NECESSARY AND SUFFICIENT CONDITIONS FOR CONTINUITY OF OPTIMAL TRANSPORT MAPS ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper we investigate the regularity of optimal transport maps for the squared distance cost on Riemannian manifolds. First of all, we provide some general necessary and sufficient conditions for a Riemannian manifold to satisfy the so-called Transport Continuity Property. Then, we show that on surfaces these conditions coincide. Finally, we give some regularity results on transport maps in some specific cases, extending in particular the results on the flat torus and the real projective space to a more general class of manifolds.

1. Introduction

Let μ, ν be two probability measures on a smooth compact connected Riemannian manifold (M, g) equipped with its geodesic distance d. Given a cost function $c : M \times M \to \mathbb{R}$, the Monge-Kantorovich problem consists in finding a transport map $T : M \to M$ which sends μ onto ν (i.e. $T_\# \mu = \nu$) and which minimizes the functional

$$\min_{S_\# \mu = \nu} \int_M c(x, S(x))\, d\mu(x).$$

In [24] McCann (generalizing [2] from the Euclidean case) proved that, if μ gives zero mass to countably $(n-1)$-rectifiable sets, then there is a unique transport map T solving the Monge-Kantorovich problem with source measure μ, target measure ν, and cost function $c = d^2/2$. Moreover, T takes the form $T(x) = \exp_x\big(\nabla_x \psi\big)$, where $\psi : M \to \mathbb{R}$ is a c-convex function (see [26, Chapter 5]). From now on, the cost function we consider will always be $c(x, y) = d(x, y)^2/2$. The purpose of this paper is to study whether the optimal map can be expected to be continuous or not.

Definition 1.1. Let (M, g) be a smooth compact connected Riemannian manifold of dimension $n \geq 2$. We say that (M, g) satisfies the *transport continuity property* (abbreviated $\mathcal{TCP}$)[1] if, whenever μ and ν are absolutely continuous measures with

[1] Compare with [13, Definition 1.1], where a slighty different definition of $\mathcal{TCP}$ is considered.

densities bounded away from zero and infinity, the unique optimal transport map T between μ and ν is continuous.

Note that the above definition makes sense, since under the above assumptions McCann's Theorem [24] ensures that the optimal transport map T from μ to ν exists and is unique. The aim of the present paper is to give necessary and sufficient conditions for $\mathcal{TCP}$.

Since this is the fourth of a series of papers [14, 15, 16] concerning the regularity of optimal maps on Riemannian manifolds and the Ma-Trudinger-Wang condition, to avoid repetition we will only introduce the main notation, referring to our previous papers for more details. For convenience of the reader, some notation from Riemannian geometry is gathered in Appendix A.

Given a smooth compact connected Riemannian manifold of dimension $n \geq 2$, for every $x \in M$, we denote by $\mathrm{I}(x) \subset T_xM$ the *injectivity domain* of the exponential map at x (see Appendix A). We will say that (M,g) satisfies **(CI)** (resp. **(SCI)**) if $\mathrm{I}(x)$ is convex (resp. strictly convex) for all $x \in M$. Let $(x,v) \in TM$ with $v \in \mathrm{I}(x)$ and $(\xi,\eta) \in T_xM \times T_xM$. Following [23, 26], the MTW *tensor* at (x,v) evaluated on (ξ,η) is defined as

$$\mathfrak{S}_{(x,v)}(\xi,\eta) = -\frac{3}{2}\,\frac{d^2}{ds^2}\Big|_{s=0}\,\frac{d^2}{dt^2}\Big|_{t=0}\,c\Big(\exp_x(t\xi),\,\exp_x(v+s\eta)\Big).$$

It is said that (M,g) satisfies the *Ma–Trudinger–Wang condition* **(MTW)** if

$$\text{(1.1)}\quad \forall\,(x,v) \in TM \text{ with } v \in \mathrm{I}(x),\ \forall\,(\xi,\eta) \in T_xM \times T_xM,$$
$$\Big[\langle \xi,\eta\rangle_x = 0 \implies \quad \mathfrak{S}_{(x,y)}(\xi,\eta) \geq 0\Big].$$

If the last inequality in (1.1) is strict unless $\xi = 0$ or $\eta = 0$, then M is said to satisfy the *strict Ma–Trudinger–Wang condition* $\mathbf{(MTW^+)}$. Our first result holds in any dimension.

Theorem 1.2. *Let (M,g) be a smooth compact connected Riemannian manifold of dimension $n \geq 2$. Then:*

(i) *If (M,g) satisfies $\mathcal{TCP}$, then* **(CI)** *and* **(MTW)** *hold.*
(ii) *If (M,g) satisfies* **(SCI)** *and* $\mathbf{(MTW^+)}$, *then $\mathcal{TCP}$ holds.*

Let us observe that, in the above result, there is a gap between the necessary and sufficient conditions for $\mathcal{TCP}$.

However, in two dimensions, we can take advantage of the following two results (and a delicate geometric argument, see Subsection 3.3) to fill the gap:
1) In $\mathbb{R}^2$, continuity of optimal maps between densities bounded away from zero and infinity is known to be true under **(MTW)** [12].
2) If **(MTW)** holds, then, for any $x \in M$, the curvature of $\mathrm{TFL}(x)$ near any point of $\mathrm{TFCL}(x)$ (see Appendix A) has to be nonnegative. (Although not explicitly stated in this way, this fact is an immediate consequence of the proof of [15, Proposition 4.1(ii)].)

Theorem 1.3. *Let (M, g) be a smooth compact Riemannian surface. Then M satisfies $\mathcal{TCP}$ if and only if* **(CI)** *and* **(MTW)** *hold.*

Since property **(CI)** is closed under C^2-convergence of the metric, as an immediate consequence of Theorem 1.3 and Remark 3.3 below, we deduce that the set of two-dimensional manifolds satisfying $\mathcal{TCP}$ is closed in C^2-topology (compare [27]).

The paper is organized as follows: In the next section, we introduce the extended MTW condition, and we provide some further regularity results (in particular, we extend the regularity results on the flat torus [5] and the real projective space to a more general class of manifolds). Moreover, we make some comments on other existing results. The proofs of Theorems 1.2 and 1.3 are given in Section 3. Finally, some notation and technical results are postponed to the appendices.

2. Further results and comments

2.1. **Extended MTW conditions.** For every $x \in M$, let us denote by $\mathrm{NF}(x) \subset T_xM$ the *nonfocal domain* at x (see Appendix A). As before, we shall say that (M, g) satisfies **(CNF)** (resp. **(SCNF)**) if $\mathrm{NF}(x)$ is convex (resp. strictly convex) for all $x \in M$. As first suggested in [13], the MTW tensor may be extended by letting v vary in the whole nonfocal domain rather than in the injectivity domain. To define this extension, we let $x \in M$, $v \in \mathrm{NF}(x)$, and $(\xi, \eta) \in T_xM \times T_xM$. Since $y = \exp_x v$ is not conjugate to x, by the Inverse Function Theorem there are an open neighborhood $\mathcal{V}$ of (x, v) in TM, and an open neighborhood $\mathcal{W}$ of (x, y) in $M \times M$, such that

$$\begin{array}{rcl} \Psi_{(x,v)} : \mathcal{V} \subset TM & \longrightarrow & \mathcal{W} \subset M \times M \\ (x', v') & \longmapsto & \big(x', \exp_{x'}(v')\big) \end{array}$$

is a smooth diffeomorphism from $\mathcal{V}$ to $\mathcal{W}$. Then we may define $\widehat{c}_{(x,v)} : \mathcal{W} \to \mathbb{R}$ by

$$\widehat{c}_{(x,v)}(x', y') = \frac{1}{2}\big|\Psi^{-1}_{(x,v)}(x', y')\big|^2_{x'} \qquad \forall\, (x', y') \in \mathcal{W}.$$

If $v \in \mathrm{I}(x)$ then for y' close to $\exp_x v$ and x' close to x we have

$$\widehat{c}_{(x,v)}(x', y') = c(x', y') = d(x', y')^2/2.$$

Let $x \in M$, $v \in \mathrm{NF}(x)$, and $(\xi, \eta) \in T_xM \times T_xM$. Following [13], the *extended* MTW *tensor* at (x, v), evaluated on (ξ, η), is defined as

$$\overline{\mathfrak{S}}_{(x,v)}(\xi, \eta) = -\frac{3}{2} \left.\frac{d^2}{ds^2}\right|_{s=0} \left.\frac{d^2}{dt^2}\right|_{t=0} \widehat{c}_{(x,v)}\Big(\exp_x(t\xi),\, \exp_x(v + s\eta)\Big).$$

It is said that (M, g) satisfies the *extended Ma–Trudinger–Wang condition* $(\overline{\mathbf{MTW}})$ if

$$\text{(2.1)} \quad \forall\,(x, v) \in TM \text{ with } v \in \mathrm{NF}(x),\ \forall\,(\xi, \eta) \in T_xM \times T_xM, \\ \Big[\langle \xi, \eta \rangle_x = 0 \implies \quad \overline{\mathfrak{S}}_{(x,y)}(\xi, \eta) \geq 0\Big].$$

As before, if the last inequality in (2.1) is strict unless $\xi = 0$ or $\eta = 0$, then M is said to satisfy the *extended strict Ma–Trudinger–Wang condition* $(\overline{\mathbf{MTW}}^+)$. Note that $(\overline{\mathbf{MTW}})$ implies $(\mathbf{MTW})$ and $(\overline{\mathbf{MTW}}^+)$ implies $(\mathbf{MTW}^+)$. The two following results follow from Theorems 1.2 and 1.3, see Appendix C.

Corollary 2.1. *Let (M, g) be a smooth compact connected Riemannian manifold of dimension $n \geq 2$. If M satisfies* **(SCNF)** *and* $(\overline{\mathbf{MTW}}^+)$*, then $\mathcal{TCP}$ holds.*

Corollary 2.2. *Let (M, g) be a smooth compact Riemannian surface. If M satisfies* **(CNF)** *and* $(\overline{\mathbf{MTW}})$*, then $\mathcal{TCP}$ holds.*

2.2. **The MTW condition without orthogonality.** We say that (M, g) satisfies $(\mathbf{MTW}^{\not\perp})$ if (1.1) holds without any orthogonality assumption, that is,

$$\text{(2.2)} \ \forall\,(x, v) \in TM \text{ with } v \in \mathrm{I}(x),\ \forall\,(\xi, \eta) \in T_xM \times T_xM, \quad \mathfrak{S}_{(x,y)}(\xi, \eta) \geq 0.$$

Let μ, ν be two σ-finite non-negative measures with positive bounded densities on a connected Riemannian manifold (M, g), and let $\psi : M \to \mathbb{R}$ be a locally semiconvex c-convex function such that the map $T : M \to M$ given by

$$T(x) = \exp_x\big(\nabla\psi(x)\big) \qquad \text{a.e. } x \in M.$$

satisfies $T_\#\mu = \nu$ (observe that in this setting we cannot talk about optimal maps, since μ may have infinite mass, and so $\int_M d(x, T(x))^2\, d\mu(x)$ may be infinite). Denote by ψ^c the c-transform of ψ and recall that

$$\partial_c\psi(x) = \{y \in M;\, \psi(x) + \psi^c(y) + c(x, y) = 0\}\,.$$

Define the contact set of $y \in M$ as

$$S(y) = \{x \in M;\ y \in \partial_c \psi(x)\} = \partial^c \psi^c(y).$$

As it will be seen in Section 3.2 (see in particular (3.15)), if (M,g) satisfies **(MTW)**, then the equality $\exp_y^{-1}(\partial^c \psi^c(y)) = \nabla^- \psi^c(y)$ holds for any $y \in M$. In particular, $\exp_y^{-1}(S(y)) \subset \overline{\mathrm{I}(y)}$ is always a convex set. The following theorem, already present in the proof of [11, Corollary 5.2], is a simple consequence of the results in [10].

Theorem 2.3. *Assume that (M,g) satisfies* **(CI)** *and* **(MTW$^{\not\perp}$)***, and suppose that there exist two positive constants $\lambda, \Lambda > 0$ such that*

$$\lambda \mathrm{vol} \le \mu \le \Lambda \mathrm{vol}\,, \qquad \lambda \mathrm{vol} \le \nu \le \Lambda \mathrm{vol}\,.$$

Then, for any $y \in M$, either $S(y)$ is a singleton or all exposed points of $\exp_y^{-1}(S(y))$ belong to $\mathrm{TCL}(y)$.

Let us recall that, according to [22], a manifold is said to have nonfocal cut locus if $\mathrm{TFCL}(x) = \emptyset$ for all $x \in M$. As an immediate corollary of the above theorem and [6, Appendix C, Theorem 3] we obtain the following result, which extends the regularity result on the flat torus $\mathbb{T}^n$ [5] and the real projective space to compact quotient of $\mathbb{S}^{n_1}_{r_1} \times \ldots \times \mathbb{S}^{n_i}_{r_i} \times \mathbb{R}^n$ with nonfocal cut locus, like for instance $M = \mathbb{RP}^{n_1}_{r_1} \times \ldots \times \mathbb{RP}^{n_i}_{r_i} \times \mathbb{T}^n$. (Let us however point out that, since [6, Appendix C, Theorem 3] is proven in the compact case, we need to slightly modify its proof in order to deal with the fact that the mass of our measures in not necessarily finite.)

Corollary 2.4. *Let (M,g) be a compact quotient of $\mathbb{S}^{n_1}_{r_1} \times \ldots \times \mathbb{S}^{n_i}_{r_i} \times \mathbb{R}^n$ with nonfocal cut locus. Then $\mathcal{TCP}$ holds. Moreover, if $\mu = f\mathrm{vol}$ and $\nu = g\mathrm{vol}$ with $f, g > 0$ and of class C^∞, then the optimal transport is C^∞ too.*

2.3. **Further comments.** Thanks to existing results in the literature and some of the above results, we can list all the known examples of compact Riemannian manifolds satisfying $\mathcal{TCP}$ (to our knowledge, the list below is exhaustive):

- Flat tori in any dimension [5].
- Round spheres in any dimension [21]
- Small C^4 deformations of round spheres in any dimension [13, 14].
- Riemannian submersions of round spheres [19].
- Products of round spheres [11].
- Quotients of all the above examples by a discrete group of isometry [6].
- Compact quotients of products of spheres and Euclidean spaces with nonfocal cut locus (Corollary 2.4).
- Compact Riemannian surfaces satisfying **(CI)** and **(MTW)** (Theorem 1.3).

By Theorem 1.2(i), any Riemannian manifold verifying $\mathcal{TCP}$ must satisfy **(CI)** and **(MTW)**. As shown by Theorem 1.3, the combination **(CI)**-**(MTW)** and $\mathcal{TCP}$ are equivalent on surfaces. We do not know if such a result holds in higher dimension.

In [7] the authors showed that small C^4 perturbations of $\mathbb{S}^2$ equipped with the round metric satisfy $(\mathbf{MTW}^{\not\perp})$. Thanks to [19, Theorem 1.2(3)], this implies that any product of them satisfy $(\mathbf{MTW})^{\not\perp}$. We dot not know if such Riemannian products satisfy $\mathcal{TCP}$.

Finally, we point out that to our knowledge there is no concrete example of a Riemannian manifold satisfying **(MTW)** but not $(\mathbf{MTW}^{\not\perp})$.

3. Proofs of Theorems 1.2 and 1.3

3.1. **Necessary conditions for $\mathcal{TCP}$.** We want to prove that Theorem 1.2(i) holds. Actually, we will show a slightly stronger result: **(CI)** is satisfied provided the cost function satisfies **Assumption (C)** (this condition first appeared in [26, page 205]):

Assumption (C): *For any c-convex function ψ and any $x \in M$, the c-subdifferential $\partial_c\psi(x)$ is pathwise connected.*

As shown in [26, Theorem 12.7], $\mathcal{TCP}$ implies **Assumption (C)**. In addition, by [26, Theorem 12.42], **Assumption (C)** and **(CI)** imply **(MTW)**. Therefore, Theorem 1.2 is a straightforward consequence of the following result:

Proposition 3.1. *Let (M,g) be a smooth compact connected Riemannian manifold of dimension $n \geq 2$ satisfying* **Assumption (C)**. *Then* **(CI)** *holds.*

Proof of Proposition 3.1. Assume by contradiction that $\mathrm{I}(\bar{x})$ (or equivalently $\overline{\mathrm{I}(\bar{x})}$) is not convex for some $\bar{x} \in M$. We need the following result, whose proof is postponed to the end:

Lemma 3.2. *Assume $\mathrm{I}(\bar{x})$ is not convex. Then there are $v_{-1}, v_1 \in \mathrm{I}(\bar{x})$ such that:*

- *$v_0 = \frac{v_{-1}+v_1}{2}$ does not belong to $\overline{\mathrm{I}(\bar{x})}$;*
- *the mapping $v \in [v_{-1}, v_1] \cap \overline{\mathrm{I}(\bar{x})} \mapsto \exp_{\bar{x}}(v) \in M$ is injective.*

Set $y_i = \exp_{\bar{x}}(v_i)$ for $i = -1, 1, 0$ (with v_{-1}, v_1, v_0 as in the above lemma), and let $\psi : M \to \mathbb{R}$ be the c-convex function defined by

$$\psi(x) = \max\Big\{c(\bar{x}, y_{-1}) - c(x, y_{-1}),\, c(\bar{x}, y_1) - c(x, y_1)\Big\} \qquad \forall\, x \in M.$$

The set of subgradients of ψ at $\bar{x}$ is given by the segment

$$\nabla^-\psi(\bar{x}) = [v_{-1}, v_1] \subset T_{\bar{x}}M,$$

which by [26, Theorem 10.25] means that

$$-\nabla^+_{\bar{x}} c(\bar{x}, y) \subset [v_{-1}, v_1] \qquad \forall\, y \in \partial_c\psi(\bar{x}),$$

where $\nabla^+_{\bar{x}} c(\bar{x}, y)$ denotes the set of supergradients of the semiconcave function $x \mapsto c(x, y)$ at $\bar{x}$. Since both v_{-1} and v_1 belong to the injectivity domain $\mathrm{I}(\bar{x})$, we have that $c(\bar{x}, \cdot)$ is differentiable at both y_{-1} and y_1, and $\nabla_{\bar{x}} c(\bar{x}, y_i) = -v_i$ for $i = -1, 1$. Moreover we observe that the mapping

$$F : y \in \partial_c\psi(\bar{x}) \longmapsto -\nabla^+_{\bar{x}} c(\bar{x}, y) \subset [v_{-1}, v_1]$$

is convex-valued and upper semicontinuous. Thanks to **Assumption (C)**, there exists a continuous curve $t \in [0, 1] \to y(t) \in M$ such that the mapping $F \circ y$ is convex-valued, upper semicontinuous, and satisfies $(F \circ y)(0) = v_{-1}$ and $(F \circ y)(1) = v_1$. By [1, Theorem 9.2.1], for every $\epsilon > 0$ we can find a Lipschitz function $f_\epsilon : [0, 1] \to [v_{-1}, v_1]$ such that

$$\mathrm{Graph}(f_\epsilon) \subset \mathrm{Graph}\big(F \circ y\big) + \epsilon B = \big\{(t, v);\, t \in [0, 1], v \in \big(F \circ y\big)(t)\big\} + \epsilon B,$$

in $[0, 1] \times [v_{-1}, v_1]$ (here B denotes the open unit ball in $\mathbb{R}^2$). By compactness and Lemma 3.2, this implies that there exists $y \in \partial_c\psi(\bar{x})$ such that

$$v_{1/2} \in -\nabla^+_{\bar{x}} c(\bar{x}, y) \setminus \overline{\mathrm{I}(\bar{x})}.$$

The set $-\nabla^+_{\bar{x}} c(\bar{x}, y)$ is the convex hull of the minimizing speeds joining $\bar{x}$ to y. Thus there are two minimizing speeds $v \neq v' \in [v_{-1}, v_1] \cap \overline{\mathrm{I}(\bar{x})}$ joining $\bar{x}$ to y, and a constant $\lambda \in (0, 1)$, such that $v_{1/2} = \lambda v + (1 - \lambda)v'$. This contradicts the fact that $v \in [v_{-1}, v_1] \cap \overline{\mathrm{I}(\bar{x})} \mapsto \exp_{\bar{x}}(v)$ is injective, and proves that (M, g) satisfies **(CI)**. □

It remains to prove Lemma 3.2.

Proof of Lemma 3.2. Without loss of generality we may assume that $g_{\bar{x}} = \mathrm{Id}_{\mathbb{R}^n}$. Therefore, it is sufficient to show that there are $v_{-1}, v_1 \in \mathrm{I}(\bar{x})$ such that $v_0 = \frac{v_{-1}+v_1}{2}$ does not belong to $\mathrm{I}(\bar{x})$ and such that the mapping

$$v \in [v_{-1}, v_1] \cap \overline{\mathrm{I}(\bar{x})} \mapsto |v| \in \mathbb{R}$$

is injective. Denote by $\tau_C : U_{\bar{x}}M \to (0, \infty)$ the restriction of the cut time $t_C(\bar{x}, \cdot)$ (see Appendix A) to the unit sphere $U_{\bar{x}}M \subset T_{\bar{x}}M$. Since $\overline{\mathrm{I}(\bar{x})}$ is not convex, there are $w_0, w_1 \in \mathrm{TCL}(\bar{x})$ and $\bar{t} \in (0, 1)$ such that $w_{\bar{t}} = (1 - \bar{t})v_{-1} + \bar{t}v_1$ does not belong to $\overline{\mathrm{I}(\bar{x})}$.

We claim that we may assume that τ_C is differentiable at $\widehat{w}_0 = w_0/|w_0|$, and that the vector $w_1 - w_0$ satisfies

$$\langle w_1 - w_0, \xi(w_0)\rangle > 0,$$

where $\xi(w_0)$ denotes the exterior unit normal to the $\mathrm{TCL}(\bar{x})$ at w_0. (Note that, once τ_C is assumed to be differentiable at $\widehat{w}_0$, then $\xi(w_0)$ exists and is unique since $\mathrm{TCL}(\bar{x})$ is given by the image of the map $\widehat{w} \in U_{\bar{x}}M \mapsto \tau_C(\widehat{w})\widehat{w} \in T_{\bar{x}}M$.)

Indeed, if not, for every $\widehat{w} \in U_{\bar{x}}M$ such that τ_C is differentiable at $\widehat{w}$ we would have

$$\overline{\mathrm{I}(\bar{x})} \subset \mathcal{S}_{\xi(w)},$$

where $\mathcal{S}_{\xi(w)} \subset T_{\bar{x}}M$ denotes the closed affine halfspace associated with the normal $\xi(w)$ at $w = \tau_C(\widehat{w})\widehat{w}$, that is,

$$\mathcal{S}_{\xi(w)} = \{w + h;\ \langle h, \xi(w)\rangle \leq 0\}.$$

However, since τ_C is Lipschitz and so differentiable a.e., we would obtain

$$\overline{\mathrm{I}(\bar{x})} \subset \overline{\mathcal{S}_{\xi(w)}} \qquad \forall\, w \in \mathrm{TCL}(\bar{x}), \tag{3.1}$$

which easily implies that $\mathrm{I}(\bar{x})$ is convex, absurd.

Now, let $w_0, w_1 \in \mathrm{TCL}(\bar{x})$ and $\bar{t} \in (0,1)$ be such that τ_C is differentiable at $\widehat{w}_0 = w_0/|w_0|$, $w_{\bar{t}} = (1-\bar{t})w_0 + \bar{t}w_1$ does not belong to $\overline{\mathrm{I}(\bar{x})}$, and $\langle w_1 - w_0, \xi(w_0)\rangle > 0$. This means that there is a maximal $\widehat{t} \in (0,1]$ such that $w_t \notin \overline{\mathrm{I}(\bar{x})}$ for all $t \in (0, \widehat{t})$. Since $\langle w_1 - w_0, \xi(w_0)\rangle > 0$ and $w_t \notin \overline{\mathrm{I}(\bar{x})}$ for small positive times, there exists $\epsilon_0 > 0$ small enough such that $(1+\epsilon)w_0 - \epsilon w_1$ belongs to $\mathrm{I}(\bar{x})$ and

$$\big[(1+\epsilon)w_0 - \epsilon w_1, w_{\widehat{t}}\big] \cap \overline{\mathrm{I}(\bar{x})} = \{w_0, w_{\widehat{t}}\}.$$

for all $\epsilon \leq \epsilon_0$. Hence, if we choose $\epsilon, \epsilon' \leq \epsilon_0$ such that $|(1+\epsilon)w_0 - \epsilon w_1| \neq |(1-\epsilon')w_{\widehat{t}}|$, then $v_{-1} = (1+\epsilon)w_0 - \epsilon w_1$ and $v_1 = (1-\epsilon')w_{\widehat{t}}$ satisfy the desired assumption. $\square$

Remark 3.3. If (M,g) satisfies **(CI)**, then **(MTW)** is equivalent to the fact that the cost $c = d^2/2$ is *regular* in the sense of [26, Definition 12.14], that is, for every $\bar{x} \in M$ and $v_0, v_1 \in \mathrm{I}(\bar{x})$ it holds

$$v_t = (1-t)v_0 + tv_1 \in \mathrm{I}(\bar{x}) \qquad \forall\, t \in [0,1], \tag{3.2}$$

and

$$c(x, y_t) - c(\bar{x}, y_t) \geq \min\Big(c(x, y_0) - c(\bar{x}, y_0),\, c(x, y_1) - c(\bar{x}, y_1)\Big), \tag{3.3}$$

for any $x \in M$, where $y_t = \exp_{\bar{x}}(v_t)$ for any $t \in [0,1]$. Such a result appeared with an incomplete proof in [26, Proof of Theorem 12.36], but it is an easy consequence of the argument given there combined with Lemma 3.5 below.

3.2. **Sufficient conditions for $\mathcal{TCP}$.** Theorem 1.2(ii) follows from the following finer result.

Proposition 3.4. *Assume that M satisfies* **(SCI)** *and* **(MTW$^+$)**. *Then, the optimal map from μ to ν is continuous whenever μ and ν satisfy:*

(i) $\lim_{r\to 0^+} \frac{\mu(B_r(x))}{r^{n-1}} = 0$ *for any* $x \in M$;

(ii) $\nu \geq c_0 \,\mathrm{vol}$ *for some constant* $c_0 > 0$.

Proof of Proposition 3.4. The proof of Proposition 3.4 is divided in three steps.

Step 1: we first show that under assumptions **(CI)** and **(MTW)**, the cost $c = d^2/2$ is "regular" (see Remark 3.3).

Lemma 3.5. *Let (M, g) be a Riemannian manifold satisfying* **(CI)**-**(MTW)**. *Fix $\bar{x} \in M$, $v_0, v_1 \in \overline{\mathrm{I}(\bar{x})}$, and let $v_t = (1-t)v_0 + tv_1 \in T_x M$. For any $t \in [0,1]$, set $y_t = \exp_x(v_t)$. Then, for any $x \in M$, for any $t \in [0,1]$,*

$$c(x, y_t) - c(\bar{x}, y_t) \geq \min\Big(c(x, y_0) - c(\bar{x}, y_0),\, c(x, y_1) - c(\bar{x}, y_1)\Big). \tag{3.4}$$

Proof of Lemma 3.5. Fix $\bar{x} \in M$ and $v_0, v_1 \in \overline{\mathrm{I}(\bar{x})}$. Note that, by continuity of c, it is sufficient to prove (3.4) with $v_0, v_1 \in \mathrm{I}(\bar{x})$. Let us fix $x \in M$ and define the function $h : [0,1] \to \mathbb{R}$ by

$$h(t) := -c(x, y_t) + c(\bar{x}, y_t) = -c(x, y_t) + \frac{1}{2}|v_t|^2_{\bar{x}} \qquad \forall\, t \in [0,1].$$

(Observe that $c(\bar{x}, y_t) = |v_t|^2_{\bar{x}}/2$, since $v_t \in I(\bar{x})$.) Our aim is to show that h can only achieve a maximum at $t = 0$ or $t = 1$. Assume that the curve $(y_t)_{0\leq t\leq 1}$ intersects $\mathrm{cut}(x)$ only a finite set of times $0 = t_0 < t_1 < \ldots < t_N < t_{N+1} = 1$, always intersects $\mathrm{cut}(x)$ transversally, and never intersects $\mathrm{fcut}(x) = \exp_x(\mathrm{TFCL}(x))$. This implies that h is smooth on the intervals (t_j, t_{j+1}) for $j = 0, \ldots, N$ and is never differentiable at $t = t_j$ for $j = 1, \ldots, N$. By semiconvexity of the function $t \mapsto -c(x, y_t)$ necessarily $\dot{h}(t_j^+) > \dot{h}(t_j^-)$, and so $h(t)$ cannot achieve a local maximum in a neighborhood of the points $t_1, \ldots, t_N$. In particular, there exists $\eta > 0$ such that $h(t)$ cannot achieve its maximum in any interval of the form $[t_j - 2\eta, t_j + 2\eta]$ with $j \in \{1, \ldots, N\}$. Let us show that h cannot have a maximum in any of the intervals $I_0 := (t_0, t_1 - \eta)$, $I_N := (t_N + \eta, t_{N+1})$, and $I_j := (t_j + \eta, t_{j+1} - \eta)$ with $j \in \{1, \ldots, N-1\}$.

Let $j \in \{0, \ldots, N\}$ be fixed. The function $y \mapsto c(x, y)$ is smooth in a neighborhood of the curve $(y_t)_{t\in I_j}$, so $q_t := -\nabla_y c(x, y_t)$ is well-defined for every $t \in I_j$. Set, for every $t \in [0,1]$, $\bar{q}_t := -\nabla_y c(\bar{x}, y_t)$. Since v_0, v_1 belongs to $\mathrm{I}(\bar{x})$ which is an open convex set (by assumption **(CI)**), the tangent vector $\bar{q}_t$ always belong to $\mathrm{I}(y_t)$ and does not intersect $\mathrm{TFL}(y_t)$. Then, arguing as in [26, Proof of Theorem 12.36], there exists a constant $C > 0$ (depensing on η) such that

$$\mathfrak{S}_{(y_t, sq_t + (1-s)\overline{q}_t)}(\dot{y}_t, q_t - \overline{q}_t) \geq -C\, |\langle \dot{y}_t, q_t - \overline{q}_t \rangle| \qquad \forall\, t \in I_j,\ \forall s \in [0,1].$$

Since (see [26, Proof of Theorem 12.36] or [13, Proof of Lemma 3.3])

$$\begin{cases} \dot{h}(t) &= \langle \dot{y}_t, q_t - \overline{q}_t \rangle \\ \ddot{h}(t) &= \frac{2}{3} \int_0^1 (1-s) \mathfrak{S}_{(y_t, sq_t + (1-s)\overline{q}_t)}(\dot{y}_t, q_t - \overline{q}_t)\, ds \end{cases}$$

for any $t \in I_j$, we get

$$\ddot{h}(t) \geq -C|\dot{h}(t)| \qquad \forall\, t \in I_j. \tag{3.5}$$

Now, as in [26, Proof of Theorem 12.36] we consider the functions $h_\varepsilon(t) = h(t) + \varepsilon(t - 1/2)^k$, with k large enough (which will be chosen below). If by contradiction h_ε attains a maximum at a time $t_0 \in I_j$ for some j, then at t_0 we get $\dot{h}_\varepsilon(t_0) = 0$ and $\ddot{h}_\varepsilon(t_0) \leq 0$, which gives

$$\dot{h}(t_0) = -\varepsilon k (t_0 - 1/2)^{k-1}, \qquad \ddot{h}(t_0) \leq -\varepsilon k(k-1)(t_0 - 1/2)^{k-2}.$$

This contradicts (3.5) for $k \geq 1 + C/2$. Moreover, since h_ε converges to h uniformly on $[0,1]$ as $\varepsilon \to 0$, for ε sufficiently small the function h_ε cannot achieve its maximum on any interval of the form I_j. This implies that $h_\varepsilon(t) \leq \max\{h_\varepsilon(0), h_\varepsilon(1)\}$ for ε small, and letting $\varepsilon \to 0$ we get (3.4).

Finally, thanks to Lemma B.2, we observe that the assumption we did on the curve $(y_t)_{0 \leq t \leq 1}$ holds generically. So (as in [17]) the result follows immediately by approximation. □

As a consequence of the above result, if $\psi : M \to \mathbb{R}$ is a c-convex function, then its c-subdifferential is always pathwise connected. As a matter of fact, if y_0, y_1 both belong to $\partial_c \psi(\bar{x})$, then there are $v_0, v_1 \in \overline{\mathrm{I}(\bar{x})}$ such that $y_0 = \exp_{\bar{x}}(v_0)$, $y_1 = \exp_{\bar{x}}(v_1)$, and

$$\psi(\bar{x}) + c(\bar{x}, y_i) = \min_{x \in M} \Big\{ \psi(x) + c(x, y_i) \Big\} \qquad \forall\, i = 0, 1.$$

The latter property can be written as

$$c(x, y_i) - c(\bar{x}, y_i) \geq \psi(\bar{x}) - \psi(x) \qquad \forall\, x \in M,\ i = 0, 1,$$

which, thanks to Lemma 3.5, implies

$$c(x, y_t) - c(\bar{x}, y_t) \geq \min\Big(c(x, y_0) - c(\bar{x}, y_0),\ c(x, y_1) - c(\bar{x}, y_1) \Big) \geq \psi(\bar{x}) - \psi(x),$$

for every $x \in M$, where $y_t = \exp_x((1-t)v_0 + tv_1)$, $t \in [0,1]$. This shows that the path $t \mapsto y_t$ belong to $\partial_c\psi(x)$, as desired.

Step 2: We strengthen locally the previous result in a quantitative way.

Lemma 3.6. *Let (M,g) be a Riemannian manifold satisfying* **(MTW)**, *$\bar{x} \in M$ be fixed, A be a compact subset of $I(\bar{x})$, and $\mathcal{B} \in M$ be a compact set containing $\bar{x}$ such that the convex envelope of $\exp_y^{-1}(\mathcal{B})$ in $T_{\bar{x}}M$ satisfies*

$$\mathrm{conv}\Big(\exp_y^{-1}(\mathcal{B})\Big) \subset \mathrm{I}(y) \qquad \forall\, y \in \exp_{\bar{x}}(A). \tag{3.6}$$

Assume that there are $K, C > 0$ such that, for any $y \in \exp_{\bar{x}}(A)$ and any $x \in \exp_y\Big(conv\big(\exp_y^{-1}(\mathcal{B})\big)\Big)$, there holds

$$\forall\,(\xi,\eta) \in T_yM \times T_yM, \quad \mathfrak{S}_{(y,x)}(\xi,\eta) \geq K|\xi|_x^2|\eta|_y^2 - C|\langle\xi,\eta\rangle_y||\xi|_x|\eta|_y. \tag{3.7}$$

Furthermore, fix $f \in C_c^\infty([0,1])$ with $f \geq 0$ and $\{f > 0\} = (1/4, 3/4)$. Then there exists $\lambda = \lambda(K,C,f) > 0$ such that for any $x \in \mathcal{B}$ and any C^2 curve $(v_t)_{0\leq t\leq 1}$ drawn in $\mathrm{I}(\bar{x})$ satisfying

$$\begin{cases} |\ddot{v}_t|_{\bar{x}} = 0 \text{ for } t \in [0,1/4] \cup [3/4,1], \\ |\ddot{v}_t|_{\bar{x}} \leq \frac{K}{8} d(\bar{x},x)|\dot{y}_t|_{y_t}^2 \text{ for } t \in [1/4,3/4], \\ v_t \in A \text{ for } t \in [1/4,3/4], \end{cases} \tag{3.8}$$

where $y_t = \exp_x(v_t)$, there holds for any $t \in [0,1]$

$$\begin{aligned} &c(x,y_t) - c(\bar{x},y) \\ &\qquad \geq \min\Big(c(x,y_0) - c(\bar{x},y_0),\ c(x,y_1) - c(\bar{x},y_1)\Big) + \lambda f(t) c(\bar{x},x). \end{aligned} \tag{3.9}$$

Proof of Lemma 3.6. We adapt the argument in Lemma 3.5, borrowing the strategy from [13, 22]: first of all, using Lemma B.2 (as we did in the proof of Lemma 3.5), up to slightly perturbing v_0 and v_1 we can assume that $v_0, v_1 \in \mathrm{I}(\bar{x})$, $(y_t)_{0\leq t\leq 1}$ intersects $\mathrm{cut}(x)$ only at a finite set of times $0 = t_0 < t_1 < \ldots < t_N < t_{N+1} = 1$, and moreover $(y_t)_{0\leq t\leq 1}$ never intersects $\mathrm{fcut}(x)$. Using the notations of Lemma 3.5, we consider the function $h : [0,1] \to \mathbb{R}$ given by

$$h(t) = -c(x,y_t) + c(\bar{x},y_t) + \lambda d(\bar{x},x)^2 f(t)/2 \qquad \forall\, t \in [0,1],$$

where $\lambda > 0$ is a positive constant to be chosen.

On the one hand, since $f(t) = 0$ and v_t is a segment for $t \in [0,1/4] \cup [3/4,1]$, arguing as in the proof of Lemma 3.5 we have that $h|_{[0,1/4]\cup[3/4,1]}$ achieve its maximum at $0, 1/4, 3/4$ or 1.

On the other hand, if $t \in [1/4, 3/4] \cap (t_j, t_{j+1})$ for some $j = 0, \dots, N$, we can argue as in the proof of [22, Theorem 3.1] (see also [13, Lemma 3.3]) to check that the identity $\dot{h}(t) = 0$ gives $\ddot{h}(t) > 0$ for $\lambda = \lambda(K, C, f)$ sufficiently small. This implies that the function h cannot have any maximum on any interval $(t_j, t_{j+1}) \cap [1/4, 3/4]$. Moreover, since $\dot{h}(t_j^+) > \dot{h}(t_j^-)$ for $j = 1, \dots, N$, h cannot achieve its maximum at any of the points t_j, $j = 1, \dots, N$.

Hence h necessarily achieves its maximum at 0 or 1, and we obtain (3.9). □

Step 3: We conclude the proof of Proposition 3.4 arguing as in [13, Theorem 3.6] and [22, Theorem 5.1].

Assumption (i) above ensures that μ gives no mass to sets with finite $(n-1)$-dimensional Hausdorff measure. Hence by McCann's Theorem [24], there exists a unique optimal transport map between μ and ν, given by $T(x) = \exp_x\big(\nabla_x \psi\big)$ where ψ is a semiconvex function. Moreover, as shown by McCann, $\nabla_x \psi \in \overline{\mathrm{I}(x)}$ at all point of differentiability of ψ. By assumption the sets $\mathrm{I}(x)$ are (strictly) convex for all x, therefore the subdifferential of ψ satisfies $\nabla^- \psi(x) \subset \overline{\mathrm{I}(x)}$ for all $x \in M$. We want to prove that ψ is C^1. To this aim, we need to show that $\nabla^- \psi(x)$ is everywhere a singleton. The proof is by contradiction.

Assume that there is $\bar{x} \in M$ such that $v_0 \neq v_1 \in \nabla^- \psi(\bar{x})$. Let $y_0 = \exp_{\bar{x}}(v_0)$, $y_1 = \exp_{\bar{x}}(v_1)$. Then $y_i \in \partial_c \psi(\bar{x})$, i.e.

$$\psi(\bar{x}) + c(\bar{x}, y_i) = \min_{x \in M} \{\psi(x) + c(x, y_i)\}, \qquad i = 0, 1.$$

In particular

$$c(x, y_i) - c(\bar{x}, y_i) \geq \psi(\bar{x}) - \psi(x), \qquad \forall\, x \in M,\ i = 0, 1. \tag{3.10}$$

For every $\delta > 0$ small, denote by $A_\delta \subset T_{\bar{x}}M$ the set swept by all the C^2 curves $t \in [0,1] \mapsto v_t \in T_{\bar{x}}M$ which satisfy $v_t = (1-t)v_0 + t v_1$ for any $t \in [0, 1/4] \cup [3/4, 1]$ and

$$|\ddot{v}_t|_{\bar{x}} \leq \delta |\dot{y}_t|_{y_t}^2 \quad \text{for } t \in [1/4, 3/4], \tag{3.11}$$

where $y_t = \exp_{\bar{x}}(v_t)$. Since the set $\overline{\mathrm{I}(\bar{x})}$ is strictly convex, the segment $I = [v_{1/4}, v_{3/4}]$ lies a positive distance away from $\mathrm{TCL}(\bar{x})$, and for every $y \in \exp_{\bar{x}}(I)$, the point $\bar{x}$ is at positive distance from $\mathrm{TCL}(y)$. Therefore, there is $\bar{\delta} > 0$ small enough such that $A_{\bar{\delta}} \subset I(\bar{x})$ and

$$\mathrm{conv}\Big(\exp_y^{-1}\big(\bar{B}_{\bar{\delta}}(\bar{x})\big)\Big) \subset \mathrm{I}(y) \qquad \forall\, y \in \exp_{\bar{x}}\big(A_{\bar{\delta}}\big). \tag{3.12}$$

By construction, any set A_δ (with $\delta \in (0,\bar\delta)$, $\bar\delta$ small) contains a parallelepiped E_δ centered at $v_{1/2} = \frac{v_0+v_1}{2}$, with one side of length $\sim |v_0 - v_1|_{\bar x}$ and the other sides of length $\sim \delta |v_0 - v_1|_{\bar x}^2$, such that all points v in such a parallelepiped can be written as v_t for some $t \in [3/8, 5/8]$. Therefore, there is $c > 0$ such that $\mathcal{L}^n(E_\delta) \geq c\delta^{n-1}$, where $\mathcal{L}^n$ denotes the Lebesgue measure on $T_{\bar x}M$. Since E_δ lies a positive distance from $\mathrm{TCL}(\bar x)$, we obtain

$$\mathrm{vol}\,(Y_\delta) \sim \mathcal{L}^n(E_\delta) \geq c\delta^{n-1}, \qquad \text{where} \qquad Y_\delta := \exp_{\bar x}(E_\delta). \tag{3.13}$$

On the other hand, by (3.12) the cost c is smooth on $\bar B_{\bar\delta}(\bar x) \times \exp_{\bar x}\big(A_{\bar\delta}\big)$ and moreover **(MTW$^+$)** holds. Hence, arguing as in [22, Lemma 2.3] we deduce that there are $K, C > 0$ such that the following property holds for any $y \in \exp_{\bar x}\big(A_{\bar\delta}\big)$ and any $x \in \exp_y\big(\mathrm{conv}\big(\exp_y^{-1}(\bar B_{\bar\delta})\big)\big)$, where $\mathrm{conv}(S)$ denotes the convex envelope of a set S:

$$\forall\,(\xi,\eta) \in T_yM \times T_yM, \quad \mathfrak{S}_{(y,x)}(\xi,\eta) \geq K|\xi|_x^2|\eta|_y^2 - C\big|\langle \xi,\eta\rangle_y\big||\xi|_x|\eta|_y.$$

By Lemma 3.6, we deduce that for any $\delta \in (0,\bar\delta)$, any $y \in Y_\delta$, and any $x \in B_{\bar\delta}(\bar x) \setminus B_{8\delta/K}(\bar x)$, there holds

$$c(x,y) - c(\bar x,y) \geq \min\Big(c(x,y_0) - c(\bar x,y_0),\, c(x,y_1) - c(\bar x,y_1)\Big) + 2\lambda m_f d(\bar x,x)^2,$$

where $m_f = \inf\{f(t);\, t \in [3/8,5/8]\} > 0$ and $\lambda = \lambda(K,C,f) > 0$. Combining this inequality with (3.10), we conclude that for any $\delta \in (0,\bar\delta)$,

$$\forall\, y \in Y_\delta,\ \forall\, x \in B_{\bar\delta}(\bar x) \setminus B_{8\delta/K}(\bar x), \qquad y \notin \partial_c\psi(x). \tag{3.14}$$

We claim that taking $\delta \in (0,\bar\delta)$ small enough, we may assume that the above property holds for any $x \in M \setminus B_{8\delta/K}(\bar x)$. Indeed, if not, there exists a sequence $\{\delta_k\} \downarrow 0$, together with sequences $\{x_k\}$ in $M \setminus B_{\bar\delta}(\bar x)$ and $\{y_k\} \in Y_{\delta_k}$, such that $y_k \in \partial_c\psi(x_k)$ for any k. By compactness, we deduce the existence of $x \in M \setminus B_{\bar\delta}(\bar x)$ and $y_t \in \exp_{\bar x}([v_0,v_1])$, with $t \in [3/8,5/8]$, such that $y_t \in \partial_c\psi(x)$. This implies that the c-convex potential $\psi^c : M \to \mathbb{R}$ satisfies $\bar x, x \in \partial_c\psi^c(y_t)$. Moreover, (3.14) gives that $\partial_c\psi^c(y_t) \cap B_{\bar\delta}(\bar x) \setminus B_{8\delta/K}(\bar x) = \emptyset$. However, by to the discussion after Lemma 3.5, we know that the set $\partial_c\psi^c(y_t)$ is pathwise connected, absurd.

In conclusion we have proved that for $\delta \in (0,\bar\delta)$ small all the mass brought into Y_δ by the optimal map comes from $B_{8\delta/K}(\bar x)$, and so

$$\mu\big(B_{8\delta/K}(\bar x)\big) \geq \nu\big(Y_\delta\big).$$

Thus, as $\mu\big(B_{8\delta/K}(\bar x)\big) \leq o(1)\delta^{n-1}$ and $\nu(Y_\delta) \geq c_0\mathrm{vol}\,(Y_\delta) \geq c'\delta^{n-1}$ (by assumption (ii) and (3.13)), we obtain a contradiction as $\delta \to 0$. $\square$

3.3. **Proof of Theorem 1.3.** Thanks to Theorem 1.2(i), we only need to prove the "if" part. As in the proof of Proposition 3.4, let ψ be a c-convex function such that $T(x) = \exp_x\big(\nabla_x\psi\big)$. We want to prove that the subdifferential of ψ is a singleton everywhere.

We begin by observing that, thanks to Lemma 3.5 and [26, Proposition 12.15], we have

$$\nabla^-\psi(x) = \exp_x^{-1}\big(\partial_c\psi(x)\big) \qquad \forall\, x \in M. \tag{3.15}$$

First of all, we claim that the subdifferential of ψ at every point is at most one-dimensional. Indeed, if $\nabla^-\psi(x)$ is a two-dimensional convex C set for some $x \in M$, then by (3.15) $\exp_x(C) = \partial_c\psi(x)$ is a set with positive volume. But then, considering the optimal transport problem from ν to μ, the set $\partial_c\psi(x)$ is sent (by $\partial^c\psi^c$) onto the point x, which implies $\mu(\{x\}) \geq \nu(\partial_c\psi(x)) > 0$, impossible.

Now, assume by contradiction that ψ is not differentiable at some point x_0. Then there exist $v_{-1} \neq v_1 \in \overline{I(x_0)}$ such that $\nabla^-\psi(x_0) \subset \overline{I(x_0)}$ is equal to the segment $[v_{-1}, v_1] = \{v_t\}_{-1\leq t\leq 1}$, $v_t = \frac{1+t}{2}v_1 + \frac{1-t}{2}v_{-1}$. (In this proof, to simplify the notation, it is more convenient to use $[v_{-1}, v_1]$ to denote $\nabla^-\psi(x_0)$ instead of $[v_0, v_1]$.) We define $y_t = \exp_{x_0}(v_t)$.

We claim that that the following holds:

(A) $[v_{\varepsilon-1}, v_{1-\varepsilon}] \subset \mathrm{I}(x_0)$ for all $\varepsilon > 0$ (i.e. $v_t \notin \mathrm{TCL}(x_0)$ for all $t \in (-1,1)$).

Since the proof of the above result is pretty involved, we postpone it to the end of this subsection.

Now the strategy is the following: by (A) we know that the cost function $c = d^2/2$ is smooth in a neighborhood of $\{x_0\} \times \{y_t;\, t \in [-3/4, 3/4]\}$ and satisfies all the assumptions of [12, Lemma 3.1] (observe that, even if that result is stated on domains of $\mathbb{R}^n$, everything is local so it holds also on manifolds), and we can deduce that propagation of singularities hold. More precisely, [12, Lemma 3.1] gives the existence of a smooth injective curve $\gamma_{x_0} \ni x_0$ contained inside the set

$$\Gamma_{-3/4,3/4} = \left\{d(\cdot, y_{-3/4})^2 - d(\cdot, y_{3/4})^2 = d(x_0, y_{-3/4})^2 - d(x_0, y_{3/4})^2\right\},$$

such that

$$2\psi - 2\psi(x_0) = -d(\cdot, y_0)^2 + d(x_0, y_0)^2 \quad \text{on } \gamma_{x_0}. \tag{3.16}$$

(Recall that $c = d^2/2$.) Moreover, restricting γ_{x_0} if necessary, we can assume that $\gamma_{x_0} \cap \mathrm{cut}(y_0) = \emptyset$.

We now observe that, since $y_t \in \exp_{x_0}([v_{-1}, v_1]) = \partial_c\psi(x_0)$, we have

$$2\psi - 2\psi(x_0) \geq d(\cdot, y_t)^2 - d(x_0, y_t)^2 \qquad \forall\, t \in [-1, 1]. \tag{3.17}$$

Moreover, thanks to Lemma 3.5,

$$-d(\cdot,y_0)^2 + d(x_0,y_0)^2 \le \max\Big(-d(\cdot,y_t)^2 + d(x_0,y_t)^2, -d(\cdot,y_\tau)^2 + d(x_0,y_\tau)^2\Big) \tag{3.18}$$

for all $-1 \le t \le 0 \le \tau \le 1$. Hence, combining (3.16), (3.17), and (3.18), we deduce that

$$2\psi - 2\psi(x_0) = d(\cdot,y_t)^2 - d(x_0,y_t)^2 \quad \text{on } \gamma_{x_0} \qquad \forall\, t \in [-1,1]$$

and

$$\gamma_{x_0} \subset \bigcap_{-1<t<\tau<1} \Gamma_{t,\tau},$$

where

$$\Gamma_{t,\tau} = \left\{ d(\cdot,y_t)^2 - d(\cdot,y_\tau)^2 = d(x_0,y_t)^2 - d(x_0,y_\tau)^2 \right\}. \tag{3.19}$$

This implies that $y_t \in \partial_c\psi(x)$ for any $x \in \gamma_{x_0}$. Moreover, if we parameterize γ_{x_0} as $s \mapsto x_s$, by differentiating with respect to s the identity

$$d(x_s,y_t)^2 - d(x_s,y_0)^2 = d(x_0,y_t)^2 - d(x_0,y_0)^2$$

we obtain

$$\dot{x}_s \cdot \left[\nabla_x^+ d(x_s,y_t)^2 - \nabla_x d(x_s,y_0)^2\right] = 0$$

for all s,t (recall that $\gamma_{x_0} \cap \mathrm{cut}(y_0) = \emptyset$), that is, for any fixed s there exists a segment which contains all elements in the superdifferential of $d(\cdot,y_t)^2$ at x_s for all $t \in [-1,1]$. Thanks to the convexity of $I(x_s)$, we can apply (A) with x_0 replaced by x_s. Hence, we can repeat the argument above starting from any point x_s, and by a topological argument as in [12, Proof of Lemma 3.1] (showing that the maximal time interval on which we can extend the curve is both open and closed) we immediately get that γ_{x_0} is a simple curve which either is closed or has infinite length.

To summarize, we finally have the following geometric picture: there exists a smooth closed curve $\gamma \subset M$, which is either closed or has infinite length, such that:

(A-a) For any $x \in \gamma$, $y_t \notin \mathrm{cut}(x)$ for all $t \in (-1,1)$.
(A-b) $\gamma \subset \bigcap_{-1<t<\tau<1} \Gamma_{t,\tau}$, where $\Gamma_{t,\tau} = \{d(\cdot,y_t)^2 - d(\cdot,y_\tau)^2 = d(x_0,y_t)^2 - d(x_0,y_\tau)^2\}$.

Let us show that the compactness of M prevents this.

By differentiating with respect to t at $t=0$ the identity

$$d(x,y_t)^2 - d(x,y_0)^2 = d(x_0,y_t)^2 - d(x_0,y_0)^2 \qquad \forall\, x \in \gamma,$$

we obtain (using (A-a))

$$\left[\nabla_y d(x,y_0)^2 - \nabla_y d(x_0,y_0)^2\right] \cdot \dot{y}_0 = 0 \qquad \forall\, x \in \gamma.$$

Observe that, since $y_0 \notin \mathrm{cut}(x)$, $\dot{y}_0 \neq 0$. Hence there exists a segment $\Sigma \subset \overline{I(y_0)}$ such that $\exp_{y_0}^{-1}(\gamma) \subset \Sigma$. This is impossible since $\gamma \subset M \setminus \mathrm{cut}(y_0)$, so $\exp_{y_0}^{-1}(\gamma)$ is either closed or it has infinite length.

This concludes the proof of the C^1 regularity of ψ. It remains to show the validity of property (A) above.

Proof of (A). To prove (A), we distinguish two cases:

(i) Either v_{-1} or v_1 does not belong to $\mathrm{TCL}(x_0)$.
(ii) Both v_{-1} and v_1 belong to $\mathrm{TCL}(x_0)$.

In case (i), by the convexity of $\mathrm{I}(x)$ we immediately get that either $[v_{-1}, v_{1-\varepsilon}] \subset \mathrm{I}(x)$ or $[v_{\varepsilon-1}, v_1] \subset \mathrm{I}(x)$ for any $\varepsilon > 0$, so (A) holds.

In case (ii), assume by contradiction that (A) is false. By the convexity of $\mathrm{I}(x_0)$ we have $[v_{-1}, v_1] \subset \mathrm{TCL}(x_0)$. Let $\bar{t} \in [-1, 1]$ be such that $|v_{\bar{t}}|_{x_0}$ is minimal on $[v_{-1}, v_1]$ (by uniform convexity of the norm, there exists a unique such point). We consider two cases:

(ii-a) $\exp_{x_0}^{-1}(y_{\bar{t}})$ is not a singleton.
(ii-b) $\exp_{x_0}^{-1}(y_{\bar{t}}) = v_{\bar{t}}$.

In case (ii-a), since $|v_{\bar{t}}|_{x_0}$ is the unique vector of minimal norm on $[v_{-1}, v_1]$, there exists $\bar{v} \in \mathrm{TCL}(x_0) \setminus [v_{-1}, v_1]$ such that $\exp_{x_0}(\bar{v}) = \exp_{x_0}(v_{\bar{t}}) = y_{\bar{t}}$ ($\bar{v}$ cannot belongs to $[v_{-1}, v_1]$ since $|\bar{v}|_{x_0} = |v_{\bar{t}}|_{x_0}$). However this is impossible since (3.15) implies $\bar{v} \in \exp_{x_0}^{-1}\big(\partial_c \psi(x_0)\big) \subset \nabla^- \psi(x_0) = [v_{-1}, v_1]$.

In case (ii-b), without loss of generality we assume that the metric at x_0 concides with the identity matrix. Let us recall that by [15, Proposition A.6] the function $w \in U_{x_0}M \mapsto t_F(x_0, w)$ has vanishing derivative at all w such that $t_F(x, w)w \in \mathrm{TFCL}(x_0)$. Hence, since the derivative of $t \mapsto |v_t|_{x_0}$ is different from 0 at every $t \neq \bar{t}$ and $|v_t|_{x_0} \leq t_F(x_0, v_t)$ for every t, we deduce that $v_t \in \mathrm{TCL}(x_0) \setminus \mathrm{TFL}(x_0)$ for all $t \neq \bar{t}$. Let us choose any time $s \neq \bar{t}$. Since $v_s \in \mathrm{TCL}(x_0) \setminus \mathrm{TFL}(x_0)$, there exists a vector v' such that $\exp_{x_0}(v') = \exp_{x_0}(v_s) \in \partial_c \psi(x_0)$. By (3.15) this implies $v' \in \nabla^- \psi(x_0) = [v_{-1}, v_1]$. Hence there exists a time $s' \neq s$ such that $v' = v_{s'}$. Moreover, since $|v_s|_{x_0} = |v_{s'}|_{x_0} > |v_{\bar{t}}|_{x_0}$, we get $|s - \bar{t}| = |s' - \bar{t}|$. Thus, by the arbitrariness of $s \in [-1, 1] \setminus \{\bar{t}\}$ we easily deduce that the only possibility is $\bar{t} = 0$, and so $y_t = y_{-t}$ for all $t \in [-1, 1]$.

By doing a change of coordinates in a neighborhood of the minimizing geodesic γ_0 going from x_0 to y_0, we can assume that $x_0 = (0, 0), y_0 = (1, 0)$, $v_0 = (1, 0)$, $[v_{-1}, v_1] = \big[(1, -1), (1, 1)\big]$, that the metric g at x_0 and y_0 is the identity matric I_2, and that the geodesic starting from x_0 with initial velocity v_0 is given by $\gamma(t) = (t, 0)$. Now, to simplify the computation, we slightly change the definition of v_δ and y_δ for

$\delta > 0$ small (this should not create confusion, since we will adopt the following notation in all the sequel of the proof): denote by v_δ the speed which belongs to the segment $[v_{-1}, v_1]$ and whose angle with the horizontal axis is δ, that is

$$v_\delta = \big(1, \tan\delta\big), \quad \bar{v}_\delta := \frac{v_\delta}{\big|v_\delta\big|} = \big(\cos\delta, \sin\delta\big), \quad t_\delta := \big|v_\delta\big| = \frac{1}{\cos\delta}.$$

Consider γ_δ the geodesic starting from x_0 with initial velocity v_δ, and set

$$y_\delta := \exp_{x_0}(v_\delta), \quad w_\delta := -\dot{\gamma}_\delta(1), \quad \text{and} \quad \bar{w}_\delta := \frac{w_\delta}{\big|w_\delta\big|}.$$

The geodesic flow sends $\big(x_0, v_\delta\big)$ to $\big(y_\delta, -w_\delta\big)$, and the linearization at $\delta = 0$ gives

$$\dot{y}_0 = 0 \quad \text{and} \quad -\dot{w}_0 = \big(0, \dot{f}_0(1)\big), \tag{3.20}$$

where f_0 denotes the solution (starting with $f_0(0) = 0, \dot{f}_0(0) = 1$) to the Jacobi equation associated with the geodesic starting from x_0 with initial velocity v_0. The curve $\delta \mapsto y_\delta$ is a smooth curve valued in a neighborhood of y_0. Moreover, since $y_\delta = y_{-\delta}$ for any small δ, y_δ has the form

$$y_\delta = y_0 + \frac{\delta^2}{2} Y + o(\delta^2)$$

for some vector Y. We now observe that, for every $\delta > 0$, the vector $\dot{y}_\delta$ satisfies (because the distance function to x_0 is semiconcave and y_δ is contained in the cut locus of x_0)

$$\langle \dot{y}_\delta, w_\delta \rangle = \langle \dot{y}_\delta, w_{-\delta} \rangle,$$

which can be written as

$$\left\langle \frac{\dot{y}_\delta}{|y_\delta|}, w_\delta - w_{-\delta} \right\rangle = 0.$$

Thanks to (3.20), we deduce that y_δ takes the form

$$y_\delta = y_0 + \frac{\delta^2}{2}(\lambda, 0) + o(\delta^2) \tag{3.21}$$

for some $\lambda \geq 0$.

We now need some notation. For every nonzero tangent vector v at x_0, we denote by $f_0(\cdot, v), f_1(\cdot, v)$ the solutions to the Jacobi equation

$$\ddot{f}(t) + k(t) f(t) = 0 \qquad \forall\, t \geq 0, \tag{3.22}$$

along the geodesic starting from x_0 with unit initial velocity $v/|v|$ which satisfy

$$f_0(0, v) = 0, \quad \dot{f}_0(0, v) = 1, \quad f_1(0, v) = 1, \quad \dot{f}_1(0, v) = 0. \tag{3.23}$$

Set now

$$\bar{v}_\delta^\perp := \big(-\sin\delta, \cos\delta\big).$$

Since

$$\dot{v}_\delta = \left(0, \frac{1}{\cos^2\delta}\right) = \frac{\sin\delta}{\cos^2\delta}\,\bar{v}_\delta + \frac{1}{\cos\delta}\,\bar{v}_\delta^\perp,$$

we have

$$\dot{y}_\delta = \frac{\sin\delta}{\cos^2\delta}\,\big(-\bar{w}_\delta\big) + f_0\big(t_\delta, v_\delta\big)\frac{1}{\cos\delta}\,\big(-\bar{w}_\delta^\perp\big). \tag{3.24}$$

Then, since $\bar{w}_\delta = (-1,0) + O(\delta)$, this means that

$$\dot{y}_\delta = \delta(1,0) + O(\delta^2),$$

which implies that the constant λ appearing in (3.21) is equal to 1. Hence $\ddot{y}_0 = (1,0)$, and we get

$$y_\delta = y_0 + \frac{\delta^2}{2}(1,0) + O(\delta^4), \tag{3.25}$$

because $y_\delta = y_{-\delta}$.

Define the curve $\delta \mapsto z_\delta$ by

$$z_\delta := \exp_{x_0}(u_\delta) \quad \text{with} \quad u_\delta := \tau_\delta \bar{v}_\delta = \big(\tau_\delta\cos\delta, \tau_\delta\sin\delta\big).$$

We now use a result from [15]: since **(MTW)** holds, then the curvature of $\mathrm{TFL}(x_0)$ near any point of $\mathrm{TFCL}(x_0)$ has to be nonnegative, see [15, Proposition 4.1(ii)]. Since $[v_{-1}, v_1] \subset \mathrm{I}(x_0) \subset \mathrm{NF}(x_0)$ and $v_0 \in \mathrm{TFCL}(x_0)$, this implies that $\tau_\delta - t_\delta = O(\delta^4)$, which also gives

$$\big|y_\delta - z_\delta\big| = O(\delta^4). \tag{3.26}$$

Denote by $\bar{a}_\delta$ the (unit) vector at time $t = \tau_\delta$ of the geodesic starting at x_0 with initial velocity $\bar{v}_\delta$. As for $\dot{y}_\delta$, we can express $\dot{z}_\delta$ in terms of $a_\delta, a_\delta^\perp$, and $f_0\big(\tau_\delta, v_\delta\big) = 0$. For that, we note that

$$\begin{aligned}
\dot{u}_\delta &= \big(\dot{\tau}_\delta\cos\delta - \tau_\delta\sin\delta, \dot{\tau}_\delta\sin\delta + \tau_\delta\cos\delta\big)\\
&= \dot{\tau}_\delta\big(\cos\delta, \sin\delta\big) + \tau_\delta\big(-\sin\delta, \cos\delta\big)\\
&= \dot{\tau}_\delta\bar{v}_\delta + \tau_\delta\bar{v}_\delta^\perp,
\end{aligned}$$

from which we deduce that

$$\dot{z}_\delta = \dot{\tau}_\delta\,\bar{a}_\delta + f_0\big(\tau_\delta, v_\delta\big)\tau_\delta\,\big(\bar{a}_\delta^\perp\big) = \dot{\tau}_\delta\,\bar{a}_\delta. \tag{3.27}$$

This gives

$$\ddot{z}_\delta = \ddot{\tau}_\delta \, \bar{a}_\delta + \dot{\tau}_\delta \, \dot{\bar{a}}_\delta$$

and

$$\dddot{z}_\delta = \dddot{\tau}_\delta \, \bar{a}_\delta + 2\ddot{\tau}_\delta \, \dot{\bar{a}}_\delta + \dot{\tau}_\delta \, \ddot{\bar{a}}_\delta.$$

Moreover, since $\tau_\delta - \frac{1}{\cos(\delta)} = \tau_\delta - t_\delta = O(\delta^4)$, we have

$$\dot{\tau}_0 = \dot{t}_0 = 0, \quad \ddot{\tau}_0 = \ddot{t}_0 = 1, \quad \dddot{\tau}_0 = \dddot{t}_0 = 0.$$

Hence we obtain

$$\dot{z}_\delta = 0, \quad \ddot{z}_\delta = \bar{a}_0 = -w_0, \quad \dddot{z}_\delta = 2\,\dot{\bar{a}}_0 (= -\dot{w}_0 \neq 0),$$

which yields

$$z_\delta = y_0 + \frac{\delta^2}{2}(1,0) + \frac{\delta^3}{3}\,\dot{\bar{a}}_0 + o(\delta^3).$$

This contradicts (3.25) and (3.26), and concludes the proof of (A). □

Appendix A. Some notations in Riemannian geometry

Given (M,g) a C^∞ compact connected Riemannian manifold of dimension $n \geq 2$, we denote by TM its tangent bundle, by UM its unit tangent bundle, and by $\exp : (x,v) \longmapsto \exp_x v$ the exponential mapping. We write $g(x) = g_x$, $g_x(v,w) = \langle v,w\rangle_x$, $g_x(v,v) = |v|_x$ and equip M with its geodesic distance d. We further define:

- $t_C(x,v)$: the *cut time* of (x,v):
$$t_C(x,v) = \max\Big\{t \geq 0;\ (\exp_x(sv))_{0\leq s\leq t} \text{ is a minimizing geodesic}\Big\}.$$
- $t_F(x,v)$: the *focalization time* of (x,v):
$$t_F(x,v) = \inf\Big\{t \geq 0;\ \det(d_{tv}\exp_x) = 0\Big\}.$$
- $\mathrm{TCL}(x)$: the *tangent cut locus* of x:
$$\mathrm{TCL}(x) = \big\{t_C(x,v)v;\ v \in T_xM \setminus \{0\}\big\}.$$
- $\mathrm{cut}(x)$: the *cut locus* of x:
$$\mathrm{cut}(x) = \exp_x(\mathrm{TCL}(x)).$$
- $\mathrm{TFL}(x)$: the *tangent focal locus* of x:
$$\mathrm{TFL}(x) = \big\{t_F(x,v)v;\ v \in T_xM \setminus \{0\}\big\}.$$

• TFCL(x): the *tangent focal cut locus* of x:

$$\mathrm{TFCL}(x) = \mathrm{TCL}(x) \cap \mathrm{TFL}(x).$$

• fcut(x): the *focal cut locus* of x:

$$\mathrm{fcut}(x) = \exp_x(\mathrm{TFCL}(x)).$$

• I(x): the *injectivity domain* of the exponential map at x:

$$\mathrm{I}(x) = \big\{tv;\ 0 \le t < t_C(x,v),\ v \in T_xM\big\}.$$

• NF(x): the *nonfocal domain* of the exponential map at x:

$$\mathrm{NF}(x) = \big\{tv;\ 0 \le t < t_F(x,v),\ v \in T_xM\big\}.$$

• $\exp^{-1}$: the *inverse of the exponential map*; by convention $\exp_x^{-1}(y)$ is the set of *minimizing* velocities v such that $\exp_x v = y$. In particular $\mathrm{TCL}(x) = \exp_x^{-1}(\mathrm{cut}(x))$, and $I(x) = \exp_x^{-1}(M \setminus \mathrm{cut}(x))$.

We notice that, for every $x \in M$, the function $t_C(x,\cdot) : U_xM \to \mathbb{R}$ is locally Lipschitz (see [4, 18, 20]) while the function $t_F(x,\cdot) : U_xM \to \mathbb{R}$ is locally semiconcave on its domain (see [4]). In particular, the regularity property of $t_C(x,\cdot)$ yields

$$\mathcal{H}^{n-1}\big(\mathrm{cut}(x)\big) < +\infty \qquad \forall\, x \in M. \tag{A.1}$$

Appendix B. On the size of the focal cut locus

Recall that, for every $x \in M$, the focal cut locus of a point x is defined as

$$\mathrm{fcut}(x) = \exp_x(\mathrm{TFCL}(x)).$$

The focal cut locus of x is always contained in its cut locus. However it is much smaller, as the following result (which we believe to be of independent interest) shows:

Proposition B.1. *For every $x \in M$ the set* fcut(x) *has Hausdorff dimension bounded by $n-2$. In particular we have*

$$\forall\, x \in M, \quad \mathcal{H}^{n-1}\big(\mathrm{fcut}(x)\big) = 0. \tag{B.1}$$

Proof. For every $k = 0, 1, \ldots, n$, denotes by Σ_x^k the set of $y \neq x \in M$ such that the convex set $\nabla_x^+ c(x,y)$ has dimension k. By [3, Corollary 4.1.13], since the function $y \mapsto c(x,y)$ is semiconcave, the set Σ_x^k is countably $(n-k)$ rectifiable for every

$k = 2, \ldots, n$, which means in particular that all the sets $\Sigma_x^2, \ldots, \Sigma_x^n$ have Hausdorff dimension bounded by $n-2$. Thus, we only need to show that the set

$$J_x = \left(J_x \cap \Sigma_x^0\right) \cup \left(J_x \cap \Sigma_x^1\right)$$

has Hausdorff dimension $\leq n-2$. The fact that $J_x^0 = J_x \cap \Sigma_x^0$ has Hausdorff dimension $\leq n-2$ is a consequence of [25, Theorem 5.1]. Now, consider $\bar{y} \in J_x^1 = J_x \cap \Sigma_x^1$. Then there are exactly two minimizing geodesics $\gamma_1, \gamma_2 : [0,1] \to M$ joining x to $\bar{y}$. By upper semicontinuity of the set of minimizing geodesics joining x to y, for $i = 1, 2$ we can modify the metric g in a small neighborhood of $\gamma_i(1/2)$ into a new metric g_i in such a way that the following holds: there exists an open neighborhood $\mathcal{V}_i$ of $\bar{y}$ such that, for any $y \in J_x^1 \cap \mathcal{V}_i$, there is only one minimizing geodesic (with respect to g_i) joining x to y. In that way, we have

$$J_x^1 \cap \mathcal{V}_1 \cap \mathcal{V}_2 \subset \left(J_x^0\right)_1 \cup \left(J_x^0\right)_2,$$

where $\left(J_x^0\right)_i$ denotes the set $J_x^0 = J_x \cap \Sigma_x^0$ with respect to the metric g_i. Hence we conclude again by [25, Theorem 5.1]. $\square$

As a corollary the following holds:

Lemma B.2. *Let $\bar{x} \in M$, $v_0, v_1 \in \mathrm{I}(\bar{x})$ and $x \in M$ be fixed. Up to slightly perturbing v_0 and v_1, we can assume that $v_0, v_1 \in \mathrm{I}(\bar{x})$, $(y_t)_{0 \leq t \leq 1}$ intersects $\mathrm{cut}(x)$ only at a finite set of times $0 < t_1 < \ldots < t_N < 1$, and moreover $(y_t)_{0 \leq t \leq 1}$ never intersects $\mathrm{fcut}(x) = \exp_x(\mathrm{TFCL}(x))$.*

Proof of Lemma B.2. The proof of this fact is a variant of argument in [17]: fix $\sigma > 0$ small enough so that

$$w \perp v_1 - v_0,\ |w|_x \leq \sigma \implies \quad v_0 + w, v_1 + w \in I(\bar{x}),$$

and consider the cylinder C_σ in $T_x M$ given by $\{v_t + w\}$, with $t \in [0,1]$ and w as above. By convexity of $\mathrm{TFL}(\bar{x})$, for σ sufficiently small we have $C_\sigma \subset \mathrm{NF}(\bar{x})$. Let us now consider the sets

$$C_\sigma^c = C_\sigma \cap \exp_{\bar{x}}^{-1}\big(\exp_{\bar{x}}(C_\sigma) \cap \mathrm{cut}(x)\big),$$

$$C_\sigma^{cf} = C_\sigma \cap \exp_{\bar{x}}^{-1}\big(\exp_{\bar{x}}(C_\sigma) \cap \mathrm{fcut}(x)\big).$$

Since $C_\sigma \subset \mathrm{NF}(\bar{x})$, $\exp_{\bar{x}}^{-1}$ is locally Lipschitz on $\exp_{\bar{x}}(C_\sigma)$, and therefore (A.1) and (B.1) imply

$$\mathcal{H}^{n-1}(C_\sigma^c) < +\infty, \quad \mathcal{H}^{n-1}(C_\sigma^{cf}) = 0.$$

We now apply the co-area formula in the following form (see [8, p. 109] and [9, Sections 2.10.25 and 2.10.26]): let $f : v_t + w \longmapsto w$ (with the notation above), then

$$\mathcal{H}^{n-1}(A) \geq \int_{f(A)} \mathcal{H}^0[A \cap f^{-1}(w)]\, \mathcal{H}^{n-1}(dw)$$

for any $A \subset C_\sigma$ Borel. Since the right-hand side is exactly $\int \#\{t;\ v_t + w \in A\}\, \mathcal{H}^{n-1}(dw)$, we immediately deduce that particular there is a sequence $w_k \to 0$ such that each $(v_t + w_k)$ intersects C_σ^c finitely many often, and $(v_t + w_k)$ never intersects C_σ^{cf}.

We now also observe that, if $y \in \mathrm{cut}(x) \setminus \mathrm{fcut}(x)$, then $\mathrm{cut}(x)$ is given in a neighborhood of y by the intersection of a finite number of smooth hypersurfaces (see for instance [22]). Thus, up to slightly perturbing v_0 and v_1, we may further assume that at the points y_{t_j} the curve $t \mapsto y_t$ intersects $\mathrm{cut}(x)$ transversally. $\square$

Appendix C. Proofs of Corollaries 2.1, 2.2, 2.4, and Theorem 2.3

C.1. **Proof of Corollary 2.1.** By Theorem 1.2(ii), it suffices to show that (M, g) satisfies **(SCI)**. We argue by contradiction. Let $v_0 \neq v_1 \in \overline{\mathrm{I}(x)}$ be such that $v_t = (1-t)v_0 + tv_1 \notin \mathrm{I}(\bar{x})$ for all $t \in (0,1)$. Then, since $\mathrm{TFL}(\bar{x})$ is strictly convex, $v_t \notin \mathrm{TFL}(\bar{x})$ for all $t \in (0,1)$. For any $t \in (0,1)$, set $y_t = \exp_{\bar{x}}(v_t)$ and $\bar{q}_t = -d_{v_t} \exp_{\bar{x}}(v_t)$ as in the proof of Lemma 3.5. Then there exists $q_t \in \overline{\mathrm{I}(y_t)}$ with $q_t \neq \bar{q}_t$ such that $\exp_{y_t}(q_t) = \exp_{y_t}(\bar{q}_t) = \bar{x}$. We now choose a sequence of points $\{x_k\} \to \bar{x}$ such that $y_t \notin \mathrm{cut}(x_k)$ and $-\nabla_y c(x_k, y_t) \to q_t$ for all $t \in [0,1]$ (see for instance [22] for such a construction). By repeating the proof of Lemma 3.5 with the smooth function $h_k(t) = -c(x_k, y_t) + |v_t|^2_{\bar{x}}/2$ over the time interval $[0,1]$ (see [13]), one can see that $\ddot{h}_k(t)$ is given by

$$\ddot{h}_k(t) = \frac{2}{3} \int_0^1 (1-s) \overline{\mathfrak{S}}_{(y_t, (1-s)\bar{q}_t - s\nabla_y c(x_k, y_t))} \big(\dot{y}_t, \bar{q}_t - [-\nabla_y c(x_k, y_t)]\big)\, ds,$$

which by $(\overline{\mathbf{MTW}}^+)$ is strictly positive whenever $\dot{h}_k(t) = \langle \dot{y}_t, \bar{q}_t - [-\nabla_y c(x_k, y_t)] \rangle_{y_t} = 0$. As in the proof Lemma 3.5, these facts implies easily that, for any $t \in (0,1)$,

$$\begin{aligned} d(x_k, y_t)^2 - |v_t|^2_{\bar{x}} = 2h_k(t) &\geq 2 \min\{h_k(0), h_k(1)\} + r(t) \\ &= \min\Big(d(x_k, y_0)^2 - d(\bar{x}, y_0)^2,\ d(x_k, y_1)^2 - d(\bar{x}, y_1)^2\Big) + r(t), \end{aligned}$$

where $r : [0,1] \mapsto [0,1]$ is a continuous function (independent of k) such that $r > 0$ on $[1/4, 3/4]$. Hence, choosing for instance $t = 1/2$ and letting $k \to \infty$ we get

$$\begin{aligned} 0 = d(\bar{x}, y_{1/2})^2 - d(\bar{x}, y_{1/2})^2 &\geq d(\bar{x}, y_{1/2})^2 - |v_{1/2}|_{\bar{x}}^2 \\ &> \min\Big(d(\bar{x}, y_0)^2 - d(\bar{x}, y_0)^2,\, d(\bar{x}, y_1)^2 - d(\bar{x}, y_1)^2\Big) = 0, \end{aligned}$$

a contradiction.

C.2. **Proof of Corollary 2.2.** By Theorem 1.3, it is sufficient to prove that **(CNF)** and **($\overline{\textbf{MTW}}$)** imply **(CI)**. Arguing as in [13], we can show that the following "extended version" of Lemma 3.5 holds:

Lemma C.1. *Let (M, g) be a Riemannian manifold satisfying* **(CNF)**-**($\overline{\textbf{MTW}}$)**. *Fix $\bar{x} \in M$, $v_0, v_1 \in \overline{\mathrm{I}(\bar{x})}$, and let $v_t = (1-t)v_0 + tv_1 \in T_x M$. For any $t \in [0,1]$, set $y_t = \exp_x(v_t)$. Then, for any $x \in M$, for any $t \in [0,1]$,*

$$c(x, y_t) - \frac{1}{2}|v_t|_{\bar{x}}^2 \geq \min\Big(c(x, y_0) - c(\bar{x}, y_0),\, c(x, y_1) - c(\bar{x}, y_1)\Big).$$

By choosing $x = \bar{x}$ we deduce that $c(x, y_t) \geq \frac{1}{2}|v_t|_{\bar{x}}^2$,, which implies that $v_t \in \overline{\mathrm{I}(\bar{x})}$, as desired.

C.3. **Proof of Theorem 2.3.** Fix $y \in M$, assume that $S(y)$ is not a singleton and suppose by contradiction that there exists $q_0 \in \exp_y^{-1}(S(y)) \cap \mathrm{I}(y) \subset T_y M$ an exposed point for $\exp_y^{-1}(S(y))$. Let us define the "c-Monge-Ampère" measure $|\partial^c \phi|$ as

$$|\partial^c \phi|(A) = \mathrm{vol}\left(\cup_{x \in A} \partial^c \phi(x)\right) \qquad \text{for all } A \subset M \text{ Borel.}$$

As shown for instance in [10, Lemma 3.1] (see also [23]), under our assumptions on μ and ν the following upper and lower bounds on $|\partial^c \phi|$ hold:

$$\frac{\lambda}{\Lambda} \mathrm{vol}\,(A) \leq |\partial^c \phi|(A) \leq \frac{\Lambda}{\lambda} \mathrm{vol}\,(A) \qquad \text{for all } A \subset M \text{ Borel.}$$

Now, let us consider the change of coordinates $x \mapsto q = -\bar{D}c(x, y)$ which sends $M \setminus \mathrm{cut}(y)$ onto $\mathrm{I}(y)$. Since $x_0 = \exp_y(q_0) \notin \mathrm{cut}(y)$, the cost $d^2/2$ is smooth in a neighborhood of $\{x_0\} \times \{y\}$. Moreover, the support of $|\partial^c \phi|$ is the whole manifold M. So, we can apply [10, Theorem 8.1 and Remark 8.2] to obtain that no exposed points can exist inside the open set $I(y)$. This gives a contradiction and concludes the proof.

C.4. **Proof of Corollary 2.4.** Let μ to ν be two probability measures such that

$$\lambda \mathrm{vol} \leq \mu \leq \Lambda \mathrm{vol}\,, \qquad \lambda \mathrm{vol} \leq \nu \leq \Lambda \mathrm{vol}$$

for two positive constants $\lambda, \Lambda > 0$. By [19], any compact quotient M of $\widetilde{M} = \mathbb{S}^{n_1}_{r_1} \times \ldots \times \mathbb{S}^{n_i}_{r_i} \times \mathbb{R}^n$ satisfies $\mathbf{MTW}^{\not\perp}$. Moreover, if $\widetilde{\pi} : \widetilde{M} \to M$ denotes the quotient map, we can use $(\widetilde{\pi})^{-1}$ to lift μ and ν onto two σ-finite measure $\widetilde{\mu}$ and $\widetilde{\nu}$ which will still satisfy the bounds

$$\lambda \widetilde{\mathrm{vol}} \leq \widetilde{\mu} \leq \Lambda \widetilde{\mathrm{vol}}\,, \qquad \lambda \widetilde{\mathrm{vol}} \leq \widetilde{\nu} \leq \Lambda \widetilde{\mathrm{vol}}\,.,$$

where $\widetilde{\mathrm{vol}}$ denotes the volume density on $\widetilde{M}$. Now, let $T = \exp_x\big(\nabla_x \psi\big) : M \to M$ denote the transport map from μ to ν, and set $\widetilde{c} = \widetilde{d}^2/2$, with $\widetilde{d}$ the Riemannian distance on $\widetilde{M}$. Observe that, since M is compact, the c-convex function is semiconvex too. Then it is easily checked that the function $\widetilde{\psi} : \widetilde{M} \to \mathbb{R}$ defined by $\widetilde{\psi} = \psi \circ \widetilde{\pi}$ is $\widetilde{c}$-convex, locally semiconvex, and $\widetilde{T} = \exp_{\widetilde{x}}\big(\nabla_{\widetilde{x}} \widetilde{\psi}\big) : \widetilde{M} \to \widetilde{M}$ sends $\widetilde{\mu}$ to $\widetilde{\nu}$. Moreover, since set of subgradients $\nabla^- \psi(x)$ at any point x belongs to $\overline{I(x)} \subset T_x M$ (see for instance [24]), by identifying the tangent spaces $T_{\widetilde{x}} M$ and $T_{\widetilde{\pi}(\widetilde{x})} M$ we obtain

$$\nabla^- \widetilde{\psi}(\widetilde{x}) = \nabla^- \psi\big(\widetilde{\pi}(\widetilde{x})\big) \subset \overline{I(\widetilde{\pi}(\widetilde{x}))}.$$

However, since $\widetilde{M}$ is a product of spheres and $\mathbb{R}^n$, thanks to the nonfocality assumption on M it is easily seen that

$$\mathrm{TCL}(\widetilde{\pi}(\widetilde{x})) \subset\subset \mathrm{I}(\widetilde{x})$$

(again we are identifying $T_{\widetilde{x}} M$ with $T_{\widetilde{\pi}(\widetilde{x})} M$). This implies that $\nabla^- \widetilde{\psi}(\widetilde{x})$ lies at a positive distance from $\mathrm{TCL}(\widetilde{x})$ for every $\widetilde{x} \in \widetilde{M}$. In particular, for every $\widetilde{y} \in \widetilde{M}$, the set

$$\widetilde{S}(\widetilde{y}) = \Big\{ \widetilde{x} \in \widetilde{M};\, \widetilde{y} \in \partial^{\widetilde{c}} \widetilde{\psi}(\widetilde{x}) \Big\}$$

cannot intersect $\mathrm{cut}(\widetilde{y})$. By Theorem 2.3 this implies that $\widetilde{S}(\widetilde{y})$ is a singleton for every $\widetilde{y}$, so $\widetilde{\psi}^{\widetilde{c}}$ is C^1. Since $\widetilde{\psi}^{\widetilde{c}}$ is the potential associated to the transport problem from $\widetilde{\nu}$ to $\widetilde{\mu}$ (that is, $\exp_{\widetilde{y}}\big(\nabla_{\widetilde{y}} \psi^{\widetilde{c}}\big)$ is the optimal map sending $\widetilde{\nu}$ onto $\widetilde{\mu}$, see for instance [26]) and the hypotheses on $\widetilde{\mu}$ and $\widetilde{\nu}$ are symmetric, we can exchange the role of $\widetilde{x}$ and $\widetilde{y}$ to deduce that $\widetilde{\psi}$ is C^1 too. This implies that also ψ is C^1, and so T is continuous as desired.

Let us also observe that, since also ψ^c is C^1, the transport map T is injective. As already observed in [11], the continuity and injectivity of T combined with the result in [LTW] implies higher regularity ($C^{1,\alpha}/C^\infty$) of optimal maps for more smooth (C^α/C^∞) densities. This concludes the proof.

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