Generic regularity of weak KAM solutions and Mañé conjecture

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Hamilton-Jacobi equations

- Let M be a smooth manifold of dimension n.
- Let $H : T^*M \longrightarrow \mathbb{R}$ be an Hamiltonian of class at least C^2.
- Let $c \in \mathbb{R}$ be fixed.
- Let $u : M \longrightarrow \mathbb{R}$ be a viscosity solution of the HJ equation

$$(HJ) \qquad\qquad H(x, d_x u) = c \qquad \forall x \in M.$$

Problem : Regularity properties of u ?

A general theorem

Additional assumptions on the Hamiltonian:

(H1) For every $K \geq 0$, there is $C^*(K) < \infty$ such that

$$\forall (x,p) \in T^*M, \quad H(x,p) \geq K\|p\| - C^*(K).$$

(H2) For every $(x,p) \in T^*M$, $\frac{\partial^2 H}{\partial p^2}(x,p)$ is positive definite.

Theorem (LR 2009)

Let $u : M \to \mathbb{R}$ be a viscosity solution of (HJ). Then u is locally semiconcave on M, its singular set $\Sigma(u)$ is nowhere dense in M, and u is $C^{1,1}_{loc}$ on the open dense set $M \setminus \overline{\Sigma(u)}$.

Here, $\Sigma(u) := \{x \in M \mid u \text{ is not differentiable at } x\}$.

Sard-type results I

Let $u : M \longrightarrow \mathbb{R}$ be a locally Lipschitz function.
We call *critical point* of u any $x \in M$ such that

$$0 \in \operatorname{conv}\left\{\lim_k d_{x_k} u \,|\, x_k \in D_u \text{ and } x_k \to x\right\},$$

where $D_u := M \setminus \Sigma(u)$.
We denote by $\mathcal{C}(u)$ the set of critical points of u.
Thanks to the Clarke Implicit Function Theorem, there holds

Proposition

For every $\lambda \in u(M) \setminus u\big(\mathcal{C}(u)\big)$, the level set

$$\{u(x) = \lambda \,|\, x \in M\}$$

is a locally Lipschitz hypersurface in M.

Sard-type results II

Theorem (Ferry 1976, Fu 1985, Bates 1993)

Assume that M has dimension 1 or 2. Then for every locally semiconcave function $u : M \to \mathbb{R}$, the set $u(\mathcal{C}(u))$ has Lebesgue measure zero.

Theorem (LR 2009)

Assume that $H : T^*M \to \mathbb{R}$ satisfies (H1)-(H2), that M has dimension 3, and that one of the following assumption holds:

- *M and H are real-analytic.*
- *H is at least C^4 and $H(x,0) = c$ for any $x \in M$.*
- *H is at least C^4 and $\{H(x,0) = c\} \subset \left\{\frac{\partial H}{\partial p}(x,0) = 0\right\}$.*

If u is viscosity solution of (HJ), then the set $u(\mathcal{C}(u))$ has Lebesgue measure zero.

HJ equations with Dirichlet conditions I

Let $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be an Hamiltonian of class $C^{k,1}$ (with $k \geq 2$) which satisfies:

(H1) For every $K \geq 0$, there is $C^*(K) < \infty$ such that

$$\forall (x,p) \in T^*M, \quad H(x,p) \geq K|p| - C^*(K).$$

(H2) For every $(x,p) \in T^*M$, $\frac{\partial^2 H}{\partial p^2}(x,p)$ is positive definite.

(H3) For every $x \in \mathbb{R}^n$, $H(x,0) < 0$.

Let Ω be an open set in $\mathbb{R}^n$ with compact boundary and $u : \overline{\Omega} \to \mathbb{R}$ be the viscosity solution of

$$\text{(HJ)} \qquad \begin{cases} H(x, \nabla_x u) = 0 & \forall x \in \Omega, \\ u_{|\partial\Omega} = 0. \end{cases}$$

HJ equations with Dirichlet conditions II

Observation:

Let $x \in \Omega$ and $p \in \{\lim_k \nabla_{x_k} u \,|\, x_k \in D_u \text{ and } x_k \to x\}$ be fixed. Then there is an extremal $\big(x(\cdot), p(\cdot)\big) : [-T, 0] \to \overline{\Omega} \times \mathbb{R}^n$ such that

$$\begin{cases} \dot{x}(t) & = & \nabla_p H\big(x(t), p(t)\big) \\ \dot{p}(t) & = & -\nabla_x H\big(x(t), p(t)\big) \end{cases} \qquad \forall t \in [-T, 0]$$

with

$$x(0) = x \quad \text{and} \quad p(0) = p.$$

Moreover, u is differentiable for every $t \in [-T, 0)$ and

$$\nabla_{x(t)} u = p(t).$$

HJ equations with Dirichlet conditions III

Theorem (Li-Nirenberg 2005, Castelpietra-LR 2008)

Assume that H and $\partial\Omega$ are of class $C^{2,1}$. Then the cut locus of u defined as $Cut(u) := \overline{\Sigma(u)}$ has a finite $(n-1)$-dimensional Hausdorff measure.

Theorem (LR 2009)

Assume that H and $\partial\Omega$ are of class C^k with $k \geq 2n^2 + 4n + 1$. Then the set $u(\mathcal{C}(u))$ has Lebesgue measure zero.

Fathi's weak KAM Theorem

From now on, M is assumed to be compact without boundary and H is an Hamiltonian of class C^2 satisfying (H1)-(H2).

Theorem (Fathi 1997)

There is a unique value $c = c[H] \in \mathbb{R}$ such that the Hamilton-Jacobi equation

$$(HJ) \qquad H(x, d_x u) = c \qquad \forall x \in M,$$

admits at least one viscosity solution.

No boundary conditions !!

The Aubry set I

Denote by $\mathcal{S}(H)$ the set of viscosity solution $u : M \to \mathbb{R}$ of the critical Hamilton-Jacobi equation

$$(HJ) \qquad H(x, d_x u) = c[H] \qquad \forall x \in M.$$

Such solutions are called weak KAM solutions or critical viscosity solutions.

Theorem

The set $\tilde{A}(H) \subset T^*M$ defined as

$$\tilde{A}(H) := \cap \{ Graph(du) \,|\, u \in \mathcal{S}(H) \}$$

is a nonempty compact set which is invariant under the Hamiltonian flow.

The Aubry set II

Let $u : M \longrightarrow \mathbb{R}$ be a critical viscosity solution. Let $x \in M$ and $p \in \{\lim_k d_{x_k} u \,|\, x_k \in D_u \text{ and } x_k \to x\}$ be fixed. Then there is an extremal $\big(x(\cdot), p(\cdot)\big) : (-\infty, 0] \longrightarrow T^*M$ such that

$$\begin{cases} \dot{x}(t) &= \nabla_p H\big(x(t), p(t)\big) \\ \dot{p}(t) &= -\nabla_x H\big(x(t), p(t)\big) \end{cases} \qquad \forall t \in (-\infty, 0]$$

with $x(0) = x$, $p(0) = p$, and $d_{x(t)} u = p(t)$, for any $t \leq 0$.

Proposition

$$\lim_{t \to -\infty} d\Big(\big(x(t), p(t)\big), \tilde{A}(H)\Big) = 0.$$

The projected Aubry set $\mathcal{A}(H) := \pi\big(\tilde{A}(H)\big) \subset M$ plays the role of the "Dirichlet" boundary.

A result by Bernard

Theorem (Bernard 2007)

Let H be a C^k Hamiltonian satisfying (H1)-(H2) with $2 \leq k \leq \infty$. Assume that the Aubry set $\tilde{A}(H)$ is either an hyperbolic periodic orbit or an hyperbolic fixed point. Then any critical viscosity solution is C^k in a neighborhood of $\mathcal{A}(H)$.

Is this kind of regularity generic ?

The Mañé Conjecture

Given a Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class C^k (with $k \geq 2$) and a potential $V : M \to \mathbb{R}$ of class C^k, we define the Hamiltonian $H_V : T^*M \to \mathbb{R}$ by

$$H_V(x,p) := H(x,p) + V(x) \qquad \forall (x,p) \in T^*M.$$

Denote by $C^k(M)$ the set of C^k potentials on M equipped with the C^k-topology.

Conjecture (Mañé 1996)

For every Tonelli Hamiltonian H of class C^k (with $k \geq 2$), there is a residual subset $\mathcal{G}$ of $C^k(M)$ such that, for every $V \in \mathcal{G}$, the Aubry set of the Hamiltonian H_V is either an hyperbolic fixed point or an hyperbolic periodic orbit.

A first step toward the Mañé Conjecture in C^2 topology

Theorem (Figalli, Rifford 2010)

Let H be a Tonelli Hamiltonian H of class C^k with $k \geq 4$ and $\epsilon > 0$ be fixed. Assume that there is a critical viscosity subsolution which is at least C^{k+1} on a neighborhood of $\mathcal{A}(H)$. Then there is a potential $V : M \to \mathbb{R}$ of class C^{k-1} with $\|V\|_{C^2} < \epsilon$ such that $c[H_V] = c[H]$ and the Aubry set of H_V is either an hyperbolic equilibrium or an hyperbolic periodic orbit.

Thank you for your attention !