The cut locus in optimal transportation

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THE CUT LOCUS
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Let $M$ be a smooth connected (compact) manifold of dimension $n$ equipped with a smooth Riemannian metric $g$. For any $x, y \in M$, we define the geodesic distance between $x$ and $y$, denoted by $d(x, y)$, as the minimum of the lengths of the curves joining $x$ to $y$. 

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Let $\mu_0$ and $\mu_1$ be **probability measures** on $M$. We call **transport map** from $\mu_0$ to $\mu_1$ any measurable map $T : M \to M$ such that $T_\#\mu_0 = \mu_1$, that is

$$\mu_1(B) = \mu_0(T^{-1}(B)), \quad \forall B \text{ measurable } \subset M.$$
Quadratic Monge’s Problem

Given two probabilities measures $\mu_0, \mu_1$ sur $M$, find a transport map $T : M \to M$ from $\mu_0$ to $\mu_1$ which minimizes the quadratic cost $(c = d^2/2)$

$$\int_M c(x, T(x))d\mu_0(x).$$

Theorem (McCann ’01)

If $\mu_0$ is absolutely continuous w.r.t. Lebesgue, then there exists a unique optimal transport map $T$ from $\mu_0$ to $\mu_1$. In fact, there is a $c$-convex function $\varphi : M \to \mathbb{R}$ satisfying

$$T(x) = \exp_x (\nabla \varphi(x)) \quad \mu_0 \text{ a.e. } x \in M.$$

(Moreover, for a.e. $x \in M$, $\nabla \varphi(x)$ belongs to the injectivity domain at $x$.)
Purpose of the talk

- Regularity of optimal transport maps
- Link with Monge-Ampère like equations
- How the geometry enters the problem
- State of the art
- Open questions
Quadratic Monge’s Problem in $\mathbb{R}^n$: Given two probability measures $\mu_0, \mu_1$ with compact supports in $\mathbb{R}^n$, we are concerned with transport maps $T: \mathbb{R}^n \to \mathbb{R}^n$ pushing forward $\mu_0$ to $\mu_1$ which minimize the transport cost

$$\int_{\mathbb{R}^n} |T(x) - x|^2 \, d\mu_0(x).$$

**Theorem (Brenier ’91)**

If $\mu_0$ is absolutely continuous with respect to the Lebesgue measure, there exists a unique optimal transport map with respect to the quadratic cost. In fact, there is a **convex** function $\psi: M \to \mathbb{R}$ such that

$$T(x) = \nabla \psi(x) \quad \mu_0 \text{ a.e. } x \in \mathbb{R}^n.$$
An obvious counterexample

\[ \text{Supp}(\mu_0) \rightarrow T \rightarrow \text{Supp}(\mu_1) \]
The convexity of the target is necessary.
The convexity of the target is necessary.
The convexity of the target is necessary

Supp($\mu_0$) \quad Supp($\mu_1$)

70% \quad 29%

49% \quad 49%
The convexity of the target is necessary

\[ T \text{ gradient of a convex function } \Rightarrow \langle y-x, T(y) - T(x) \rangle \geq 0 \]

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Caffarelli’s Regularity Theory

If $\mu_0$ and $\mu_1$ are associated with densities $f_0, f_1$ w.r.t. Lebesgue, then

$$T_#\mu_0 = \mu_1 \iff \int_{\mathbb{R}^n} \zeta(T(x)) f_0(x) \, dx = \int_{\mathbb{R}^n} \zeta(y) f_1(y) \, dy \quad \forall \zeta.$$ 

$\Rightarrow \psi$ weak solution of the Monge-Ampère equation:

$$\det \left( \nabla^2 \psi(x) \right) = \frac{f_0(x)}{f_1(\nabla \psi(x))}.$$ 

Theorem (Caffarelli ’90s)

Let $\Omega_0, \Omega_1$ be connected and bounded open sets in $\mathbb{R}^n$ and $f_0, f_1$ be probability densities resp. on $\Omega_0$ and $\Omega_1$ such that $f_0, f_1, 1/f_0, 1/f_1$ are essentially bounded. Assume that $\mu_0$ and $\mu_1$ have respectively densities $f_0$ and $f_1$ w.r.t. Lebesgue and that $\Omega_1$ is convex. Then the quadratic optimal transport map from $\mu_0$ to $\mu_1$ is continuous.
Given two probabilities measures $\mu_0, \mu_1$ sur $M$, find a transport map $T : M \rightarrow M$ from $\mu_0$ to $\mu_1$ which minimizes the quadratic cost ($c = d^2/2$)

$$\int_M c(x, T(x))d\mu_0(x).$$

**Definition**

We say that the Riemannian manifold $(M, g)$ satisfies the **Transport Continuity Property (TCP)** if for any pair of probability measures $\mu_0, \mu_1$ associated locally with **continuous positive densities** $\rho_0, \rho_1$, that is

$$\mu_0 = \rho_0 \text{vol}_g, \quad \mu_1 = \rho_1 \text{vol}_g,$$

the optimal transport map from $\mu_0$ to $\mu_1$ is **continuous**.
Let \( x \in M \) be fixed.

- For every \( v \in T_x M \), we define the **exponential** of \( v \) by

  \[
  \exp_x(v) = \gamma_{x,v}(1),
  \]

  where \( \gamma_{x,v} : [0, 1] \to M \) is the unique geodesic starting at \( x \) with speed \( v \).

- We call **injectivity domain** at \( x \), the subset of \( T_x M \) defined by

  \[
  \mathcal{I}(x) := \left\{ v \in T_x M \mid \exists t > 1 \text{ s.t. } \gamma_{tv} \text{ is the unique minim. geod. between } x \text{ and } \exp_x(tv) \right\}
  \]

  It is a star-shaped (w.r.t. \( 0 \in T_x M \)) bounded open set with Lipschitz boundary.
The distance to the cut locus

Using the exponential mapping, we can associate to each unit tangent vector its **distance to the cut locus**.

In that way, we define the so-called **injectivity domain** $\mathcal{I}(x)$ whose boundary is the **tangent cut locus** $\text{TCL}(x)$. 
A necessary condition for TCP

**Theorem (Figalli-R-Villani ’10)**

Let \((M, g)\) be a smooth compact Riemannian manifold satisfying TCP. Then following properties hold:

- all the injectivity domains are convex,
- for any \(x, x' \in M\), the function

\[ F_{x,x'} : v \in \mathcal{I}(x) \mapsto c(x, \exp_x(v)) - c(x', \exp_x(v)) \]

**is quasiconvex** (its sublevel sets are always convex).

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Lemma (Sufficient condition)

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class $C^2$. Assume that for every $v \in U$ and every $w \in \mathbb{R}^n \setminus \{0\}$ the following property holds:

$$\langle \nabla_v F, w \rangle = 0 \implies \langle \nabla^2_v F w, w \rangle > 0.$$

Then $F$ is quasiconvex.

Lemma (Necessary condition)

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class $C^2$. Assume that $F$ is quasiconvex. Then for every $v \in U$ and every $w \in \mathbb{R}^n \setminus \{0\}$ the following property holds:

$$\langle \nabla_v F, w \rangle = 0 \implies \langle \nabla^2_v F w, w \rangle \geq 0.$$
Proof.

Let $\nu_0, \nu_1 \in U$ be fixed. Set $\nu_t := (1 - t)\nu_0 + t\nu_1$, for every $t \in [0, 1]$. Define $h : [0, 1] \to \mathbb{R}$ by

$$h(t) := F(\nu_t) \quad \forall t \in [0, 1].$$

If $h \not\leq \max\{h(0), h(1)\}$, there is $\tau \in (0, 1)$ such that

$$h(\tau) = \max_{t \in [0, 1]} h(t) > \max\{h(0), h(1)\}.$$

There holds

$$\dot{h}(\tau) = \langle \nabla_{\nu_\tau} F, \dot{\nu}_\tau \rangle \quad \text{et} \quad \ddot{h}(\tau) = \langle \nabla^2_{\nu_\tau} F \dot{\nu}_\tau, \dot{\nu}_\tau \rangle.$$

Since $\tau$ is a local maximum, one has $\dot{h}(\tau) = 0$.

Contradiction !!
Assume that all the injectivity domains are convex and fix $x, x' \in M$. Recall that

$$F_{x,x'}(v) = F(v) = c(x, \exp_x(v)) - c(x', \exp_x(v)).$$

- $F$ is not smooth.
- For generic segments, $t \mapsto F(v_t)$ is smooth outside a finite set of ”convex” times.
- If it is smooth at $v$, then $\nabla^2_v F$ has the form

$$\nabla^2_v F(h, h) = - \int_0^1 (1 - t) \frac{\partial^4 c}{\partial^2 x \partial^2 y}(\ast)(\ast) \, dt$$
The Ma-Trudinger-Wang tensor is defined as

$$\mathcal{S}_{(x,v)}(\xi, \eta) = -\frac{3}{2} \left. \frac{d^2}{ds^2} \right|_{s=0} \left. \frac{d^2}{dt^2} \right|_{t=0} c(\exp_x(t\xi), \exp_x(v + s\eta)),$$

for every \(x \in M\), \(v \in \mathcal{I}(x)\), and \(\xi, \eta \in T_xM\).

**Proposition (Villani '09, Figalli-R-Villani '10)**

Let \((M, g)\) be such that all injectivity domains are convex. Then the following properties are equivalent:

- All the functions \(F_{x,x'}\) are quasiconvex.
- The **MTW** tensor \(\mathcal{S}\) is \(\succeq 0\), that is for any \(x \in M\), \(v \in \mathcal{I}(x)\), and \(\xi, \eta \in T_xM\),

$$\langle \xi, \eta \rangle_x = 0 \implies \mathcal{S}_{(x,v)}(\xi, \eta) \geq 0.$$
When geometry enters the problem

Theorem (Loeper ’06)

For every \( x \in M \) and for any pair of unit orthogonal tangent vectors \( \xi, \eta \in T_x M \), there holds

\[
S_{(x,0)}(\xi, \eta) = \kappa_x(\xi, \eta),
\]

where the latter denotes the sectional curvature of \( M \) at \( x \) along the plane spanned by \( \{\xi, \eta\} \).

Corollary (Loeper ’06)

\[
TCP \implies \mathcal{S} \succeq 0 \implies \kappa \geq 0.
\]

Caution!!! \( \kappa \geq 0 \nRightarrow \mathcal{S} \succeq 0 \).
Sufficient conditions for TCP

**Theorem (Figalli-R-Villani '10)**

Let \((M, g)\) be a compact smooth Riemannian surface. It satisfies TCP if and only if the two following properties hold:

- all the injectivity domains are convex,
- \(\mathcal{G} \succeq 0\).

**Theorem (Figalli-R-Villani '10)**

Assume that \((M, g)\) satisfies the two following properties:

- all its injectivity domains are strictly convex,
- the MTW tensor \(\mathcal{G}\) is \(\succ 0\), that is for any \(x \in M\), \(v \in \mathcal{I}(x)\), and \(\xi, \eta \in T_x M\),

\[
\langle \xi, \eta \rangle_x = 0 \implies \mathcal{G}_{(x,v)}(\xi, \eta) > 0.
\]

Then, it satisfies TCP.

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Examples:
- Flat tori (Cordero-Erausquin ’99)
- Round spheres (Loeper ’06)
- Quotients of the above objects
- Oblate ellipsoids of revolution (Bonnard-Caillau-R ’10)
- Perturbations of round spheres (Figalli-R-Villani ’12)
- Product of spheres (Figalli-Kim-McCann ’13)

Counterexamples:
Questions

- Does $\mathcal{G} \succeq 0 \implies \textbf{TCP}$ in dimension $\geq 3$?
- Smoother data imply further regularity?
- How is the set of metrics satisfying $\mathcal{G} \succ 0$?
- Does $\mathcal{G} \succeq 0$ imply convexity of injectivity domains?
- Does $\mathcal{G} \succeq 0$ imply more topological obstructions than $\kappa \geq 0$?

Focalization is the major obstacle
Focalization

Definition
Let $x \in M$ and $v$ be a unit tangent vector in $T_x M$. The vector $v$ is \textbf{not conjugate} at time $t \geq 0$ if for any $t' \in [0, t + \delta]$ ($\delta > 0$ small) the geodesic from $x$ to $\gamma(t')$ is locally minimizing.

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Again, using the exponential mapping, we can associate to each unit tangent vector its **distance to the conjugate locus**.

In that way, we define the so-called **nonfocal domain** $\mathcal{N}(x)$ whose boundary is the **tangent conjugate locus** $TFL(x)$. 
The following inclusion holds

injectivity domain $\subset$ nonfocal domain.

Moreover, on a surface the nonfocal domain has smooth boundary !!!
The convex Earth theorem

Theorem (Figalli-R ’09)

Any small deformation of $\mathbb{S}^2$ in $C^5$ topology satisfies $\mathcal{S} \geq 1/2$, has convex injectivity domains and satisfies TCP.
Theorem (Loeper ’06)

The MTW tensor on the round (unit) sphere $S^2$ satisfies $\mathcal{G} \succeq 1$, that is for any $x \in S^2$, $v \in I(x)$ and $\xi, \eta \in T_x S^2$,

$$\langle \xi, \eta \rangle_x = 0 \implies \mathcal{G}_{(x,v)}(\xi, \eta) \geq |\xi|^2 |\eta|^2.$$ 

In particular, the round sphere $S^2$ satisfies TCP.

Two issues for stability

- Stability of the (uniform) convexity of injectivity domains.
- Stability of properties of the form $\mathcal{G} \succeq K$ with $K > 0$. 
As a matter of fact, on $\mathbb{S}^2$, the **MTW** tensor is given by

\[
\mathfrak{g}_{(x,v)}(\xi, \xi^\perp) = 3 \left[ \frac{1}{r^2} - \frac{\cos(r)}{r \sin(r)} \right] \xi_1^4 + 3 \left[ \frac{1}{\sin^2(r)} - \frac{r \cos(r)}{\sin^3(r)} \right] \xi_2^4 + \frac{3}{2} \left[ -\frac{6}{r^2} + \frac{\cos(r)}{r \sin(r)} + \frac{5}{\sin^2(r)} \right] \xi_1^2 \xi_2^2,
\]

with $x \in \mathbb{S}^2$, $v \in \mathcal{I}(x)$, $r := |v|$, $\xi = (\xi_1, \xi_2)$, $\xi^\perp = (-\xi_2, \xi_1)$.
Sketch of proof

We extend the Ma-Trudinger-Wang tensor beyond boundaries of injectivity domains.

The uniform convexity of the nonfocal domains $\mathcal{NF}(x)$ is stable.

The positivity of the extended tensor $\overline{\mathcal{S}}$ is stable.

$\overline{\mathcal{S}} \succ 0 + \text{(uniform) convexity of the } \mathcal{NF}(x)\text{'s} \implies \mathcal{S} \succ 0 + \text{(uniform) convexity of the } \mathcal{I}(x)\text{'s}.$
Thank you for your attention !!