Introduction to Sub-Riemannian Geometry

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Outline

- Lecture 1:

  A controllability result: The Chow-Rashevsky Theorem

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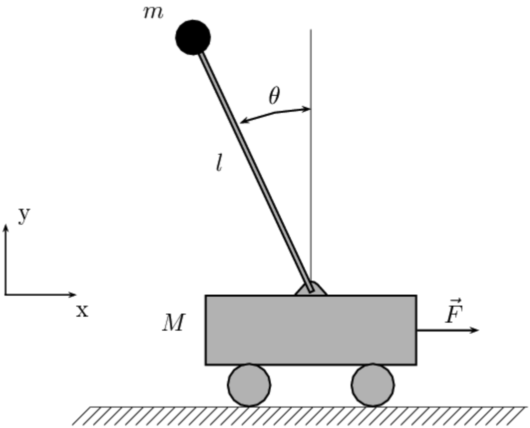
- Lecture 2:

  An optimal control study: Sub-Riemannian geodesics

Lecture 1

A controllability result:
The Chow-Rashevsky Theorem

Control of an inverted pendulum



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Proposition

Under classical assumptions on the datas, for every $x \in M$ and every measurable control $u : [0, T] \to U$ the Cauchy problem

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t)\big) & a.e.\ t \in [0, T], \\ x(0) = x \end{cases}$$

admits a unique solution

$$x(\cdot) = x(\cdot\,; x, u) : [0, T] \longrightarrow M.$$

Controllability issues

Given two points x_1, x_2 in the state space M and $T > 0$, can we find a control u such that the solution of

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t)\big) & \text{a.e. } t \in [0, T] \\ x(0) = x_1 \end{cases}$$

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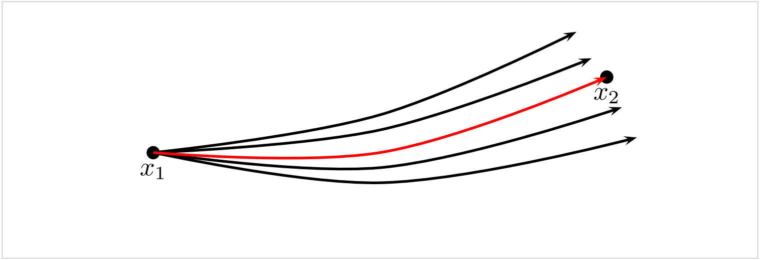

Controllability of linear control systems in $\mathbb{R}^n$

An autonomous linear control system in $\mathbb{R}^n$ has the form

$$\dot{\xi} = A\xi + B\,u,$$

with $\xi \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in M_n(\mathbb{R})$, $B \in M_{n,m}(\mathbb{R})$.

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Theorem

The following assertions are equivalent:

(i) *For any $T > 0$ and any $\xi_1, \xi_2 \in \mathbb{R}^n$, there is $u \in L^1([0,T];\mathbb{R}^m)$ such that*

$$\xi\left(T;\xi_1,u\right) = \xi_2.$$

(ii) *The Kalman rank condition is satisfied:*

$$rk\left(B, AB, A^2B, \cdots, A^{n-1}B\right) = n.$$

Proof of the theorem

Duhamel's formula

$$\xi\big(T;\xi,u\big) = e^{TA}\,\xi + e^{TA}\int_0^T e^{-tA}\,B\,u(t)dt.$$

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Then the controllability property (i) is equivalent to the surjectivity of the mappings

$$\mathcal{F}^T \; : \; u \in L^1([0,T];\mathbb{R}^m) \longmapsto \int_0^T e^{-tA} B\, u(t)dt.$$

Proof of (ii) $\Rightarrow$ (i)

If $\mathcal{F}^T$ is not onto (for some $T > 0$), there is $p \neq 0_n$ such that

$$\left\langle p, \int_0^T e^{-tA} B\, u(t) dt \right\rangle = 0 \qquad \forall u \in L^1([0,T];\mathbb{R}^m).$$

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Using the linearity of $\langle \cdot, \cdot \rangle$ and taking $u(t) = B^* e^{-tA^*} p$, we infer that

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Application to local controllability

Let $\dot{x} = f(x, u)$ be a nonlinear control system with $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ of class C^1.

Theorem

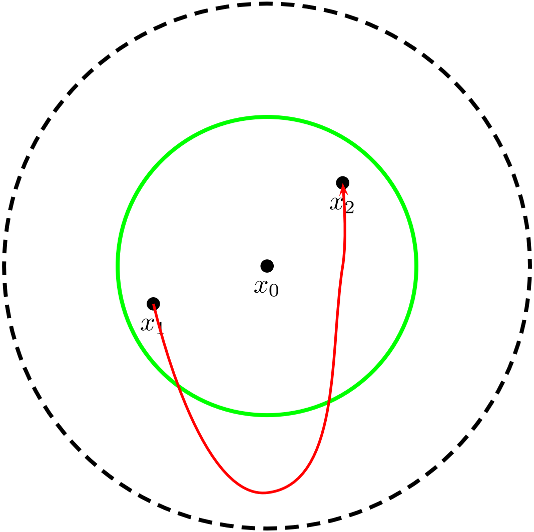
Assume that $f(x_0, 0) = 0$ and that the pair

$$A = \frac{\partial f}{\partial x}(x_0, 0), \quad B = \frac{\partial f}{\partial u}(x_0, 0),$$

satisfies the Kalman rank condition. Then for there is $\delta > 0$ such that for any x_1, x_2 with $|x_1 - x_0|, |x_2 - x_0| < \delta$, there is $u : [0, 1] \to \mathbb{R}^m$ smooth satisfying

$$x\big(1; x_1, u\big) = x_2.$$

Local controllability around x_0



Proof of the Theorem

Define $\mathcal{G} : \mathbb{R}^n \times L^1([0,1];\mathbb{R}^m) \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ by

$$\mathcal{G}\big(x, u\big) := \big(x, x(1; x, u)\big).$$

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The mapping $\mathcal{G}$ is a C^1 submersion at $(0,0)$. Thus there are n controls $u^1, \cdots, u^n$ in $L^1([0,1];\mathbb{R}^m)$ such that

$$\begin{array}{rcccl} \tilde{\mathcal{G}} & : & \mathbb{R}^n \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \times \mathbb{R}^n \\ & & (x, \lambda) & \longmapsto & \mathcal{G}\left(x, \sum_{i=k}^n \lambda_k u^k\right) \end{array}$$

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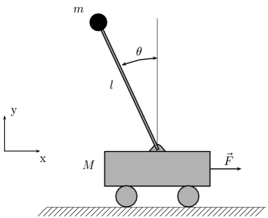
Back to the inverted pendulum



The equations of motion are given by

$$
\begin{aligned}
(M+m)\,\ddot{x} + m\ell\,\ddot{\theta}\cos\theta - m\ell\,\dot{\theta}^2\,\sin\theta &= u \\
m\ell^2\,\ddot{\theta} - mg\ell\,\sin\theta + m\ell\,\ddot{x}\cos\theta &= 0.
\end{aligned}
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Back to the inverted pendulum

The linearized control system at $x = \dot{x} = \theta = \dot{\theta} = 0$ is given by

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It can be written as a control system

$$\dot{\xi} = A\,\xi + B\,u,$$

with $\xi = (x, \dot{x}, \theta, \dot{\theta})$,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{M\ell} & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{M\ell} \end{pmatrix}.$$

Back to the inverted pendulum

The Kalman matrix (B, AB, A^2, A^3B) equals

$$\begin{pmatrix} 0 & \frac{1}{M} & 0 & \frac{mg}{M^2\ell} \\ \frac{1}{M} & 0 & \frac{mg}{M^2\ell} & 0 \\ 0 & -\frac{1}{M\ell} & 0 & -\frac{(M+m)g}{M^2\ell^2} \\ -\frac{1}{M\ell} & 0 & -\frac{(M+m)g}{M^2\ell^2} & 0 \end{pmatrix}.$$

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Its determinant equals

$$-\frac{g^2}{M^4\ell^4} < 0$$

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In conclusion, the inverted pendulum is locally controllable around $(0,0,0,0)^*$.

Movie

The Chow-Rashevsky Theorem

Theorem (Chow 1939, Rashevsky 1938)

Let M be a smooth manifold and $X^1, \cdots, X^m$ be m smooth vector fields on M. Assume that

$$Lie\left\{X^1, \ldots, X^m\right\}(x) = T_x M \qquad \forall x \in M.$$

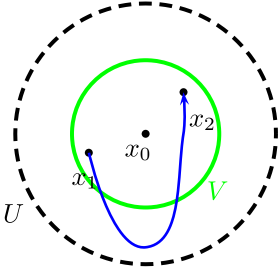
Then the control system

$$\dot{x} = \sum_{i=1}^{m} u_i X^i(x)$$

is locally controllable in any time at every point of M.

Comment I

The local controllability in any time at every point means that for every $x_0 \in M$, every $T > 0$ and every neighborhood U of x_0, there is a neighborhood $V \subset U$ of x_0 such that for any $x_1, x_2 \in V$, there is a control $u \in L^1([0,T];\mathbb{R}^m)$ such that the trajectory $x(\cdot; x_1, u) : [0,T] \to M$ remains in U and steers x_1 to x_2, *i.e.* $x(T; x_1, u) = x_2$.



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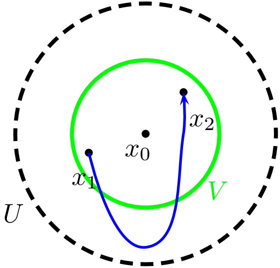

Local controllability in time $T > 0$

$\Rightarrow$ Local controllability in time $T' > 0, \quad \forall T' > 0$

Comment II

If M is **connected** then

Local controllability $\Rightarrow$ Global controllability

Let $x \in M$ be fixed. Denote by $\mathcal{A}(x)$ the accessible set from x, that is

$$\begin{aligned} \mathcal{A}(x) &:= \left\{ x\left(T; x, u\right) \mid T \geq 0, u \in L^1 \right\} \\ &= \left\{ x\left(1; x, u\right) \mid u \in L^1 \right\}. \end{aligned}$$

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- By local controllability, $\mathcal{A}(x)$ is open.
- Let y be in the closure of $\mathcal{A}(x)$. The set $\mathcal{A}(y)$ contains a small ball centered at y and there are points of $\mathcal{A}(x)$ in that ball. Then $\mathcal{A}(x)$ is closed.

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By connectedness of M, we infer that $\mathcal{A}(x) = M$ for every $x \in M$, and in turn that the control system is globally controllable in any time.

The Chow-Rashevsky Theorem

Theorem (Chow 1939, Rashevsky 1938)

Let M be a smooth manifold and $X^1, \cdots, X^m$ be m smooth vector fields on M. Assume that

$$Lie\left\{X^1, \ldots, X^m\right\}(x) = T_x M \qquad \forall x \in M.$$

Then the control system $\dot{x} = \sum_{i=1}^{m} u_i X^i(x)$ is locally controllable in any time at every point of M.

The condition in red is called Hörmander's condition or bracket generating condition. Families of vector fields satisfying that condition are called nonholonomic, completely nonholonomic, or totally nonholonomic.

Comment III

Definition

Given two smooth vector fields X, Y on $\mathbb{R}^n$, the Lie bracket $[X, Y]$ at $x \in \mathbb{R}^n$ is defined by

$$[X, Y](x) = DY(x)X(x) - DX(x)Y(x).$$

The Lie brackets of two smooth vector fields on M can be defined in charts with the above formula.

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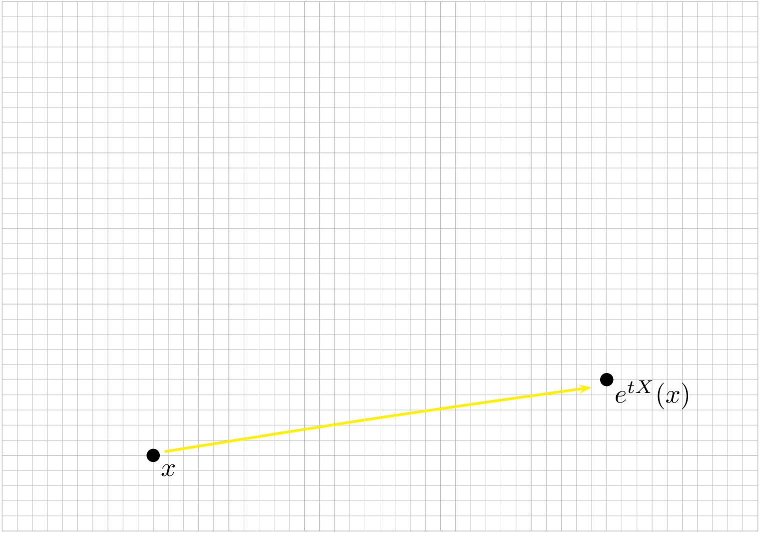
The Lie brackets of two smooth vector fields on M can be defined in charts with the above formula.

Given a family $\mathcal{F}$ of smooth vector fields on M, we denote by $\mathrm{Lie}\{\mathcal{F}\}$ the Lie algebra generated by $\mathcal{F}$. It is the smallest vector subspace S of smooth vector fields containing $\mathcal{F}$ that also satisfies

$$[X, Y] \in S \qquad \forall X \in \mathcal{F}, \forall Y \in S.$$

Comment III



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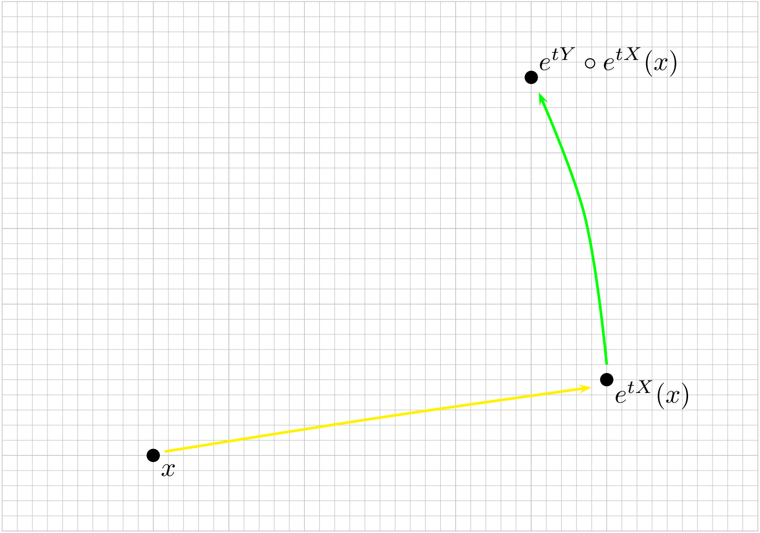

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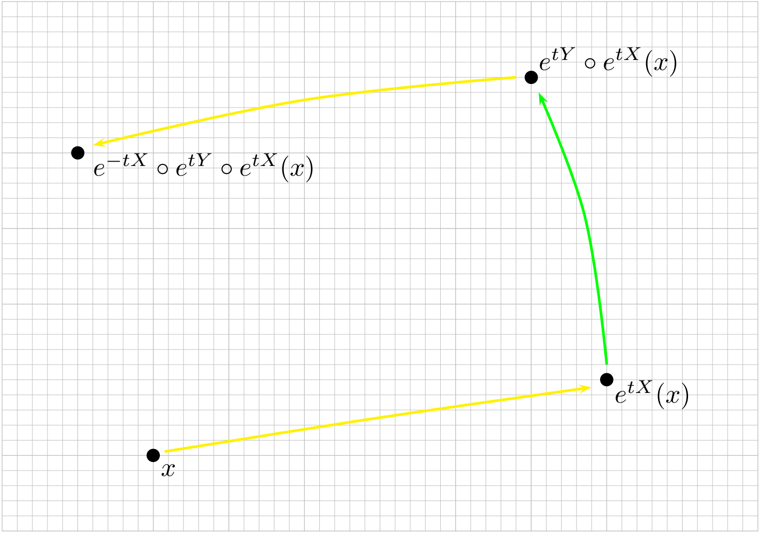

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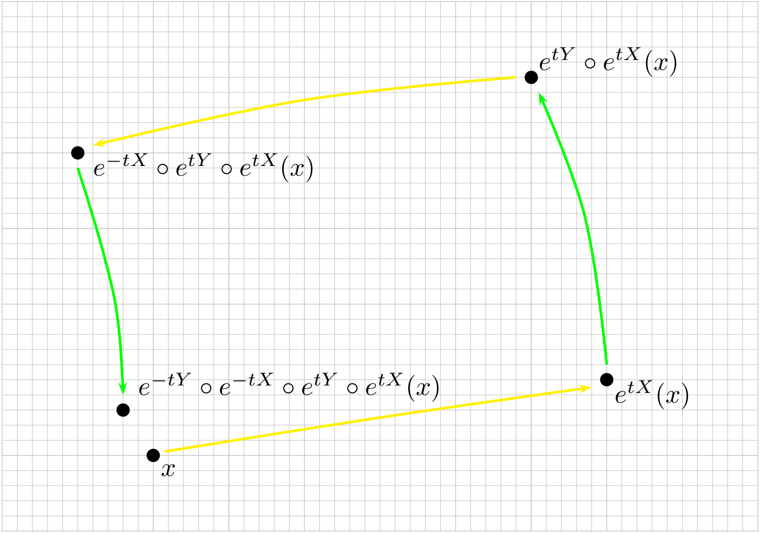

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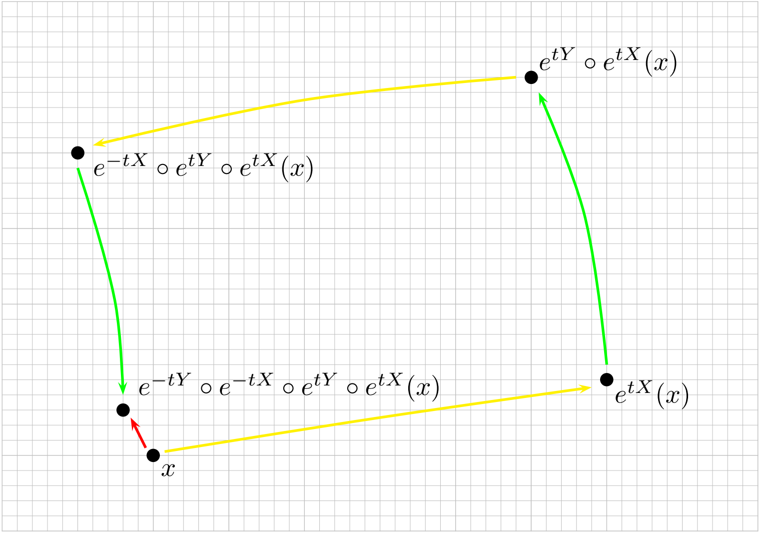
x

Comment III



$e^{tY} \circ e^{tX}(x)$
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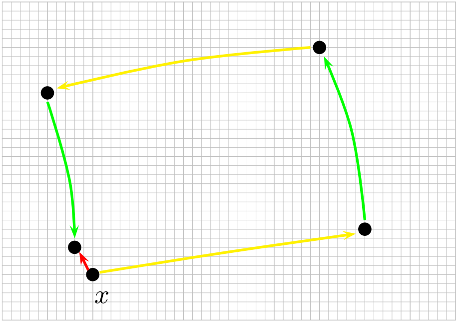
$e^{tX}(x)$
x

Comment III

Exercise

We have

$$[X, Y](x) = \lim_{t \downarrow 0} \frac{\left(e^{-tY} \circ e^{-tX} \circ e^{tY} \circ e^{tX}\right)(x) - x}{t^2}.$$



Comment III

Given a family $\mathcal{F}$ of smooth vector fields on M, we set $\mathrm{Lie}^1(\mathcal{F}) := \mathrm{Span}(\mathcal{F})$, and define recursively $\mathrm{Lie}^k(\mathcal{F})$ $(k = 2, 3, \ldots)$ by

$$\mathrm{Lie}^{k+1}(\mathcal{F}) := \mathrm{Span}\Big(\mathrm{Lie}^k(\mathcal{F}) \cup \Big\{[X, Y] \,|\, X \in \mathcal{F}, Y \in \mathrm{Lie}^k(\mathcal{F})\Big\}\Big).$$

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For example, the Lie algebra $\mathrm{Lie}\{X^1, \ldots, X^m\}$ is the vector subspace of smooth vector fields which is spanned by all the brackets (made from $X^1, \ldots, X^m$) of length $1, 2, 3, \ldots$.

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Since M has finite dimension, for every $x \in M$, there is $r = r(x) \geq 1$ (called degree of nonholonomy at x) such that

$$T_x M \supset \mathrm{Lie}\{X^1, \ldots, X^m\}(x) = \mathrm{Lie}^r\{X^1, \ldots, X^m\}(x).$$

Comment IV

We can prove the Chow-Rashevsky Theorem in the contact case in $\mathbb{R}^3$ as follows:

Exercise

Let X^1, X^2 be two smooth vector fields in $\mathbb{R}^3$ such that

$$Span\Big\{X^1(0), X^2(0), [X^1, X^2](0)\Big\} = \mathbb{R}^3.$$

Then the mapping $\varphi_\lambda : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\varphi_\lambda(t_1, t_2, t_3) = e^{\lambda X^1} \circ e^{t_3 X^2} \circ e^{-\lambda X^1} \circ e^{t_2 X^2} \circ e^{t_1 X^1}(0)$$

is a local diffeomorphism at the origin for $\lambda > 0$ small.

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$\rightsquigarrow$ Ball-Box Theorem

The End-Point mapping

Given a control system of the form

$$\dot{x} = \sum_{i=1}^{m} u_i X^i(x) \qquad (x \in M, u \in \mathbb{R}^m),$$

we define the **End-Point mapping** from x in time $T > 0$ as

$$\begin{array}{rcl} E^{x,T} \ : \ L^2\big([0,T];\mathbb{R}^m\big) & \longrightarrow & M \\ u & \longmapsto & x\big(T;x,u\big) \end{array}$$

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Proposition

The mapping $E^{x,T}$ is of class C^1 (on its domain) and

$$D_u E^{x,T}(v) = \xi(T), \quad \textit{where}$$

$$\dot{\xi} = \left(\sum_{i=1}^{m} u_i D_{x_u} X^i \right) \cdot \xi + \sum_{i=1}^{m} v_i X^i(x_u), \quad \xi(0) = 0.$$

Linearized control system

Remark

Setting for every $t \in [0, T]$, $A_u(t) := \sum_{i=1}^m u_i(t) D_{x_u(t)} X^i$, *we have*

$$D_u E^{x,T}(v) = S_u(T) \int_0^T S_u(t)^{-1} \sum_{i=1}^m v_i(t) X^i(x_u(t))\, dt$$

with S_u *solution of* $\dot{S}_u = A_u S_u$ *a.e.* $t \in [0, T]$, $S_u(0) = I_n$.

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Proposition

For every $u \in L^2([0,T];\mathbb{R}^m)$ *and any* $i = 1,\ldots,m$, *we have*

$$X^i\Big(E^{x,T}(u)\Big) \in D_u E^{x,T}\Big(L^2([0,T];\mathbb{R}^m)\Big).$$

Regular controls vs. Singular controls

Definition

A control $u \in L^2\big([0,T];\mathbb{R}^m\big)$ is called **regular** with respect to $E^{x,T}$ if $E^{x,T}$ is a submersion at u. If not, u is called **singular**.

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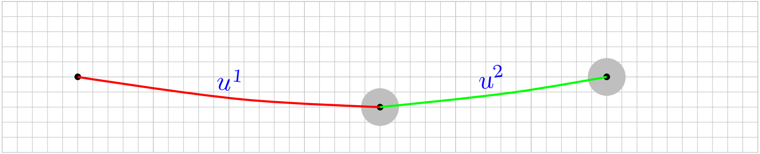
Exercise

The concatenations $u^1 * u^2$ and $u^2 * u^1$ of a regular control u^1 with another control u^2 are regular.



Rank of a control

Definition

The rank of a control $u \in L^2([0,T];\mathbb{R}^m)$ (with respect to $E^{x,T}$) is defined as the dimension of the image of the linear mapping $D_u E^{x,T}$. We denote it by $\mathrm{rank}^{x,T}(u)$.

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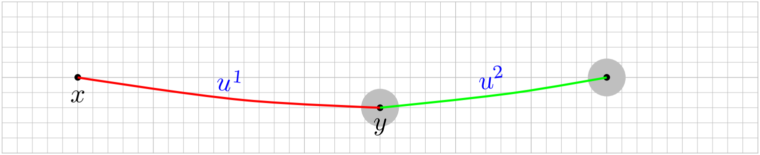
Exercise

The following properties hold:

- $rank^{x,T_1+T_2}(u^1 * u^2) \geq \max\{rank^{x,T_1}(u^1), rank^{y,T_2}(u^2)\}$.
- $rank^{y,T_1}(\check{u}^1) = rank^{x,T_1}(u^1)$.



Openness: Statement

The Chow-Rashevsky will follow from the following result:

Proposition

Let M be a smooth manifold and $X^1, \cdots, X^m$ be m smooth vector fields on M. Assume that

$$Lie\left\{X^1, \ldots, X^m\right\}(x) = T_x M \qquad \forall x \in M.$$

Then, for every $x \in M$ and every $T > 0$, the End-Point mapping

$$\begin{array}{rrcl} E^{x,T} : & L^2\big([0,T];\mathbb{R}^m\big) & \longrightarrow & M \\ & u & \longmapsto & x\big(T; x, u\big) \end{array}$$

is open (on its domain).

Openness: Sketch of proof

Let $x \in M$ and $T > 0$ be fixed. Set for every $\epsilon > 0$,

$$d(\epsilon) = \max\Big\{\operatorname{rank}^{x,\epsilon}(u) \mid \|u\|_{L^2} < \epsilon\Big\}.$$

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If not, we have $d(\epsilon) = d_0 \in \{1, \ldots, n-1\}$ for some $\epsilon > 0$. Given u^ϵ s.t. $\mathrm{rank}^{x,\epsilon}(u^\epsilon) = d_0$, there are d_0 controls $v^1, \ldots, v^{d_0}$ such that the mapping

$$\mathcal{E} \,:\, \lambda = (\lambda^1, \ldots, \lambda^{d_0}) \in \mathbb{R}^{d_0} \mapsto E^{x,\epsilon}\left(u^\epsilon + \sum_{j=1}^{d_0} \lambda^j v^j\right)$$

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is an immersion near 0. Thus, its local image N is a d_0 dimensional submanifold of M of class C^1 such that

$$X^i\big(\mathcal{E}(\lambda)\big) \in \mathrm{Im}\big(D_\lambda \mathcal{E}\big) = T_y N.$$

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$$X^i\big(\mathcal{E}(\lambda)\big) \in \mathrm{Im}\big(D_\lambda \mathcal{E}\big) = T_y N. \qquad \textit{Contradiction!!!}$$

Openness: Sketch of proof (the return method)

To conclude, we pick (for any $\epsilon > 0$ small) a regular control u^ϵ in $L^2([0,\epsilon];\mathbb{R}^m)$ and define $\tilde{u} \in L^2([0, T+2\epsilon];\mathbb{R}^m)$ by

$$\tilde{u} := u^\epsilon * \check{u}^\epsilon * u.$$

Up to reparametrizing u into a control v on $[0, T-2\epsilon]$, the new control $\tilde{v} = u^\epsilon * \check{u}^\epsilon * v$ is regular, close to u in L^2 provided $\epsilon > 0$ is small, and steers x to $E^{x,T}(u)$.

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The openness follows from the Inverse Function Theorem.

Remarks

Proposition

Let M be a smooth manifold and $X^1, \cdots, X^m$ be m smooth vector fields on M. Assume that

$$Lie\left\{X^1, \ldots, X^m\right\}(x) = T_x M \qquad \forall x \in M.$$

Then, for every $x \in M$ and every $T > 0$, the set of controls which are regular w.r.t. $E^{x,T}$ is open and dense in L^2.

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The above result holds indeed in the smooth topology.

Proposition (Sontag)

Under the same assumptions, the set of controls which are regular w.r.t. $E^{x,T}$ is open and dense in C^∞.

Example: The baby stroller

$$\begin{cases} \dot{x} = u_1 \cos\theta \\ \dot{y} = u_1 \sin\theta \\ \dot{\theta} = u_2 \end{cases}$$

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$$\mathrm{Span}\Big\{X(\xi), Y(\xi), [X,Y](\xi)\Big\} = \mathbb{R}^3 \qquad \forall \xi = (x,y,\theta).$$

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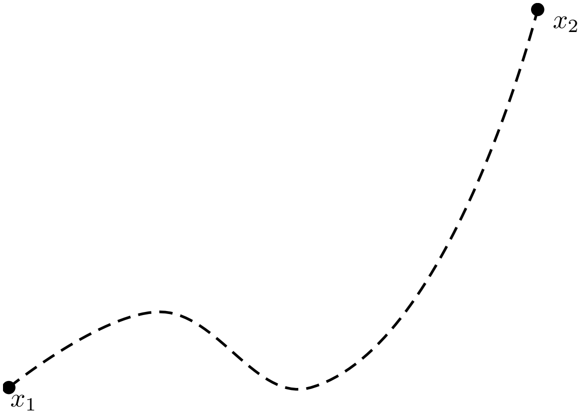

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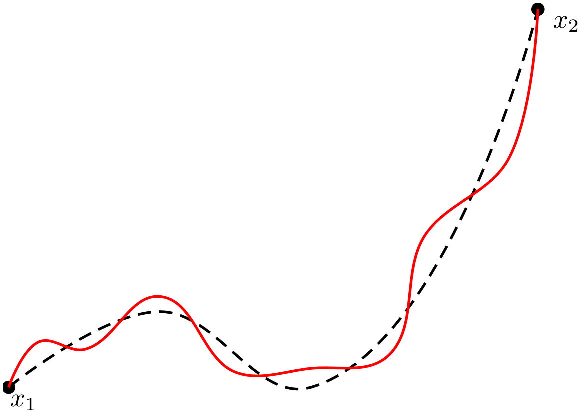

Thank you for your attention !!

Lecture 2

Sub-Riemannian geodesics

Sub-Riemannian structures

Let M be a smooth connected manifold of dimension $n \geq 2$.

Sub-Riemannian structures

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Definition

A sub-Riemannian structure on M is a pair (Δ, g) where:

- Δ is a **totally nonholonomic distribution** of rank $m \in [2, n]$, that is it is defined locally as

$$\Delta(x) = \mathrm{Span}\Big\{X^1(x), \ldots, X^m(x)\Big\} \subset T_xM,$$

  where $X^1, \ldots, X^m$ are m linearly independent vector fields satisfying the Hörmander condition.
- g_x is a **scalar product** on $\Delta(x)$.

Remark

- *In general* Δ *does not admit a global frame. However we can always construct* $k = m \cdot (n+1)$ *smooth vector fields* $Y^1, \ldots, Y^k$ *such that*

$$\Delta(x) = Span\Big\{ Y^1(x), \ldots, Y^k(x) \Big\} \qquad \forall x \in M.$$

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- *If* (M, g) *is a Riemannian manifold, then any totally nonholomic distribution* Δ *gives rise to a SR structure* (Δ, g) *on* M.

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- *If (M,g) is a Riemannian manifold, then any totally nonholonomic distribution Δ gives rise to a SR structure (Δ, g) on M.*

Example (Heisenberg)

Take in $\mathbb{R}^3$, $\Delta = \mathrm{Span}\{X^1, X^2\}$ with

$$X^1 = \partial_x - \frac{y}{2}\partial_z, \quad X^2 = \partial_y + \frac{x}{2}\partial_z \text{ and } g = dx^2 + dy^2.$$

The Chow-Rashevsky Theorem

Definition

We call **horizontal path** any path $\gamma \in W^{1,2}([0,1];M)$ satisfying

$$\dot{\gamma}(t) \in \Delta(\gamma(t)) \qquad \text{a.e. } t \in [0,1].$$

We observe that if $\Delta = \text{Span}\{Y^1, \ldots, Y^k\}$, for any $x \in M$ and any control $u \in L^2([0,1];\mathbb{R}^k)$, the solution to

$$\dot{\gamma} = \sum_{i=1}^{k} u_i \, Y^i(\gamma), \quad \gamma(0) = x$$

is an horizontal path joining x to $\gamma(1)$.

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Theorem (Chow-Rashevsky)

Let Δ be a totally nonholonomic distribution on M then any pair of points can be joined by an horizontal path.

The sub-Riemannian distance

The **length** (w.r.t g) of an horizontal path γ is defined as

$$\mathrm{length}^g(\gamma) := \int_0^T |\dot{\gamma}(t)|^g_{\gamma(t)} \, dt$$

Definition

Given $x, y \in M$, the **sub-Riemannian distance** between x and y is

$$d_{SR}(x,y) := \inf\Big\{ \mathrm{length}^g(\gamma) \,|\, \gamma \text{ hor.}, \gamma(0) = x, \gamma(1) = y \Big\}.$$

Proposition

The manifold M equipped with the distance d_{SR} is a metric space whose topology coincides with the topology of M (as a manifold).

Minimizing horizontal paths and geodesics

Definition

Given $x, y \in M$, we call **minimizing horizontal path** between x and y any horizontal path $\gamma : [0, T] \to M$ connecting x to y such that

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The **sub-Riemannian energy** between x and y is defined as

$$e_{SR}(x, y) := \inf \left\{ \text{energy}^g(\gamma) := \int_0^1 \left(|\dot{\gamma}(t)|^g_{\gamma(t)} \right)^2 dt \mid \gamma \ldots \right\}.$$

Definition

We call **minimizing geodesic** between x and y any horizontal path $\gamma : [0, 1] \to M$ connecting x to y such that

$$e_{SR}(x, y) = \text{energy}^g(\gamma).$$

A SR Hopf-Rinow Theorem

Theorem

Let (Δ, g) be a sub-Riemannian structure on M. Assume that (M, d_{SR}) is a complete metric space. Then the following properties hold:

- *The closed balls $\bar{B}_{SR}(x, r)$ are compact (for any $r \geq 0$).*
- *For every $x, y \in M$, there exists at least one minimizing geodesic joining x to y.*

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Remark

Given a complete Riemannian manifold (M, g), for any totally nonholonomic distribution Δ on M, the SR structure (Δ, g) is complete.

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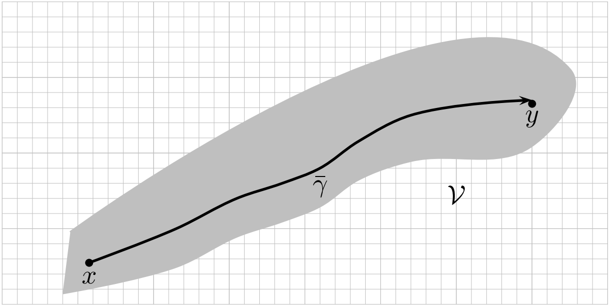
Remark

Given a complete Riemannian manifold (M, g), for any totally nonholonomic distribution Δ on M, the SR structure (Δ, g) is complete. As a matter of fact, since $d_g \leq d_{SR}$ any Cauchy sequence w.r.t. d_{SR} is Cauchy w.r.t. d_g.

The Hamiltonian geodesic equation

Let $x, y \in M$ and a **minimizing geodesic** $\bar{\gamma}$ joining x to y be fixed. The SR structure admits **an orthonormal frame** along $\bar{\gamma}$, that is there is an open neighborhood $\mathcal{V}$ of $\bar{\gamma}([0,1])$ and an orthonormal family of m vector fields $X^1, \ldots, X^m$ such that

$$\Delta(z) = \mathrm{Span}\left\{X^1(z), \ldots, X^m(z)\right\} \qquad \forall z \in \mathcal{V}.$$



The Hamiltonian geodesic equation

There is a control $\bar{u} \in L^2\big([0,1];\mathbb{R}^m\big)$ such that

$$\dot{\bar{\gamma}}(t) = \sum_{i=1}^{m} \bar{u}_i(t)\, X^i\big(\bar{\gamma}(t)\big) \qquad \text{a.e. } t \in [0,1].$$

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$$\dot{\bar{\gamma}}(t) = \sum_{i=1}^{m} \bar{u}_i(t)\, X^i\big(\bar{\gamma}(t)\big) \qquad \text{a.e. } t \in [0,1].$$

Moreover, on the one hand any control $u \in \mathcal{U} \subset L^2([0,1];\mathbb{R}^m)$ (u sufficiently close to $\bar{u}$) gives rise to a trajectory γ_u solution of

$$\dot{\gamma}_u = \sum_{i=1}^{m} u^i\, X^i(\gamma_u) \quad \text{on } [0,T], \quad \gamma_u(0) = x.$$

The Hamiltonian geodesic equation

There is a control $\bar{u} \in L^2\big([0,1];\mathbb{R}^m\big)$ such that

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On the other hand, for any horizontal path $\gamma : [0,1] \rightarrow \mathcal{V}$ there is a (unique) control $u \in L^2\big([0,1];\mathbb{R}^m\big)$ for which the equation in red is satisfied.

The Hamiltonian geodesic equation

So, considering as previously the **End-Point mapping**

$$E^{x,1} \,:\, L^2\big([0,1];\mathbb{R}^m\big) \longrightarrow M$$

defined by

$$E^{x,1}(u) := \gamma_u(1),$$

and setting $C(u) = \|u\|_{L^2}^2$, we observe that $\bar{u}$ is solution to the following **optimization problem with constraints**:

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(Since the family $X^1, \ldots, X^m$ is orthonormal, we have

$$\text{energy}^g(\gamma_u) = C(u) \qquad \forall u \in \mathcal{U}.)$$

The Hamiltonian geodesic equation

Proposition (Lagrange Multipliers)

There are $p \in T_y^*M \simeq (\mathbb{R}^n)^*$ *and* $\lambda_0 \in \{0,1\}$ *with* $(\lambda_0, p) \neq (0,0)$ *such that*

$$p \cdot D_{\bar{u}} E^{x,1} = \lambda_0 D_{\bar{u}} C.$$

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Proof.

The mapping $\Phi : \mathcal{U} \to \mathbb{R} \times M$ defined by

$$\Phi(u) := \big(C(u), E^{x,1}(u) \big)$$

cannot be a submersion at $\bar{u}$. As a matter of fact, if $D_{\bar{u}}\Phi$ is surjective, then it is open at $\bar{u}$, so it must contain elements of the form $(C(\bar{u}) - \delta, y)$ for $\delta > 0$ small. □

$\rightsquigarrow$ two cases: $\lambda_0 = 0$ or $\lambda_0 = 1$.

The Hamiltonian geodesic equation

First case: $\lambda_0 = 0$

Then we have

$$p \cdot D_{\bar{u}} E^{x,1} = 0 \text{ with } p \neq 0.$$

So $\bar{u}$ is **singular** (w.r.t. x and $T = 1$).

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- *If there are no nontrivial singular control, then this case cannot occur.*

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- *If Δ has rank n, that is $\Delta = TM$ (Riemannian case), then there are no singular control. So this case cannot occur.*
- *If there are no nontrivial singular control, then this case cannot occur.*
- *If there are no nontrivial singular minimizing control, then this case cannot occur.*

The Hamiltonian geodesic equation

Second case: $\lambda_0 = 1$

Define the Hamiltonian $H : \mathcal{V} \times (\mathbb{R}^n)^* \to \mathbb{R}$ by

$$H(x,p) := \frac{1}{2} \sum_{i=1}^{m} \left(p \cdot X^i(x)\right)^2 .$$

Proposition

There is a smooth arc $p : [0,1] \to (\mathbb{R}^n)^*$ with $p(1) = p/2$ such that

$$\begin{cases} \dot{\bar{\gamma}} & = & \frac{\partial H}{\partial p}(\bar{\gamma}, p) = \sum_{i=1}^{m} \left[p \cdot X^i(\bar{\gamma})\right] X^i(\bar{\gamma}) \\ \dot{p} & = & -\frac{\partial H}{\partial x}(\bar{\gamma}, p) = -\sum_{i=1}^{m} \left[p \cdot X^i(\bar{\gamma})\right] p \cdot D\bar{\gamma} X^i \end{cases}$$

for a.e. $t \in [0,1]$ and $\bar{u}_i(t) = p \cdot X^i(\bar{\gamma}(t))$ for a.e. $t \in [0,1]$ and any i.

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for a.e. $t \in [0,1]$ *and* $\bar{u}_i(t) = p \cdot X^i(\bar{\gamma}(t))$ *for a.e.* $t \in [0,1]$ *and any* i*. In particular, the path* $\bar{\gamma}$ *is smooth on* $[0,1]$*.*

The Hamiltonian geodesic equation

Proof.

We have $D_{\bar{u}}C(v) = 2\langle \bar{u}, v\rangle_{L^2}$ and we remember that

$$D_{\bar{u}}E^{x,T}(v) = S(1)\int_0^1 S(t)^{-1}B(t)v(t)\,dt$$

with

$$\begin{cases} A(t) = \sum_{i=1}^m u_i(t) D_{\bar{\gamma}(t)}X^i, \\ B(t) = (X^1(\bar{\gamma}(t)),\ldots,X^m(\bar{\gamma}(t))) \end{cases} \quad \forall t\in[0,1],$$

and S solution of

$$\dot{S}(t) = A(t)S(t) \text{ for a.e. } t\in[0,1], \quad S(0) = I_n.$$

□

The Hamiltonian geodesic equation

Proof.

Then $p \cdot D_{\bar{u}} E^{x,1} = \lambda_0 D_{\bar{u}} C$ yields

$$\int_0^1 \left[p \cdot S(1) S(t)^{-1} B(t) - 2\bar{u}(t)^* \right] v(t)\, dt = 0 \quad \forall v \in L^2.$$

We infer that

$$\bar{u}(t) = \frac{1}{2} \left(p \cdot S(1) S(t)^{-1} B(t) \right)^* \quad \text{a.e. } t \in [0,1],$$

and that the absolutely continuous arc $p : [0,1] \to (\mathbb{R}^n)^*$ defined by

$$p(t) := \frac{1}{2} p \cdot S(1) S(t)^{-1}$$

satisfies the desired equations. □

The Hamiltonian geodesic equation

Define the **Hamiltonian** $H : T^*M \to \mathbb{R}$ by

$$H(x,p) = \frac{1}{2} \max \left\{ \frac{p(v)^2}{g_x(v,v)} \,|\, v \in \Delta_x \setminus \{0\} \right\}.$$

We call **normal extremal** any curve $\psi : [0,T] \to T^*M$ satisfying

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Theorem

Let $\gamma : [0,1] \to M$ be a minimizing geodesic. One of the two following non-exclusive cases occur:

- *γ is singular.*
- *γ admits a normal extremal lift.*

Examples

Example 1: The Riemannian case

Let $\Delta(x) = T_x M$ for any $x \in M$ so that ANY curve is horizontal. There are no singular curve, so any minimizing geodesic is the projection of a normal extremal.

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Let $\Delta(x) = T_xM$ for any $x \in M$ so that ANY curve is horizontal. There are no singular curve, so any minimizing geodesic is the projection of a normal extremal.

Example 2: Heisenberg

Recall that in $\mathbb{R}^3$, $\Delta = \mathrm{Span}\{X^1, X^2\}$ with

$$X^1 = \partial_x - \frac{y}{2}\partial_z, \quad X^2 = \partial_y + \frac{x}{2}\partial_z \text{ and } g = dx^2 + dy^2.$$

Examples

Any horizontal path has the form $\gamma_u = (x, y, z) : [0, 1] \to \mathbb{R}^3$ with

$$\begin{cases} \dot{x}(t) & = & u_1(t) \\ \dot{y}(t) & = & u_2(t) \\ \dot{z}(t) & = & \frac{1}{2}\left(u_2(t)x(t) - u_1(t)y(t)\right), \end{cases}$$

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$$z(1) - z(0) = \int_\alpha \frac{1}{2}\left(xdy - ydx\right),$$

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$$z(1) - z(0) = \int_{\mathcal{D}} dx \wedge dy + \int_c \frac{1}{2}\left(xdy - ydx\right)$$

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Examples

Let $\gamma_u = (x, y, z) : [0,1] \to \mathbb{R}^3$ be a minimizing geodesic from $P_1 := \gamma_u(0)$ to $P_2 := \gamma_u(1) \neq P_1$. Since u is necessarily regular, there is a smooth arc $p : [0,1] \to (\mathbb{R}^3)^*$ s.t.

$$\begin{cases} \dot{x} &= p_x - \frac{y}{2} p_z \\ \dot{y} &= p_y + \frac{x}{2} p_z \\ \dot{z} &= \frac{1}{2}\left(\left(p_y + \frac{x}{2} p_z\right) x - \left(p_x - \frac{y}{2} p_z\right) y\right), \end{cases} \qquad \begin{cases} \dot{p}_x &= -\left(p_y + \frac{x}{2} p_z\right) \frac{p_z}{2} \\ \dot{p}_y &= \left(p_x - \frac{y}{2} p_z\right) \frac{p_z}{2} \\ \dot{p}_z &= 0. \end{cases}$$

Hence $p_z = \bar{p}_z$ for every t. Which implies that

$$\ddot{x} = -\bar{p}_z \dot{y} \quad \text{and} \quad \ddot{y} = \bar{p}_z \dot{x}.$$

If $\bar{p}_z = 0$, then the geodesic from P_1 to P_2 is a segment with constant speed. If $\bar{p}_z \neq 0$, we have or

$$\dddot{x} = -\bar{p}_z^2 \dot{x} \quad \text{and} \quad \dddot{y} = -\bar{p}_z^2 \dot{y}.$$

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Which means that the curve $t \mapsto (x(t), y(t))$ is a circle.

Examples

Example 3: The Martinet distribution

In $\mathbb{R}^3$, let $\Delta = \mathrm{Span}\{X^1, X^2\}$ with X^1, X^2 fo the form

$$X^1 = \partial_{x_1} \quad \text{and} \quad X^2 = \left(1 + x_1\phi(x)\right)\partial_{x_2} + x_1^2\partial_{x_3},$$

where ϕ is a smooth function and g be a smooth metric on Δ.

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Theorem

There is $\bar{\epsilon} > 0$ such that for every $\epsilon \in (0, \bar{\epsilon})$, the (singular) horizontal path given by

$$\gamma(t) = (0, t, 0) \qquad \forall t \in [0, \epsilon],$$

minimizes the length (w.r.t. g) among all horizontal paths joining 0 to $(0, \epsilon, 0)$.

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minimizes the length (w.r.t. g) among all horizontal paths joining 0 to $(0, \epsilon, 0)$. Moreover if $\{X^1, X^2\}$ is orthonormal w.r.t. g and $\phi(0) \neq 0$, then γ can not be the projection of a normal extremal.

The SR exponential mapping

Denote by $\psi_{x,p} : [0,1] \to T^*M$ the solution of

$$\dot{\psi}(t) = \vec{H}\big(\psi(t)\big) \qquad \forall t \in [0,1], \quad \psi(0) = (x,p)$$

and let

$$\mathcal{E}_x := \Big\{ p \in T_x^*M \,|\, \psi_{x,p} \text{ defined on } [0,1] \Big\}.$$

Definition

The **sub-Riemannian exponential map** from $x \in M$ is defined by

$$\begin{array}{rccl} \exp_x \,:\, \mathcal{E}_x \subset T_x^*M & \longrightarrow & M \\ p & \longmapsto & \pi\big(\psi_{x,p}(1)\big). \end{array}$$

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Proposition

Assume that (M, d_{SR}) is complete. Then for every $x \in M$, $\mathcal{E}_x = T_x^*M$.

On the image of the exponential mapping

Proposition (Agrachev-Trélat-LR)

Assume that (M, d_{SR}) *is complete. Then for every* $x \in M$, *the set* $\exp_x(T_x^*M)$ *is open and dense.*

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Proposition (Agrachev-Trélat-LR)

Assume that (M, d_{SR}) *is complete. Then for every* $x \in M$*, the set* $\exp_x(T_x^*M)$ *is open and dense.*

Lemma

Let $y \neq x$ *in* M *be such that there is a function* $\phi : M \to \mathbb{R}$ *differentiable at* y *such that*

$$\phi(y) = d_{SR}^2(x,y) \quad \text{and} \quad d_{SR}^2(x,z) \geq \phi(z) \quad \forall z \in M.$$

Then there is a unique minimizing geodesic $\gamma : [0,1] \to M$ *between* x *and* y*. It is the projection of a normal extremal* $\psi : [0,1] \to T^*M$ *satisfying* $\psi(1) = (y, \frac{1}{2}D_y\phi)$*. In particular* $x = \exp_y(-\frac{1}{2}D_y\phi)$*.*

On the image of the exponential mapping

Proof.

Let $y \neq x$ in M satisfying the assumption and $\bar{\gamma} = \gamma_{\bar{u}} : [0,1] \rightarrow M$ be a minimizing geodesic from x to y.

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Let $y \neq x$ in M satisfying the assumption and $\bar{\gamma} = \gamma_{\bar{u}} : [0,1] \rightarrow M$ be a minimizing geodesic from x to y. We have for every $u \in \mathcal{U} \subset L^2([0,1];\mathbb{R}^m)$ (close to $\bar{u}$),

$$\|u\|_{L^2}^2 = C(u) \geq e_{SR}\left(x, E^{x,1}(u)\right) \qquad ,$$

with equality if $u = \bar{u}$.

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$$\bar{u} \text{ minimizes } C(u) - \phi\left(E^{x,1}(u)\right) \text{ among all } u \in \mathcal{U}.$$

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We infer that there is $p \neq 0$ such that

$$p \cdot D_u E^{x,1} = D_u C \quad \text{with } p = D_{E^{x,1}(u)}\phi$$

and in turn get the result. □

On the image of the exponential mapping

Remark

If (M, d_{SR}) is complete and there are no singular minimizing curves, then $\exp_x(T_x^*M) = M$.

Examples:

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- Heisenberg.

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If (M, d_{SR}) is complete and there are no strictly singular minimizing curves, then $\exp_x(T_x^* M) = M$.

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- For generic SR structures of rank ≥ 3.

Remark

If (M, d_{SR}) is complete and there are no strictly singular minimizing curves, then $\exp_x(T_x^*M) = M$.

$\rightsquigarrow$ Medium fat distributions.

Open problems in SR geometry I: The Sard conjecture

Let M be a smooth connected manifold of dimension n and $\mathcal{F} = \{X^1, \ldots, X^k\}$ be a family of smooth vector fields on M satisfying the Hörmander condition. Given $x \in M$ and $T > 0$, the **End-Point mapping** $E^{x,T}$ is defined as

$$\begin{array}{rrcl} E^{x,T} : & L^2\big([0,T];\mathbb{R}^m\big) & \longrightarrow & M \\ & u & \longmapsto & x\big(T;x,u\big) \end{array}$$

where $x(\cdot) = x(\cdot; x, u) : [0,T] \longrightarrow M$ is solution to the Cauchy problem

$$\dot{x} = \sum_{i=1}^{m} u_i X^i(x), \quad x(0) = x.$$

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Proposition

The map $E^{x,T}$ is smooth on its domain.

The Sard Conjecture

Theorem (Morse 1939, Sard 1942)

Let $f : \mathbb{R}^d \to \mathbb{R}^p$ be a function of class C^k, then

$$k \geq \max\{1, d-p+1\} \quad \Longrightarrow \quad \mathcal{L}^p\left(f\left(Crit(f)\right)\right) = 0,$$

where $Crit(f)$ is the set of critical points of f, i.e. the points where $D_x f$ is not onto.

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Conjecture

The set $E^{x,T}\left(Sing_{\mathcal{F}}^{x,T}\right) \subset M$ has Lebesgue measure zero.

Open problems in SR geometry II: Regularity of minimizing geodesics

Let (Δ, g) be complete SR structure on a smooth manifold M.

Open Question

Do the minimizing geodesics enjoy some regularity ? Are they at least of class C^1 ?

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$\rightsquigarrow$ Very partial results by Monti, Leonardi and later Monti.

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Thank you for your attention !!