Remarks on Input to State Stabilization

Michael Malisoff [1]
Department of Mathematics
304 Lockett Hall
Louisiana State University
and A. & M. College
Baton Rouge LA 70803-4918 USA
malisoff@lsu.edu

Ludovic Rifford
Institut Girard Desargues
Université Lyon 1
Bâtiment Braconnier
21 Avenue Claude Bernard
69622 Villeurbanne Cedex France
rifford@desargues.univ-lyon1.fr

Eduardo Sontag [2]
Department of Mathematics
Rutgers-New Brunswick
Hill Center-Busch Campus
110 Frelinghuysen Road
Piscataway NJ 08854-8019 USA
sontag@control.rutgers.edu

***Abstract*— We announce a new construction of a stabilizing feedback law for nonlinear globally asymptotically controllable (GAC) systems. Given a control affine GAC system, our feedback renders the closed loop system input to state stable with respect to actuator errors and small observation noise. We also announce a variant of our result for fully nonlinear GAC systems.**

I. INTRODUCTION

The theory of input to state stable (ISS) systems forms the basis for much current research in mathematical control theory (cf. [6], [7], [17]). The ISS property was introduced in [15]. In the past decade, there has been a great deal of research done to find ISS stabilizing control laws (cf. [5], [6], [7], [9]). In this note, we study the ISS stabilizability of control affine systems of the form

$$\dot{x} = f(x) + G(x)u \tag{1}$$

where f and G are locally Lipschitz vector fields on $\mathbb{R}^n$, $f(0) = 0$, and the control u is valued in $\mathbb{R}^m$ (but see also §V for extensions for fully nonlinear systems). We assume throughout this note that the system (1) is globally asymptotically controllable (GAC), and we construct a feedback $K : \mathbb{R}^n \to \mathbb{R}^m$ for which

$$\dot{x} = f(x) + G(x)K(x) + G(x)u \tag{2}$$

is ISS (cf. §II for the relevant definitions). As pointed out in [2], [17], a *continuous* (time-invariant) stabilizing feedback K fails to exist in general.

This fact forces us to consider *discontinuous* feedbacks K, so our solutions will be interpreted in the more general sense of sampling and Euler solutions for dynamics which are discontinuous in the state (cf. [3], [17]). By an Euler solution, we mean a uniform limit of sampling solutions, taken as the frequency of sampling becomes infinite (cf. §II for precise definitions). Our construction extends [15], [16], which show how to make C^o-stabilizable systems ISS to actuator errors. In particular, our feedback applies to the nonholonomic integrator (cf. [2], [12], and §IV below) and other applications where Brockett's necessary condition is not satisfied, and which therefore cannot be stabilized by continuous state feedbacks (cf. [2], [17]).

Our results also *strengthen* [3], which constructed feedbacks for GAC systems which render the systems globally asymptotically stable. Our main tool will be the recent constructions of semiconcave control Lyapunov functions (CLF's) for GAC systems from [11], [12], [13]. Our construction also applies in the more general situation where measurement noise may occur. In particular, our feedback K has the additional feature that the *perturbed* system

$$\dot{x} = f(x) + G(x)K(x + e) + G(x)u \tag{3}$$

is ISS (with respect to the actuator error u) when the observation error $e : [0, \infty) \to \mathbb{R}^n$ in the controller is *sufficiently small* (cf. the definitions below). In this context, the precise value of $e(t)$ is unknown to the controller, but information about upper bounds on the magnitude of $e(t)$ can be used to design the feedback. The following theorem is shown in [8]:

Theorem 1: If (1) is GAC, then there exists a feedback K for which (3) is ISS for Euler solutions.

Theorem 1 characterizes the uniform limits of sampling solutions of (3) (cf. §II for the definitions of Euler and sampling solutions). From a computational standpoint, it is also desirable to know how frequently to sample in order to achieve ISS for sampling solutions. This information is provided by the following theorem:

Theorem 2: If (1) is GAC, then there exists a feedback K for which (3) is ISS for sampling solutions.

This note is organized as follows. In §II, we review the relevant background on CLF's, GAC and ISS systems, nonsmooth analysis, and discontinuous feedbacks. In §III, we sketch the proofs of the above theorems (cf. [8] for their detailed proofs). This is followed in §IV by a comparison of our feedback construction with the known feedback constructions for C^o-stabilizable systems, and an application of our results to the nonholonomic integrator. We close in §V by announcing an extension of our results for fully nonlinear systems.

II. DEFINITIONS AND MAIN LEMMAS

We let $\mathcal{K}_\infty$ denote the set of all continuous functions $\rho : [0, \infty) \to [0, \infty)$ for which (i) $\rho(0) = 0$ and (ii) ρ is

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strictly increasing and unbounded. Let $\mathcal{KL}$ denote the set of all continuous $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ for which (1) $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each $t \geq 0$, (2) $\beta(s, \cdot)$ is nonincreasing for each $s \geq 0$, and (3) $\beta(s,t) \to 0$ as $t \to +\infty$ for each $s \geq 0$.

For each $k \in \mathbb{N}$ and $r > 0$, we use the control sets

$$\begin{aligned}&\mathcal{M}^k := \{\text{measurable } u : [0,\infty) \to \mathbb{R}^k : |u|_\infty < \infty\}\\ &\mathcal{M}^k_r := \{u \in \mathcal{M}^k : |u|_\infty \leq r\},\end{aligned}$$

where $|\cdot|_\infty$ is the essential supremum. We let $\|u(s)\|_I$ denote the essential supremum of the restriction of a function u to an interval I. Let $|\cdot|$ denote the Euclidean norm, and

$$r\mathcal{B}_k := \{x \in \mathbb{R}^k : |x| < r\}$$

for each $k \in \mathbb{N}$ and $r > 0$. The closure of $r\mathcal{B}_k$ is denoted by $r\bar{\mathcal{B}}_k$, and $\mathrm{bd}(\cdot)$ denotes the boundary operator. We set

$$\begin{aligned}&\mathcal{O} := \{e : [0,\infty) \to \mathbb{R}^n\}, \quad \sup(e) := \sup\{|e(t)| : t \geq 0\}\\ &\mathcal{O}_\eta := \{e \in \mathcal{O} : \sup(e) \leq \eta\}\end{aligned}$$

for all $e \in \mathcal{O}$ and $\eta > 0$. For any compact set $\mathcal{F} \subseteq \mathbb{R}^n$ and $\varepsilon > 0$, we define the compact set

$$\mathcal{F}^\varepsilon := \{x \in \mathbb{R}^n : \min\{|x - p| : p \in \mathcal{F}\} \leq \varepsilon\},$$

i.e., the "ε-enlargement of $\mathcal{F}$". Given a continuous function $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n : (x,u) \mapsto h(x,u)$ which is locally Lipschitz in x uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}^m$, we let $\phi_h(\cdot, x_o, u)$ denote the trajectory of $\dot{x} = h(x,u)$ starting at $x_o \in \mathbb{R}^n$ for each $u \in \mathcal{M}^m$. In this case, $\phi_h(\cdot, x_o, u)$ is defined on some maximal interval $[0, t)$, with $t > 0$ depending on u and x_o. When we say that a function defined on $\mathbb{R}^n$ is C^o (resp., C^1), we mean that it is continuous (resp., continuously differentiable). We say that a function $\alpha : \mathbb{R}^k \to [0,\infty)$ is *proper* (a.k.a. *radially unbounded*) provided $\alpha(x) \to +\infty$ as $|x| \to +\infty$. We say that a function $\alpha : \mathbb{R}^k \to [0,\infty)$ is *positive definite* provided $\alpha(x) = 0$ iff $x = 0$. We use the following controllability notion:

Definition 2.1: A control system $\dot{x} = h(x,u)$ is called *globally asymptotically controllable (GAC)* provided there exist a nondecreasing $\mathcal{N} : [0,\infty) \to [0,\infty)$ and a function $\beta \in \mathcal{KL}$ satisfying the following: For each $x_o \in \mathbb{R}^n$, there exists $u \in \mathcal{M}^m$ such that

(a) $|\phi_h(t, x_o, u)| \leq \beta(|x_o|, t)$ for all $t \geq 0$; and
(b) $|u(t)| \leq \mathcal{N}(|x_o|)$ for a.e. $t \geq 0$.

We call $\mathcal{N}$ the *GAC modulus* of h.

In this note, we allow discontinuous feedbacks, so the dynamics are discontinuous in the state variable. This produces the technical problem of precisely defining what is meant by a solution, since the standard existence theorems for solutions would not apply. We will resolve this problem by forming our trajectories through sampling solutions and their uniform limits, as follows.

We say that $\pi = \{t_o, t_1, t_2, \ldots\} \subset [0,\infty)$ is a *partition* provided $t_o = 0$, $t_i < t_{i+1}$ for all $i \geq 0$, and $t_i \to +\infty$ as $i \to +\infty$. We denote the set of all partitions by Par. Let $F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n : (x,p,u) \mapsto F(x,p,u)$ be a continuous function which is locally Lipschitz in x uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$. A (state) *feedback* (for F) is defined to be any locally bounded function $K : \mathbb{R}^n \to \mathbb{R}^m$ for which $K(0) = 0$. In particular, we allow discontinuous feedbacks. The arguments x, p, and u in F represent the state, feedback value, and actuator error, respectively.

Given a feedback $K : \mathbb{R}^n \to \mathbb{R}^m$, $x_o \in \mathbb{R}^n$, a partition

$$\pi = \{t_o, t_1, t_2, \ldots\} \in \mathrm{Par},$$

$e \in \mathcal{O}$, and $u \in \mathcal{M}^m$, the *sampling solution* for the initial value problem

$$\dot{x}(t) = F(x(t), K(x(t) + e(t)), u(t)) \tag{4}$$

$$x(0) = x_o \tag{5}$$

is the continuous function $x(\cdot)$ defined by recursively solving

$$\dot{x}(t) = F(x(t), K(x(t_i) + e(t_i)), u(t))$$

from the initial time $t = t_i$ up until the maximal time

$$s_i := t_i \vee \sup\{s \in [t_i, t_{i+1}] : x(\cdot) \text{ is defined on } [t_i, s)\},$$

where $x(0) = x_o$. (The $t_i \vee$ term in the formula for s_i is used to allow the possibility that $x(\cdot)$ is not defined at all on $[t_i, t_{i+1}]$, in which case the supremum in the definition of s_i is by definition $-\infty$.) In this case, the sampling solution of (4)-(5) is defined from time zero up to the maximal time $\bar{t} = \inf\{s_i : s_i < t_{i+1}\}$. We denote this sampling solution by $t \mapsto x_\pi(t; x_o, u, e)$ to exhibit its dependence on $\pi \in \mathrm{Par}$, $x_o \in \mathbb{R}^n$, $u \in \mathcal{M}^m$, and $e \in \mathcal{O}$, or simply by x_π where the dependence is clear. In particular, if $s_i = t_{i+1}$ for all i, then $\bar{t} = +\infty$, so in that case, x_π is defined on $[0,\infty)$.

We also define the *upper diameter* and *lower diameter* of $\pi = \{t_o, t_1, t_2, \ldots\} \in \mathrm{Par}$ by

$$\overline{\mathbf{d}}(\pi) := \sup_{i \geq 0}(t_{i+1} - t_i), \quad \underline{\mathbf{d}}(\pi) := \inf_{i \geq 0}(t_{i+1} - t_i)$$

respectively. We let

$$\mathrm{Par}(\delta) := \left\{\pi \in \mathrm{Par} : \overline{\mathbf{d}}(\pi) < \delta\right\}$$

for each $\delta > 0$. We say that a function $y : [0,\infty) \to \mathbb{R}^n$ is an *Euler solution (robust to small observation errors)* of (4)-(5) for $u \in \mathcal{M}^m$ provided there are sequences $\pi_r \in \mathrm{Par}$ and $e_r \in \mathcal{O}$ such that

(a) $\overline{\mathbf{d}}(\pi_r) \to 0$; (b) $\sup(e_r)/\underline{\mathbf{d}}(\pi_r) \to 0$; and
(c) $t \mapsto x_{\pi_r}(t; x_o, u, e_r)$ converges uniformly to y

as $r \to +\infty$. In this note, we design feedbacks which make GAC systems ISS with respect to actuator errors in the following generalized sense:

Definition 2.2: We say that (4) is *ISS for sampling solutions* provided there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ satisfying: For each $\varepsilon, M, N > 0$ with $0 < \varepsilon < M$, there exist positive $\delta = \delta(\varepsilon, M, N)$ and $\kappa = \kappa(\varepsilon, M, N)$ such that for each $\pi \in \mathrm{Par}(\delta)$, $x_o \in M\bar{\mathcal{B}}_n$, $u \in \mathcal{M}^m_N$, and $e \in \mathcal{O}$ for which $\sup(e) \leq \kappa \underline{\mathbf{d}}(\pi)$,

$$|x_\pi(t; x_o, u, e)| \leq \beta(M, t) + \gamma(N) + \varepsilon \tag{6}$$

for all $t \geq 0$.

Roughly speaking, condition (6) says that the system is ISS, modulo small overflows, if the sampling is done 'quickly enough' (but not 'too quickly'). On the other hand, our results are new, even for the particular case where the observation error $e = 0$. Moreover, if we restrict to the case where $e = 0$, then the condition on $\underline{\mathbf{d}}(\pi)$ in Definition 2.2 is no longer needed. Notice that the bounds on e are in the supremum, not the essential supremum. We also use the following analog of Definition 2.2 for Euler solutions:

Definition 2.3: We say that (4) is *ISS for Euler solutions* provided there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ satisfying: If $u \in \mathcal{M}^m$ and $x_o \in \mathbb{R}^n$, and if $t \mapsto x(t)$ is an Euler solution of (4)-(5), then

$$|x(t)| \leq \beta(|x_o|, t) + \gamma(|u|_\infty) \tag{7}$$

for all $t \geq 0$.

Our main tools in this note will be nonsmooth analysis and nonsmooth Lyapunov functions. We use the following definitions (cf. [4], [17]), in which Ω is an arbitrary open subset of $\mathbb{R}^n$.

Definition 2.4: Let $g : \Omega \to \mathbb{R}$ be a continuous function on Ω; it is said to be *semiconcave* on Ω provided for any point $x_o \in \Omega$, there exist $\rho, C > 0$ such that

$$g(x) + g(y) - 2g\left(\frac{x+y}{2}\right) \leq C||x-y||^2$$

for all $x, y \in x_o + \rho\mathcal{B}_n$.

The *proximal superdifferential* (resp., *proximal subdifferential*) of a function $V : \Omega \to \mathbb{R}$ at $x \in \Omega$, which is denoted by $\partial^P V(x)$ (resp., $\partial_P V(x)$), is defined to be the set of all $\zeta \in \mathbb{R}^n$ for which there exist $\sigma, \eta > 0$ such that

$$\begin{array}{l} V(y) - V(x) - \sigma|y-x|^2 \leq \langle \zeta, y - x\rangle \\ (\text{resp., } V(y) - V(x) - \sigma|y-x|^2 \geq \langle \zeta, y - x\rangle) \end{array}$$

for all $y \in x + \eta\mathcal{B}_n$. The *limiting subdifferential* of a continuous function $V : \Omega \to \mathbb{R}$ at $x \in \Omega$ (cf. [10]) is

$$\partial_L V(x) := \left\{ \begin{array}{l} q \in \mathbb{R}^n : \text{ there exist } x_n \to x \text{ and} \\ q_n \in \partial_P V(x_n) \text{ such that } q_n \to q. \end{array} \right\}.$$

Assume $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n : (x, u) \mapsto h(x, u)$ is continuous and locally Lipschitz in x uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}^m$, and $h(0, 0) = 0$. The next definition was introduced in [14] and reformulated in proximal terms in [17]:

Definition 2.5: A *control-Lyapunov function (CLF)* for

$$\dot{x} = h(x, u) \tag{8}$$

is defined to be any continuous, positive definite, proper function $V : \mathbb{R}^n \to \mathbb{R}$ for which there exist a continuous, positive definite function $W : \mathbb{R}^n \to \mathbb{R}$, and a nondecreasing function $\alpha : [0, \infty) \to [0, \infty)$, satisfying

$$\forall \zeta \in \partial_P V(x), \quad \inf_{|u| \leq \alpha(|x|)} \langle \zeta, h(x, u)\rangle \leq -W(x)$$

for all $x \in \mathbb{R}^n$.

Recall the following lemmas (cf. [13]):

Lemma 2.6: If (8) is GAC, then there exists a CLF V for (8) which is semiconcave on $\mathbb{R}^n \setminus \{0\}$, and a nondecreasing function $\alpha : [0, \infty) \to [0, \infty)$, that satisfy

$$\forall \zeta \in \partial_L V(x), \quad \min_{|u| \leq \alpha(|x|)} \langle \zeta, h(x, u)\rangle \leq -V(x) \tag{9}$$

for all $x \in \mathbb{R}^n$.

Lemma 2.7: Let $V : \Omega \to \mathbb{R}$ be semiconcave. Then V is locally Lipschitz, and $\emptyset \neq \partial_L V(x) \subseteq \partial^P V(x)$ for all $x \in \Omega$. Moreover, for each compact set $Q \subset \Omega$, there exist constants $\sigma, \mu > 0$ such that

$$V(y) - V(x) - \sigma|y-x|^2 \leq \langle \zeta, y - x\rangle$$

for all $y \in x + \mu\mathcal{B}_n$, all $x \in Q$, and all $\zeta \in \partial^P V(x)$.

Remark 2.8: In [13], the control variable u takes all its values in a given compact metric space U. The version of the CLF existence theorem in [13] is the same as Lemma 2.6 above except that the minimum in (9) is replaced by the minimum over all $u \in U$. Lemma 2.6 follows from a minor modification of the arguments of [12], [13], using the GAC modulus (see Definition 2.1). The existence theory [12] for semiconcave CLF's strengthens the proof that continuous CLF's exist for any GAC system (see [14]).

III. DISCUSSION OF PROOFS OF THEOREMS

In this section, we sketch the proofs of our theorems. Due to space restrictions, we only sketch the proof of Theorem 2 for the special case where the observation error $e = 0$. For complete proofs of our theorems, see [8]. We begin by outlining the proof of Theorem 2.

Let $M, N > 0$ be given, and V be a CLF satisfying the requirements of Lemma 2.6 for the dynamics

$$h(x, u) := f(x) + G(x)u. \tag{10}$$

Define the functions $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ by

$$\begin{array}{l} \underline{\alpha}(s) := \min\{s, \min\{|x| : V(x) \geq s\}\} \text{ and} \\ \overline{\alpha}(s) := \max\{|x| : V(x) \leq s\}. \end{array} \tag{11}$$

Let $x \mapsto \zeta(x)$ be any selection of $\partial_L V(x)$ on $\mathbb{R}^n \setminus \{0\}$ and $\zeta(0) \in \mathbb{R}^n$ be arbitrary. It follows that

$$\forall x \in \mathbb{R}^n, \quad \underline{\alpha}(V(x)) \leq |x| \text{ and } \overline{\alpha}(V(x)) \geq |x|. \tag{12}$$

For each $x \in \mathbb{R}^n$, we can choose $u = u_x \in \alpha(|x|)\mathcal{B}_m$ that satisfies the inequality in (9) for (10) and $\zeta = \zeta(x)$. Define the feedback $K_1 : \mathbb{R}^n \to \mathbb{R}^m$ by

$$K_1(x) := u_x$$

for all $x \neq 0$ and $K_1(0) := 0$. We use the functions

$$\begin{array}{lcl} a(x) & := & \langle \zeta(x), f(x) + G(x)K_1(x)\rangle \\ b_j(x) & := & \langle \zeta(x), g_j(x)\rangle \text{ for } j = 1, 2, \ldots, m \\ K_2(x) & := & -V(x)(\text{sgn}\{b_1(x)\}, \ldots, \text{sgn}\{b_m(x)\})^T \end{array} \tag{13}$$

where g_j is the jth column of G and

$$\text{sgn}\{s\} := \left\{ \begin{array}{rl} 1, & s > 0 \\ -1, & s < 0 \\ 0, & s = 0 \end{array} \right. .$$

Therefore, $K := K_1 + K_2 : \mathbb{R}^n \to \mathbb{R}^m$ is a feedback for the dynamics

$$F(x, p, u) := f(x) + G(x)(p + u). \tag{14}$$

We claim that K satisfies the requirements of the theorem. To see why this is the case, first choose

$$S := \{x \in \mathbb{R}^n : V(x) \leq \underline{\alpha}^{-1}(N)\}$$

and $\varepsilon \in (0, \min\{1, M\})$ for which $(2\varepsilon)\mathcal{B}_n \subseteq S$. Set

$$Q := \left\{ \left[\overline{\alpha} \circ \underline{\alpha}^{-1}(N + M) + 1\right] \bar{\mathcal{B}}_n \right\} \setminus \varepsilon\mathcal{B}_n,$$
$$\lambda_- := \min \left\{ V(p) : p \in Q^{\varepsilon/2} \right\},$$
$$\lambda_+ := \max \left\{ V(p) : p \in Q^{\varepsilon} \right\}.$$

It follows from the estimates (12) that $S \subseteq Q^\varepsilon$. We can then choose $\tilde{\varepsilon} \in (0, \varepsilon)$ for which

$$\overline{\alpha}\left(p + \frac{\mathcal{L}_\varepsilon}{4}\tilde{\varepsilon}\right) \leq \overline{\alpha}(p) + \frac{\varepsilon}{8} \quad \forall p \in \left[0, \underline{\alpha}^{-1}(N) + \lambda_+\right] \tag{15}$$

where $\mathcal{L}_\varepsilon > 1$ is the Lipschitz constant for V on $Q^{\varepsilon/2}$ guaranteed by Lemma 2.7. By again applying Lemma 2.7, we can also find $\sigma, \mu > 0$ such that

$$V(y) - V(x) \leq \langle \zeta(x), y - x \rangle + \sigma|y - x|^2 \tag{16}$$

for all $y \in x + \mu\mathcal{B}_n$ and $x \in Q^{\varepsilon/2}$. We can then choose

$$\delta = \delta(\varepsilon, M, N) \in \left(0, \frac{\tilde{\varepsilon}}{16 + \lambda_+ + 16\lambda_+}\right) \tag{17}$$

such that

$$\begin{aligned} &|x_\pi(t; x_o, u, 0) - x_i| \\ &\leq \min\left\{\mu, \frac{\tilde{\varepsilon}}{16(1 + \mathcal{L}_\varepsilon)}, \sqrt{\frac{\lambda_-}{8\sigma}(t - t_i)}\right\} \end{aligned} \tag{18}$$

(where $x_i = x_\pi(t_i; x_o, u, 0)$) and

$$\begin{aligned} &\|\zeta(x_i) \cdot (F(x_i, K(x_i), u(s)) - f(x_\pi(s)) \\ &-G(x_\pi(s))[u(s) + K(x_i)])\|_{[t_i, t_{i+1}]} \leq \tfrac{\lambda_-}{8} \end{aligned} \tag{19}$$

for all $u \in \mathcal{M}_N^m$, $t \in [t_i, t_{i+1}]$, $\pi \in \text{Par}(\delta)$, and all i such that $x_i \in Q^{\varepsilon/2}$. Defining $J(t) := 16/(16 + t)$, and defining $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ by

$$\beta(s, t) := \overline{\alpha}\left(\underline{\alpha}^{-1}(s)J(t)\right), \quad \gamma(s) := \overline{\alpha} \circ \underline{\alpha}^{-1}(s), \tag{20}$$

we can then use the estimates (12) and (18)-(19) to conclude that the sampling ISS estimate (6) holds for all $x_o \in M\bar{\mathcal{B}}_n$, $u \in \mathcal{M}_N^m$, $\pi \in \text{Par}(\delta)$, and $e = 0$. This gives the conclusion of Theorem 2 for the case of zero observation errors.

We turn next to Theorem 1. We need to show the ISS property (7) for all Euler solutions $x(t)$ of (4)-(5) with the choice (14). To this end, choose $u \in \mathcal{M}^m$ and $x_o \in \mathbb{R}^n$. Using the conclusion of Theorem 2 that (4) is ISS for sampling solutions, we can let

$$\delta_r := \delta\left(\frac{1}{r}, |x_o|, |u|_\infty\right) \quad \text{and} \quad \kappa_r := \kappa\left(\frac{1}{r}, |x_o|, |u|_\infty\right)$$

be the constants from Definition 2.2 for large $r \in \mathbb{N}$. Let $x(t)$ be an Euler solution of (4)-(5), and let π_r and e_r satisfy the requirements of the Euler solution definition. It follows from the definition that there is a subsequence $(\pi_{r'}, e_{r'})$ of (π_r, e_r) such that

$$\overline{\mathbf{d}}(\pi_{r'}) \leq \delta_r, \quad \sup(e_{r'}) \leq \kappa_r \underline{\mathbf{d}}(\pi_{r'})$$

for all r, r'. It follows from estimate (6) that

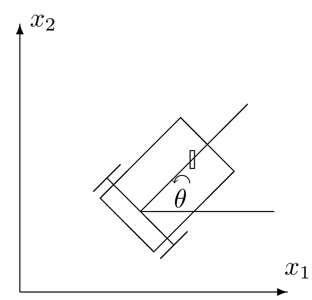
$$|x_{\pi_{r'}}(t; x_o, u, e_{r'})| \leq \beta(|x_o|, t) + \gamma(|u|_\infty) + \frac{1}{r} \tag{21}$$

for all $t \geq 0$ and $r, r' \in \mathbb{N}$, where β and γ are in (20). The ISS condition (7) now follows by passing to the limit in (21) as $r', r \to \infty$. This gives the conclusion of Theorem 1.

IV. ISS STABILIZATION OF THE NONHOLONOMIC INTEGRATOR

In this section, we apply the feedback construction from §III to Brockett's nonholonomic integrator control system (cf. [2], [12], [17]). The nonholonomic integrator was introduced in [2], as an example of a system which cannot be stabilized using continuous state feedback. It is well-known that if the state space of a system contains topological obstacles (e.g., if the state space is $\mathbb{R}^2 \setminus (-1, +1)^2$), then the system cannot be stabilized by a continuous state feedback; this follows from a theorem of Milnor (cf. [17]). Brockett's example illustrates how there may still be obstacles to continuous stabilization, even if the state space is all of $\mathbb{R}^n$. In Brockett's example, the system is 'nonholonomic' in the sense that it is impossible to move *instantly* in some directions, even though it is possible to move *eventually* in every direction.

The underlying physical model for Brockett's example is as follows. Consider a three-wheeled shopping cart whose front wheel acts as a castor. The state variable is $(x_1, x_2, \theta)^T$, where $(x_1, x_2)^T$ is the midpoint of the rear axle of the cart, and θ is the cart's orientation. The front wheel, which is a castor, is free to rotate, but there is a "non-slipping" constraint that $(\dot{x}_1, \dot{x}_2)^T$ must always be parallel to $(\cos(\theta), \sin(\theta))^T$. The following figure from [17] illustrates the model:



The equations for the model are therefore

$$\dot{x}_1 = u_1 \cos(\theta), \quad \dot{x}_2 = u_1 \sin(\theta), \quad \dot{\theta} = u_2 \tag{22}$$

where u_1 is interpreted as a "drive" command and u_2 is a steering command. Using a feedback transformation (cf. [17]) brings the equations (22) into the form

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_1 u_2 - x_2 u_1 \tag{23}$$

which is called the nonholonomic integrator system.

One can show that (23) is a GAC system. However, since Brockett's necessary condition is not satisfied for (23) (cf. [2], [17]), the system has no continuous state feedback stabilizer. While there does not exist a C^1 CLF for (23), it is now well-known that every GAC system admits a continuous CLF (cf. [14]). In fact, it was shown in [11] that (23) has the nonsmooth CLF

$$V(x) = \max\left\{\sqrt{x_1^2 + x_2^2}, |x_3| - \sqrt{x_1^2 + x_2^2}\right\}. \tag{24}$$

For the case of the dynamics (23) and CLF (24), the feedback $K = K_1 + K_2$ we constructed in §III is as follows.

We use the radius

$$r(x) := \sqrt{x_1^2 + x_2^2} \quad \forall x = (x_1, x_2, x_3)^T \in \mathbb{R}^3.$$

The sets

$$S_o := \{x \in \mathbb{R}^3 :\ x_3 \neq 0,\ r(x) = 0\},$$
$$S_+ := \{x \in \mathbb{R}^3 :\ x_3^2 \geq 4r^2(x) > 0\},$$
$$S_- := \{x \in \mathbb{R}^3 :\ x_3^2 < 4r^2(x)\}$$

form a partition of $\mathbb{R}^3 \setminus \{0\}$. Note that $V(x) = r(x)$ on S_- and $V(x) = |x_3| - r(x)$ on $\mathbb{R}^3 \setminus S_-$. To find our selection $\zeta(x) \in \partial_L V(x)$, we choose $\zeta(0) = 0$, and we set

$$\zeta(x) = (0, -1, \mathrm{sgn}\{x_3\})^T$$

for all $x \in S_o$. Using $b(x) = (b_1(x), b_2(x))^T$ and the notation of (13), this gives

$$b(x) = \begin{cases} \begin{pmatrix} -x_2\mathrm{sgn}\{x_3\} - x_1/r(x) \\ x_1\mathrm{sgn}\{x_3\} - x_2/r(x) \end{pmatrix}, & x \in S_+ \\ \dfrac{1}{r(x)}\,(x_1, x_2)^T, & x \in S_- \\ (0, -1)^T, & x \in S_o \end{cases}$$

and

$$K_1(x) = \begin{cases} \mu_1(x) \begin{pmatrix} -x_2\mathrm{sgn}\{x_3\} - x_1/r(x) \\ x_1\mathrm{sgn}\{x_3\} - x_2/r(x) \end{pmatrix}, & x \in S_+ \\ -(x_1, x_2)^T, & x \in S_- \\ (0, |x_3|)^T, & x \in S_o \end{cases}$$

with $b(0) = K_1(0) = 0$, where

$$\mu_1(x) := \frac{r(x) - |x_3|}{r^2(x) + 1}.$$

In this case, we have taken

$$K_1(x) = -b(x)V(x)/|b(x)|^2$$

for $x \neq 0$, and K_1 is continuous at the origin. On the other hand, our feedback K_2 from (13) becomes

$$K_2(x) = -\begin{cases} \begin{pmatrix} \mu_2(x_1, -x_2, x) \\ \mu_2(x_2, x_1, x) \end{pmatrix}, & x \in S_+ \\ r(x)\,(\,\mathrm{sgn}\{x_1\},\ \mathrm{sgn}\{x_2\}\,)^T, & x \in S_- \\ |x_3|\,(0,\ -1)^T, & x \in S_0 \end{cases}$$

with $K_2(0) = 0$, where

$$\mu_2(a, b, x)\ :=\ (|x_3| - r(x))\,\mathrm{sgn}\{\,b\,r(x)\,\mathrm{sgn}\{x_3\} - a\,\}.$$

Since V is semiconcave on $\Omega := \mathbb{R}^3 \setminus \mathrm{bd}(S_-)$, the argument from §III applies to sampling solutions that satisfy the additional requirement that the corresponding perturbed solution $\tilde{x}_\pi$ (cf. [8], [17]) remains in Ω. It follows that the nonholonomic integrator system (23) can be stabilized for actuator errors and small observation errors (for this restricted set of sampling solutions), using the combined feedback $K = K_1 + K_2$.

Remark 4.1: In this example, we used the CLF (24) because it has been explicitly proven in [11] to be a CLF for the control system (23). The example illustrates how to apply our feedback construction to more general CLF's that may not be semiconcave on $\mathbb{R}^3 \setminus \{0\}$. On the other hand, one can show that (23) also has the CLF

$$\tilde{V}(x) = \left(\sqrt{x_1^2 + x_2^2} - |x_3|\right)^2 + x_3^2,$$

which is semiconcave on $\mathbb{R}^3 \setminus \{0\}$. The fact that $\tilde{V}$ is a CLF for the system follows from a slight variant of the change of coordinate arguments used to show that (24) is a CLF. To check that $\hat{V}$ is semiconcave on $\mathbb{R}^3 \setminus \{0\}$, it suffices to verify this semiconcavity for $S(x) = -r(x)|x_3|$, which in turn follows from the semiconcavity of

$$(r, s) \mapsto -|rs| = \min\{\pm rs\}$$

(cf. [9] for details). Therefore, if we use $\tilde{V}$ to form our feedbacks for (23), instead of the CLF (24), then our theorems apply directly, without any state restrictions on the sampling solutions.

Remark 4.2: Notice that our choice of K_2 in (13) is continuous at the origin. On the other hand, the nonsmooth analog

$$\tilde{K}(x) := -\zeta(x)G(x)$$

of the usual Lie derivative ISS stabilizing feedback (where $\zeta(x) \in \partial_L V(x)$ for all $x \neq 0$ and $\zeta(0) = 0$) for the dynamics (23) and the CLF (24) is easily shown to be discontinuous at the origin. This can be seen by comparing the values of

$$\tilde{K}\left((\varepsilon, \varepsilon, 0)^T\right) = -\left(1/\sqrt{2}, 1/\sqrt{2}\right)^T,$$

$$\tilde{K}\left((\varepsilon, \varepsilon, 3\sqrt{2}\varepsilon)^T\right) = \left(1/\sqrt{2}, 1/\sqrt{2}\right)^T + \varepsilon(1, -1)^T$$

for small $\varepsilon > 0$.

V. ISS FOR FULLY NONLINEAR GAC SYSTEMS

We conclude with an extension of our results that can be applied to *fully nonlinear* GAC systems

$$\dot{x} = f(x, u). \tag{25}$$

We assume for simplicity throughout this section that all observation errors in the controller are zero, and that

$$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n : (x, u) \mapsto f(x, u)$$

is continuous and locally Lipschitz in x uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}^m$ and $f(0, 0) = 0$. It is natural to ask whether these hypotheses are sufficient for the existence of a state feedback $K(x)$ for which

$$\dot{x} = f(x, K(x) + u)$$

is ISS for Euler solutions. However, one can easily construct examples for which such feedbacks cannot exist. Here is an example of a GAC system from [16] where this situation occurs:

Example 5.1: Consider the system

$$\dot{x} = -x + u^2x^2$$

on $\mathbb{R}$. If $K(x)$ is any feedback for which

$$\dot{x} = -x + (K(x) + u)^2x^2 \tag{26}$$

is ISS for sampling solutions, then

$$|K(x)| < x^{-1/2}$$

for sufficiently large $x > 0$. It follows that each Euler solution of

$$\dot{x} = -x + (K(x)+1)^2 x^2$$

starting at $x(0) = 4$ is unbounded. Therefore, there does not exist a feedback K for which (26) is ISS for Euler solutions.

On the other hand, one *can* find a (possibly discontinuous) feedback that makes (25) ISS, in an appropriate weaker sense. We use the following weaker sense of ISS for fully nonlinear systems that was introduced in [16]:

Definition 5.2: We say that (25) is *input to state stabilizable (ISSable) in the weak sense* provided there exist a feedback K, and an $m \times m$ matrix G of continuously differentiable functions which is invertible at each point, such that $\dot{x} = F(x, K(x), u)$ is ISS for sampling and Euler solutions, where $F(x, p, u) = f(x, p + G(x)u)$.
The following result is shown in [8]:

Proposition 5.3: If (25) is GAC, then (25) is also ISSable in the weak sense.

The preceding proposition allows us to characterize GAC for fully nonlinear systems in terms of feedback equivalence, as follows. Recall that two systems

$$\dot{x} = f(x, u), \quad \dot{x} = h(x, u)$$

evolving on $\mathbb{R}^n \times \mathbb{R}^m$ are called *feedback equivalent* provided there exist a feedback $K : \mathbb{R}^n \to \mathbb{R}^m$ and an everywhere invertible function $G : \mathbb{R}^n \to \mathbb{R}^{m \times m}$ for which

$$h(x, u) = f(x, K(x) + G(x)u) \;\; \forall x \in \mathbb{R}^n, u \in \mathbb{R}^m;$$

in this case, we also say that $\dot{x} = f(x, u)$ is feedback equivalent to (4) with $e \equiv 0$ and $F(x, p, u) := f(x, p + G(x)u)$ (cf. §II). The following elegant statement follows directly from Theorem 1 in [3] and Proposition 5.3:

Corollary 5.4: The fully nonlinear control system (25) is GAC if and only if it is feedback equivalent to a system which is ISS for sampling and Euler solutions.

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