Geometric control theory, closing lemma, and weak KAM theory

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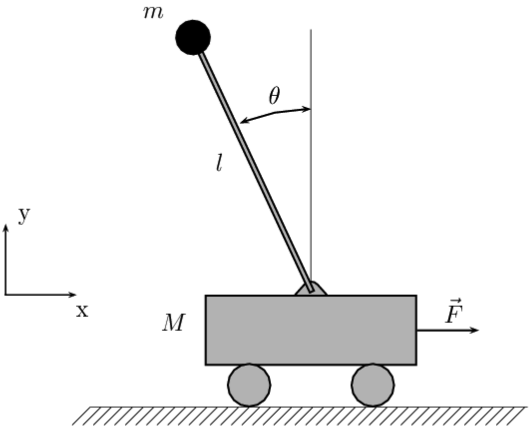
Outline

- Lecture 1: Geometric control methods
- Lecture 2: Applications to Hamiltonian dynamics
- Lecture 3: Weak KAM theory (an introduction)
- Lecture 4: Closing Aubry sets

Lecture 1

Geometric control methods

Control of an inverted pendulum



Control systems

A general control system has the form

$$\dot{x} = f(x, u)$$

where

- x is the state in M
- u is the control in U

Proposition

Under classical assumptions on the datas, for every $x \in M$ and every measurable control $u : [0,T] \to U$ the Cauchy problem

$$\begin{cases} \dot{x}(t) = f\left(x(t), u(t)\right) & a.e. \ t \in [0,T], \\ x(0) = x \end{cases}$$

admits a unique solution

$$x(\cdot) = x(\cdot\,; x, u) : [0,T] \longmapsto M.$$

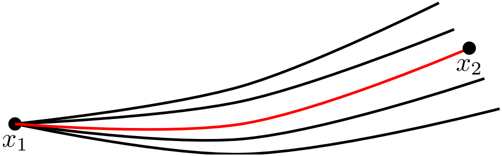
Controllability issues

Given two points x_1, x_2 in the state space M and $T > 0$, can we find a control u such that the solution of

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t)\big) & \text{a.e. } t \in [0, T] \\ x(0) = x_1 \end{cases}$$

satisfies

$$x(T) = x_2 \quad ?$$



Controllability of linear control systems

A (nonautonomous) linear control system has the form

$$\dot{\xi} = A\xi + B\,u,$$

with $\xi \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in M_n(\mathbb{R})$, $B \in M_{n,m}(\mathbb{R})$.

Theorem

The following assertions are equivalent:

(i) *For any* $T > 0$ *and any* $\xi_1, \xi_2 \in \mathbb{R}^n$, *there is* $u \in L^1([0,T];\mathbb{R}^m)$ *such that*

$$\xi\left(T;\xi_1,u\right) = \xi_2.$$

(ii) *The Kalman rank condition is satisfied:*

$$rk\left(B, AB, A^2B, \cdots, A^{n-1}B\right) = n.$$

Proof of the theorem

Duhamel's formula

$$\xi(T;\xi,u) = e^{TA}\xi + e^{TA}\int_0^T e^{-tA}B\,u(t)dt.$$

Then the controllability property (i) is equivalent to the surjectivity of the mappings

$$\mathcal{F}^T \; : \; u \in L^1([0,T];\mathbb{R}^m) \longmapsto \int_0^T e^{-tA}B\,u(t)dt.$$

Proof of (ii) $\Rightarrow$ (i)

If $\mathcal{F}^T$ is not onto (for some $T > 0$), there is $p \neq 0_n$ such that

$$\left\langle p, \int_0^T e^{-tA} B\, u(t) dt \right\rangle = 0 \qquad \forall u \in L^1([0,T];\mathbb{R}^m).$$

Using the linearity of $\langle \cdot, \cdot \rangle$ and taking $u(t) = B^* e^{-tA^*} p$, we infer that

$$p^* e^{-tA} B = 0 \qquad \forall t \in [0,T].$$

Derivating n times at $t = 0$ yields

$$p^* B = p^* A B = p^* A^2 B = \cdots = p^* A^{n-1} B = 0.$$

Which means that p is orthogonal to the image of the $n \times mn$ matrix

$$\left(B, AB, A^2 B, \cdots, A^{n-1} B\right).$$

Contradiction !!!

Proof of (i) $\Rightarrow$ (ii)

If

$$\mathrm{rk}\left(B, AB, A^2B, \cdots, A^{n-1}B\right) < n,$$

there is a nonzero vector p such that

$$p^* B = p^* A B = p^* A^2 B = \cdots = p^* A^{n-1} B = 0.$$

By the Cayley-Hamilton Theorem, we deduce that

$$p^* A^k B = 0 \qquad \forall k \geq 1,$$

and in turn

$$p^* e^{-tA} B = 0 \qquad \forall t \geq 0.$$

We infer that

$$\left\langle p, \int_0^T e^{-tA} B u(t) dt \right\rangle = 0 \qquad \forall u \in L^1([0, T]; \mathbb{R}^m), \forall T > 0.$$

Contradiction !!!

Application to local controllability

Let $\dot{x} = f(x, u)$ be a nonlinear control system with $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ and $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ of class C^1.

Theorem

Assume that $f(x_0, 0) = 0$ and that the pair

$$A = \frac{\partial f}{\partial x}(x_0, 0), \quad B = \frac{\partial f}{\partial u}(x_0, 0),$$

satisfies the Kalman rank condition. Then for there is $\delta > 0$ such that for any x_1, x_2 with $|x_1 - x_0|, |x_2 - x_0| < \delta$, there is $u : [0, 1] \longrightarrow \mathbb{R}^m$ smooth satisfying

$$x\big(1; x_1, u\big) = x_2.$$

Proof of the Theorem

Define $\mathcal{G} : \mathbb{R}^n \times L^1([0,1];\mathbb{R}^m) \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ by

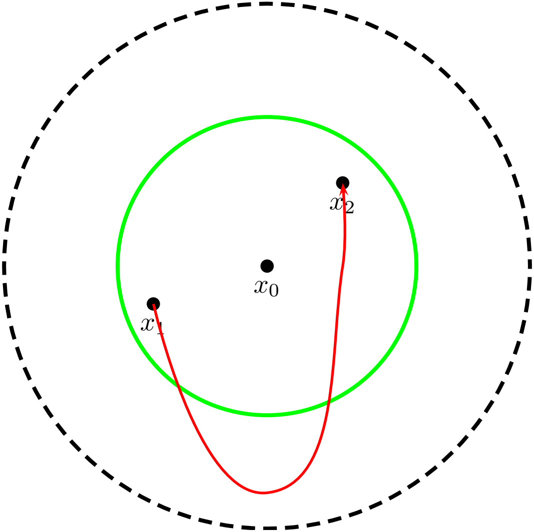
$$\mathcal{G}(x, u) := (x, x(1; x, u)).$$

The mapping $\mathcal{G}$ is a C^1 submersion at $(0,0)$. Thus there are n controls $u^1, \cdots, u^n$ in $L^1([0,1];\mathbb{R}^m)$ such that

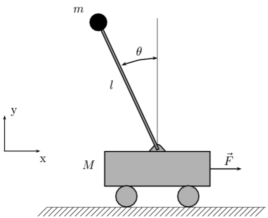
$$\begin{array}{rcccl} \tilde{\mathcal{G}} & : & \mathbb{R}^n \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \times \mathbb{R}^n \\ & & (x, \lambda) & \longmapsto & \mathcal{G}\left(x, \sum_{i=k}^{n} \lambda_k u^k\right) \end{array}$$

is a C^1 diffeomorphism at $(0,0)$. Since the set of smooth controls is dense in $L^1([0,1];\mathbb{R}^m)$, we can take $u^1, \ldots, u^n$ to be smooth. We apply the Inverse Function Theorem.

Local controllability around x_0



Back to the inverted pendulum



The equations of motion are given by

$$\begin{aligned} (M+m)\,\ddot{x} + m\ell\,\ddot{\theta}\cos\theta - m\ell\,\dot{\theta}^2\,\sin\theta &= u \\ m\ell^2\,\ddot{\theta} - mg\ell\,\sin\theta + m\ell\,\ddot{x}\cos\theta &= 0. \end{aligned}$$

Back to the inverted pendulum

The linearized control system at $x = \dot{x} = \theta = \dot{\theta} = 0$ is given by

$$\begin{aligned} (M+m)\,\ddot{x} + m\ell\,\ddot{\theta} &= u \\ m\ell^2\,\ddot{\theta} - mg\ell\,\theta + m\ell\,\ddot{x} &= 0. \end{aligned}$$

It can be written as a control system

$$\dot{\xi} = A\,\xi + B\,u,$$

with $\xi = (x, \dot{x}, \theta, \dot{\theta})$,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{M\ell} & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{M\ell} \end{pmatrix}.$$

Back to the inverted pendulum

The Kalman matrix (B, AB, A^2, A^3B) equals

$$\begin{pmatrix} 0 & \frac{1}{M} & 0 & \frac{mg}{M^2\ell} \\ \frac{1}{M} & 0 & \frac{mg}{M^2\ell} & 0 \\ 0 & -\frac{1}{M\ell} & 0 & -\frac{(M+m)g}{M^2\ell^2} \\ -\frac{1}{M\ell} & 0 & -\frac{(M+m)g}{M^2\ell^2} & 0 \end{pmatrix}.$$

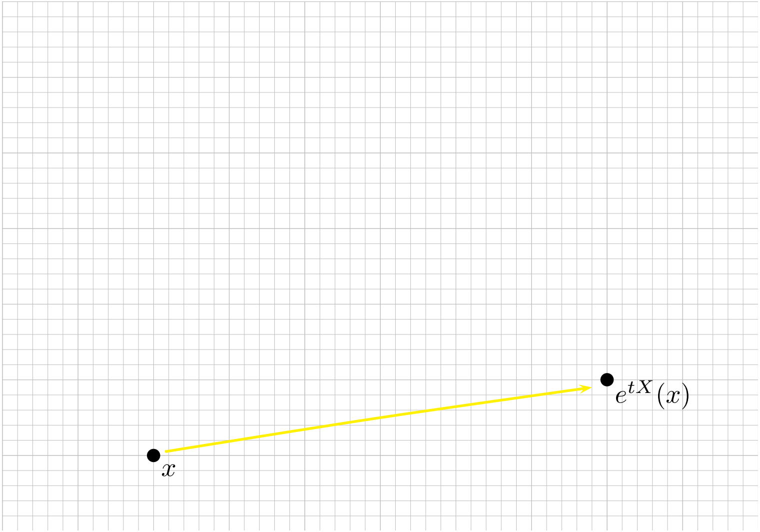
Its determinant equals

$$-\frac{g^2}{M^4\ell^4} < 0$$

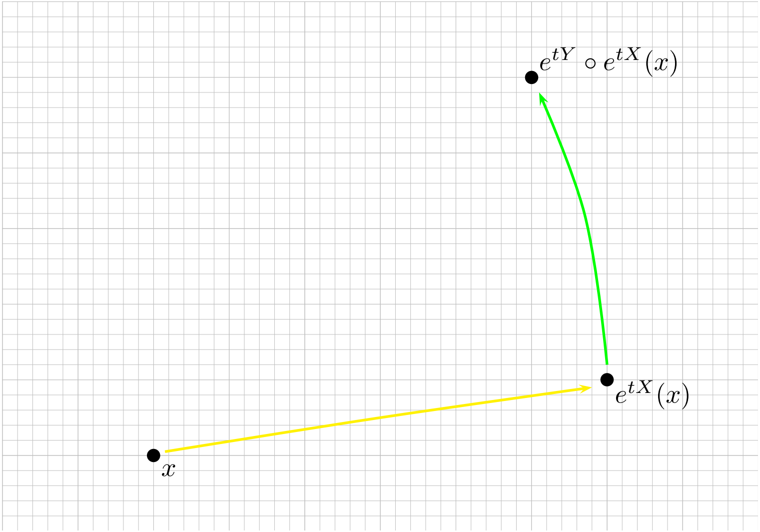
In conclusion, the inverted pendulum is locally controllable around $(0,0,0,0)^*$.

Movie

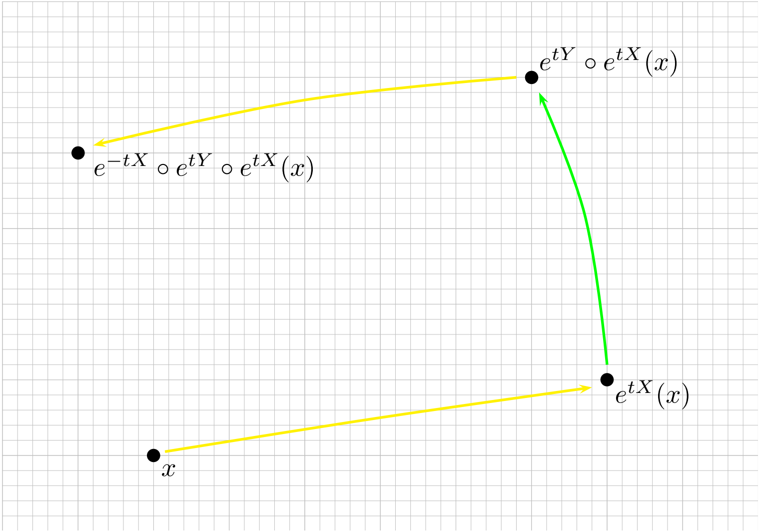
Lie brackets



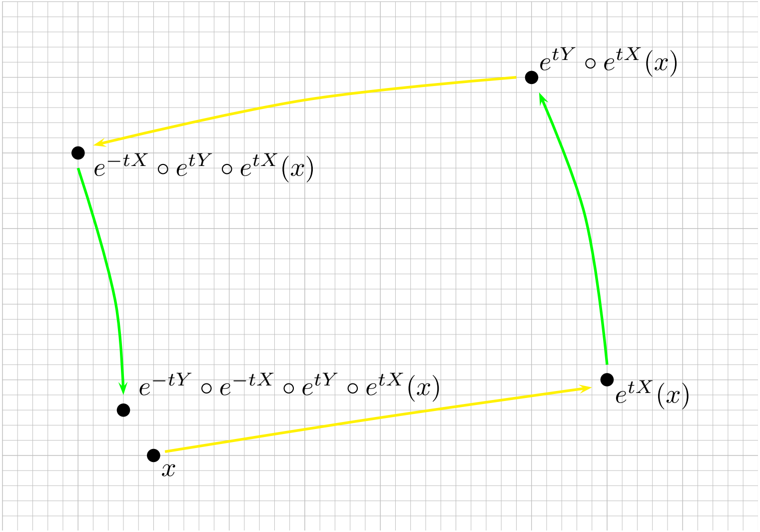
Lie brackets



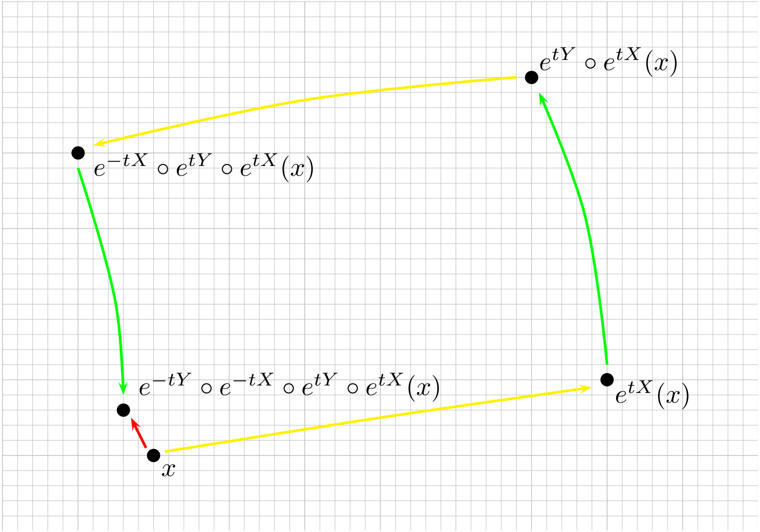
Lie brackets



Lie brackets



Lie brackets



Lie brackets

Definition

Let M be a smooth manifold. Given two smooth vector fields X, Y on M, the Lie bracket $[X, Y]$ is the smooth vector field on M defined by

$$[X, Y](x) = \lim_{t \downarrow 0} \frac{\left(e^{-tY} \circ e^{-tX} \circ e^{tY} \circ e^{tX}\right)(x) - x}{t^2},$$

for every $x \in M$.

Given a family $\mathcal{F}$ of smooth vector fields on M, we denote by Lie$\{\mathcal{F}\}$ the Lie algebra generated by $\mathcal{F}$. It is the smallest vector subspace S of smooth vector fields containing $\mathcal{F}$ that also satisfies

$$[X, Y] \in S \qquad \forall X \in \mathcal{F}, \forall Y \in S.$$

Lie brackets : Examples

In $\mathbb{R}^n$:

- Let X, Y be two smooth vector fields, then

$$[X,Y](x) = DY(x)X(x) - DX(x)Y(x).$$

- If $X(x) = A\,x$, $Y(x) = B\,x$ with $A, B \in M_n(\mathbb{R})$, then

$$[X,Y](x) = [A,B]\,x = (BA - AB)\,x.$$

- If $X(x) = A\,x$, $Y(x) = b$ with $A \in M_n(\mathbb{R})$, $b \in \mathbb{R}^n$, then

$$[X,Y](x) = -Ab, \quad [X,[X,Y]](x) = A^2b, \quad \cdots$$

$$\Longrightarrow \qquad \mathrm{Lie}\{X,Y\} = \mathrm{Span}\left\{Ax, b, Ab, A^2b, A^3b, \ldots\right\}.$$

The Chow-Rashevsky Theorem

Theorem (Chow 1939, Rashevsky 1938)

Let M be a smooth connected manifold and $X^1, \cdots, X^m$ be m smooth vector fields on M. Assume that

$$Lie\left\{X^1, \ldots, X^m\right\}(x) = T_x M \qquad \forall x \in M.$$

Then the control system

$$\dot{x} = \sum_{i=1}^{m} u_i \, X^i(x)$$

is globally controllable on M.

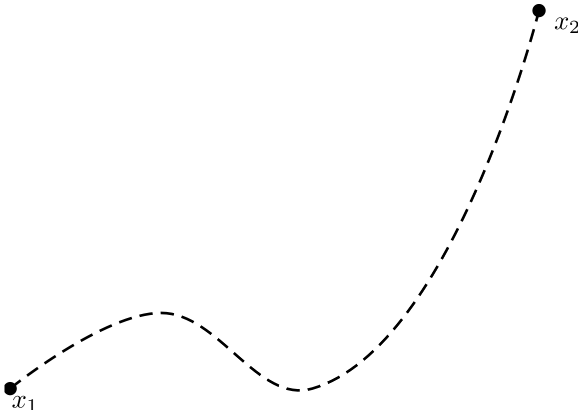
Example: The baby stroller

$$\begin{cases} \dot{x} = u_1 \cos\theta \\ \dot{y} = u_1 \sin\theta \\ \dot{\theta} = u_2 \end{cases}$$

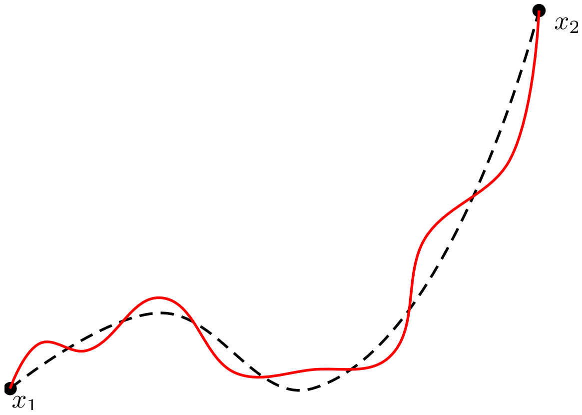
$$X = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad [X,Y] = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

$$\mathrm{Span}\Big\{X(\xi), Y(\xi), [X,Y](\xi)\Big\} = \mathbb{R}^3 \qquad \forall \xi = (x,y,\theta).$$

Example: The baby stroller



Example: The baby stroller



The End-Point mapping

Given a control system of the form

$$\dot{x} = \sum_{i=1}^{m} u_i \, X^i(x) \qquad (x \in M, , u \in \mathbb{R}^m),$$

we define the **End-Point mapping** from x in time $T > 0$ as

$$\begin{array}{rcl} E^{x,T} \; : \; L^1\big([0,T];\mathbb{R}^m\big) & \longrightarrow & M \\ u & \longmapsto & x\big(T;x,u\big) \end{array}$$

Under appropriate assumptions, it is a C^1 mapping.

Theorem

Assume that

$$Lie\left\{X^1,\ldots,X^m\right\}(x) = T_x M \qquad \forall x \in M.$$

Then for any $x \in M$, $T > 0$, $E^{x,T}$ is an open mapping.

Proof of the Chow-Rashevsky Theorem

Let $x \in M$ be fixed. Denote by $\mathcal{A}(x)$ the accessible set from x, that is

$$\mathcal{A}(x) := \left\{ x\big(T; x, u\big) \mid T \geq 0, u \in L^1 \right\}.$$

- By openness of the $E^{x,T}$'s, $\mathcal{A}(x)$ is open.
- Let y be in the closure of $\mathcal{A}(x)$. The set $\mathcal{A}(y)$ contains a small ball centered at y and there are points of $\mathcal{A}(x)$ in that ball. Then $\mathcal{A}(x)$ is closed.

We conclude easily by connectedness of M.

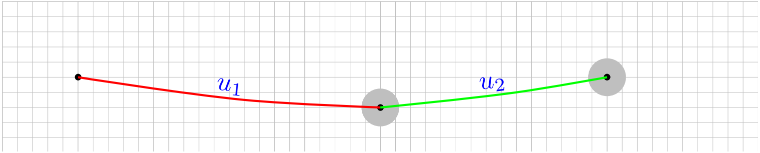
Regular controls vs. Singular controls

Definition

A control $u \in L^1\big([0,T];\mathbb{R}^m\big)$ is called **regular** with respect to $E^{x,T}$ if $E^{x,T}$ is a submersion at u. If not, u is called **singular**.

Remark

The concatenations $u_1 * u_2$ and $u_2 * u_1$ of a regular control u_1 with another control u_2 are regular.



Openness: Sketch of proof

Lemma

Assume that

$$Lie\left\{X^1,\ldots,X^m\right\}(x) = T_xM \qquad \forall x \in M.$$

Then for every $x \in M$ and every $T > 0$, the set of regular controls (w.r.t. $E^{x,T}$) is generic.

Then we apply the so-called Return Method: Given $x \in M$ and $T > 0$, we pick (for any $\alpha > 0$ small) a regular control u^α in $L^1([0,\alpha];\mathbb{R}^m)$. Then for every $u \in L^1([0,T];\mathbb{R}^m)$, the control $\tilde{u}$ defined by

$$\tilde{u} := u^\alpha * \check{u}^\alpha * u$$

is regular and steers x to $E^{x,T}(u)$ in time $T + 2\alpha$.

Then, we can apply the Inverse Function Theorem...

Thank you for your attention !!

Lecture 2

Applications to Hamiltonian dynamics

Setting

Let $n \geq 2$ be fixed. Let $\mathbb{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Hamiltonian of class C^k, with $k \geq 2$, satisfying the following properties:

(H1) **Superlinear growth:**
For every $K \geq 0$, there is $C^*(K) \in \mathbb{R}$ such that

$$H(x,p) \geq K|p| + C^*(K) \qquad \forall x, p.$$

(H2) **Uniform convexity:**
For every x, p, $\frac{\partial^2 H}{\partial p^2}(x,p)$ is positive definite.

(H3) **Uniform boundedness in the bers:**
For every $R \geq 0$,

$$A^*(R) := \sup \left\{ H(x,p) \,|\, |p| \leq R \right\} < +\infty.$$

Under these assumptions, H generates a flow ϕ_t^H which is of class C^{k-1} and complete.

A connecting problem

Let be given two solutions

$$\big(x_i, p_i\big) \;:\; [0,\tau] \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \qquad i = 1, 2,$$

of the Hamiltonian system

$$\begin{cases} \dot{x}(t) &= \nabla_p H\big(x(t), p(t)\big) \\ \dot{p}(t) &= -\nabla_x H\big(x(t), p(t)\big). \end{cases}$$

Question

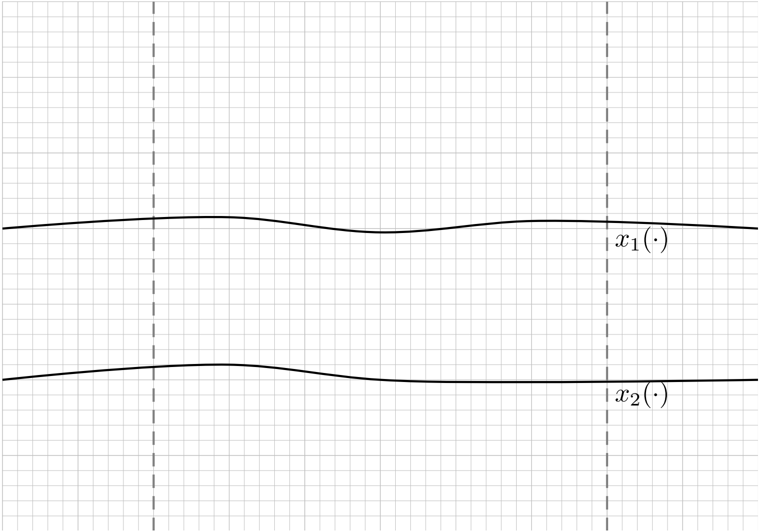
Can I add a potential V to the Hamiltonian H in such a way that the solution of the new Hamiltonian system

$$\begin{cases} \dot{x}(t) &= \nabla_p H\big(x(t), p(t)\big) \\ \dot{p}(t) &= -\nabla_x H\big(x(t), p(t)\big) - \nabla V(x(t)), \end{cases}$$

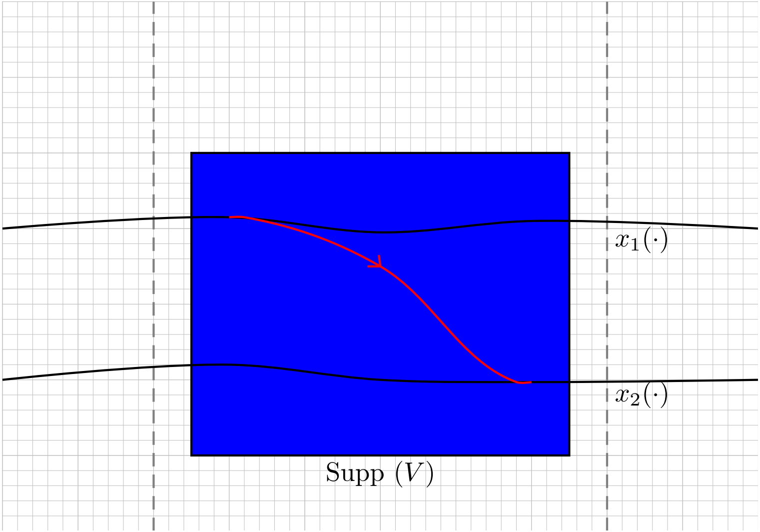
starting at $\big(x_1(0), p_1(0)\big)$ satisfies

$$\big(x(\tau), p(\tau)\big) = \big(x_2(\tau), p_2(\tau)\big)?$$

Picture



Picture



Control approach

Study the mapping

$$\begin{array}{rcl} E \,:\, L^1([0,\tau];\mathbb{R}^n) & \longrightarrow & \mathbb{R}^n \times \mathbb{R}^n \\ u & \longmapsto & \big(x_u(\tau), p_u(\tau)\big) \end{array}$$

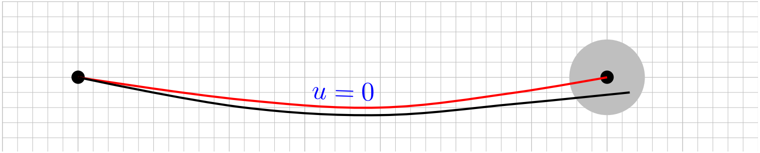
where

$$\big(x_u, p_u\big) : [0,\tau] \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$$

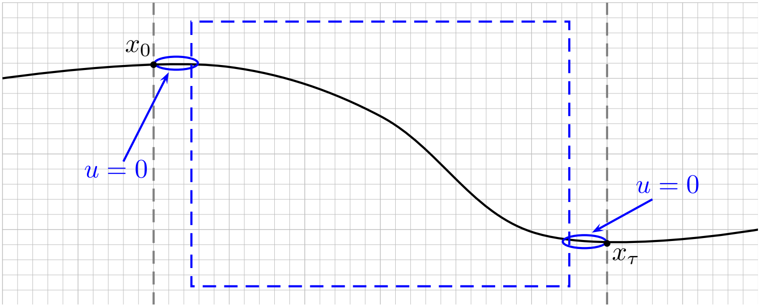
is the solution of

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \nabla_p H\big(x(t), p(t)\big) \\ \dot{p}(t) & = & -\nabla_x H\big(x(t), p(t)\big) - u(t), \end{array} \right.$$

starting at $\big(x_1(0), p_1(0)\big)$.



Exercise



Exercise

Given $x, u : [0,\tau] \to \mathbb{R}^n$ *as above, does there exists a function* $V : \mathbb{R}^n \to \mathbb{R}$ *whose the support is included in the dashed blue square above and such that*

$$\nabla V(x(t)) = u(t) \qquad \forall t \in [0,\tau]\,?$$

Exercise (solution)

There is a necessary condition

$$\int_0^\tau \langle \dot{x}(t), u(t) \rangle dt = 0.$$

As a matter of fact,

$$\begin{aligned} \int_0^\tau \langle \dot{x}(t), u(t) \rangle dt &= \int_0^\tau \langle \dot{x}(t), \nabla V(x(t)) \rangle dt \\ &= V(x_\tau) - V(x_0) = 0. \end{aligned}$$

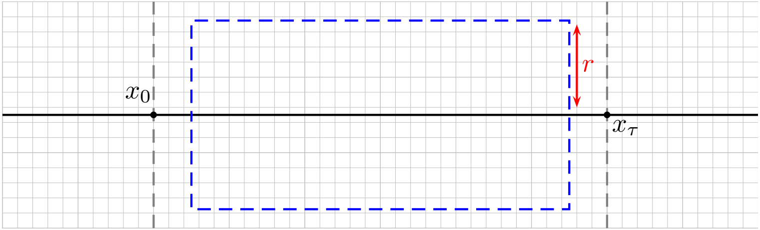
Proposition

If the above necessary condition is satisfied, then there is $V : \mathbb{R}^n \rightarrow \mathbb{R}$ *satisfying the desired properties such that*

$$\|V\|_{C^1} \leq \frac{K}{r} \|u\|_\infty.$$

Exercise (solution)

If $x(t) = (t, 0)$, that is



then we set

$$V(t, y) := \phi(|y|/r) \left[\int_0^t u_1(s)\, ds + \sum_{i=1}^{n-1} \int_0^{y_i} u_{i+1}(t+s)\, ds \right],$$

for every (t, y), with $\phi : [0, \infty) \to [0, 1]$ satisfying

$$\phi(s) = 1 \quad \forall s \in [0, 1/3] \quad \text{and} \quad \phi(s) = 0 \quad \forall s \geq 2/3.$$

Control approach

Study the mapping

$$\begin{array}{rcl} E \ : \ L^1([0,\tau];\mathbb{R}^n) & \longrightarrow & \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \\ u & \longmapsto & \big(x_u(\tau), p_u(\tau), \xi_u(\tau)\big) \end{array}$$

where $\big(x_u, p_u, \xi_u\big) : [0,\tau] \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ is the solution of

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \nabla_p H\big(x(t), p(t)\big) \\ \dot{p}(t) & = & -\nabla_x H\big(x(t), p(t)\big) - u(t) \\ \dot{\xi}(t) & = & \langle \nabla_p H\big(x(t), p(t)\big), u(t) \rangle, \end{array} \right.$$

starting at $\big(x_1(0), p_1(0), 0\big)$.

Objective: Showing that E is a submersion at $u \equiv 0$.

Control approach

Assume that $E : L^1([0,\tau];\mathbb{R}^n) \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ is a submersion at $u \equiv 0$.

- There are $\ell = 2n+1$ controls $u^1, \cdots, u^\ell$ in $L^1([0,\tau];\mathbb{R}^n)$ such that

$$\begin{array}{rcl} \tilde{E} : \mathbb{R}^l & \longrightarrow & \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \\ \lambda & \longmapsto & E\left(\sum_{k=1}^{\ell} \lambda_k u^k\right) \end{array}$$

  is a C^1 diffeomorphism at 0.
- The set of controls $u \in L^1([0,\tau];\mathbb{R}^n)$ such that

$$u \text{ is smooth and } \mathrm{Supp}(u) \subset (0,\tau)$$

  is dense.

We are done.

E submersion ?

No !!

- If $u \in L^1([0,\tau];\mathbb{R}^n)$ with $\int_0^\tau \langle \dot{x}(t), u(t)\rangle dt = 0$, then

$$H\big(x_u(\tau), p_u(\tau)\big) = H\big(x_u(0), p_u(0)\big) = 0.$$

  The final state $\big(x_u(\tau), p_u(\tau)\big)$ must belong to the same level set of H as the initiale state $\big(x_u(0), p_u(0)\big)$. We need to suppress one degree of freedom in the p variable.

- It is not sufficient to get the local controllability.
  We also need to allow free time.

A local controllability result

Given $N, m \geq 1$, let us a consider a nonlinear control system in $\mathbb{R}^N$ of the form

$$\dot{\xi} = F_0(\xi) + \sum_{i=1}^{m} u_i \, F_i(\xi),$$

$G : \mathbb{R}^N \longrightarrow \mathbb{R}^k$ be a function of class C^1, and $\bar{\xi} : [0, T] \longrightarrow \mathbb{R}^N$ be a solution associated with $\bar{u} \equiv 0$.

Our aim is to give sufficient conditions on $F_0, F_1, \ldots, F_m$, and G to have the following property:

For any neighborhood $\mathcal{V}$ of $\bar{u} \equiv 0$ in $L^1\big([0, \bar{T}]; \mathbb{R}^m\big)$, the set

$$\Big\{ G\big(\xi_{\bar{\xi}(0),u}(T)\big) \mid u \in \mathcal{V} \Big\}$$

is a neighborhood of $G\big(\xi_{\bar{\xi}(0),\bar{u}}(T)\big)$.

A local controllability result

Denote by $E^{\bar{\xi}(0),T}$ the **End-Point mapping**

$$u \in L^1([0,T];\mathbb{R}^m) \longmapsto \xi_{\bar{\xi}(0),u}(T),$$

where $\xi_{\bar{\xi}(0),u}$ is the trajectory of the control system associated with u and starting at $\bar{\xi}(0)$.

Proposition

If G is a submersion at $\bar{\xi}(T)$, and

$$\mathrm{Span}\Big\{ F_i(\bar{\xi}(\bar{T})), \, [F_0, F_i](\bar{\xi}(\bar{T})) \mid i = 1, \ldots, m \Big\} + \mathrm{Ker}\left(dG\left(\bar{\xi}(\bar{T})\right)\right) = \mathbb{R}^N,$$

then $G \circ E^{\bar{\xi}(0),T}$ is a submersion at $\bar{u} \equiv 0$.

Thanks to the uniform convexity of H in the p variable, the above result applies to our control problem.

An alternative method

Let

$$\left(x_i, p_i\right) \; : \; [0,\tau] \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \qquad i = 1, 2,$$

be two solutions of the Hamiltonian system

$$\begin{cases} \dot{x}(t) & = & \nabla_p H\big(x(t), p(t)\big) \\ \dot{p}(t) & = & -\nabla_x H\big(x(t), p(t)\big). \end{cases}$$

Question

Given an arc $x : [0,\tau] \rightarrow \mathbb{R}^n$ *such that*

$$x(t) = x_1(t) \, \forall t \in [0,\delta] \text{ and } x(t) = x_2(t) \, \forall t \in [\tau - \delta, \tau],$$

does there exist $p, u : [0,\tau] \rightarrow \mathbb{R}^n$ *such that*

$$\begin{cases} \dot{x}(t) & = & \nabla_p H\big(x(t), p(t)\big) \\ \dot{p}(t) & = & -\nabla_x H\big(x(t), p(t)\big) - u(t) \end{cases} \qquad \forall t \in [0, t] \qquad ?$$

An alternative method (flatness)

Yes !!

Remark

For every x, $p \mapsto \frac{\partial H}{\partial p}(x,p)$ is a diffeomorphism.

Then we can set

$$\begin{cases} p(t) & := & \left(\frac{\partial H}{\partial p}(x(t),\cdot)\right)^{-1}(x(t),\dot{x}(t)) \\ u(t) & := & -\frac{\partial H}{\partial x}(x(t),p(t)) - \dot{p}(t) \end{cases} \qquad \forall t \in [0,\tau],$$

By construction there holds

$$\begin{cases} \dot{x}(t) & = & \nabla_p H(x(t),p(t)) \\ \dot{p}(t) & = & -\nabla_x H(x(t),p(t)) - u(t) \end{cases} \qquad \forall t \in [0,t]$$

$$\text{and} \qquad u(t) = 0 \qquad \forall t \in [0,\delta] \cup [\tau-\delta,\tau].$$

To get $\int_0^t \langle \dot{x}(s), u(s) ds = 0 \forall t$, we reparametrize in time.

A connecting problem fitting the action

Let be given two solutions

$$\big(x_i, p_i\big) \ : \ [0,\tau] \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \qquad i=1,2$$

of the Hamiltonian system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \nabla_p H\big(x(t),p(t)\big) \\ \dot{p}(t) & = & -\nabla_x H\big(x(t),p(t)\big). \end{array} \right.$$

Question

Can I add a potential V to the Hamiltonian H in such a way that the solution $\big(x,p\big) \ : \ [0,\tau] \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$ of the new Hamiltonian system associated with $H_V := H + V$ starting at $\big(x_1(0),p_1(0)\big)$ satisfies $\big(x(\tau),p(\tau)\big) = \big(x_2(\tau),p_2(\tau)\big)$

$$\text{and} \qquad \int_0^\tau L\big(x(t),\dot{x}(t)\big) - V\big(x(t)\big)\,dt = \textit{data} \qquad ?$$

A connecting problem fitting the action

This can be done !!

Study the mapping

$$\begin{array}{rcl} E \;:\; L^1([0,\tau];\mathbb{R}^n) & \longrightarrow & \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \\ u & \longmapsto & \big(x_u(\tau), p_u(\tau), \xi_u(\tau), \ell_u(\tau)\big) \end{array}$$

where $\big(x_u, p_u, \xi_u, \ell_u\big) : [0,\tau] \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ is the solution of

$$\begin{cases} \dot{x}(t) & = & \nabla_p H\big(x(t), p(t)\big) \\ \dot{p}(t) & = & -\nabla_x H\big(x(t), p(t)\big) - u(t) \\ \dot{\xi}(t) & = & \langle \nabla_p H\big(x(t), p(t)\big), u(t) \rangle \\ \dot{\ell}(t) & = & \langle p(t), \nabla_p H\big(x(t), p(t)\big) \rangle, \end{cases}$$

starting at $\big(x_1(0), p_1(0), 0, 0\big)$.
Again, we need to relax time.
(It works provided some algebraic condition is satisfied.)

Controlling the differential of an Hamiltonian flow

Let be given a solution

$$\big(x,p\big)\ :\ [0,\tau] \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$$

of the Hamiltonian system

$$\begin{cases} \dot{x}(t) & = & \nabla_p H\big(x(t),p(t)\big) \\ \dot{p}(t) & = & -\nabla_x H\big(x(t),p(t)\big). \end{cases}$$

Question

Can I add a potential V to H in such a way that:

- (x,p) *is still solution of the new Hamiltonian system associated with* $H_V := H + V$.
- *The differential of* ϕ_τ^H *at* $(x(0),p(0))$ *equals a data.*

Controlling the differential of an Hamiltonian flow

This can be done !!

Take a potential V satisfying

$$V(x(t)) = 0 \text{ and } \nabla V(x(t)) = 0 \qquad \forall t \in [0, \tau].$$

Then (x, p) is still solution of the new Hamiltonian system associated with H_V.

The Control is:

$$u(t) = \mathrm{Hess}_{x(t)} V$$

(indeed $\mathrm{Hess}_{x(t)} V$ restricted to a space transverse to $\dot{x}(t)$).

Controlling the differential of an Hamiltonian flow

Study the mapping

$$\begin{array}{rcl} E \ : \ L^1\left([0,\tau];\mathbb{R}^{n(n-1)/2}\right) & \longrightarrow & \mathrm{Sp}(n) \\ u & \longmapsto & D_u(\tau) \end{array}$$

where $D_u : [0,\tau] \longrightarrow \mathrm{Sp}(n)$ is the resolvent of the linearized system

$$\left\{ \begin{array}{rcl} \dot{h}(t) & = & \nabla_{px} H\, h(t) + \nabla_{pp} H\, v(t) \\ \dot{v}(t) & = & -\nabla_{xx} H\, h(t) - \nabla_{xp} H\, v(t) - u(t) \end{array} \right. \qquad \forall t \in [0,\tau],$$

starting at I_{2n}.

Indeed we need to work in $\mathrm{Sp}(n-1)$.

Controlling the differential of an Hamiltonian flow

Our control system has the form

$$\dot{X}(t) = A(t)X(t) + \sum_{i=1}^{k} u_i(t)B_i(t)X(t),$$

where the state X belongs to $M_n(\mathbb{R})$ and

$$A, B_1, \ldots, B_k : [0, \tau] \longrightarrow M_n(\mathbb{R}) \quad \text{are smooth.}$$

Indeed we are interested in trajectories starting at I_{2m} and valued in the **symplectic group**

$$\mathrm{Sp}(m) = \{X \mid X^* \mathbb{J} X = \mathbb{J}\}.$$

Assumption: $A(t), B_i(t) \in T_{I_{2m}}\mathrm{Sp}(m)$ for any $t \in [0, \tau]$.

Controlling the differential of an Hamiltonian flow

Define the k sequences of smooth mappings

$$\{B_1^j\}, \ldots, \{B_k^j\} : [0,\tau] \longrightarrow T_{I_{2m}}\mathrm{Sp}(m)$$

by

$$\begin{cases} B_i^0(t) := B_i(t) \\ B_i^j(t) := \dot{B}_i^{j-1}(t) + B_i^{j-1}(t)A(t) - A(t)B_i^{j-1}(t), \end{cases}$$

for every $t \in [0,\tau]$ and every $i \in \{1,\ldots,k\}$.

Theorem

Assume that there is some $t \in [0,\tau]$ such that

$$Span\Big\{B_i^j(t) \,|\, i \in \{1,\,\ldots,k\}, j \in \mathbb{N}\Big\} = T_{I_{2m}}Sp(m).$$

Then the End-Point mapping $E^{I_{2m},\tau} : L^1\big([0,\tau];\mathbb{R}^k\big) \longrightarrow Sp(m)$ is a submersion at $u \equiv 0$.

Thank you for your attention !!

Lecture 3

Weak KAM theory
(an introduction)

Setting

Let M be a smooth compact manifold of dimension $n \geq 2$ be fixed. Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian of class C^k, with $k \geq 2$, satisfying the following properties:

(H1) **Superlinear growth:**
For every $K \geq 0$, there is $C^*(K) \in \mathbb{R}$ such that

$$H(x,p) \geq K|p| + C^*(K) \qquad \forall (x,p) \in T^*M.$$

(H2) **Uniform convexity:**
For every $(x,p) \in T^*M$, $\frac{\partial^2 H}{\partial p^2}(x,p)$ is positive definite.

For sake of simplicity, we may assume that $M = \mathbb{T}^n$, that is that $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies (H1)-(H2) and is periodic with respect to the x variable.

Critical value of H

Definition

We call **critical value** of H the constant $c = c[H]$ defined as

$$c[H] := \inf_{u \in C^1(M;\mathbb{R})} \left\{ \max_{x \in M} \left\{ H\big(x, du(x)\big) \right\} \right\}.$$

In other terms, $c[H]$ is the infimum of numbers $c \in \mathbb{R}$ such that there is a C^1 function $u : M \to \mathbb{R}$ satisfying

$$H\big(x, du(x)\big) \leq c \qquad \forall x \in M.$$

Note that

$$\min_{x \in M} \{H(x,0)\} \leq c[H] \leq \max_{x \in M} \{H(x,0)\}.$$

Critical subsolutions of H

Definition

We call **critical subsolution** any Lipschitz function $u : M \to \mathbb{R}$ such that

$$H\big(x, du(x)\big) \leq c[H] \qquad \text{for a.e. } x \in M.$$

Proposition

The set of critical subsolutions is nonempty.

Proof.

- Any C^1 function $u : M \to \mathbb{R}$ such that $H(\cdot, du(\cdot)) \leq c$ is $L(c)$-Lipschitz with $L(c)$ depending only on c.
- Arzelà-Ascoli Theorem.
- If $u_k \to u$ then $\mathrm{Graph}(du) \subset \liminf_{k\to\infty} \mathrm{Graph}(du_k)$.

□

Characterization of critical subsolutions

Let $L : TM \to \mathbb{R}$ be the Tonelli Lagrangian associated with H by Legendre-Fenchel duality, that is

$$L(x,v) := \max_{p \in T_x^* M} \left\{ p \cdot v - H(x,p) \right\} \qquad \forall (x,v) \in TM.$$

Proposition

A Lipschitz function $u : M \to \mathbb{R}$ is a critical subsolution if and only if

$$u\big(\gamma(b)\big) - u\big(\gamma(a)\big) \leq \int_a^b L\big(\gamma(t), \dot{\gamma}(t)\big)\, ds + c\,(b-a),$$

for every Lipschitz curve $\gamma : [a,b] \to M$.

It is a consequence of the inequality

$$p \cdot v \leq L(x,v) + H(x,p) \qquad \forall x, v, p.$$

Characterization of critical subsolutions

Proof.

If u is C^1, then

$$\begin{aligned} u(\gamma(b)) - u(\gamma(a)) &= \int_a^b du(\gamma(t)) \cdot \dot{\gamma}(t)\, dt \\ &\leq \int_a^b L(\gamma(t), \dot{\gamma}(t))\, dt \\ &\qquad + \int_a^b H(\gamma(t), du(\gamma(t)))\, dt \\ &\leq \int_a^b L(\gamma(t), \dot{\gamma}(t))\, ds + c\,(b-a). \end{aligned}$$

If u is not C^1 then regularize it by (classical) convolution. The function $u * \rho_\epsilon$ is subsolution of $H \leq c + \alpha\epsilon$. Apply the above argument to $u * \rho_\epsilon$ and pass to the limit. □

Lax-Oleinik semigroups $\{\mathcal{T}_t\}$ and $\{\check{\mathcal{T}}_t\}$

Definition

Given $u : M \to \mathbb{R}$ and $t \geq 0$, the Lipschitz functions $\mathcal{T}_t u, \check{\mathcal{T}}_t u$ are defined by

$$\begin{aligned} \mathcal{T}_t u(x) &:= \min_{y \in M} \{u(y) + A_t(y,x)\} \\ \check{\mathcal{T}}_t u(x) &:= \max_{y \in M} \{u(y) - A_t(x,y)\}, \end{aligned}$$

$$\text{with} \quad A_t(z,z') := \inf \left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s))\, ds + c\, t \right\},$$

where the infimum is taken over the Lipschitz curves $\gamma : [0,t] \to M$ such that $\gamma(0) = z$ and $\gamma(t) = z'$.

The set of critical subsolutions is invariant with respect to both $\{\mathcal{T}_t\}$ and $\{\check{\mathcal{T}}_t\}$.

The weak KAM Theorem

Theorem (Fathi, 1997)

There is a critical subsolution $u : M \to \mathbb{R}$ such that

$$\mathcal{T}_t u = u \qquad \forall t \geq 0.$$

It is called a **critical** *or a* **weak KAM solution** *of H.*

Given a critical solution $u : M \to \mathbb{R}$, for every $x \in M$, there is a curve

$$\gamma : (-\infty, 0] \to M \quad \text{with} \quad \gamma(0) = x$$

such that, for any $a < b \leq 0$,

$$u\big(\gamma(b)\big) - u\big(\gamma(a)\big) = \int_a^b L\big(\gamma(s), \dot{\gamma}(s)\big)\, ds + c(b-a).$$

Therefore, any restriction of γ minimizes the action between its end-points. Then, it satisfies the Euler-Lagrange equations.

The classical Dirichlet problem

Let Ω be an open set in $\mathbb{R}^n$ with compact boundary and $H : \mathbb{R}^n \to \mathbb{R}$ of class C^2 satisfying (H1),(H2) and

(H3) For every $x \in \bar{\Omega}$, $H(x,0) < 0$.

Proposition

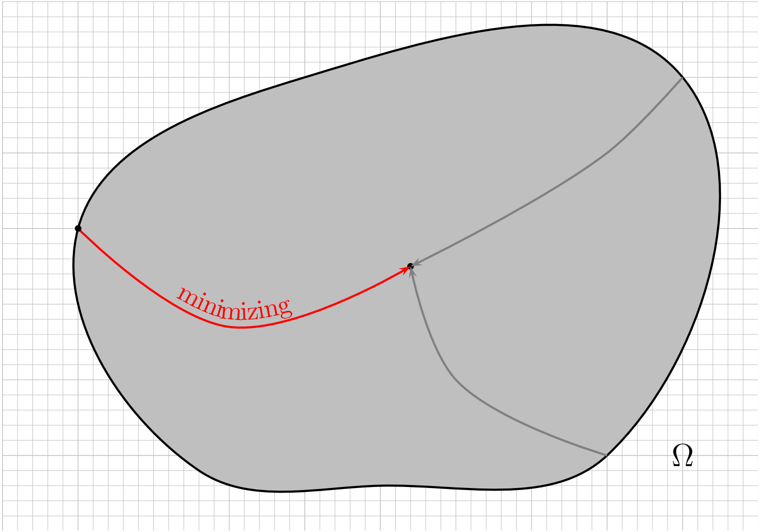
The continuous function $u : \bar{\Omega} \to \mathbb{R}$ given by

$$u(x) := \inf \left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s))\, ds \right\},$$

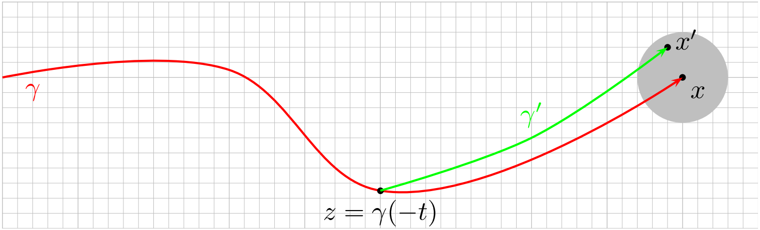
where the infimum is taken among Lipschitz curves $\gamma : [0,t] \to \bar{\Omega}$ with $\gamma(0) \in \partial\Omega, \gamma(t) = x$ is the unique viscosity solution to the Dirichlet problem

$$\begin{cases} H(x, du(x)) = 0 & \forall x \in \Omega, \\ u(x) = 0 & \forall x \in \partial\Omega. \end{cases}$$

The classical Dirichlet problem (picture)



Semiconcavity of critical solutions



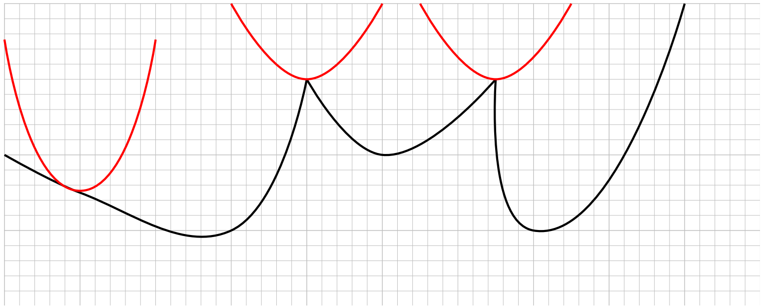
$$u(x) = u(z) + \int_{-t}^{0} L\big(\gamma(s), \dot\gamma(s)\big)\, ds + ct$$

$$u(x') \leq u(z) + \int_{-t}^{0} L\big(\gamma'(s), \dot\gamma'(s)\big)\, ds + ct$$

Thus

$$u(x') \leq u(x) + \int_{-t}^{0} L\big(\gamma'(s), \dot\gamma'(s)\big) - L\big(\gamma(s), \dot\gamma(s)\big)\, ds$$

Semiconcavity of critical solutions

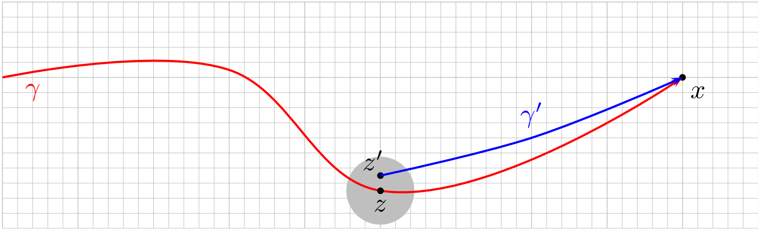


Any critical solution $u : M \longrightarrow \mathbb{R}$ is **semiconcave**, that is it can be written locally (in charts) as

$$u = g + h,$$

the sum of a smooth function g and a concave function h.

Regularity along minimizing curves



$$u(x) = u(z) + \int_{-t}^{0} L\big(\gamma(s), \dot\gamma(s)\big)\, ds + ct.$$

$$u(x) \leq u(z') + \int_{-t}^{0} L\big(\gamma'(s), \dot\gamma'(s)\big)\, ds + ct.$$

Thus

$$u(z') \geq u(z) + \int_{-t}^{0} L\big(\gamma(s), \dot\gamma(s)\big) - L\big(\gamma'(s), \dot\gamma'(s)\big)\, ds$$

Regularity along minimizing curves

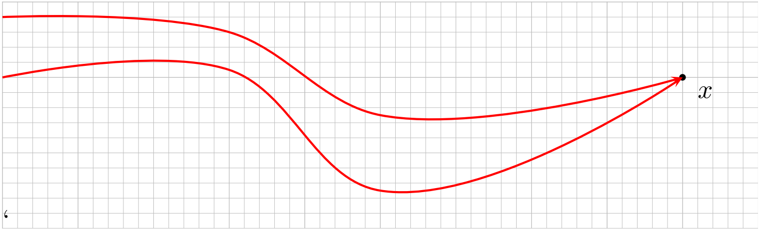
Distribution of calibrated curves

Let u be a critical solution. For every $x \in M$, we define the **limiting di erential** of u at x by

$$d^* u(x) := \{\lim du(x_k) \,|\, x_k \to x, u \text{ diff at } x_k\}.$$

It is a nonempty compact subset satisfying

$$H\big(x, du^*(x)\big) = c \qquad \forall x \in M.$$



Remark

Let $u : M \to \mathbb{R}$ be a critical solution, $x \in M$ be fixed and $\gamma : (-\infty, 0] \to \mathbb{R}$ be a **calibrated curve** with $\gamma(0) = x$. Fix

$$x_\infty \in \bigcap_{t \leq 0} \overline{\gamma\big((-\infty, t]\big)}.$$

We can check that

$$\liminf_{t \to +\infty} \left\{ A_t \left(x_\infty, x_\infty \right) \right\} = 0.$$

Proposition

The critical value of H satisfies

$$c[H] = -\inf \left\{ \frac{1}{T} \int_0^T L\big(\gamma(t), \dot{\gamma}(t)\big)\, dt \right\},$$

where the infimum is taken over the Lipschitz curves $\gamma : [0, T] \to M$ such that $\gamma(0) = \gamma(T)$.

Projected Aubry set and Aubry set

Definition and Proposition

- The **projected Aubry set** of H defined as

$$\mathcal{A}(H) = \{x \in M \,|\, A_t(x,x) = 0\}\,.$$

  is compact and nonempty.
- Any critical subsolution u is C^1 at any point of $\mathcal{A}(H)$ and satisfies $H\big(x, du(x)\big) = c[H], \forall x \in \mathcal{A}(H)$.
- For every $x \in \mathcal{A}(H)$, the differential of a critical subsolution at x does not depend on u.
- The **Aubry set** of H defined by

$$\tilde{\mathcal{A}(H)} := \big\{\big(x, du(x)\big) \,|\, x \in \mathcal{A}(H), u \text{ crit. subsol.}\big\} \subset T^*M$$

  is compact, invariant by ϕ_t^H, and is a Lipschitz graph over $\mathcal{A}(H)$.

Examples (in $\mathbb{T}^n$)

Let $H : T(\mathbb{T}^n)^* \to \mathbb{R}$ be the Hamiltonian defined by

$$H(x,p) = \frac{1}{2}|p|^2 + V(x) \qquad \forall (x,p) \in \mathbb{T}^n \times \mathbb{R}^n.$$

- $L(x,v) = \frac{1}{2}|v|^2 - V(x)$.
- $H(x,0) \leq \max_M V \Longrightarrow c[H] \leq \max_M V$.
- Let $x_{\max} \in \mathbb{T}^n$ be such that $V(x_{max}) = \max_M V$, then

$$\frac{1}{T}\int_0^T L\big(x_{\max}, 0\big)\, dt = -\max_M V.$$

  Thus $c[H] \geq \max_M V$.
- In conclusion $c[H] = \max_M V$ and

$$\tilde{\mathcal{A}}(H) = \left\{ (x,0) \,|\, V(x) = \max_M V \right\}.$$

Examples (in $\mathbb{T}^1 = \mathbb{S}^1$)

Let $H : T(\mathbb{S}^1)^* \to \mathbb{R}$ be the Hamiltonian defined by

$$H(x,p) = \frac{1}{2}(p - f(x))^2 \qquad \forall (x,p) \in \mathbb{S}^1 \times \mathbb{R}.$$

- $u : \mathbb{S}^1 \to \mathbb{R}$ defined by (set $\alpha := \left(\int_0^1 f(r)dr\right)$)

$$u(x) = \int_0^x f(r)dr - \alpha\, x \qquad \forall x \in \mathbb{S}^1,$$

  is a smooth solution of $H(x, du(x)) = \alpha^2/2$ for any x.
  Then $c[H] = \alpha^2/2$.
- Along characteristics, there holds ($p(t) := du(x(t))$)

$$\begin{cases} \dot{x}(t) &= \frac{\partial H}{\partial p}(x(t), p(t))) = p(t) - f(x(t)) \\ \dot{p}(t) &= -\frac{\partial H}{\partial x}(x(t), p(t)) = \big(f(x(t)) - p(t)\big) f'(x(t)). \end{cases}$$

  Then $x \in \mathcal{A}(H) \Longrightarrow \dot{x} = (f(x) - \alpha) - f(x) = \alpha.$
- Either equilibria everywhere or one orbit.

Examples (in $\mathbb{S}^1 = \mathbb{R}/[0,\pi]$)

Let $H : T(\mathbb{S}^1)^* \to \mathbb{R}$ be the Hamiltonian defined by

$$H(x,p) = \frac{1}{2}(p-\omega)^2 - V(x) \qquad \forall (x,p) \in \mathbb{S}^1 \times \mathbb{R},$$

with $V(x) = \sin^2(x)$ and $\omega = -\int_0^\pi 2\sqrt{V(r)}dr = -4.$

- $H(x,0) \leq 0 \Longrightarrow c[H] \leq 0.$
- $L(x,v) = v^2/2 + \omega v + V(x) \Rightarrow L(0,0) = 0 \Rightarrow c[H] \geq 0.$
- Let u be a critical subsolution. Then there holds a.e.

$$(u' - \omega)^2 \leq 2V \Longrightarrow u' - \omega \leq 2\sqrt{V} \Longrightarrow u' \leq \omega + 2\sqrt{V}.$$

  In conclusion, $u(x) = \int_0^x 2\sqrt{V(r)}dr + \omega x$ for any x.
- The Aubry set consists in one equilibria and one orbit.

Examples (Mañé's Lagrangians)

Let X be a smooth vector field on M and $L : TM \longrightarrow \mathbb{R}$ defined by

$$L_X(x,v) = \frac{1}{2}|v - X(x)|^2 \qquad \forall (x,v) \in TM.$$

- $H_X(x,p) = \frac{1}{2}|p|^2 + p \cdot X(x)$.
- $H(x,0) = 0$ for any $x \in M$. Then $c[H] = 0$.
- Characteristics of $u = 0$ satisfy

$$\dot{x}(t) = X(x(t)), \quad p(t) = 0.$$

- The projected Aubry set always contains the set of recurrent points.

Two theorems by Bernard

Theorem (Bernard, 2006)

There exists a critical subsolution of class $C^{1,1}$.

Idea of the proof: Use a Lasry-Lions type convolution.
If u is a given critical solution, then $\left(\mathcal{T}_s \circ \check{\mathcal{T}}_t\right)(u)$ is $C^{1,1}$ provided $s, t > 0$ are small enough.

Theorem (Bernard, 2007)

Assume that the Aubry set is exactly one hyperbolic periodic orbit, then any critical solution is "smooth" in a neighborhood of $\mathcal{A}(H)$. As a consequence, there is a "smooth" critical subsolution.

Idea of the proof: The Aubry set is the boundary at infinity, that is any calibrated curve $\gamma : (-\infty, 0] \to M$ tends to $\mathcal{A}(H)$ as t tends to $-\infty$. Indeed, for every $p \in d^* u(x)$ there is such a calibrated curve such that $\dot{\gamma}(0) = \frac{\partial H}{\partial p}(\gamma(0), p)$.

Thank you for your attention !!

Lecture 4

Closing Aubry sets

Setting

Let M be a smooth compact manifold of dimension $n \geq 2$ be fixed. Let $\mathbb{H} : T^*M \to \mathbb{R}$ be a Hamiltonian of class C^k, with $k \geq 2$, satisfying the following properties:

(H1) **Superlinear growth:**
For every $K \geq 0$, there is $C^*(K) \in \mathbb{R}$ such that

$$H(x,p) \geq K|p| + C^*(K) \qquad \forall (x,p) \in T^*M.$$

(H2) **Uniform convexity:**
For every $(x,p) \in T^*M$, $\frac{\partial^2 H}{\partial p^2}(x,p)$ is positive definite.

For sake of simplicity, we may assume that $M = \mathbb{T}^n$, that is that $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies (H1)-(H2) and is periodic with respect with the x variable.

The Mañé Conjecture

Conjecture (Mañé, 96)

For every Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class C^k (with $k \geq 2$), there is a residual subset (i.e., a countable intersection of open and dense subsets) $\mathcal{G}$ of $C^k(M)$ such that, for every $V \in \mathcal{G}$, the Aubry set of the Hamiltonian $H_V := H + V$ is either an equilibrium point or a periodic orbit.

Strategy of proof:

- Density result.
- Stability result.

Mañé's density Conjecture

Conjecture (Mañé's density conjecture)

For every Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class C^k (with $k \geq 2$) there exists a dense set $\mathcal{D}$ in $C^k(M)$ such that, for every $V \in \mathcal{D}$, the Aubry set of the Hamiltonian H_V is either an equilibrium point or a periodic orbit.

Proposition (Contreras-Iturriaga, 1999)

Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian of class C^k (with $k \geq 3$) whose Aubry set is an equilibrium point (resp. a periodic orbit). Then, there is a smooth potential $V : M \to \mathbb{R}$, with $\|V\|_{C^k}$ as small as desired, such that the Aubry set of H_V is a hyperbolic equilibrium (resp. a hyperbolic periodic orbit).

Stability

Proposition (Contreras-Iturriaga, 1999)

Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian of class C^k (with $k \geq 3$). If V is a potential of class C^2 such that $\tilde{A}(H_V)$ is a hyperbolic equilibrium or a hyperbolic periodic orbit, then there exists $\epsilon > 0$ such that the same property holds for every $W : M \to \mathbb{R}$ with $\|W - V\|_{C^2} < \epsilon$.

Proof.

- If $V_k \to V$, then $\tilde{A}(H_{V_k}) \to \tilde{A}(H_V)$ for the Hausdorff topology in T^*M.
- The existence of a hyperbolic periodic orbit is persistent under small perturbations.

□

Mañé's density Conjecture

We are reduced to prove the

Conjecture (Mañé's density conjecture)

For every Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class C^k (with $k \geq 2$) there exists a dense set $\mathcal{D}$ in $C^k(M)$ such that, for every $V \in \mathcal{D}$, the Aubry set of the Hamiltonian H_V is either an equilibrium point or a periodic orbit.

Remark

If we show that generically the Aubry set contains an equilibrium or a periodic orbit we are done.

From now on, we assume that a given Hamiltonian H of class C^k ($k \geq 2$) satisfies $c[H] = 0$ and that $\tilde{\mathcal{A}}(H)$ contains no equilibrium.

Preliminaries

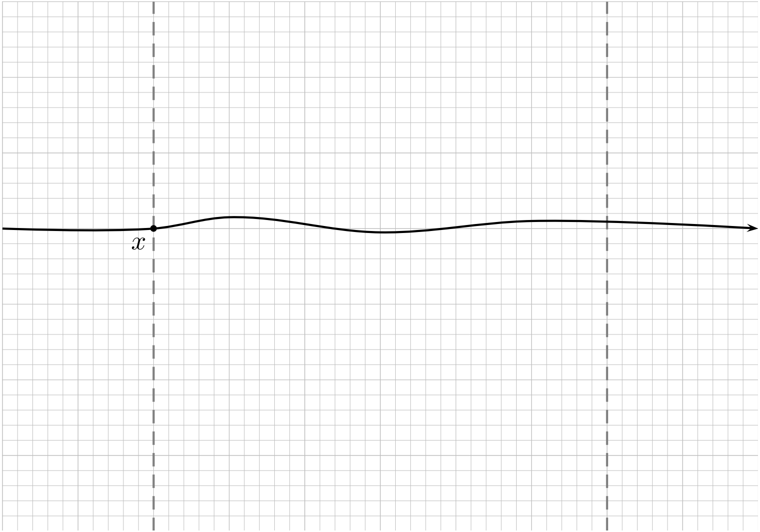
We need to find:

- a potential $V : M \to \mathbb{R}$ **small**,
- a periodic orbit $\gamma : [0, T] \to M$ ($\gamma(0) = \gamma(T)$),
- a Lipschitz function $v : M \to \mathbb{R}$,

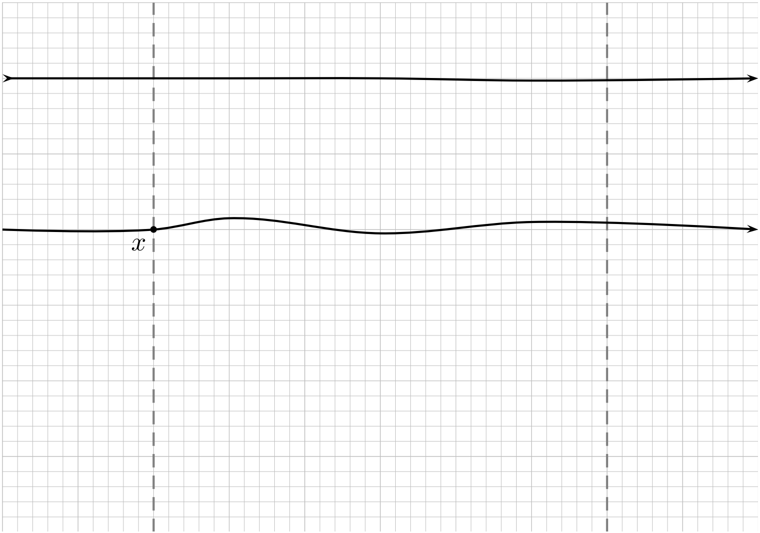
in such a way that the following properties are satisfied:

- $H_V\big(x, dv(x)\big) \leq 0$ for a.e. $x \in M$, $(\Rightarrow c[H_V] \leq 0)$
- $\int_0^T L_V\big(\gamma(t), \dot{\gamma}(t)\big)\, dt = 0.$ $(\Rightarrow c[H_V] \geq 0)$

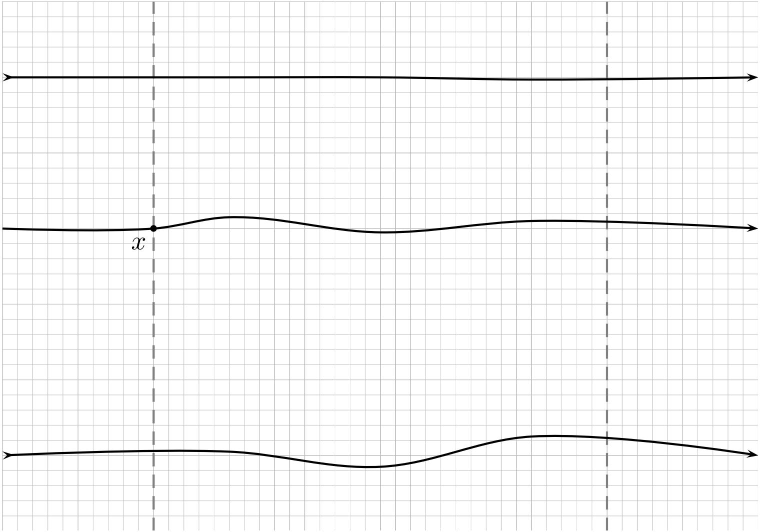
The strategy (picture)



The strategy (picture)



The strategy (picture)



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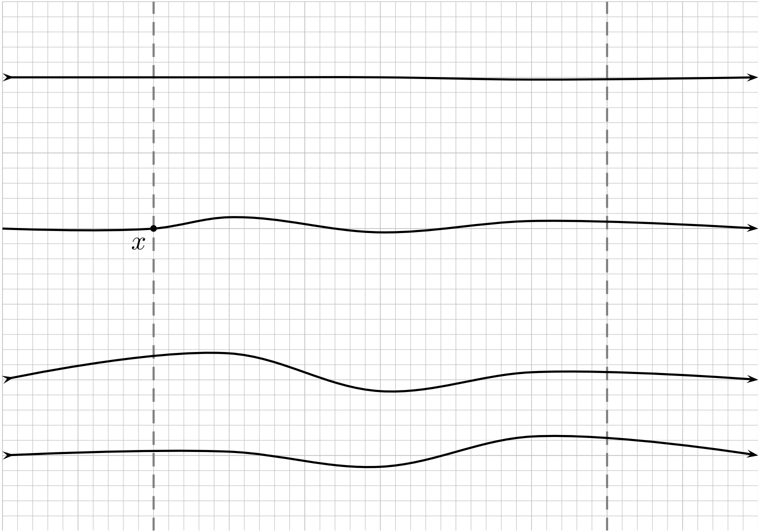
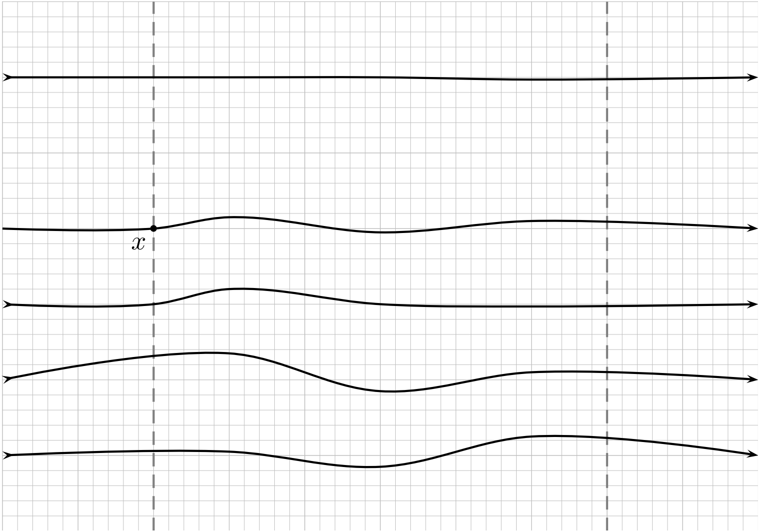
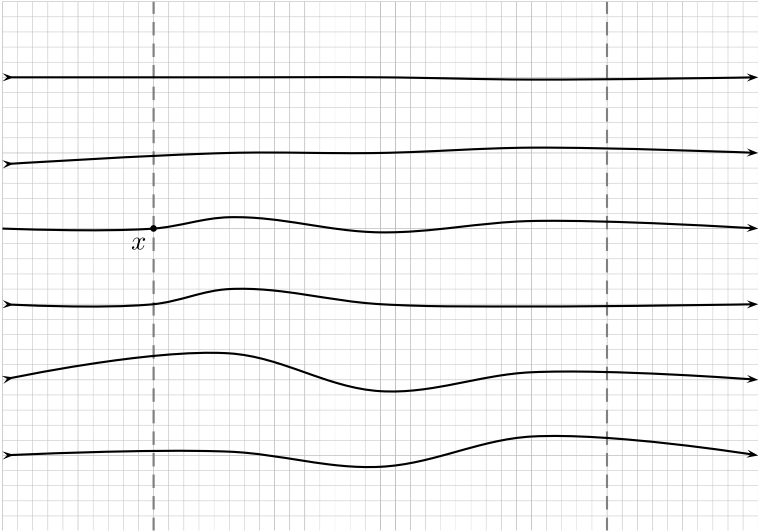


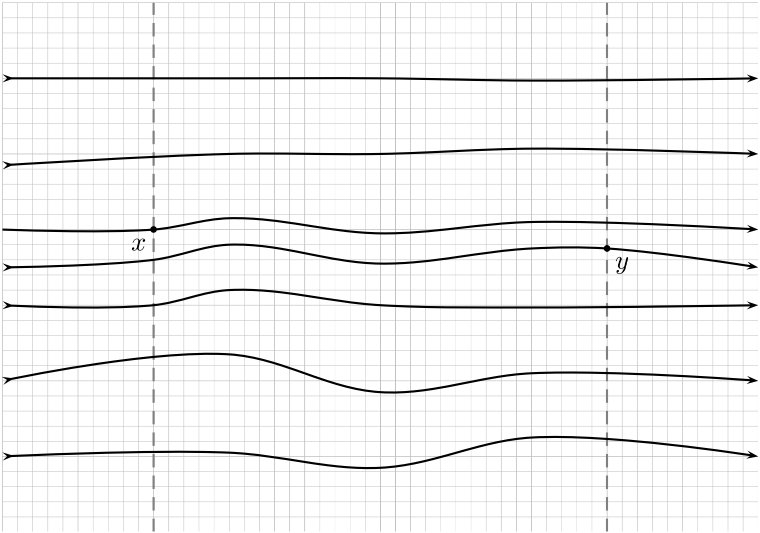
The strategy (picture)



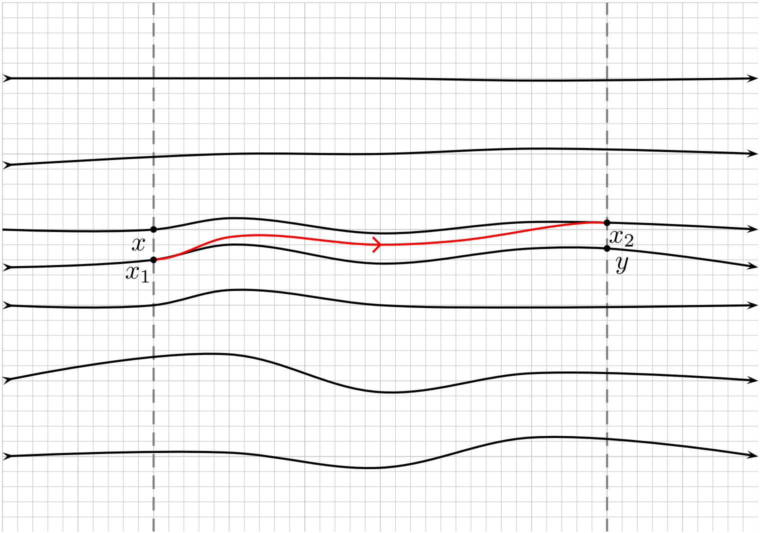
The strategy (picture)



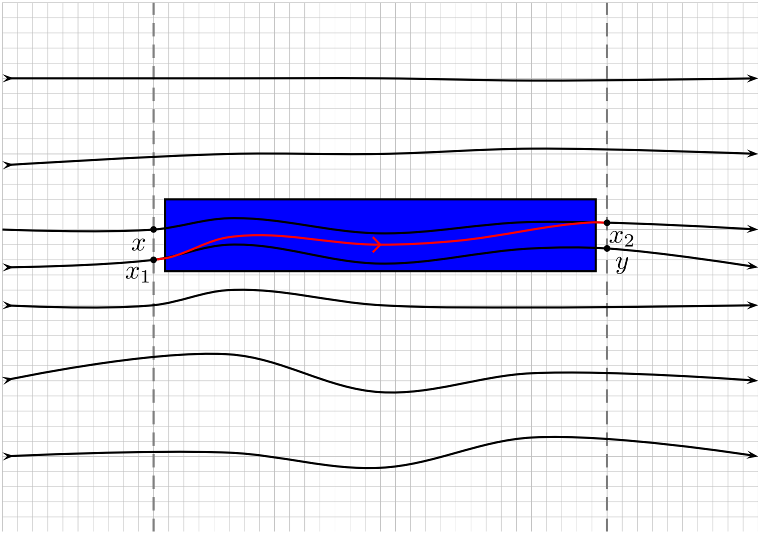
The strategy (picture)



The strategy (picture)



The strategy (picture)



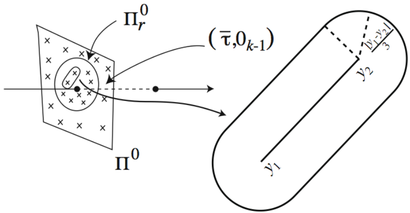
Picture

Given $y_1, y_2 \in \mathbb{R}^m$, set

$$\mathrm{Cyl}(y_1; y_2) := \bigcup_{s \in [0,1]} B^m\Big((1-s)y_1 + sy_2, \big|y_1 - y_2\big|/3\Big).$$

Lemma

Let $r > 0$ and Y be a finite set in $\mathbb{R}^m$ such that $B_{r/12} \cap Y$ contains at least two points. Then, there are $y_1 \neq y_2 \in Y$ such that the cylinder $\mathrm{Cyl}(y_1; y_2)$ is included in B_r and does not intersect $Y \setminus \{y_1, y_2\}$.



Closing Aubry sets in C^1 topology

Theorem (Figalli-R, 2010)

Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class C^k with $k \geq 4$, and fix $\epsilon > 0$. Then there exists a potential $V : M \to \mathbb{R}$ of class C^{k-2}, with $\|V\|_{C^1} < \epsilon$, such that $c[H_V] = c[H]$ and the Aubry set of H_V is either an (hyperbolic) equilibrium point or a (hyperbolic) periodic orbit.

The above result is not satisfactory. The property "having an Aubry set which is an hyperbolic closed orbit" is not stable under C^1 perturbations.

Closing lemmas

Let X be a smooth vector field on a compact manifold M and $x \in M$ be a recurrent point w.r.t to the flow of X.

Proposition

For every $\epsilon > 0$, there is a smooth vector field Y having x as a periodic point such that $\|Y - X\|_{C^0} < \epsilon$.

Theorem (Pugh, 1967)

For every $\epsilon > 0$, there is a smooth vector field Y having x as a periodic point such that $\|Y - X\|_{C^1} < \epsilon$.

Ref: M.-C. Arnaud. Le "closing lemma" en topologie C^1.

No Lipschitz closing lemma !!!!

Partial density results I

Theorem (Figalli-R, 2010)

Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian of class C^k with $k \geq 2$, and fix $\epsilon > 0$. Assume that there are a recurrent point $\bar{x} \in \mathcal{A}(H)$, a critical viscosity subsolution $u : M \to \mathbb{R}$, and an open neighborhood $\mathcal{V}$ of $\mathcal{O}^+(\bar{x})$ such that the following properties are satisfied:

(i) *u is of class $C^{1,1}$ in $\mathcal{V}$;*

(ii) *$H(x, du(x)) = c[H]$ for every $x \in \mathcal{V}$;*

(iii) *$\mathcal{H}ess^g u(\bar{x})$ is a singleton.*

Then there exists a potential $V : M \to \mathbb{R}$ of class C^k, with $\|V\|_{C^2} < \epsilon$, such that $c[H_V] = c[H]$ and the Aubry set of H_V is either an equilibrium point or a periodic orbit.

Application to Mañé's Lagrangians

Recall that given X a C^k-vector field on M with $k \geq 2$, the Mañé Lagrangian $L_X : TM \to \mathbb{R}$ associated to X is defined by

$$L_X(x,v) := \frac{1}{2}\|v - X(x)\|_x^2 \qquad \forall (x,v) \in TM,$$

while the Mañé Hamiltonian $H_X : TM \to \mathbb{R}$ is given by

$$H_X(x,p) = \frac{1}{2}\|p\|_x^2 + \langle p, X(x)\rangle \qquad \forall (x,p) \in T^*M.$$

Corollary (Figalli-R, 2010)

Let X be a vector field on M of class C^k with $k \geq 2$. Then for every $\epsilon > 0$ there is a potential $V : M \to \mathbb{R}$ of class C^k, with $\|V\|_{C^2} < \epsilon$, such that the Aubry set of $H_X + V$ is either an equilibrium point or a periodic orbit.

Partial density results II

Theorem (Figalli-R, 2010)

Assume that $\dim M \geq 3$. *Let* $H : T^*M \to \mathbb{R}$ *be a Tonelli Hamiltonian of class* C^k *with* $k \geq 4$, *and fix* $\epsilon > 0$. *Assume that there are a recurrent point* $\bar{x} \in \mathcal{A}(H)$, *a critical viscosity subsolution* $u : M \to \mathbb{R}$, *and an open neighborhood* $\mathcal{V}$ *of* $\mathcal{O}^+(\bar{x})$ *such that*

$$u \text{ is at least } C^{k+1} \text{ on } \mathcal{V}.$$

Then there exists a potential $V : M \to \mathbb{R}$ *of class* C^{k-1}, *with* $\|V\|_{C^2} < \epsilon$, *such that* $c[H_V] = c[H]$ *and the Aubry set of* H_V *is either an (hyperbolic) equilibrium point or a (hyperbolic) periodic orbit.*

Discussion

Thanks to the Bernard Theorem about the regularity of weak KAM solutions in a neighborhood of the projected Aubry set whenever the Aubry set is an hyperbolic periodic orbit, we infer that the Mañé density conjecture is equivalent to the:

Conjecture (Regularity Conjecture for critical subsolutions)

For every Tonelli Hamiltonian $H : T^*M \to \mathbb{R}$ of class C^∞ there is a set $\mathcal{D} \subset C^\infty(M)$ which is dense in $C^2(M)$ (with respect to the C^2 topology) such that the following holds: For every $V \in \mathcal{D}$, there are a recurrent point $\bar{x} \in \mathcal{A}(H)$, a critical viscosity subsolution $u : M \to \mathbb{R}$, and an open neighborhood $\mathcal{V}$ of $\mathcal{O}^+(\bar{x})$ such that u is of class C^∞ on $\mathcal{V}$.

Thank you for your attention !!