The intrinsic dynamics of optimal transport

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Workshop Analysis in Lyon
ENS Lyon, 2015

In honour of Luigi Ambrosio

School matching around a lake

Find a **transport map** ($T_\sharp \mu_X = \mu_Y$)

$$T : X = \{pupils\} = \mathbb{S}^1 \longrightarrow Y = \{schools\} = \mathbb{S}^1$$

which minimizes the **transportation cost**

$$\int_X c\,(x, T(x))\; d\mu_X(x)$$

for some **cost** $c : \mathbb{S}^1 \times \mathbb{S}^1 \to [0, \infty)$.

Kingsley lake, FL

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Euclidean cost

$$c(x, y) = |y - x|^2$$

The (quadratic) geodesic cost

Let (M, g) be a smooth compact Riemannian manifold, given two probabilities measures μ_0, μ_1 on M, find a transport map $T : M \to M$ from μ_0 to μ_1 which minimizes

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- **No smooth costs satisfy Sub-TWIST**

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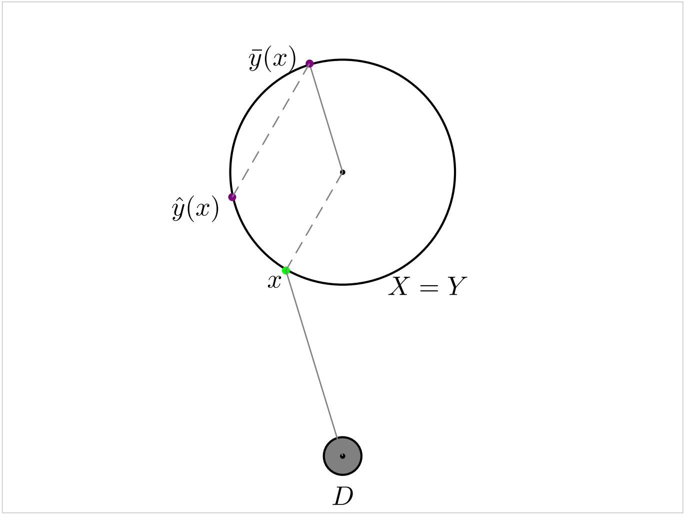


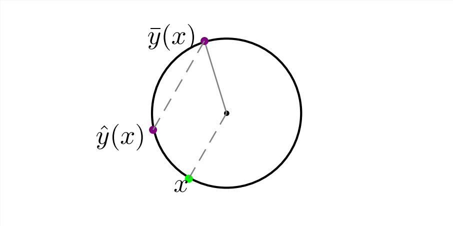
The (quadratic) Euclidean cost

Let $\tilde{\psi}$ be the distance function to the disc D, then set

$$\psi(x) := \tilde{\psi}(x) - \frac{1}{2}|x|^2 \text{ and } \phi(y) := \min_x \{\psi(x) + c(x,y)\}.$$

We check that for x close to the south pole, we have

$$\begin{aligned} \partial_c \psi(x) &:= \{(x,y) \,|\, c(x,y) = \phi(y) - \psi(x)\} \\ &= \{\bar{y}(x), \hat{y}(x)\}. \end{aligned}$$



The (quadratic) Euclidean cost

Let us consider an absolutely continuous probability measure μ_0 on $X = \mathbb{S}^1$ whose support is close to the south pole. Then define the measures $\bar{\nu}, \hat{\nu}$ on N by

$$\bar{\nu} := \frac{1}{2}\bar{y}_\sharp\mu_0, \ \hat{\nu} := \frac{1}{2}\hat{y}_\sharp\mu_0, \text{ and set } \mu_1 := \bar{\nu} + \hat{\nu}.$$

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Any plan γ with marginals μ_0 and μ_1 satisfies

$$\int_{X\times Y} c(x,y)\, d\gamma(x,y) \quad \geq \quad \int_{X\times Y} [\phi(y) - \psi(x)]\; d\gamma(x,y)$$

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$$\begin{aligned} \int_{X\times Y} c(x,y)\, d\gamma(x,y) &\geq \int_{X\times Y} \left[\phi(y) - \psi(x)\right]\, d\gamma(x,y) \\ &= \int_Y \phi(y)\, d\mu_1(y) - \int_X \psi(x)\, d\mu_0(x) \end{aligned}$$

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with equality in the first inequality if and only if $\gamma = \bar{\gamma}$ with $\bar{\gamma} := \frac{1}{2}\left(Id, \bar{y}\right)_\sharp \mu_0 + \frac{1}{2}\left(Id, \hat{y}\right)_\sharp \mu_0$.

Non-genericity of twist

Theorem (McCann, LR)

Let M, N be smooth compact manifolds of dimensions $n \geq 1$ and $c : M \times N \to [0, \infty)$ a cost function of class C^2. Assume that

$$\exists (\bar{x}, \bar{y}) \in M \times N \quad \text{such that} \quad \frac{\partial^2 c}{\partial x \partial y}(\bar{x}, \bar{y}) \quad \text{is invertible.} \qquad (1)$$

Then there is a pair μ_0, μ_1 of probability measures respectively on M and N which are both absolutely continuous w.r.t. Lebesgue for which **there is a unique optimal transport plan** *and such that this plan is* **not supported on a graph**.

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Then there is a pair μ_0, μ_1 of probability measures respectively on M and N which are both absolutely continuous w.r.t. Lebesgue for which **there is a unique optimal transport plan** *and such that this plan is* **not supported on a graph**. **The set of costs c satisfying (1) is open and dense in $C^2(M \times N; \mathbb{R})$.**

Purpose of the talk

- Study sufficient conditions for smooth costs that insure uniqueness of Kantorovitch optimizers (minimizing transport plans).

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- Exhibit such costs on arbitrary manifolds.

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- Exhibit such costs on arbitrary manifolds.
- Study the size of the set of such costs (genericity for some topology)

References

- K. Hestir and S. C. Williams. Supports of doubly stochastic measures (1995)
- S. Bianchini and L. Caravenna. On the extremality, uniqueness, and optimality of transference plans (2009)

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- W. Gangbo and R. J. McCann. Shape recognition via Wasserstein distance (2000)
- P.-A. Chiappori, R. McCann and L. Nesheim. Hedonic price equilibria, stable matching, and optimal transport: equivalence, topology, and uniqueness (2010)
- R. J. McCann and LR. The intrinsic dynamics of optimal transport (2015)

Setting

- M, N be smooth compact manifolds of dimensions ≥ 1.
- $c : M \times N \to [0, \infty)$ of class C^1.
- Given two probabilities measures μ_0, μ_1 on M, denote by $\Pi(\mu_0, \mu_1)$, the set of probability measures on $M \times N$ having marginals μ_0 and ν_0.
- A transport plan $\gamma \in \Pi(\mu_0, \mu_1)$ is called optimal if it minimizes the transportation cost

$$\int_{M \times N} c(x, y) d\gamma(x, y).$$

Alternant chains

Definition

We call *L-chain* in S ($L \geq 1$) any ordered family of pairs

$$\Big((x_1, y_1), \ldots, (x_L, y_L)\Big) \in (M \times N)^L$$

such that:

- The set $\{(x_1, y_1), \ldots (x_L, y_L)\}$ is c-**cyclically monotone**.

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such that:

- The set $\{(x_1, y_1), \ldots (x_L, y_L)\}$ is c-**cyclically monotone**.
- For every $l = 1, \ldots, L-1$ odd,

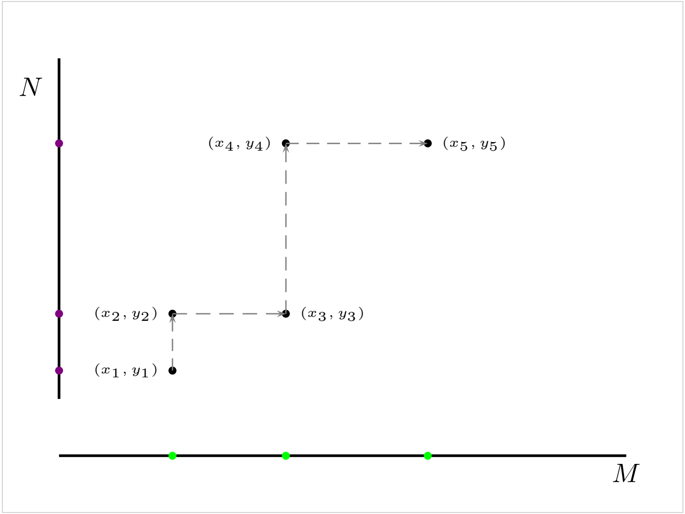
$$x_l = x_{l+1},\ y_l \neq y_{l+1},\ \frac{\partial c}{\partial x}(x_l, y_l) = \frac{\partial c}{\partial x}(x_l, y_{l+1}),$$

- For every $l = 1, \ldots, L-1$ even,

$$y_l = y_{l+1},\ x_l \neq x_{l+1},\ \frac{\partial c}{\partial y}(x_l, y_l) = \frac{\partial c}{\partial y}(x_{l+1}, y_l).$$

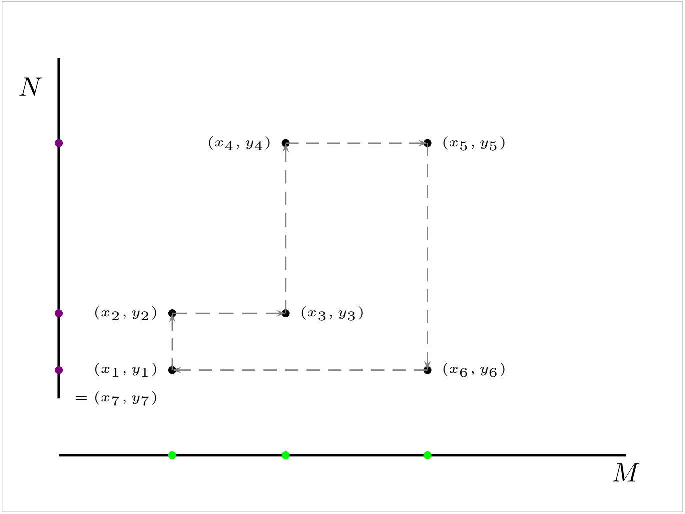
Alternant chains (picture)

A 5-chain



Alternant chains (picture)

Cyclic chains $\rightsquigarrow$ infinite chains



Optimal transport is unique if long chains are rare

Theorem

Let μ_0, μ_1 *be probability measures respectively on* M *and* N *which are both absolutely continuous w.r.t. Lebesgue. Denote by* $\mathcal{S}^\infty$ *the set of points in* $M \times N$ *which occur in L-chains for arbitrarily large L and assume that* $\mu_0\left(\pi^M(\mathcal{S}^\infty)\right) = 0$ *or* $\mu_1\left(\pi^M(\mathcal{S}^\infty)\right) = 0$.

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Theorem

Let μ_0, μ_1 be probability measures respectively on M and N which are both absolutely continuous w.r.t. Lebesgue. Denote by $\mathcal{S}^\infty$ the set of points in $M \times N$ which occur in L-chains for arbitrarily large L and assume that $\mu_0\left(\pi^M(\mathcal{S}^\infty)\right) = 0$ or $\mu_1\left(\pi^M(\mathcal{S}^\infty)\right) = 0$. Then there is a **unique** *optimal transport plan.*

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Comments:

- The theorem applies if there is a uniform bound on the length of all chains in $M \times N$.
- The theorem does not apply if there are cyclic chains on a set of positive measure.

Sktech of proof

Given μ_0, μ_1, there is a c-**cyclically monotone** set $\mathcal{S}$ and Lipschitz potentials $\psi : M \to \mathbb{R}$ and $\phi : M \to \mathbb{R}$ which satisfy

$$\psi(x) = \max_y \{\phi(y) - c(x,y)\}, \ \phi(y) = \min_x \{\psi(x) + c(x,y)\},$$

$$\mathcal{S} \subset \partial_c \psi := \Big\{ (x,y) \in M \times N \,|\, c(x,y) = \phi(y) - \psi(x) \Big\},$$

such that $\gamma \in \Pi(\mu,\nu)$ is optimal if and only if $\mathrm{Supp}(\gamma) \subset \mathcal{S}$.

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such that $\gamma \in \Pi(\mu, \nu)$ is optimal if and only if $\mathrm{Supp}(\gamma) \subset \mathcal{S}$.

Observation:
If ψ is differentiable at x, then

$$y \in \partial_c \psi(x) \Longrightarrow \frac{\partial c}{\partial x}(x,y) = -d_x \psi.$$

Sktech of proof

The previous observation allows to decompose $\mathcal{S}$ into a **numbered limb system** consisting of Borel graph and antigraphs (apart from a set of measure zero).

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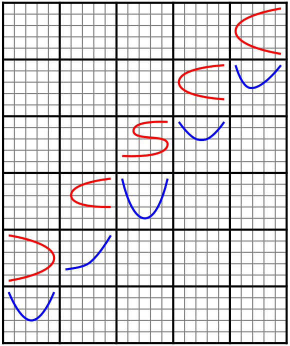
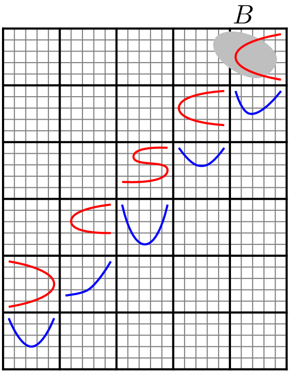


Sktech of proof

Then the result follows from uniqueness of transport plans in $\Pi(\mu_0, \mu_1)$ concentrated on the numbered limb system.



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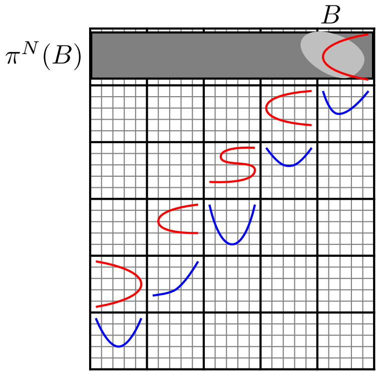

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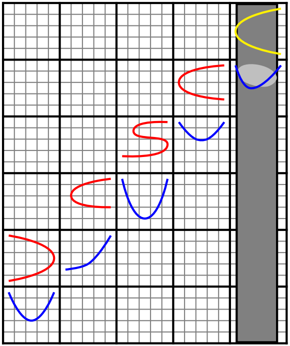
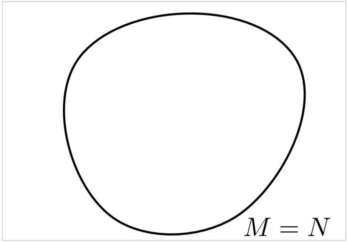


Examples: Strictly convex sets

Setting: $M = N =$ smooth strictly convex compact hypersurface in $\mathbb{R}^n$, $c(x,y) = |y-x|^2$.



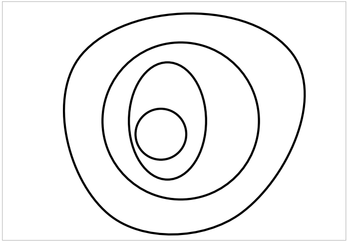
Lemma

There is no chain of length ≥ 4.

$\rightsquigarrow$ Uniqueness of optimal transport plans

Examples: Nested strictly convex sets

Setting: $M = N = \cup_{k=1}^{K} \mathcal{C}_k$ nested family of smooth strictly convex compact hypersurfaces in $\mathbb{R}^n$, $c(x,y) = |y-x|^2$.



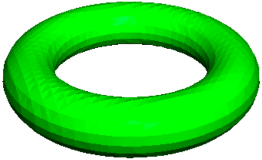
Lemma

There is no chain of length $\geq 4K+1$.

$\rightsquigarrow$ Uniqueness of optimal transport plans

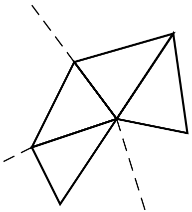
Examples: On arbitrary manifold

Setting: $M = N$ smooth compact manifold of dimension n



Let us consider a triangulation of the manifold.

Examples: On arbitrary manifold



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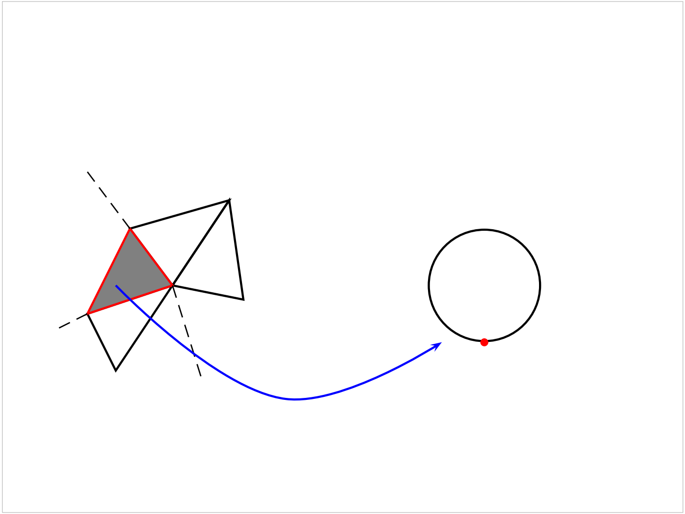

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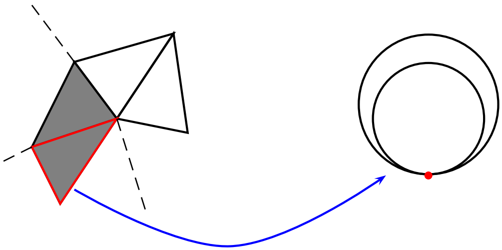

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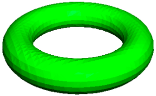
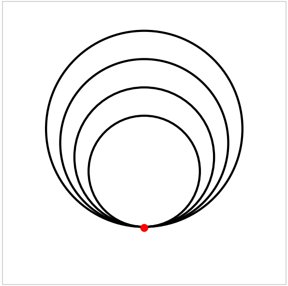


F

$\longrightarrow$

smooth



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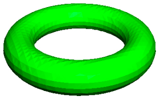
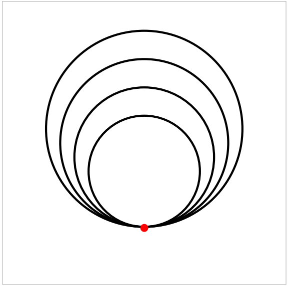


$$\xrightarrow[\text{smooth}]{F}$$



Then we define

$$c(x,y) = |F(y) - F(x)|^2$$

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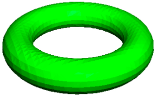

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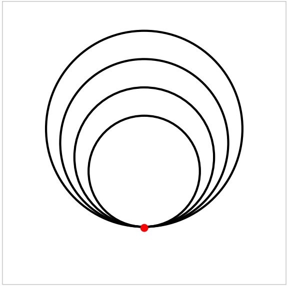

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Thank you for your attention !!