The cut locus in optimal transportation

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THE CUT LOCUS
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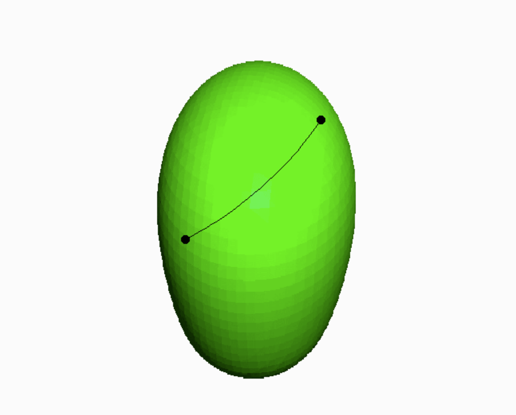
The framework

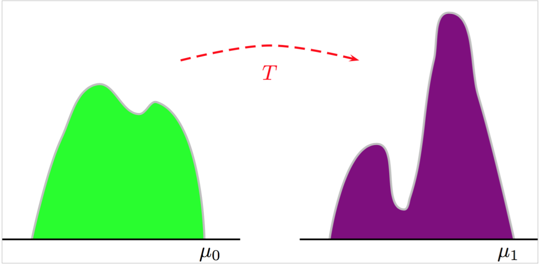
Let M be a **smooth connected (compact) manifold of dimension** n equipped with **a smooth Riemannian metric** g. For any $x, y \in M$, we define the **geodesic distance** between x and y, denoted by $d(x,y)$, as the minimum of the lengths of the curves joining x to y.



Transport maps

Let μ_0 and μ_1 be **probability measures** on M. We call **transport map** from μ_0 to μ_1 any measurable map $T : M \to M$ such that $T_\sharp \mu_0 = \mu_1$, that is

$$\mu_1(B) = \mu_0\big(T^{-1}(B)\big), \quad \forall B \text{ measurable } \subset M.$$



Quadratic Monge's Problem

Given two probabilities measures μ_0, μ_1 sur M, find a transport map $T : M \to M$ from μ_0 to μ_1 which **minimizes** the quadratic cost ($c = d^2/2$)

$$\int_M c(x, T(x)) d\mu_0(x).$$

Theorem (McCann '01)

*If μ_0 is absolutely continuous w.r.t. Lebesgue, then there exists a unique optimal transport map T from μ_0 to μ_1. In fact, there is a c-**convex** function $\varphi : M \to \mathbb{R}$ satisfying*

$$T(x) = \exp_x(\nabla\varphi(x)) \qquad \mu_0 \text{ a.e. } x \in M.$$

(Moreover, for a.e. $x \in M$, $\nabla\varphi(x)$ belongs to the injectivity domain at x.)

Purpose of the talk

- Regularity of optimal transport maps
- Link with Monge-Ampère like equations
- How the geometry enters the problem
- State of the art
- Open questions

Back to $\mathbb{R}^n$

Quadratic Monge's Problem in $\mathbb{R}^n$: Given two probability measures μ_0, μ_1 with compact supports in $\mathbb{R}^n$, we are concerned with transport maps $T : \mathbb{R}^n \to \mathbb{R}^n$ pushing forward μ_0 to μ_1 which **minimize** the transport cost

$$\int_{\mathbb{R}^n} |T(x) - x|^2 \, d\mu_0(x).$$

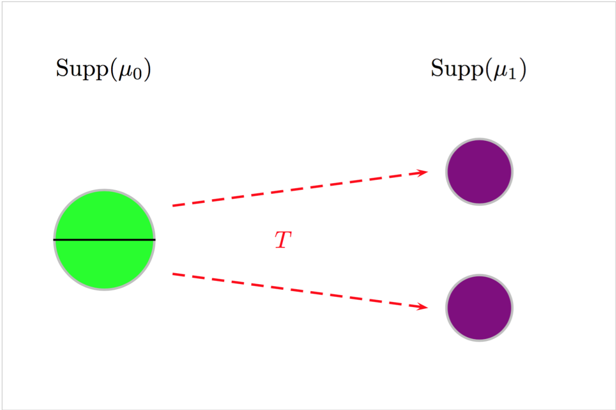
Theorem (Brenier '91)

If μ_0 is absolutely continuous with respect to the Lebesgue measure, there exists a unique optimal transport map with respect to the quadratic cost. In fact, there is a **convex** *function $\psi : M \to \mathbb{R}$ such that*

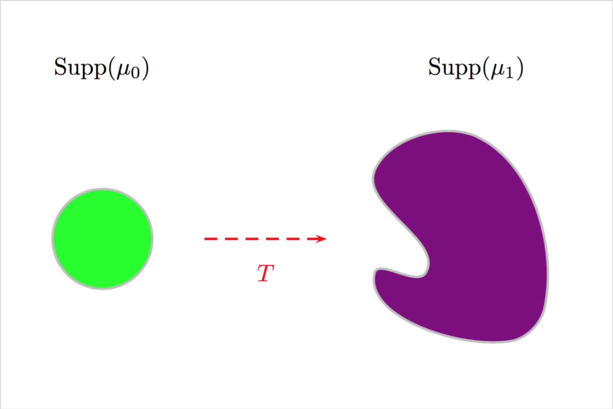
$$T(x) = \nabla\psi(x) \qquad \mu_0 \text{ a.e. } x \in \mathbb{R}^n.$$

Necessary and sufficient conditions for regularity ?

An obvious counterexample



The convexity of the targer is necessary



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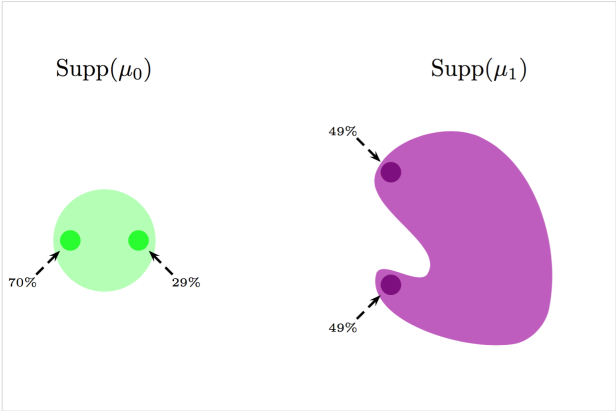

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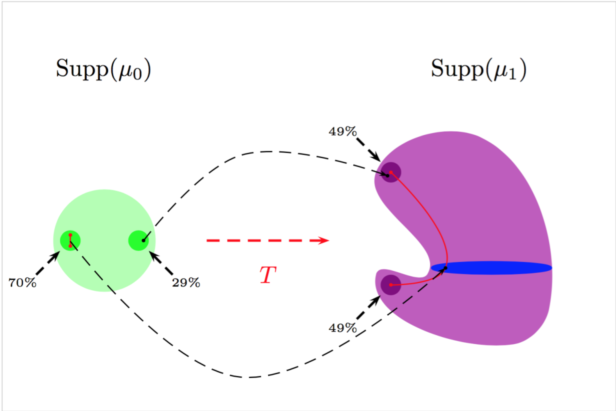

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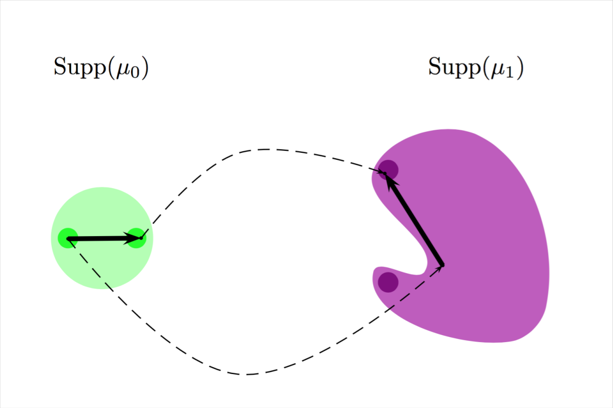


T gradient of a convex function $\Longrightarrow \langle y-x, T(y)-T(x)\rangle \geq 0$!!!

Caffarelli's Regularity Theory

If μ_0 and μ_1 are associated with densities f_0, f_1 w.r.t. Lebesgue, then

$$T_\sharp \mu_0 = \mu_1 \iff \int_{\mathbb{R}^n} \zeta\big(T(x)\big) f_0(x)dx = \int_{R^n} \zeta(y) f_1(y)dy \quad \forall \zeta.$$

$\rightsquigarrow$ ψ weak solution of the **Monge-Ampère equation** :

$$\det\left(\nabla^2\psi(x)\right) = \frac{f_0(x)}{f_1(\nabla\psi(x))}.$$

Theorem (Caffarelli '90s)

Let Ω_0, Ω_1 be connected and bounded open sets in $\mathbb{R}^n$ and f_0, f_1 be probability densities resp. on Ω_0 and Ω_1 such that $f_0, f_1, 1/f_0, 1/f_1$ are **essentially bounded***. Assume that μ_0 and μ_1 have respectively densities f_0 and f_1 w.r.t. Lebesgue and that Ω_1 is* **convex***. Then the quadratic optimal transport map from μ_0 to μ_1 is continuous.*

Back to Riemannian manifolds

Given two probabilities measures μ_0, μ_1 sur M, find a transport map $T : M \to M$ from μ_0 to μ_1 which minimizes the quadratic cost ($c = d^2/2$)

$$\int_M c(x, T(x)) d\mu_0(x).$$

Definition

We say that the Riemannian manifold (M, g) satisfies the **Transport Continuity Property (TCP)** if for any pair of probability measures μ_0, μ_1 associated locally with **continuous positive densities** ρ_0, ρ_1, that is

$$\mu_0 = \rho_0 \text{vol}_g, \quad \mu_1 = \rho_1 \text{vol}_g,$$

the optimal transport map from μ_0 to μ_1 is **continuous**.

Exponential mapping and injectivity domains

Let $x \in M$ be fixed.

- For every $v \in T_xM$, we define the **exponential** of v by

$$\exp_x(v) = \gamma_{x,v}(1),$$

  where $\gamma_{x,v} : [0,1] \to M$ is the unique geodesic starting at x with speed v.

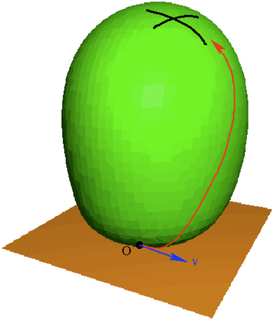
- We call **injectivity domain** at x, the subset of T_xM defined by

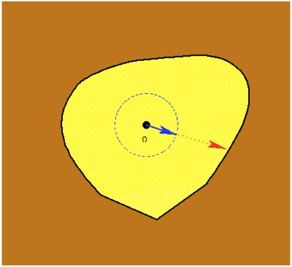
$$\mathcal{I}(x) := \left\{ v \in T_xM \,\middle|\, \begin{array}{c} \exists t > 1 \text{ s.t. } \gamma_{tv} \text{ is the unique} \\ \text{minim. geod. between } x \text{ and } \exp_x(tv) \end{array} \right\}$$

  It is a star-shaped (w.r.t. $0 \in T_xM$) bounded open set with Lipschitz boundary.

The distance to the cut locus

Using the exponential mapping, we can associate to each unit tangent vector its **distance to the cut locus**.





In that way, we define the so-called **injectivity domain** $\mathcal{I}(x)$ whose boundary is the **tangent cut locus** $\mathrm{TCL}(x)$.

A necessary condition for **TCP**

Theorem (Figalli-R-Villani '10)

Let (M, g) be a smooth compact Riemannian manifold satisfying **TCP**. *Then following properties hold:*

- *all the injectivity domains are convex,*
- *for any $x, x' \in M$, the function*

$$F_{x,x'} \; : \; v \in \mathcal{I}(x) \longmapsto c\big(x, \exp_x(v)\big) - c\big(x', \exp_x(v)\big)$$

  is **quasiconvex** *(its sublevel sets are always convex).*

Almost characterization of quasiconvex functions

Lemma (Sufficient condition)

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class C^2. Assume that for every $v \in U$ and every $w \in \mathbb{R}^n \setminus \{0\}$ the following property holds :

$$\langle \nabla_v F, w \rangle = 0 \Longrightarrow \langle \nabla_v^2 F\, w, w \rangle > 0.$$

Then F is quasiconvex.

Lemma (Necessary condition)

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class C^2. Assume that F is quasiconvex. Then for every $v \in U$ and every $w \in \mathbb{R}^n \setminus \{0\}$ the following property holds :

$$\langle \nabla_v F, w \rangle = 0 \Longrightarrow \langle \nabla_v^2 F\, w, w \rangle \geq 0.$$

Proof of the easy lemma

Proof.

Let $v_0, v_1 \in U$ be fixed. Set $v_t := (1-t)v_0 + tv_1$, for every $t \in [0,1]$. Define $h : [0,1] \to \mathbb{R}$ by

$$h(t) := F(v_t) \qquad \forall t \in [0,1].$$

If $h \not\leq \max\{h(0), h(1)\}$, there is $\tau \in (0,1)$ such that

$$h(\tau) = \max_{t\in[0,1]} h(t) \; > \; \max\{h(0), h(1)\}.$$

There holds

$$\dot{h}(\tau) = \langle \nabla_{v_\tau} F, \dot{v}_\tau \rangle \quad \text{et} \quad \ddot{h}(\tau) = \langle \nabla^2_{v_\tau} F \, \dot{v}_\tau, \dot{v}_\tau \rangle.$$

Since τ is a local maximum, one has $\dot{h}(\tau) = 0$.
Contradiction !! □

Back to our problem

Assume that all the injectivity domains are convex and fix $x, x' \in M$. Recall that

$$F_{x,x'}(v) = F(v) = c\big(x, \exp_x(v)\big) - c\big(x', \exp_x(v)\big).$$

- F is not smooth.
- For generic segments, $t \mapsto F(v_t)$ is smooth outside a finite set of "convex" times.
- If it is smooth at v, then $\nabla_v^2 F$ has the form

$$\nabla_v^2 F(h,h) = -\int_0^1 (1-t) \frac{\partial^4 c}{\partial^2 x \partial^2 y}(*)(*)\, dt$$

The Ma-Trudinger-Wang tensor

The **MTW** tensor $\mathfrak{S}$ is defined as

$$\mathfrak{S}_{(x,v)}(\xi,\eta) = -\frac{3}{2}\,\frac{d^2}{ds^2}\Big|_{s=0}\,\frac{d^2}{dt^2}\Big|_{t=0}\,c\left(\exp_x(t\xi),\exp_x(v+s\eta)\right),$$

for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi,\eta \in T_xM$.

Proposition (Villani '09, Figalli-R-Villani '10)

Let (M,g) be such that all injectivity domains are convex. Then the following properties are equivalent:

- *All the functions $F_{x,x'}$ are quasiconvex.*
- *The* **MTW** *tensor $\mathfrak{S}$ is $\succeq 0$, that is for any $x \in M$, $v \in \mathcal{I}(x)$, and $\xi,\eta \in T_xM$,*

$$\langle \xi,\eta\rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi,\eta) \geq 0.$$

When geometry enters the problem

Theorem (Loeper '06)

For every $x \in M$ and for any pair of unit orthogonal tangent vectors $\xi, \eta \in T_xM$, there holds

$$\mathfrak{S}_{(x,0)}(\xi,\eta) = \kappa_x(\xi,\eta),$$

where the latter denotes the **sectional curvature** *of M at x along the plane spanned by $\{\xi,\eta\}$.*

Corollary (Loeper '06)

$$\textbf{TCP} \Longrightarrow \mathfrak{S} \succeq 0 \Longrightarrow \kappa \geq 0.$$

Caution!!! $\kappa \geq 0 \not\Rightarrow \mathfrak{S} \succeq 0.$

Sufficient conditions for **TCP**

Theorem (Figalli-R-Villani '10)

Let (M, g) be a compact smooth Riemannian **surface**. *It satisfies* **TCP** *if and only if the two following properties holds:*

- *all the injectivity domains are convex,*
- $\mathfrak{S} \succeq 0$.

Theorem (Figalli-R-Villani '10)

Assume that (M, g) satisfies the two following properties:

- *all its injectivity domains are* **strictly convex**,
- *the* **MTW** *tensor $\mathfrak{S}$ is $\succ 0$, that is for any $x \in M, v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM$,*

$$\langle \xi, \eta \rangle_x = 0 \Longrightarrow \mathfrak{S}_{(x,v)}(\xi, \eta) > 0.$$

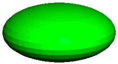
Then, it satisfies **TCP**.

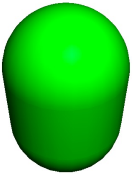
Examples and counterexamples

Examples:

- Flat tori (Cordero-Erausquin '99)
- Round spheres (Loeper '06)
- Quotients of the above objects
- Oblate ellipsoids of revolution (Bonnard-Caillau-R '10)
- Perturbations of round spheres (Figalli-R-Villani '12)
- Product of spheres (Figalli-Kim-McCann '13)

Counterexamples:





Questions

- Does $\mathfrak{S} \succeq 0 \Longrightarrow$ **TCP** in dimension ≥ 3 ?
- Smoother datas imply further regularity ?

- How is the set of metrics satisfying $\mathfrak{S} \succ 0$?
- Does $\mathfrak{S} \succeq 0$ imply convexity of injectivity domains ?
- Does $\mathfrak{S} \succeq 0$ imply more topological obstructions than $\kappa \geq 0$?

Focalization is the major obstacle

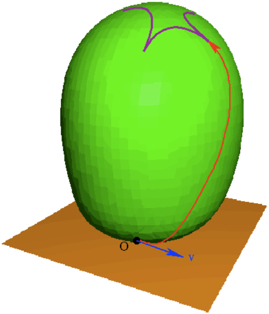
Focalization

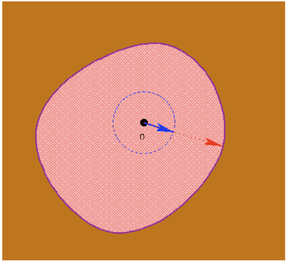
Definition

Let $x \in M$ and v be a unit tangent vector in $T_x M$. The vector v is **not conjugate** at time $t \geq 0$ if for any $t' \in [0, t+\delta]$ ($\delta > 0$ small) the geodesic from x to $\gamma(t')$ is locally minimizing.

The distance to the conjugate locus

Again, using the exponential mapping, we can associate to each unit tangent vector its **distance to the conjugate locus**.



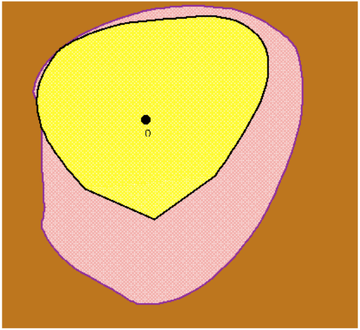


In that way, we define the so-called **nonfocal domain** $\mathcal{NF}(x)$ whose boundary is the **tangent conjugate locus** $\mathrm{TFL}(x)$.

Fundamental inclusion and remark

The following inclusion holds

$$\text{injectivity domain} \subset \text{nonfocal domain.}$$



Moreover, on a surface the nonfocal domain has smooth boundary !!!

The convex Earth theorem

Theorem (Figalli-R '09)

Any small deformation of $\mathbb{S}^2$ *in* C^5 *topology satisfies* $\mathfrak{S} \succeq 1/2$, *has convex injectivity domains and satisfies* **TCP**.

Spheres

Theorem (Loeper '06)

The **MTW** *tensor on the round (unit) sphere* $\mathbb{S}^2$ *satisfies* $\mathfrak{S} \succeq 1$, *that is for any* $x \in \mathbb{S}^2, v \in \mathcal{I}(x)$ *and* $\xi, \eta \in T_x\mathbb{S}^2$,

$$\langle \xi, \eta \rangle_x = 0 \implies \mathfrak{S}_{(x,v)}(\xi, \eta) \geq |\xi|^2 |\eta|^2.$$

In particular, the round sphere $\mathbb{S}^2$ *satisfies* **TCP**.

Two issues for stability

- Stability of the (uniform) convexity of injectivity domains.
- Stability of properties of the form $\mathfrak{S} \succeq K$ with $K > 0$.

Issues

As a matter of fact, on $\mathbb{S}^2$, the **MTW** tensor is given by

$$\begin{aligned}\mathfrak{S}_{(x,v)}\left(\xi,\xi^{\perp}\right) &= 3\left[\frac{1}{r^2}-\frac{\cos(r)}{r\sin(r)}\right]\xi_1^4+3\left[\frac{1}{\sin^2(r)}-\frac{r\cos(r)}{\sin^3(r)}\right]\xi_2^4\\ &\qquad+\frac{3}{2}\left[-\frac{6}{r^2}+\frac{\cos(r)}{r\sin(r)}+\frac{5}{\sin^2(r)}\right]\xi_1^2\xi_2^2,\end{aligned}$$

with $x\in\mathbb{S}^2, v\in\mathcal{I}(x), r:=|v|, \xi=(\xi_1,\xi_2), \xi^{\perp}=(-\xi_2,\xi_1)$.

Sketch of proof

- We extend the Ma-Trudinger-Wang tensor beyond boundaries of injectivity domains.
- The uniform convexity of the nonfocal domains $\mathcal{NF}(x)$ is stable.
- The positivity of the extended tensor $\overline{\mathfrak{S}}$ is stable.
- $\overline{\mathfrak{S}} \succ 0$ + (uniform) convexity of the $\mathcal{NF}(x)$'s
  $\Longrightarrow$ $\mathfrak{S} \succ 0$ + (uniform) convexity of the $\mathcal{I}(x)$'s.

Thank you for your attention !!