Geometric control theory, closing lemma, and weak KAM theory

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Outline

- Lecture 1: Geometric control methods
- Lecture 2: Applications to Hamiltonian dynamics
- Lecture 3: Weak KAM theory (an introduction)
- Lecture 4: Closing Aubry sets
Lecture 1

Geometric control methods
Control of an inverted pendulum

\[ m \]

\[ l \]

\[ \theta \]

\[ F \]
A general control system has the form
\[ \dot{x} = f(x, u) \]
where
- \( x \) is the state in \( M \)
- \( u \) is the control in \( U \)

**Proposition**

*Under classical assumptions on the data, for every \( x \in M \) and every measurable control \( u : [0, T] \rightarrow U \) the Cauchy problem*

\[
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t)) & \text{a.e. } t \in [0, T], \\
x(0) &= x
\end{aligned}
\]

*admits a unique solution*

\[
x(\cdot) = x(\cdot; x, u) : [0, T] \hookrightarrow M.
\]
Controllability issues

Given two points $x_1, x_2$ in the state space $M$ and $T > 0$, can we find a control $u$ such that the solution of

\[
\begin{cases}
    \dot{x}(t) = f(x(t), u(t)) & \text{a.e. } t \in [0, T] \\
    x(0) = x_1
\end{cases}
\]

satisfies

\[x(T) = x_2\]
A (nonautonomous) linear control system has the form

\[ \dot{\xi} = A \xi + B u, \]

with \( \xi \in \mathbb{R}^n, u \in \mathbb{R}^m, A \in M_n(\mathbb{R}), B \in M_{n,m}(\mathbb{R}) \).

**Theorem**

The following assertions are equivalent:

(i) For any \( T > 0 \) and any \( \xi_1, \xi_2 \in \mathbb{R}^n \), there is \( u \in L^1([0, T]; \mathbb{R}^m) \) such that

\[ \xi(T; \xi_1, u) = \xi_2. \]

(ii) The Kalman rank condition is satisfied:

\[ \text{rk} \left( B, AB, A^2 B, \cdots, A^{n-1} B \right) = n. \]
Proof of the theorem

**Duhamel’s formula**

\[ \xi(T; \xi, u) = e^{TA} \xi + e^{TA} \int_0^T e^{-tA} B u(t) dt. \]

Then the controllability property (i) is equivalent to the surjectivity of the mappings

\[ \mathcal{F}^T : u \in L^1([0, T]; \mathbb{R}^m) \mapsto \int_0^T e^{-tA} B u(t) dt. \]
Proof of (ii) $\Rightarrow$ (i)

If $\mathcal{F}^T$ is not onto (for some $T > 0$), there is $p \neq 0_n$ such that

$$\left\langle p, \int_0^T e^{-tA} B u(t) dt \right\rangle = 0 \quad \forall u \in L^1([0, T]; \mathbb{R}^m).$$

Using the linearity of $\langle \cdot, \cdot \rangle$ and taking $u(t) = B^* e^{-tA^*} p$, we infer that

$$p^* e^{-tA} B = 0 \quad \forall t \in [0, T].$$

Derivating $n$ times at $t = 0$ yields

$$p^* B = p^* A B = p^* A^2 B = \cdots = p^* A^{n-1} B = 0.$$

Which means that $p$ is orthogonal to the image of the $n \times mn$ matrix

$$(B, AB, A^2 B, \cdots, A^{n-1} B).$$

Contradiction !!!
Proof of \((i) \Rightarrow (ii)\)

If 
\[
\text{rk} \left( B, AB, A^2B, \cdots, A^{n-1}B \right) < n,
\]
there is a nonzero vector \(p\) such that 
\[
p^* B = p^* AB = p^* A^2B = \cdots = p^* A^{n-1}B = 0.
\]

By the Cayley-Hamilton Theorem, we deduce that 
\[
p^* A^k B = 0 \quad \forall k \geq 1,
\]
and in turn 
\[
p^* e^{-tA} B = 0 \quad \forall t \geq 0.
\]

We infer that 
\[
\left\langle p, \int_0^T e^{-tA} B u(t)dt \right\rangle = 0 \quad \forall u \in L^1([0, T]; \mathbb{R}^m), \forall T > 0.
\]

Contradiction !!!
Let $\dot{x} = f(x, u)$ be a nonlinear control system with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ of class $C^1$.

**Theorem**

Assume that $f(x_0, 0) = 0$ and that the pair

$$A = \frac{\partial f}{\partial x}(x_0, 0), \quad B = \frac{\partial f}{\partial u}(x_0, 0),$$

satisfies the Kalman rank condition. Then for there is $\delta > 0$ such that for any $x_1, x_2$ with $|x_1 - x_0|, |x_2 - x_0| < \delta$, there is $u : [0, 1] \rightarrow \mathbb{R}^m$ smooth satisfying

$$x(1; x_1, u) = x_2.$$
Define \( \mathcal{G} : \mathbb{R}^n \times L^1([0, 1]; \mathbb{R}^m) \to \mathbb{R}^n \times \mathbb{R}^n \) by

\[
\mathcal{G}(x, u) := (x, x(1; x, u)).
\]

The mapping \( \mathcal{G} \) is a \( C^1 \) submersion at \((0, 0)\). Thus there are \( n \) controls \( u^1, \ldots, u^n \) in \( L^1([0, 1]; \mathbb{R}^m) \) such that

\[
\tilde{\mathcal{G}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n
\]

\[
(x, \lambda) \mapsto \mathcal{G}(x, \sum_{k=1}^n \lambda_k u^k)
\]

is a \( C^1 \) diffeomorphism at \((0, 0)\). Since the set of smooth controls is dense in \( L^1([0, 1]; \mathbb{R}^m) \), we can take \( u^1, \ldots, u^n \) to be smooth. We apply the Inverse Function Theorem.
Local controllability around $x_0$
The equations of motion are given by

\[
\begin{align*}
(M + m) \ddot{x} + ml \dot{\theta} \cos \theta & - ml \dot{\theta}^2 \sin \theta = u \\
ml^2 \ddot{\theta} & - mg \ell \sin \theta + ml \ddot{x} \cos \theta = 0.
\end{align*}
\]
Back to the inverted pendulum

The linearized control system at $x = \dot{x} = \theta = \dot{\theta} = 0$ is given by

\[
(M + m)\ddot{x} + m\ell\ddot{\theta} = u
\]

\[
ml^2\ddot{\theta} - mg\ell\theta + m\ell\ddot{x} = 0.
\]

It can be written as a control system

\[
\dot{\xi} = A\xi + B\ u,
\]

with $\xi = (x, \dot{x}, \theta, \dot{\theta})$,

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & \frac{-mg}{M} & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & \frac{(M+m)g}{M\ell} & 0
\end{pmatrix}
\]

and

\[
B = \begin{pmatrix}
0 \\
\frac{1}{M} \\
0 \\
-\frac{1}{M\ell}
\end{pmatrix}.
\]
The Kalman matrix \((B, AB, A^2, A^3B)\) equals

\[
\begin{pmatrix}
0 & \frac{1}{M} & 0 & \frac{mg}{M^2\ell} \\
\frac{1}{M} & 0 & \frac{mg}{M^2\ell} & 0 \\
0 & -\frac{1}{M\ell} & 0 & -\frac{(M+m)g}{M^2\ell^2} \\
-\frac{1}{M\ell} & 0 & -\frac{(M+m)g}{M^2\ell^2} & 0
\end{pmatrix}.
\]

Its determinant equals

\[-\frac{g^2}{M^4\ell^4} < 0\]

In conclusion, the inverted pendulum is locally controllable around \((0, 0, 0, 0)^*\).
Lie brackets

\[ x \rightarrow e^{tX}(x) \]
Lie brackets

\( e^{tY} \circ e^{tX}(x) \)

\( x \)
Lie brackets

\[ \begin{align*}
&\mathbf{e}^t \mathbf{X}(x) \\
&\mathbf{e}^{-t} \mathbf{X} \circ \mathbf{e}^t \mathbf{Y} \circ \mathbf{e}^t \mathbf{X}(x) \\
&\mathbf{e}^t \mathbf{Y} \circ \mathbf{e}^t \mathbf{X}(x)
\end{align*} \]
Lie brackets

\[ e^{-tX} \circ e^{tY} \circ e^{tX}(x) \]

\[ e^{tY} \circ e^{tX}(x) \]

\[ e^{-tY} \circ e^{-tX} \circ e^{tY} \circ e^{tX}(x) \]

\[ e^{tX}(x) \]

\[ e^{-tX} \circ e^{tY} \circ e^{tX}(x) \]
Lie brackets

**Definition**

Let $M$ be a smooth manifold. Given two smooth vector fields $X, Y$ on $M$, the Lie bracket $[X, Y]$ is the smooth vector field on $M$ defined by

$$[X, Y](x) = \lim_{t \to 0} \frac{\left( e^{-tY} \circ e^{-tX} \circ e^{tY} \circ e^{tX} \right) (x) - x}{t^2},$$

for every $x \in M$.

Given a family $\mathcal{F}$ of smooth vector fields on $M$, we denote by $\text{Lie}\{\mathcal{F}\}$ the Lie algebra generated by $\mathcal{F}$. It is the smallest vector subspace $S$ of smooth vector fields containing $\mathcal{F}$ that also satisfies

$$[X, Y] \in S \quad \forall X \in \mathcal{F}, \forall Y \in S.$$
Lie brackets: Examples

In $\mathbb{R}^n$:

- Let $X, Y$ be two smooth vector fields, then

$$[X, Y](x) = DY(x)X(x) - DX(x)Y(x).$$

- If $X(x) = Ax, Y(x) = Bx$ with $A, B \in M_n(\mathbb{R})$, then

$$[X, Y](x) = [A, B]x = (BA - AB)x.$$

- If $X(x) = Ax, Y(x) = b$ with $A \in M_n(\mathbb{R}), b \in \mathbb{R}^n$, then

$$[X, Y](x) = -Ab, \quad [X, [X, Y]](x) = A^2b, \quad \cdots$$

$$\Rightarrow \quad \text{Lie}\{X, Y\} = \text{Span}\{Ax, b, Ab, A^2b, A^3b, \ldots\}.$$
The Chow-Rashevsky Theorem

Theorem (Chow 1939, Rashevsky 1938)

Let $M$ be a smooth connected manifold and $X^1, \cdots, X^m$ be $m$ smooth vector fields on $M$. Assume that

$$\text{Lie} \{X^1, \ldots, X^m\} (x) = T_x M \quad \forall x \in M.$$ 

Then the control system

$$\dot{x} = \sum_{i=1}^{m} u_i X^i(x)$$

is globally controllable on $M$. 
Example: The baby stroller

\[
\begin{align*}
\dot{x} &= u_1 \cos \theta \\
\dot{y} &= u_1 \sin \theta \\
\dot{\theta} &= u_2
\end{align*}
\]

\[
X = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad [X, Y] = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}
\]

\[
\text{Span}\left\{X(\xi), Y(\xi), [X, Y](\xi)\right\} = \mathbb{R}^3 \quad \forall \xi = (x, y, \theta).
\]
Example: The baby stroller
Example: The baby stroller
Given a control system of the form

\[ \dot{x} = \sum_{i=1}^{m} u_i X^i(x) \quad (x \in M, u \in \mathbb{R}^m), \]

we define the \textbf{End-Point mapping} from \( x \) in time \( T > 0 \) as

\[ E^{x,T} : L^1([0, T]; \mathbb{R}^m) \longrightarrow M \]

\[ u \longmapsto x(T; x, u) \]

Under appropriate assumptions, it is a \( C^1 \) mapping.

\textbf{Theorem}

Assume that

\[ \text{Lie} \{ X^1, \ldots, X^m \} (x) = T_x M \quad \forall x \in M. \]

Then for any \( x \in M, T > 0, E^{x,T} \) is an open mapping.
Let $x \in M$ be fixed. Denote by $A(x)$ the accessible set from $x$, that is

$$A(x) := \{ x(T; x, u) \mid T \geq 0, u \in L^1 \}.$$ 

- By openness of the $E^{x,T}$'s, $A(x)$ is open.
- Let $y$ be in the closure of $A(x)$. The set $A(y)$ contains a small ball centered at $y$ and there are points of $A(x)$ in that ball. Then $A(x)$ is closed.

We conclude easily by connectedness of $M$. 
Regular controls vs. Singular controls

**Definition**
A control \( u \in L^1([0, T]; \mathbb{R}^m) \) is called \textit{regular} with respect to \( E^{x, T} \) if \( E^{x, T} \) is a submersion at \( u \). If not, \( u \) is called \textit{singular}. 

**Remark**
The concatenations \( u_1 \ast u_2 \) and \( u_2 \ast u_1 \) of a regular control \( u_1 \) with another control \( u_2 \) are regular.
Lemma

Assume that

\[ \text{Lie} \{ X^1, \ldots, X^m \} (x) = T_x M \quad \forall x \in M. \]

Then for every \( x \in M \) and every \( T > 0 \), the set of regular controls (w.r.t. \( E^{x,T} \)) is generic.

Then we apply the so-called Return Method: Given \( x \in M \) and \( T > 0 \), we pick (for any \( \alpha > 0 \) small) a regular control \( u^\alpha \) in \( L^1([0, \alpha]; \mathbb{R}^m) \). Then for every \( u \in L^1([0, T]; \mathbb{R}^m) \), the control \( \tilde{u} \) defined by

\[ \tilde{u} := u^\alpha * \tilde{u}^\alpha * u \]

is regular and steers \( x \) to \( E^{x,T} (u) \) in time \( T + 2\alpha \).

Then, we can apply the Inverse Function Theorem...
Thank you for your attention !!
Lecture 2

Applications to Hamiltonian dynamics
Let $n \geq 2$ be fixed. Let $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Hamiltonian of class $C^k$, with $k \geq 2$, satisfying the following properties:

(H1) **Superlinear growth:**
For every $K \geq 0$, there is $C^*(K) \in \mathbb{R}$ such that
\[
H(x, p) \geq K|p| + C^*(K) \quad \forall x, p.
\]

(H2) **Uniform convexity:**
For every $x, p$, $\frac{\partial^2 H}{\partial p^2}(x, p)$ is positive definite.

(H3) **Uniform boundedness in the fibers:**
For every $R \geq 0$,
\[
A^*(R) := \sup \{H(x, p) | |p| \leq R\} < +\infty.
\]

Under these assumptions, $H$ generates a flow $\phi^H_t$ which is of class $C^{k-1}$ and complete.
A connecting problem

Let be given two solutions

$$(x_i, p_i) : [0, \tau] \rightarrow \mathbb{R}^n \times \mathbb{R}^n \quad i = 1, 2,$$

of the Hamiltonian system

$$\begin{cases} 
\dot{x}(t) &= \nabla_p H(x(t), p(t)) \\
\dot{p}(t) &= -\nabla_x H(x(t), p(t)) 
\end{cases}.$$ 

**Question**

Can I add a potential $V$ to the Hamiltonian $H$ in such a way that the solution of the new Hamiltonian system

$$\begin{cases} 
\dot{x}(t) &= \nabla_p H(x(t), p(t)) \\
\dot{p}(t) &= -\nabla_x H(x(t), p(t)) - \nabla V(x(t)) 
\end{cases},$$

starting at $(x_1(0), p_1(0))$ satisfies

$$(x(\tau), p(\tau)) = (x_2(\tau), p_2(\tau))?$$
Weak KAM Theory in Italy
Control approach

Study the mapping

\[ E : L^1([0, \tau]; \mathbb{R}^n) \rightarrow \mathbb{R}^n \times \mathbb{R}^n \]

\[ u \mapsto (x_u(\tau), p_u(\tau)) \]

where

\[ (x_u, p_u) : [0, \tau] \rightarrow \mathbb{R}^n \times \mathbb{R}^n \]

is the solution of

\[
\begin{cases}
\dot{x}(t) &= \nabla_p H(x(t), p(t)) \\
\dot{p}(t) &= -\nabla_x H(x(t), p(t)) - u(t),
\end{cases}
\]

starting at \( (x_1(0), p_1(0)) \).

\[ u = 0 \]
Exercise

Given \( x, u : [0, \tau] \to \mathbb{R}^n \) as above, does there exists a function \( V : \mathbb{R}^n \to \mathbb{R} \) whose the support is included in the dashed blue square above and such that

\[
\nabla V(x(t)) = u(t) \quad \forall t \in [0, \tau]
\]
Exercise (solution)

There is a necessary condition

$$\int_0^\tau \langle \dot{x}(t), u(t) \rangle dt = 0.$$

As a matter of fact,

$$\int_0^\tau \langle \dot{x}(t), u(t) \rangle dt = \int_0^\tau \langle \dot{x}(t), \nabla V(x(t)) \rangle dt$$

$$= V(x_\tau) - V(x_0) = 0.$$

Proposition

If the above necessary condition is satisfied, then there is $V : \mathbb{R}^n \to \mathbb{R}$ satisfying the desired properties such that

$$\| V \|_{C^1} \leq \frac{K}{r} \| u \|_\infty.$$
Exercise (solution)

If $x(t) = (t, 0)$, that is

then we set

$$V(t, y) := \phi(|y|/r) \left[ \int_0^t u_1(s) \, ds + \sum_{i=1}^{n-1} \int_0^{y_i} u_{i+1}(t + s) \, ds \right],$$

for every $(t, y)$, with $\phi : [0, \infty) \to [0, 1]$ satisfying

$$\phi(s) = 1 \quad \forall s \in [0, 1/3] \quad \text{and} \quad \phi(s) = 0 \quad \forall s \geq 2/3.$$
Control approach

Study the mapping

\[ E : L^1([0, \tau]; \mathbb{R}^n) \rightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \]

\[ u \mapsto (x_u(\tau), p_u(\tau), \xi_u(\tau)) \]

where \((x_u, p_u, \xi_u) : [0, \tau] \rightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\) is the solution of

\[
\begin{aligned}
\dot{x}(t) &= \nabla_p H(x(t), p(t)) \\
\dot{p}(t) &= -\nabla_x H(x(t), p(t)) - u(t) \\
\dot{\xi}(t) &= \langle \nabla_p H(x(t), p(t)), u(t) \rangle,
\end{aligned}
\]

starting at \((x_1(0), p_1(0), 0)\).

Objective: Showing that \(E\) is a submersion at \(u \equiv 0\).
Assume that $E : L^1([0, \tau]; \mathbb{R}^n) \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ is a submersion at $u \equiv 0$.

- There are $\ell = 2n + 1$ controls $u^1, \cdots, u^{\ell}$ in $L^1([0, \tau]; \mathbb{R}^n)$ such that
  \[
  \tilde{E} : \mathbb{R}^{\ell} \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}
  \]
  \[
  \lambda \longmapsto E \left( \sum_{k=1}^{\ell} \lambda_k u^k \right)
  \]
  is a $C^1$ diffeomorphism at 0.

- The set of controls $u \in L^1([0, \tau]; \mathbb{R}^n)$ such that
  \[
  u \text{ is smooth and } \text{Supp}(u) \subset (0, \tau)
  \]
  is dense.

We are done.
No !!

- If \( u \in L^1([0, \tau]; \mathbb{R}^n) \) with \( \int_0^\tau \langle \dot{x}(t), u(t) \rangle \, dt = 0 \), then
  \[
  H(x_u(\tau), p_u(\tau)) = H(x_u(0), p_u(0)) = 0.
  \]

  The final state \((x_u(\tau), p_u(\tau))\) must belong to the same level set of \(H\) as the initial state \((x_u(0), p_u(0))\). We need to suppress one degree of freedom in the \(p\) variable.

- It is not sufficient to get the local controllability. We also need to allow free time.
A local controllability result

Given $N, m \geq 1$, let us consider a nonlinear control system in $\mathbb{R}^N$ of the form

$$\dot{\xi} = F_0(\xi) + \sum_{i=1}^{m} u_i F_i(\xi),$$

$G : \mathbb{R}^N \to \mathbb{R}^k$ be a function of class $C^1$, and $\bar{\xi} : [0, T] \to \mathbb{R}^N$ be a solution associated with $\bar{u} \equiv 0$.

Our aim is to give sufficient conditions on $F_0, F_1, \ldots, F_m$, and $G$ to have the following property:

For any neighborhood $\mathcal{V}$ of $\bar{u} \equiv 0$ in $L^1([0, \bar{T}]; \mathbb{R}^m)$, the set

$$\left\{ G(\xi_{\bar{\xi}(0)}, u(T)) \mid u \in \mathcal{V} \right\}$$

is a neighborhood of $G(\xi_{\bar{\xi}(0)}, \bar{u}(T))$. 
Denote by $E^{\xi(0), T}$ the End-Point mapping

$$u \in L^1([0, T]; \mathbb{R}^m) \mapsto \xi_{\xi(0), u}(T),$$

where $\xi_{\xi(0), u}$ is the trajectory of the control system associated with $u$ and starting at $\bar{\xi}(0)$.

**Proposition**

If $G$ is a submersion at $\bar{\xi}(T)$, and

$$\text{Span}\left\{ F_i(\bar{\xi}(\bar{T})), \ [F_0, F_i](\bar{\xi}(\bar{T})) \mid i = 1, \ldots, m \right\}$$

$$+ \text{Ker} \left( dG(\bar{\xi}(\bar{T})) \right) = \mathbb{R}^N,$$

then $G \circ E^{\xi(0), T}$ is a submersion at $\bar{u} \equiv 0$.

Thanks to the uniform convexity of $H$ in the $p$ variable, the above result applies to our control problem.
An alternative method

Let 

\[(x_i, p_i) : [0, \tau] \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \quad i = 1, 2,\]

be two solutions of the Hamiltonian system

\[
\begin{cases}
\dot{x}(t) = \nabla_p H(x(t), p(t)) \\
\dot{p}(t) = -\nabla_x H(x(t), p(t)).
\end{cases}
\]

Question

Given an arc \( x : [0, \tau] \rightarrow \mathbb{R}^n \) such that

\[x(t) = x_1(t) \forall t \in [0, \delta] \text{ and } x(t) = x_2(t) \forall t \in [\tau - \delta, \tau],\]

does there exist \( p, u : [0, \tau] \rightarrow \mathbb{R}^n \) such that

\[
\begin{cases}
\dot{x}(t) = \nabla_p H(x(t), p(t)) \\
\dot{p}(t) = -\nabla_x H(x(t), p(t)) - u(t)
\end{cases} \quad \forall t \in [0, t]
\]
An alternative method (flatness)

Remark

For every \( x, p \mapsto \frac{\partial H}{\partial p}(x, p) \) is a diffeomorphism.

Then we can set

\[
\begin{align*}
    p(t) &:= \left( \frac{\partial H}{\partial p}(x(t), \cdot) \right)^{-1}(x(t), \dot{x}(t)) \\
    u(t) &:= -\frac{\partial H}{\partial x}(x(t), p(t)) - \dot{p}(t)
\end{align*}
\]

By construction there holds

\[
\begin{align*}
    \dot{x}(t) &= \nabla_p H(x(t), p(t)) \\
    \dot{p}(t) &= -\nabla_x H(x(t), p(t)) - u(t)
\end{align*}
\]

and

\[ u(t) = 0 \quad \forall t \in [0, \delta] \cup [\tau - \delta, \tau]. \]

To get \( \int_0^t \langle \dot{x}(s), u(s) \rangle ds = 0 \forall t \), we reparametrize in time.
Let be given two solutions 

$$(x_i, p_i) : [0, \tau] \rightarrow \mathbb{R}^n \times \mathbb{R}^n \quad i = 1, 2$$

of the Hamiltonian system

$$\begin{align*}
\dot{x}(t) &= \nabla_p H(x(t), p(t)) \\
\dot{p}(t) &= -\nabla_x H(x(t), p(t)).
\end{align*}$$

Question

Can I add a potential $V$ to the Hamiltonian $H$ in such a way that the solution $(x, p) : [0, \tau] \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ of the new Hamiltonian system associated with $H_V := H + V$ starting at $(x_1(0), p_1(0))$ satisfies $(x(\tau), p(\tau)) = (x_2(\tau), p_2(\tau))$

and

$$\int_0^\tau L(x(t), \dot{x}(t)) - V(x(t)) \, dt = \text{data} \quad ?$$
Study the mapping

\[ E : L^1([0, \tau]; \mathbb{R}^n) \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \]

where \((x_u, p_u, \xi_u, \ell_u) : [0, \tau] \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \) is the solution of

\[
\begin{align*}
\dot{x}(t) &= \nabla_p H(x(t), p(t)) \\
\dot{p}(t) &= -\nabla_x H(x(t), p(t)) - u(t) \\
\dot{\xi}(t) &= \langle \nabla_p H(x(t), p(t)), u(t) \rangle \\
\dot{\ell}(t) &= \langle p(t), \nabla_p H(x(t), p(t)) \rangle,
\end{align*}
\]

starting at \((x_1(0), p_1(0), 0, 0)\).

Again, we need to relax time.

(It works provided some algebraic condition is satisfied.)
Controlling the differential of an Hamiltonian flow

Let be given a solution

$$(x, p) : [0, \tau] \rightarrow \mathbb{R}^n \times \mathbb{R}^n$$

of the Hamiltonian system

$$\begin{cases} 
\dot{x}(t) &= \nabla_p H(x(t), p(t)) \\
\dot{p}(t) &= -\nabla_x H(x(t), p(t))
\end{cases}$$

Question

Can I add a potential $V$ to $H$ in such a way that:

- $(x, p)$ is still solution of the new Hamiltonian system associated with $H_V := H + V$.
- The differential of $\phi^H_\tau$ at $(x(0), p(0))$ equals a data.
This can be done !!

Take a potential \( V \) satisfying

\[
V(x(t)) = 0 \text{ and } \nabla V(x(t)) = 0 \quad \forall t \in [0, \tau].
\]

Then \((x, p)\) is still solution of the new Hamiltonian system associated with \( H_V \).

The Control is:

\[
u(t) = \text{Hess}_{x(t)} V
\]

(Indeed \( \text{Hess}_{x(t)} V \) restricted to a space transverse to \( \dot{x}(t) \)).
Controlling the differential of an Hamiltonian flow

Study the mapping

\[ E : L^1 \left([0, \tau]; \mathbb{R}^{n(n-1)/2}\right) \rightarrow \text{Sp}(n) \]

\[ u \mapsto D_u(\tau) \]

where \( D_u : [0, \tau] \rightarrow \text{Sp}(n) \) is the resolvent of the linearized system

\[
\begin{align*}
\dot{h}(t) &= \nabla_{px} H h(t) + \nabla_{pp} H v(t) \\
\dot{v}(t) &= -\nabla_{xx} H h(t) - \nabla_{xp} H v(t) - u(t)
\end{align*}
\]

\( \forall t \in [0, \tau], \)

starting at \( l_{2n} \).

Indeed we need to work in \( \text{Sp}(n-1) \).
Our control system has the form

\[ \dot{X}(t) = A(t)X(t) + \sum_{i=1}^{k} u_i(t)B_i(t)X(t), \]

where the state \( X \) belongs to \( M_n(\mathbb{R}) \) and

\[ A, B_1, \ldots, B_k : [0, \tau] \rightarrow M_n(\mathbb{R}) \text{ are smooth.} \]

Indeed we are interested in trajectories starting at \( l_{2m} \) and valued in the symplectic group

\[ \text{Sp}(m) = \{ X \mid X^* J X = J \}. \]

Assumption: \( A(t), B_i(t) \in T_{l_{2m}} \text{Sp}(m) \) for any \( t \in [0, \tau] \).
Define the $k$ sequences of smooth mappings
\[ \{B^j_1\}, \ldots, \{B^j_k\} : [0, \tau] \to T_{l_2m}Sp(m) \]
by
\[
\begin{align*}
B^0_i(t) & := B_i(t) \\
B^j_i(t) & := \dot{B}^j_{i-1}(t) + B^j_{i-1}(t)A(t) - A(t)B^j_{i-1}(t),
\end{align*}
\]
for every $t \in [0, \tau]$ and every $i \in \{1, \ldots, k\}$.

**Theorem**

Assume that there is some $t \in [0, \tau]$ such that
\[
\text{Span}\left\{ B^j_i(t) \mid i \in \{1, \ldots, k\}, j \in \mathbb{N} \right\} = T_{l_2m}Sp(m).
\]
Then the End-Point mapping
\[ E^{l_2m,\tau} : L^1([0, \tau]; \mathbb{R}^k) \to Sp(m) \]
is a submersion at $u \equiv 0$. 
Thank you for your attention !!
Lecture 3

Weak KAM theory
(an introduction)
Let $M$ be a smooth compact manifold of dimension $n \geq 2$ be fixed. Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian of class $C^k$, with $k \geq 2$, satisfying the following properties:

**(H1) Superlinear growth:**
For every $K \geq 0$, there is $C^*(K) \in \mathbb{R}$ such that

$$H(x, p) \geq K|p| + C^*(K) \quad \forall (x, p) \in T^*M.$$ 

**(H2) Uniform convexity:**
For every $(x, p) \in T^*M$, $\frac{\partial^2 H}{\partial p^2}(x, p)$ is positive definite.

For sake of simplicity, we may assume that $M = \mathbb{T}^n$, that is that $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies (H1)-(H2) and is periodic with respect to the $x$ variable.
Definition

We call critical value of $H$ the constant $c = c[H]$ defined as

$$c[H] := \inf \{ \max_{x \in M} \{ H(x, du(x)) \} \}.$$ 

In other terms, $c[H]$ is the infimum of numbers $c \in \mathbb{R}$ such that there is a $C^1$ function $u : M \to \mathbb{R}$ satisfying

$$H(x, du(x)) \leq c \quad \forall x \in M.$$ 

Note that

$$\min_{x \in M} \{ H(x, 0) \} \leq c[H] \leq \max_{x \in M} \{ H(x, 0) \}.$$
Critical subsolutions of $H$

**Definition**

We call critical subsolution any Lipschitz function $u : M \rightarrow \mathbb{R}$ such that

$$H(x, du(x)) \leq c[H] \quad \text{for a.e. } x \in M.$$

**Proposition**

The set of critical subsolutions is nonempty.

**Proof.**

- Any $C^1$ function $u : M \rightarrow \mathbb{R}$ such that $H(\cdot, du(\cdot)) \leq c$ is $L(c)$-Lipschitz with $L(c)$ depending only on $c$.
- Arzelà-Ascoli Theorem.
- If $u_k \rightarrow u$ then $\text{Graph}(du) \subset \lim \inf_{k \rightarrow \infty} \text{Graph}(du_k)$. 

Ludovic Rifford  Weak KAM Theory in Italy
Characterization of critical subsolutions

Let \( L : TM \to \mathbb{R} \) be the Tonelli Lagrangian associated with \( H \) by Legendre-Fenchel duality, that is
\[
L(x, v) := \max_{p \in T^*_x M} \left\{ p \cdot v - H(x, p) \right\} \quad \forall (x, v) \in TM.
\]

**Proposition**

A Lipschitz function \( u : M \to \mathbb{R} \) is a critical subsolution if and only if
\[
u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L(\gamma(t), \dot{\gamma}(t)) \, ds + c (b - a),
\]
for every Lipschitz curve \( \gamma : [a, b] \to M \).

It is a consequence of the inequality
\[
p \cdot v \leq L(x, v) + H(x, p) \quad \forall x, v, p.
\]
**Proof.**

If $u$ is $C^1$, then

$$u(\gamma(b)) - u(\gamma(a)) = \int_a^b du(\gamma(t)) \cdot \dot{\gamma}(t) \, dt$$

$$\leq \int_a^b L(\gamma(t), \dot{\gamma}(t)) \, dt$$

$$\quad + \int_a^b H(\gamma(t), du(\gamma(t))) \, dt$$

$$\leq \int_a^b L(\gamma(t), \dot{\gamma}(t)) \, ds + c (b - a).$$

If $u$ is not $C^1$ then regularize it by (classical) convolution. The function $u \ast \rho_\epsilon$ is subsolution of $H \leq c + \alpha \epsilon$. Apply the above argument to $u \ast \rho_\epsilon$ and pass to the limit.
Lax-Oleinik semigroups \( \{ T_t \} \) and \( \{ \tilde{T}_t \} \)

**Definition**

Given \( u : M \to \mathbb{R} \) and \( t \geq 0 \), the Lipschitz functions \( T_t u, \tilde{T}_t u \) are defined by

\[
T_t u(x) := \min_{y \in M} \{ u(y) + A_t(y, x) \}
\]

\[
\tilde{T}_t u(x) := \max_{y \in M} \{ u(y) - A_t(x, y) \},
\]

with \( A_t(z, z') := \inf \left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s)) \, ds + c \, t \right\}, \)

where the infimum is taken over the Lipschitz curves \( \gamma : [0, t] \to M \) such that \( \gamma(0) = z \) and \( \gamma(t) = z' \).

The set of critical subsolutions is invariant with respect to both \( \{ T_t \} \) and \( \{ \tilde{T}_t \} \).
The weak KAM Theorem

**Theorem (Fathi, 1997)**

There is a critical subsolution $u : M \to \mathbb{R}$ such that

$$\mathcal{T}_t u = u \quad \forall t \geq 0.$$ 

It is called a critical or a weak KAM solution of $H$.

Given a critical solution $u : M \to \mathbb{R}$, for every $x \in M$, there is a curve

$$\gamma : (-\infty, 0] \to M \quad \text{with} \quad \gamma(0) = x$$

such that, for any $a < b \leq 0$,

$$u(\gamma(b)) - u(\gamma(a)) = \int_a^b L(\gamma(s), \dot{\gamma}(s)) \, ds + c(b - a).$$

Therefore, any restriction of $\gamma$ minimizes the action between its end-points. Then, it satisfies the Euler-Lagrange equations.
The classical Dirichlet problem

Let $\Omega$ be an open set in $\mathbb{R}^n$ with compact boundary and $H : \mathbb{R}^n \to \mathbb{R}$ of class $C^2$ satisfying (H1),(H2) and (H3) For every $x \in \bar{\Omega}$, $H(x, 0) < 0$.

**Proposition**

The continuous function $u : \bar{\Omega} \to \mathbb{R}$ given by

$$u(x) := \inf \left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s)) \, ds \right\},$$

where the infimum is taken among Lipschitz curves $\gamma : [0, t] \to \bar{\Omega}$ with $\gamma(0) \in \partial \Omega$, $\gamma(t) = x$ is the unique viscosity solution to the Dirichlet problem

$$\begin{cases} 
H(x, du(x)) = 0 & \forall x \in \Omega, \\
u(x) = 0 & \forall x \in \partial \Omega.
\end{cases}$$
The classical Dirichlet problem (picture)
Semiconcavity of critical solutions

\[ z = \gamma(-t) \]

\[
\begin{align*}
    u(x) &= u(z) + \int_{-t}^{0} L(\gamma(s), \dot{\gamma}(s)) \, ds + ct \\
    u(x') &\leq u(z) + \int_{-t}^{0} L(\gamma'(s), \dot{\gamma}'(s)) \, ds + ct
\end{align*}
\]

Thus

\[
    u(x') \leq u(x) + \int_{-t}^{0} L(\gamma'(s), \dot{\gamma}'(s)) - L(\gamma(s), \dot{\gamma}(s)) \, ds
\]
Any critical solution $u : M \to \mathbb{R}$ is semiconcave, that is it can be written locally (in charts) as

$$u = g + h,$$

the sum of a smooth function $g$ and a concave function $h$. 
Regularity along minimizing curves

\[ u(x) = u(z) + \int_{-t}^{0} L(\gamma(s), \dot{\gamma}(s)) \, ds + ct. \]

\[ u(x) \leq u(z') + \int_{-t}^{0} L(\gamma'(s), \dot{\gamma}'(s)) \, ds + ct. \]

Thus

\[ u(z') \geq u(z) + \int_{-t}^{0} L(\gamma(s), \dot{\gamma}(s)) - L(\gamma'(s), \dot{\gamma}'(s)) \, ds \]
Regularity along minimizing curves
Let $u$ be a critical solution. For every $x \in M$, we define the limiting differential of $u$ at $x$ by

$$d^* u(x) := \{ \lim du(x_k) | x_k \to x, \ u \text{ diff at } x_k \}.$$ 

It is a nonempty compact subset satisfying

$$H(x, du^*(x)) = c \quad \forall x \in M.$$
Remark

Let $u : M \to \mathbb{R}$ be a critical solution, $x \in M$ be fixed and $\gamma : (-\infty, 0] \to \mathbb{R}$ be a calibrated curve with $\gamma(0) = x$. Fix

$$x_\infty \in \bigcap_{t \leq 0} \gamma\left((-\infty, t]\right).$$

We can check that

$$\liminf_{t \to +\infty} \left\{ A_t\left(x_\infty, x_\infty\right) \right\} = 0.$$

Proposition

The critical value of $H$ satisfies

$$c[H] = -\inf \left\{ \frac{1}{T} \int_0^T L(\gamma(t), \dot{\gamma}(t)) \, dt \right\},$$

where the infimum is taken over the Lipschitz curves $\gamma : [0, T] \to M$ such that $\gamma(0) = \gamma(T)$. 
Definition and Proposition

- The **projected Aubry set** of $H$ defined as
  \[ \mathcal{A}(H) = \{ x \in M \mid A_t(x, x) = 0 \} \]
is compact and nonempty.

- Any critical subsolution $u$ is $C^1$ at any point of $\mathcal{A}(H)$ and satisfies $H(x, du(x)) = c[H], \forall x \in \mathcal{A}(H)$.

- For every $x \in \mathcal{A}(H)$, the differential of a critical subsolution at $x$ does not depend on $u$.

- The **Aubry set** of $H$ defined by
  \[ \tilde{\mathcal{A}}(H) := \{ (x, du(x)) \mid x \in \mathcal{A}(H), u \text{ crit. subsol.} \} \subset T^*M \]
is compact, invariant by $\phi^H_t$, and is a Lipschitz graph over $\mathcal{A}(H)$. 

Ludovic Rifford
Weak KAM Theory in Italy
Let $H : T(T^n)^* \to \mathbb{R}$ be the Hamiltonian defined by

$$H(x, p) = \frac{1}{2} |p|^2 + V(x) \quad \forall (x, p) \in T^n \times \mathbb{R}^n.$$ 

- $L(x, v) = \frac{1}{2} |v|^2 - V(x)$.
- $H(x, 0) \leq \max_M V \implies c[H] \leq \max_M V$.
- Let $x_{\text{max}} \in T^n$ be such that $V(x_{\text{max}}) = \max_M V$, then

$$\frac{1}{T} \int_0^T L(x_{\text{max}}, 0) \, dt = -\max_M V.$$ 

Thus $c[H] \geq \max_M V$.

- In conclusion $c[H] = \max_M V$ and

\[ \tilde{A}(H) = \left\{ (x, 0) \mid V(x) = \max_M V \right\} . \]
Examples (in $\mathbb{T}^1 = \mathbb{S}^1$)

Let $H : T(\mathbb{S}^1)^* \to \mathbb{R}$ be the Hamiltonian defined by

$$H(x, p) = \frac{1}{2} (p - f(x))^2 \quad \forall (x, p) \in \mathbb{S}^1 \times \mathbb{R}.$$ 

- $u : \mathbb{S}^1 \to \mathbb{R}$ defined by (set $\alpha := \left(\int_0^1 f(r)dr\right)$)

  $$u(x) = \int_0^x f(r)dr - \alpha x \quad \forall x \in \mathbb{S}^1,$$

is a smooth solution of $H(x, du(x)) = \alpha^2/2$ for any $x$. Then $c[H] = \alpha^2/2$.

- Along characteristics, there holds ($p(t) := du(x(t))$)

  $$\begin{cases}
  \dot{x}(t) &= \frac{\partial H}{\partial p}(x(t), p(t)) = p(t) - f(x(t)) \\
  \dot{p}(t) &= -\frac{\partial H}{\partial x}(x(t), p(t)) = (f(x(t)) - p(t))f'(x(t)).
  \end{cases}$$

  Then $x \in A(H) \implies \dot{x} = (f(x) - \alpha) - f(x) = \alpha$.

- Either equilibria everywhere or one orbit.
Examples (in $S^1 = \mathbb{R}/[0, \pi]$)

Let $H : T(S^1)^* \rightarrow \mathbb{R}$ be the Hamiltonian defined by

$$H(x, p) = \frac{1}{2}(p - \omega)^2 - V(x) \quad \forall (x, p) \in S^1 \times \mathbb{R},$$

with $V(x) = \sin^2(x)$ and $\omega = -\int_0^\pi 2\sqrt{V(r)}dr = -4$.

- $H(x, 0) \leq 0 \implies c[H] \leq 0$.
- $L(x, v) = v^2/2 + \omega v + V(x) \implies L(0, 0) = 0 \implies c[H] \geq 0$.
- Let $u$ be a critical subsolution. Then there holds a.e.

$$(u' - \omega)^2 \leq 2V \implies u' - \omega \leq 2\sqrt{V} \implies u' \leq \omega + 2\sqrt{V}.$$ 

In conclusion, $u(x) = \int_0^x 2\sqrt{V(r)}dr + \omega x$ for any $x$.

- The Aubry set consists in one equilibria and one orbit.
Examples (Mañé’s Lagrangians)

Let $X$ be a smooth vector field on $M$ and $L : TM \to \mathbb{R}$ defined by

$$L_X(x, v) = \frac{1}{2} |v - X(x)|^2 \quad \forall (x, v) \in TM.$$

- $H_X(x, p) = \frac{1}{2} |p|^2 + p \cdot X(x)$.
- $H(x, 0) = 0$ for any $x \in M$. Then $c[H] = 0$.
- Characteristics of $u = 0$ satisfy

$$\dot{x}(t) = X(x(t)), \quad p(t) = 0.$$

- The projected Aubry set always contains the set of recurrent points.
Two theorems by Bernard

**Theorem (Bernard, 2006)**

*There exists a critical subsolution of class $C^{1,1}$.*

Idea of the proof: Use a Lasry-Lions type convolution. If $u$ is a given critical solution, then $(\mathcal{T}_s \circ \mathcal{T}_t)(u)$ is $C^{1,1}$ provided $s, t > 0$ are small enough.

**Theorem (Bernard, 2007)**

*Assume that the Aubry set is exactly one hyperbolic periodic orbit, then any critical solution is "smooth" in a neighborhood of $\mathcal{A}(H)$. As a consequence, there is a "smooth" critical subsolution.*

Idea of the proof: The Aubry set is the boundary at infinity, that is any calibrated curve $\gamma : (-\infty, 0] \rightarrow M$ tends to $\mathcal{A}(H)$ as $t$ tends to $-\infty$. Indeed, for every $p \in d^* u(x)$ there is such a calibrated curve such that $\dot{\gamma}(0) = \frac{\partial H}{\partial p}(\gamma(0), p)$. 
Thank you for your attention !!
Lecture 4

Closing Aubry sets
Let $M$ be a smooth compact manifold of dimension $n \geq 2$ be fixed. Let $\mathcal{H} : T^* M \to \mathbb{R}$ be a Hamiltonian of class $C^k$, with $k \geq 2$, satisfying the following properties:

(H1) **Superlinear growth:**
For every $K \geq 0$, there is $C^*(K) \in \mathbb{R}$ such that
\[
H(x, p) \geq K|p| + C^*(K) \quad \forall (x, p) \in T^* M.
\]

(H2) **Uniform convexity:**
For every $(x, p) \in T^* M$, \(\frac{\partial^2 H}{\partial p^2}(x, p)\) is positive definite.

For sake of simplicity, we may assume that $M = \mathbb{T}^n$, that is that $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies (H1)-(H2) and is periodic with respect with the $x$ variable.
Conjecture (Mañé, 96)

For every Tonelli Hamiltonian $H : T^* M \to \mathbb{R}$ of class $C^k$ (with $k \geq 2$), there is a residual subset (i.e., a countable intersection of open and dense subsets) $\mathcal{G}$ of $C^k(M)$ such that, for every $V \in \mathcal{G}$, the Aubry set of the Hamiltonian $H_V := H + V$ is either an equilibrium point or a periodic orbit.

Strategy of proof:
- Density result.
- Stability result.
Mañé’s density Conjecture

Conjecture (Mañé’s density conjecture)

For every Tonelli Hamiltonian $H : T^* M \to \mathbb{R}$ of class $C^k$ (with $k \geq 2$) there exists a dense set $\mathcal{D}$ in $C^k(M)$ such that, for every $V \in \mathcal{D}$, the Aubry set of the Hamiltonian $H_V$ is either an equilibrium point or a periodic orbit.

Proposition (Contreras-Iturriaga, 1999)

Let $H : T^* M \to \mathbb{R}$ be a Hamiltonian of class $C^k$ (with $k \geq 3$) whose Aubry set is an equilibrium point (resp. a periodic orbit). Then, there is a smooth potential $V : M \to \mathbb{R}$, with $\|V\|_{C^k}$ as small as desired, such that the Aubry set of $H_V$ is a hyperbolic equilibrium (resp. a hyperbolic periodic orbit).
Proposition (Contreras-Iturriaga, 1999)

Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian of class $C^k$ (with $k \geq 3$). If $V$ is a potential of class $C^2$ such that $\tilde{A}(H_V)$ is a hyperbolic equilibrium or a hyperbolic periodic orbit, then there exists $\varepsilon > 0$ such that the same property holds for every $W : M \to \mathbb{R}$ with $\|W - V\|_{C^2} < \varepsilon$.

Proof.

- If $V_k \to V$, then $\tilde{A}(H_{V_k}) \to \tilde{A}(H_V)$ for the Hausdorff topology in $T^*M$.
- The existence of a hyperbolic periodic orbit is persistent under small perturbations.
Mañé’s density Conjecture

We are reduced to prove the

Conjecture (Mañé’s density conjecture)

For every Tonelli Hamiltonian \( H : T^* M \to \mathbb{R} \) of class \( C^k \) (with \( k \geq 2 \)) there exists a dense set \( \mathcal{D} \) in \( C^k(M) \) such that, for every \( V \in \mathcal{D} \), the Aubry set of the Hamiltonian \( H_V \) is either an equilibrium point or a periodic orbit.

Remark

If we show that generically the Aubry set contains an equilibrium or a periodic orbit we are done.

From now on, we assume that a given Hamiltonian \( H \) of class \( C^k \) (\( k \geq 2 \)) satisfies \( c[H] = 0 \) and that \( \tilde{A}(H) \) contains no equilibrium.
We need to find:

- a potential $V : M \rightarrow \mathbb{R}$ small,
- a periodic orbit $\gamma : [0, T] \rightarrow M$ ($\gamma(0) = \gamma(T)$),
- a Lipschitz function $v : M \rightarrow \mathbb{R}$,

in such a way that the following properties are satisfied:

- $H_V(x, dv(x)) \leq 0$ for a.e. $x \in M$, ($\Rightarrow c[H_V] \leq 0$)
- $\int_0^T L_V(\gamma(t), \dot{\gamma}(t)) \, dt = 0$. ($\Rightarrow c[H_V] \geq 0$)
The strategy (picture)
The strategy (picture)
The strategy (picture)

\[ x \]
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The strategy (picture)
Given \( y_1, y_2 \in \mathbb{R}^m \), set
\[
\text{Cyl}(y_1; y_2) := \bigcup_{s \in [0,1]} B^m((1 - s)y_1 + sy_2, |y_1 - y_2|/3).
\]

**Lemma**

Let \( r > 0 \) and \( Y \) be a finite set in \( \mathbb{R}^m \) such that \( B_{r/12} \cap Y \) contains at least two points. Then, there are \( y_1 \neq y_2 \in Y \) such that the cylinder \( \text{Cyl}(y_1; y_2) \) is included in \( B_r \) and does not intersect \( Y \setminus \{y_1, y_2\} \).
Closing Aubry sets in $C^1$ topology

Theorem (Figalli-R, 2010)

Let $H : T^* M \to \mathbb{R}$ be a Tonelli Hamiltonian of class $C^k$ with $k \geq 4$, and fix $\epsilon > 0$. Then there exists a potential $V : M \to \mathbb{R}$ of class $C^{k-2}$, with $\|V\|_{C^1} < \epsilon$, such that $c[H_V] = c[H]$ and the Aubry set of $H_V$ is either an (hyperbolic) equilibrium point or a (hyperbolic) periodic orbit.

The above result is not satisfactory. The property ”having an Aubry set which is an hyperbolic closed orbit” is not stable under $C^1$ perturbations.
Let $X$ be a smooth vector field on a compact manifold $M$ and $x \in M$ be a recurrent point w.r.t to the flow of $X$.

**Proposition**

For every $\epsilon > 0$, there is a smooth vector field $Y$ having $x$ as a periodic point such that $\|Y - X\|_{C^0} < \epsilon$.

**Theorem (Pugh, 1967)**

For every $\epsilon > 0$, there is a smooth vector field $Y$ having $x$ as a periodic point such that $\|Y - X\|_{C^1} < \epsilon$.

Ref: M.-C. Arnaud. Le ”closing lemma” en topologie $C^1$.

No Lipschitz closing lemma !!!!
Theorem (Figalli-R, 2010)

Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class $C^k$ with $k \geq 2$, and fix $\epsilon > 0$. Assume that there are a recurrent point $\bar{x} \in \mathcal{A}(H)$, a critical viscosity subsolution $u : M \to \mathbb{R}$, and an open neighborhood $\mathcal{V}$ of $O^+(\bar{x})$ such that the following properties are satisfied:

(i) $u$ is of class $C^{1,1}$ in $\mathcal{V}$;
(ii) $H(x, du(x)) = c[H]$ for every $x \in \mathcal{V}$;
(iii) $\mathcal{Hess}^g u(\bar{x})$ is a singleton.

Then there exists a potential $V : M \to \mathbb{R}$ of class $C^k$, with $\|V\|_{C^2} < \epsilon$, such that $c[H_V] = c[H]$ and the Aubry set of $H_V$ is either an equilibrium point or a periodic orbit.
Application to Mañé’s Lagrangians

Recall that given $X$ a $C^k$-vector field on $M$ with $k \geq 2$, the Mañé Lagrangian $L_X : TM \to \mathbb{R}$ associated to $X$ is defined by

$$L_X(x, v) := \frac{1}{2} \| v - X(x) \|^2_x \quad \forall (x, v) \in TM,$$

while the Mañé Hamiltonian $H_X : TM \to \mathbb{R}$ is given by

$$H_X(x, p) = \frac{1}{2} \| p \|^2_x + \langle p, X(x) \rangle \quad \forall (x, p) \in T^*M.$$

Corollary (Figalli-R, 2010)

Let $X$ be a vector field on $M$ of class $C^k$ with $k \geq 2$. Then for every $\epsilon > 0$ there is a potential $V : M \to \mathbb{R}$ of class $C^k$, with $\| V \|_{C^2} < \epsilon$, such that the Aubry set of $H_X + V$ is either an equilibrium point or a periodic orbit.
Theorem (Figalli-R, 2010)

Assume that \( \dim M \geq 3 \). Let \( H : T^*M \to \mathbb{R} \) be a Tonelli Hamiltonian of class \( C^k \) with \( k \geq 4 \), and fix \( \epsilon > 0 \). Assume that there are a recurrent point \( \bar{x} \in A(H) \), a critical viscosity subsolution \( u : M \to \mathbb{R} \), and an open neighborhood \( \mathcal{V} \) of \( O^+(\bar{x}) \) such that

\[
\text{u is at least } C^{k+1} \text{ on } \mathcal{V}.
\]

Then there exists a potential \( V : M \to \mathbb{R} \) of class \( C^{k-1} \), with \( \|V\|_{C^2} < \epsilon \), such that \( c[H_V] = c[H] \) and the Aubry set of \( H_V \) is either an (hyperbolic) equilibrium point or a (hyperbolic) periodic orbit.
Thanks to the Bernard Theorem about the regularity of weak KAM solutions in a neighborhood of the projected Aubry set whenever the Aubry set is an hyperbolic periodic orbit, we infer that the Mañe density conjecture is equivalent to the:

Conjecture (Regularity Conjecture for critical subsolutions)

For every Tonelli Hamiltonian $H : T^* M \to \mathbb{R}$ of class $C^\infty$ there is a set $\mathcal{D} \subset C^\infty(M)$ which is dense in $C^2(M)$ (with respect to the $C^2$ topology) such that the following holds: For every $V \in \mathcal{D}$, there are a recurrent point $\bar{x} \in \mathcal{A}(H)$, a critical viscosity subsolution $u : M \to \mathbb{R}$, and an open neighborhood $\mathcal{V}$ of $\mathcal{O}^+(\bar{x})$ such that $u$ is of class $C^\infty$ on $\mathcal{V}$. 
Thank you for your attention !!