample and equiregular
- with geodesic growth vector $G = (2, 3)$.

If one fix two geodesics $\gamma_\lambda(t), \gamma_\eta(s)$ corresponding to two covectors $\lambda, \eta$

- $C(t, s) := \frac{1}{2} d^2(\gamma_\lambda(t), \gamma_\eta(s))$ is not $C^2$ at zero!

Still we can determine the main expansion

\[ Q_\lambda(t) \simeq \frac{1}{t^2} I_\lambda + \frac{1}{3} R_\lambda + O(t), \quad \text{for } t \to 0 \]
The Heisenberg case

We compute it on the orthonormal basis $v := \dot{\gamma}(0)$ and $v^\perp := \dot{\gamma}(0)^\perp$ for $\mathcal{D}_{x_0}$.

The matrices representing $I_\lambda$ and $R_\lambda$ in the basis $\{v, v^\perp\}$ of $\mathcal{D}_{x_0}$ are

$$I_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \quad R_\lambda = \frac{2}{5} \begin{pmatrix} 0 & 0 \\ 0 & h_z^2 \end{pmatrix},$$  

(1)

where $\lambda$ has coordinates $(h_x, h_y, h_z)$ dual to o.n. basis $X, Y +$ Reeb $Z$

- anisotropy of the different directions on $\mathcal{D}_{x_0}$
- curvature is always zero in the direction of $\dot{\gamma}$
- curvature of lines $\{h_z = 0\}$ is zero.
- curvature of lift of circles $\{h_z \neq 0\}$ is not bounded (nor constant)

$\rightarrow$ trace($I_\lambda$) = 5 $\leftrightarrow$ is related to MCP(0,5).
Contact 3D case

Every (non trivial) geodesic is ample and equiregular with geodesic growth vector $\mathcal{G} = (2, 3)$.

The matrices representing $I_\lambda$ and $R_\lambda$ in the basis $\{\dot{\gamma}, \dot{\gamma}^\perp\}$ of $\mathcal{D}_{x_0}$ are

$$ I_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \quad R_\lambda = \frac{2}{5} \begin{pmatrix} 0 & 0 \\ 0 & r_\lambda \end{pmatrix}, \quad (2) $$

where

$$ r_\lambda = h_z^2 + \kappa (h_x^2 + h_y^2) + 3\chi (h_x^2 - h_y^2) $$

and $\lambda$ has coordinates $(h_x, h_y, h_z)$ dual to o.n. basis $X, Y + $ Reeb $Z$.

- $\chi$ and $\kappa$ are two functions on $M$ ($\chi = \kappa = 0$ in Heisenberg)
- In high dim contact case it is computed/computable ($\rightarrow$ in preparation)
- Using generalization of Jacobi fields
Assume now that the geodesic is also equiregular: \[\text{[← no dependence on } t\text{]}\]

- Define \(d_i := k_i - k_{i-1} = \dim \mathcal{F}_\gamma^i - \dim \mathcal{F}_\gamma^{i-1}\)
- By equiregularity one gets \(d_1 \geq d_2 \geq \ldots \geq d_m\).

Denote \(n_j := \text{length of the } j\text{-th row } (j = 1, \ldots, k)\)
The operator $\mathcal{I}_\lambda$

Being an operator on a Euclidean space $\mathcal{I}_\lambda : \mathcal{D}_{x_0} \to \mathcal{D}_{x_0}$ has well defined eigenvalues and trace.

**Theorem**

Assume the geodesic is **ample and equiregular**. Then

- $\text{spec} \mathcal{I}_\lambda = \{n_1^2, \ldots, n_k^2\}$
- $\text{tr} \mathcal{I}_\lambda = n_1^2 + \ldots + n_k^2$.

*In particular we can rewrite*

$$\text{tr} \mathcal{I}_\lambda = d_1 + 3d_2 + 5d_3 + \ldots + (2m-1)d_m$$

- Notice that $k_1 = \dim \mathcal{D}_{x_0}$ is the dimension of the distribution.
- In the Riemannian we recover $\mathcal{I}_\lambda$ is the identity.
- In the Heisenberg (and any 3D contact) $\text{tr} \mathcal{I}_\lambda = 5$ for every geodesic.
- Looks like an “Hausdorff dimension” with odd coeff.
Why “a variational approach”?

Two main reasons:

- Tonelli Lagrangian $\rightarrow$ include also Finsler case, and not restrict to (sub)-Riemannian situations
- Drift $\rightarrow$ To have more space to find constant curvature model

In our setting, even Heisenberg has no constant curvature ($\mathcal{R}_\lambda$ not constant with respect to $\lambda$)

Linear-Quadratic control system in $\mathbb{R}^n$

\[
\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^k, \quad J_T(x(\cdot)) = \frac{1}{2} \int_0^T |u|^2 - \langle Qx, x \rangle \, dt.
\]

with $A, B$ in normal form*, has constant curvature equal to $Q$.

(*) $A, B$ are in Brunowsky normal form [i.e. systems of the form $x^{(n)} = u$]

(**) controllability (Kalman) $\rightarrow$ any geodesic is ample and equiregular

\[\dim \mathcal{F}^i = \text{rank}(B, AB, \ldots, A^i B).\]

(***) $\text{spec}(I_\lambda) = \text{Kronecker indices of Brunovsky normal form}$
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SR case and sub-Laplacian operator

A sub-Riemannian manifold $M$ with o.n. frame $\mathcal{D}_x = \text{span}_x \{X_1, \ldots, X_k\}$

- horizontal gradient $\nabla f = \sum_{i=1}^k X_i(f) X_i$

Given a smooth measure $\mu$ we can introduce a sub-Laplacian operator

$$ \Delta_\mu = \text{div}_\mu \nabla = \sum_{i=1}^k X_i^2 + (\text{div}_\mu X_i) X_i $$

- it is essentially self-adjoint on $C^\infty_0(M)$ w.r.t. $\mu$
- if the structure is Riemannian, this is the Laplace-Beltrami operator

**Corollary (of the Main Theorem)**

$$ \Delta \dot{c}_t \big|_{x_0} \simeq \frac{\text{tr} I_\lambda}{t^2} + \frac{1}{3} R\text{ic}(\lambda) + O(t), \quad \text{for } t \to 0 $$

→ does not depend on $\mu$ since $\dot{c}_t$ has critical point at $x_0$.
→ it is just the trace of the main expansion $\Delta \dot{c}_t \big|_{x_0} = \text{trace}(d_{x_0}^2 \dot{c}_t)$
Application: sub-Laplacian of SR distance

We want a formula for the sub-Laplacian of the distance from a geodesic.

\[ f_t(\cdot) = \frac{1}{2} d^2(\cdot, \gamma(t)) = -tc_t(\cdot), \quad t \in (0, \varepsilon) \]

- \( f_t \) has no critical point at \( x_0 \) \( \rightarrow \) the volume should appear

**Theorem**

Assume \( \dim \mathcal{D}_x \) locally constant at \( x_0 \) and \( \gamma(t) = \exp_{x_0}(t\lambda) \) equiregular. Then there exists a smooth n-form \( \omega \) defined along \( \gamma \), such that for any smooth volume \( \mu \) s.t. \( \mu_{\gamma}(t) = e^{g(t)} \omega_{\gamma}(t) \), we have

\[ \Delta_{\mu} f_t \big|_{x_0} = \text{tr} \mathcal{I}_\lambda - \dot{g}(0)t - \frac{1}{3} \text{Ric}(\lambda)t^2 + O(t^3), \]

**Theorem**

Let \( \gamma \) be an equiregular geodesic with initial covector \( \lambda \in T^*_{x_0} M \). Then there exists a smooth n-form \( \omega \) defined along \( \gamma \), such that for any smooth volume \( \mu \) s.t. \( \mu_{\gamma}(t) = e^{g(t)} \omega_{\gamma}(t) \), we have
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Sub-Riemannian homotheties

- $M$ is a complete, connected, orientable sub-Riemannian manifold
- $\mu$ smooth volume form on $M$.

Denote $\Sigma_{x_0}$ is the open and dense set where the function $\tilde f = \frac{1}{2} d^2(x_0, \cdot)$ is smooth.

- The sub-Riemannian homothety is the map $\phi_t : \Sigma_{x_0} \to M$
  $$\phi_t(x) = \gamma_{x_0,x}(t) = \pi \circ e^{(t-1)\tilde H}(d_x\tilde f).$$

- Fix $\Omega \subset \Sigma_{x_0}$ be a bounded, measurable set, with $0 < \mu(\Omega) < +\infty$
- let $\Omega_{x_0,t} \doteq \phi_t(\Omega)$.

- The map $t \mapsto \mu(\Omega_{x_0,t})$ is smooth
- $\mu(\Omega_{x_0,t}) \to 0$ for $t \to 0$.

At which order $\mu(\Omega_{x_0,t}) \sim t$?

- the order does not depend on the smooth measure $\mu$!
Figure: Sub-Riemannian homothety of the set $\Omega$ with center $x_0$. 
Riemannian vs sub-Riemannian

Two basic examples

\((\mathcal{R})\) For a Riemannian structure, it is well known that

\[ \mu(\Omega_{x_0}, t) \sim t^{\dim M}, \quad \text{for } t \to 0, \]

[here \( f(t) \sim g(t) \) means \( f(t) = g(t)(C + o(1)) \) for \( t \to 0 \) and \( C > 0 \)]

\((\mathcal{SR})\) In the Heisenberg group it follows from [Juillet, 2009] that

\[ \mu(\Omega_{x_0}, t) \sim t^5, \quad \text{for } t \to 0, \]

5 is neither the topological (\( \dim = 3 \)) nor the metric dimension (\( \dim_H = 4 \)).

\( \rightarrow \) The exponent is a different dimensional invariant, associated with behavior of geodesics based at \( x_0 \).
Geodesic dimension

Definition

Let \( \gamma(t) = \text{Exp}(t, \lambda) \) be ample (at \( t = 0 \)) with \( G_\gamma = \{k_1, k_2, \ldots, k_m\} \). Then we define

\[
\mathcal{N}_\lambda \triangleq \sum_{i=1}^{m} (2i - 1)(k_i - k_{i-1}),
\]

and \( \mathcal{N}_\lambda \triangleq +\infty \) if the geodesic is not ample.

\[ \rightarrow \text{If } \lambda \text{ is associated with an equiregular geodesic } \gamma \text{ then } \mathcal{N}_\lambda = \text{trace } \mathcal{I}_\lambda. \]

The function \( \lambda \mapsto \mathcal{N}_\lambda \) is constant a.e. on \( T_{x_0}^* M \), assuming its minimum value.

\[
\mathcal{N}_{x_0} \triangleq \min\{\mathcal{N}_\lambda | \lambda \in T_{x_0}^* M\} < +\infty.
\]

We call \( \mathcal{N}_{x_0} \) the geodesic dimension of \( M \) at \( x_0 \).
Main result

- For every $x_0 \in M$ we have the inequality $\mathcal{N}_{x_0} \geq \text{dim } M$ and the equality holds if and only if the structure is Riemannian at $x_0$.
- If the distribution is equiregular at $x_0$ we have $\mathcal{N}_{x_0} > \text{dim}_\mathcal{H} M$ from Mitchell’s formula.

**Theorem**

*Let $\mu$ be a smooth volume. For any bounded, measurable set $\Omega \subset \Sigma_{x_0}$, with $0 < \mu(\Omega) < +\infty$ we have*

$$\mu(\Omega_{x_0}, t) \sim t^{\mathcal{N}_{x_0}}, \quad \text{for } t \to 0.$$  \hspace{1cm} (3)

- In general this number can depend on the point
- If the structure is left invariant is it does not.
- $\mathcal{N}_{x_0} = 5$ in 3D contact and $\mathcal{N}_{x_0} = 2n + 3$ in $(2n + 1)$-dim contact manifolds
An open question: MCP on Carnot Groups

We say that an $n$-dim manifold $M$ satisfies $MCP(0, D)$ if

$$\mu(\Omega_{x_0, t}) \geq t^D \mu(\Omega), \quad \text{for } t \in [0, 1]. \quad (4)$$

- this is a global estimate!
- Riemannian with $Ric \geq 0$ satisfy $MCP(0, n)$
- [Julliet, 2009] the Heisenberg group satisfies $MCP(0, 5)$
- [Rifford, 2013] Any ideal Carnot group satisfies $MCP(0, D)$ for some integer $D \geq Q + n - k$.

It is consistent with our theorem since $N \geq Q + n - k$.

- Conjecture: Carnot groups satisfy $MCP(0, N)$?
- For Step 2 free Carnot groups $Q + n - k \sim k^2$ and $N \sim k^3$
- at the moment $\rightarrow$ MCP(0,14) for the (3,6) Carnot group. (w/ L.Rizzi)
Free step 2 Carnot groups

These are Carnot groups with growth vector \((k, n)\) with

\[
n = k + \frac{k(k - 1)}{2}
\]

The generic geodesic is ample and equiregular with geodesic growth vector:

\[
\begin{array}{cccc}
& & & \\
& & k^2 & \\
& (k - 1)^2 & & \\
\vdots & & \vdots & \\
1^2 & & \cdots & 1 \\
k & & k - 1 & 2 \\
& & \cdots & \\
& & & 1
\end{array}
\]

\[
\rightarrow \text{spec}(\mathcal{I}_\lambda) = \{1^2, 2^2, \ldots, k^2\} \quad \text{and} \quad \mathcal{N} = \frac{k(k+1)(2k+1)}{6} \sim k^3
\]
Final remarks

→ Final remarks on MCP

• What about non nilpotent situations?
• Higher order terms in $\mu(\Omega_{x_0}, t)$? Bounds on curvature?

→ Open questions

• What about other kind of comparison? (Laplacian eigenvalues, etc.)
• Comparison for solution of heat equation? (the function $c_t$ appears in the heat kernel expression)

\[
p_t(x, y) = \frac{1}{t^{n/2}} \exp\left(-\frac{d^2(x, y)}{4t}\right)\left(C + O(t)\right)
\]

• study of the heat kernel along geodesics
THANK YOU FOR YOUR ATTENTION!