Regularity of optimal transport maps on Riemannian manifolds

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Recent Development of nonlinear PDEs
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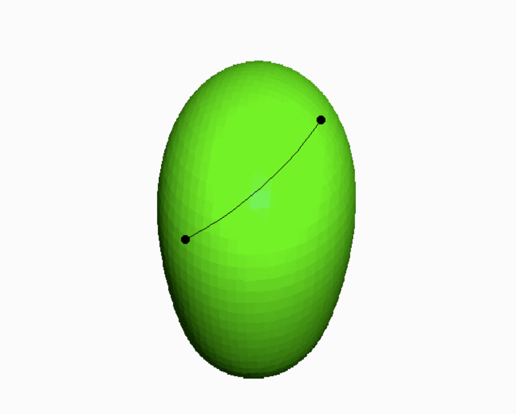
The framework

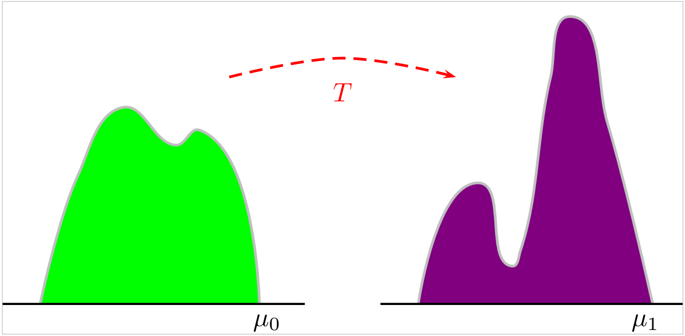
Let M be a **smooth connected (compact) manifold of dimension** n equipped with **a smooth Riemannian metric** g. For any $x, y \in M$, we define the **geodesic distance** between x and y, denoted by $d(x, y)$, as the minimum of the lengths of the curves joining x to y.



Transport maps

Let μ_0 and μ_1 be **probability measures** on M. We call **transport map** from μ_0 to μ_1 any measurable map $T : M \rightarrow M$ such that $T_\sharp \mu_0 = \mu_1$, that is

$$\mu_1(B) = \mu_0\big(T^{-1}(B)\big), \quad \forall B \text{ measurable } \subset M.$$



Quadratic Monge's Problem

Given two probabilities measures μ_0, μ_1 sur M, find a transport map $T : M \longrightarrow M$ from μ_0 to μ_1 which **minimizes** the quadratic cost ($c = d^2/2$)

$$\int_M c(x, T(x)) d\mu_0(x).$$

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Theorem (McCann '01)

*If μ_0 is absolutely continuous w.r.t. Lebesgue, then there exists a unique optimal transport map T from μ_0 to μ_1. In fact, there is a c-**convex** function $\varphi : M \longrightarrow \mathbb{R}$ satisfying*

$$T(x) = \exp_x (\nabla \varphi(x)) \qquad \mu_0 \text{ a.e. } x \in M.$$

(Moreover, for a.e. $x \in M$, $\nabla \varphi(x)$ belongs to the injectivity domain at x.)

Purpose of the talk

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- Open questions

Back to $\mathbb{R}^n$

Quadratic Monge's Problem in $\mathbb{R}^n$: Given two probability measures μ_0, μ_1 with compact supports in $\mathbb{R}^n$, we are concerned with transport maps $T : \mathbb{R}^n \to \mathbb{R}^n$ pushing forward μ_0 to μ_1 which **minimize** the transport cost

$$\int_{\mathbb{R}^n} |T(x) - x|^2 \, d\mu_0(x).$$

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Theorem (Brenier '91)

If μ_0 is absolutely continuous with respect to the Lebesgue measure, there exists a unique optimal transport map with respect to the quadratic cost. In fact, there is a **convex** *function $\psi : M \to \mathbb{R}$ such that*

$$T(x) = \nabla \psi(x) \qquad \mu_0 \text{ a.e. } x \in \mathbb{R}^n.$$

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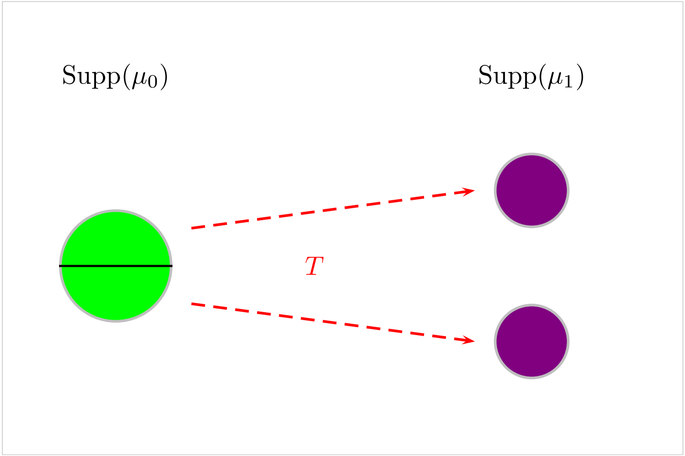
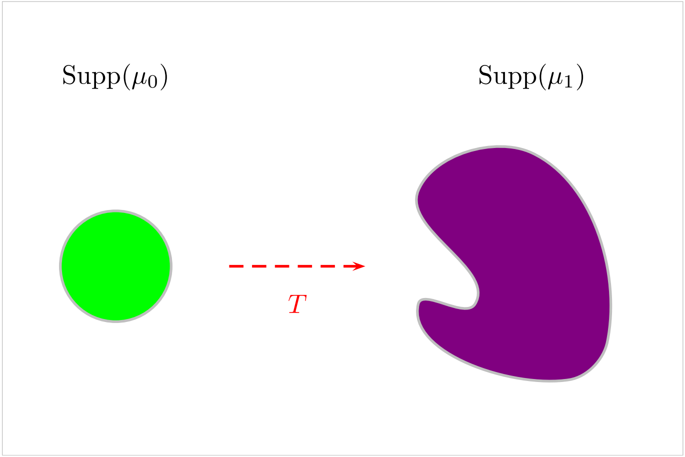
Necessary and sufficient conditions for regularity ?

An obvious counterexample



The convexity of the targer is necessary



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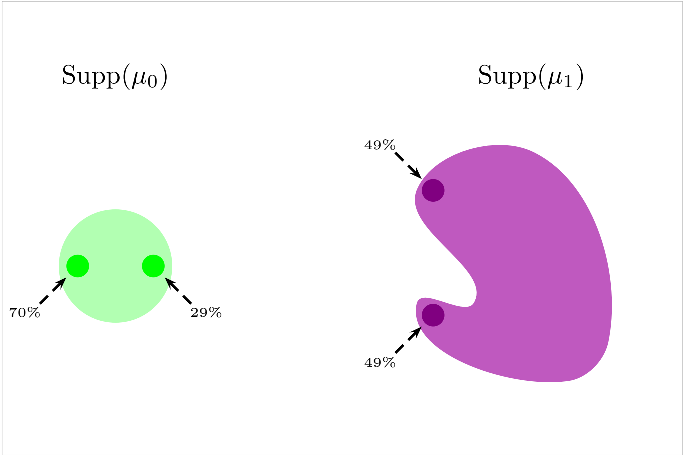

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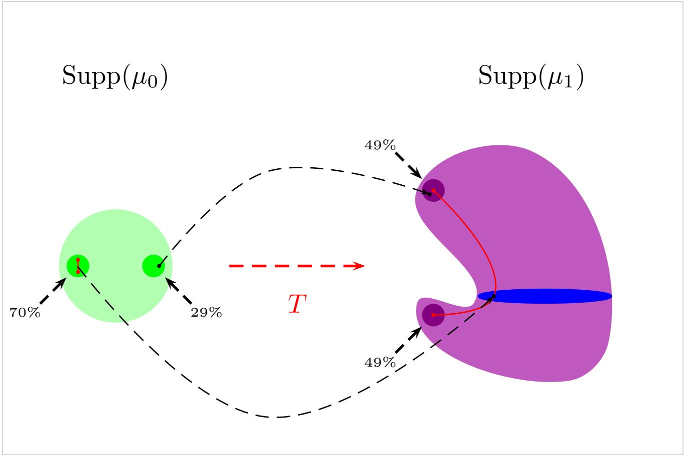

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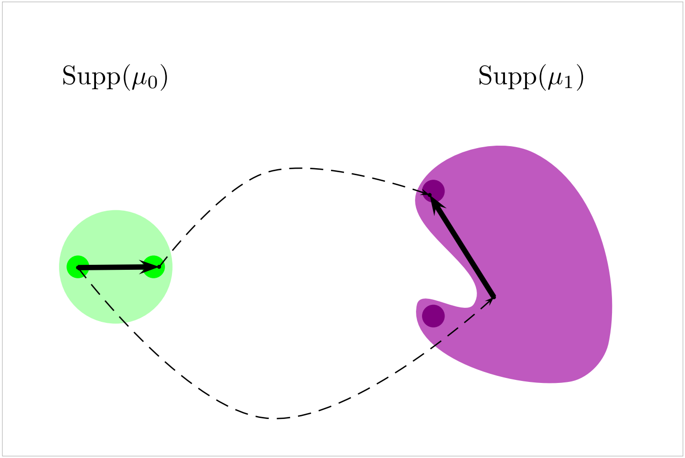

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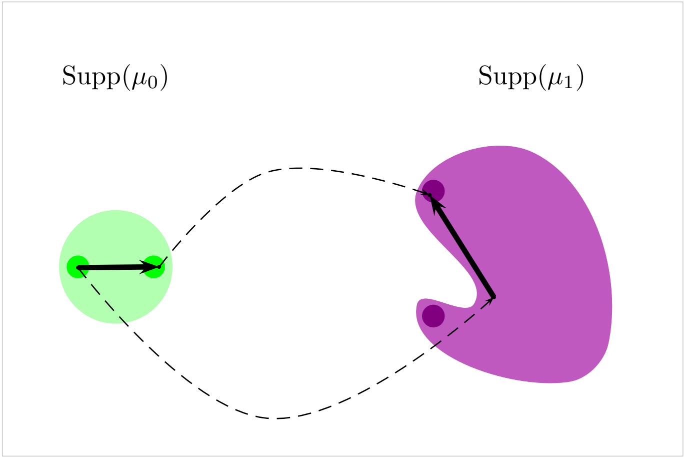

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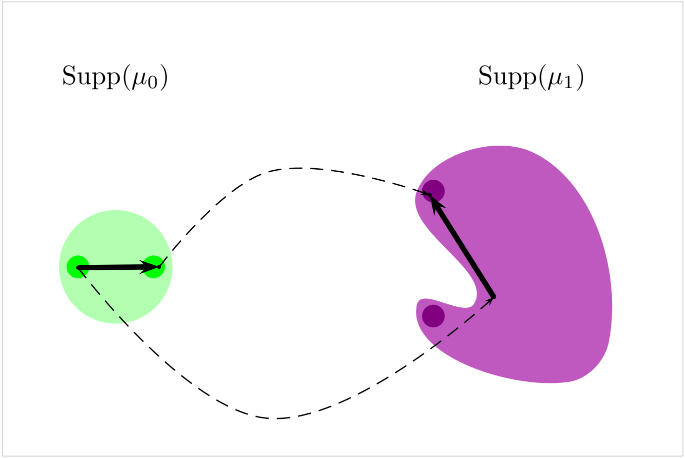

T gradient of a convex function $\Longrightarrow \langle y - x, T(y) - T(x) \rangle \geq 0$!!!

Caffarelli's Regularity Theory

If μ_0 and μ_1 are associated with densities f_0, f_1 w.r.t. Lebesgue, then

$$T_\sharp \mu_0 = \mu_1 \Longleftrightarrow \int_{\mathbb{R}^n} \zeta\big(T(x)\big) f_0(x) dx = \int_{R^n} \zeta(y) f_1(y) dy \quad \forall \zeta.$$

$\rightsquigarrow$ ψ weak solution of the **Monge-Ampère equation** :

$$\det\left(\nabla^2 \psi(x)\right) = \frac{f_0(x)}{f_1(\nabla \psi(x))}.$$

Theorem (Caffarelli '90s)

Let Ω_0, Ω_1 be connected and bounded open sets in $\mathbb{R}^n$ and f_0, f_1 be probability densities resp. on Ω_0 and Ω_1 such that $f_0, f_1, 1/f_0, 1/f_1$ are **essentially bounded**. *Assume that μ_0 and μ_1 have respectively densities f_0 and f_1 w.r.t. Lebesgue and that Ω_1 is* **convex**. *Then the quadratic optimal transport map from μ_0 to μ_1 is continuous.*

Back to Riemannian manifolds

Given two probabilities measures μ_0, μ_1 sur M, find a transport map $T : M \to M$ from μ_0 to μ_1 which minimizes the quadratic cost ($c = d^2/2$)

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Definition

We say that the Riemannian manifold (M, g) satisfies the **Transport Continuity Property (TCP)** if for any pair of probability measures μ_0, μ_1 associated locally with **continuous positive densities** ρ_0, ρ_1, that is

$$\mu_0 = \rho_0 \mathrm{vol}_g, \quad \mu_1 = \rho_1 \mathrm{vol}_g,$$

the optimal transport map from μ_0 to μ_1 is **continuous**.

Exponential mapping and injectivity domains

Let $x \in M$ be fixed.

- For every $v \in T_xM$, we define the **exponential** of v by

$$\exp_x(v) = \gamma_{x,v}(1),$$

  where $\gamma_{x,v} : [0,1] \longrightarrow M$ is the unique geodesic starting at x with speed v.

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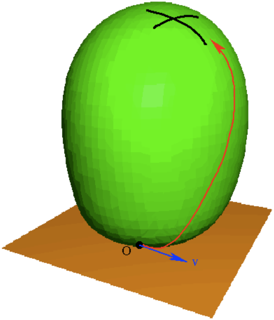
- We call **injectivity domain** at x, the subset of T_xM defined by

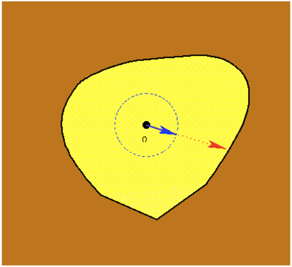
$$\mathcal{I}(x) := \left\{ v \in T_xM \;\middle|\; \begin{array}{c} \exists t > 1 \text{ s.t. } \gamma_{tv} \text{ is the unique} \\ \text{minim. geod. between } x \text{ and } \exp_x(tv) \end{array} \right\}$$

  It is a star-shaped (w.r.t. $0 \in T_xM$) bounded open set with Lipschitz boundary.

The distance to the cut locus

Using the exponential mapping, we can associate to each unit tangent vector its **distance to the cut locus**.





In that way, we define the so-called **injectivity domain** $\mathcal{I}(x)$ whose boundary is the **tangent cut locus** $\mathrm{TCL}(x)$.

A necessary condition for **TCP**

Theorem (Figalli-R-Villani '10)

Let (M,g) be a smooth compact Riemannian manifold satisfying **TCP**. *Then following properties hold:*

- *all the injectivity domains are convex,*
- *for any $x, x' \in M$, the function*

$$F_{x,x'} \,:\, v \in \mathcal{I}(x) \longmapsto c\big(x, \exp_x(v)\big) - c\big(x', \exp_x(v)\big)$$

is **quasiconvex** *(its sublevel sets are always convex).*

Almost characterization of quasiconvex functions

Lemma (Sufficient condition)

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class C^2. Assume that for every $v \in U$ and every $w \in \mathbb{R}^n \setminus \{0\}$ the following property holds :

$$\langle \nabla_v F, w \rangle = 0 \implies \langle \nabla_v^2 F\, w, w \rangle > 0.$$

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Lemma (Necessary condition)

Let $U \subset \mathbb{R}^n$ be an open convex set and $F : U \to \mathbb{R}$ be a function of class C^2. Assume that F is quasiconvex. Then for every $v \in U$ and every $w \in \mathbb{R}^n \setminus \{0\}$ the following property holds :

$$\langle \nabla_v F, w \rangle = 0 \implies \langle \nabla_v^2 F\, w, w \rangle \geq 0.$$

Proof of the easy lemma

Proof.

Let $v_0, v_1 \in U$ be fixed. Set $v_t := (1-t)v_0 + tv_1$, for every $t \in [0,1]$. Define $h : [0,1] \to \mathbb{R}$ by

$$h(t) := F(v_t) \qquad \forall t \in [0,1].$$

If $h \not\leq \max\{h(0), h(1)\}$, there is $\tau \in (0,1)$ such that

$$h(\tau) = \max_{t \in [0,1]} h(t) \, > \, \max\{h(0), h(1)\}.$$

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There holds

$$\dot{h}(\tau) = \langle \nabla_{v_\tau} F, \dot{v}_\tau \rangle \quad \text{et} \quad \ddot{h}(\tau) = \langle \nabla^2_{v_\tau} F\, \dot{v}_\tau, \dot{v}_\tau \rangle.$$

Since τ is a local maximum, one has $\dot{h}(\tau) = 0$.
Contradiction !! □

The Ma-Trudinger-Wang tensor

The **MTW** tensor $\mathfrak{S}$ is defined as

$$\mathfrak{S}_{(x,v)}(\xi,\eta) = -\frac{3}{2}\,\frac{d^2}{ds^2}\Big|_{s=0}\,\frac{d^2}{dt^2}\Big|_{t=0} c\left(\exp_x(t\xi), \exp_x(v+s\eta)\right),$$

for every $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM$.

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Proposition (Villani '09, Figalli-R-Villani '10)

Let (M,g) be such that all injectivity domains are convex. Then the following properties are equivalent:

- *All the functions $F_{x,x'}$ are quasiconvex.*
- *The **MTW** tensor $\mathfrak{S}$ is $\succeq 0$, that is for any $x \in M$, $v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM$,*

$$\langle \xi, \eta \rangle_x = 0 \Longrightarrow \mathfrak{S}_{(x,v)}(\xi,\eta) \geq 0.$$

When geometry enters the problem

Theorem (Loeper '06)

For every $x \in M$ and for any pair of unit orthogonal tangent vectors $\xi, \eta \in T_xM$, there holds

$$\mathfrak{S}_{(x,0)}(\xi, \eta) = \kappa_x(\xi, \eta),$$

where the latter denotes the **sectional curvature** *of M at x along the plane spanned by $\{\xi, \eta\}$.*

Corollary (Loeper '06)

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Caution!!! $\kappa \geq 0 \not\Rightarrow \mathfrak{S} \succeq 0$.

Sufficient conditions for **TCP**

Theorem (Figalli-R-Villani '10)

Let (M,g) *be a compact smooth Riemannian* **surface**. *It satisfies* **TCP** *if and only if the two following properties holds:*

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Theorem (Figalli-R-Villani '10)

Assume that (M, g) satisfies the two following properties:

- *all its injectivity domains are* **strictly convex**,
- *the* **MTW** *tensor $\mathfrak{S}$ is $\succ 0$, that is for any $x \in M, v \in \mathcal{I}(x)$, and $\xi, \eta \in T_xM$,*

$$\langle \xi, \eta \rangle_x = 0 \Longrightarrow \mathfrak{S}_{(x,v)}(\xi, \eta) > 0.$$

Then, it satisfies **TCP**.

Examples and counterexamples

Examples:

- Flat tori (Cordero-Erausquin '99)
- Round spheres (Loeper '06)
- Product of spheres (Figalli-Kim-McCann '13)
- Quotients of the above objects
- Perturbations of round spheres (Figalli-R-Villani '12)

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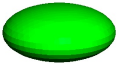

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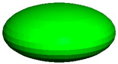

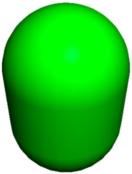

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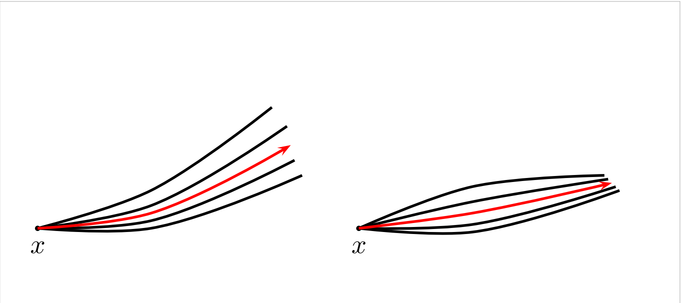
Focalization is the major obstacle

Focalization

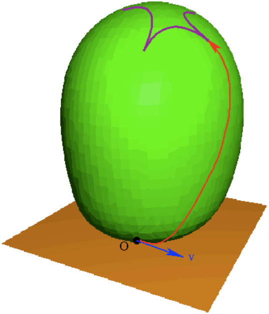
Definition

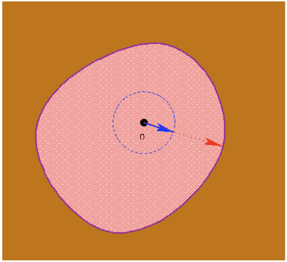
Let $x \in M$ and v be a unit tangent vector in T_xM. The vector v is **not conjugate** at time $t \geq 0$ if for any $t' \in [0, t+\delta]$ ($\delta > 0$ small) the geodesic from x to $\gamma(t')$ is locally minimizing.



The distance to the conjugate locus

Again, using the exponential mapping, we can associate to each unit tangent vector its **distance to the conjugate locus**.



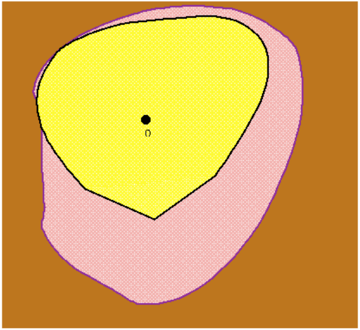


In that way, we define the so-called **nonfocal domain** $\mathcal{NF}(x)$ whose boundary is the **tangent conjugate locus** $\mathrm{TFL}(x)$.

Fundamental inclusion

The following inclusion holds

$$\text{injectivity domain} \subset \text{nonfocal domain}.$$



Stay away property

Theorem

Let (M, g) be a compact Riemannian manifold satisfying **TCP** *and μ_0, μ_1 two probability measures associated with* **smooth positive densities**. *Assume that the continuous transport map $T : M \to M$ satisfies*

$$T(x) \notin TFL(x) \qquad \forall x \in M.$$

Then T is smooth.

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Then T is smooth.

The potential $\psi : M \to \mathbb{R}$ satisfying $T(x) = \exp_x(\nabla\psi(x))$ is the solution of the Monge-Ampère like equation $(c = d^2)$

$$\det\left(\nabla^2\psi(x) + \nabla_{xx}c\left(x, T(x)\right)\right)$$
$$= \left|\det\left(\nabla_{xy}c\left(x, T(x)\right)\right)\right| \frac{f_0(x)}{f_1(T(x))}.$$

Thank you for your attention !!