Sard conjectures and measures contraction properties in sub-Riemannian geometry

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Throughout all the talk, $M$ is a smooth connected manifold of dimension $n$ equipped with a **sub-Riemannian structure of rank** $m$ in $M$ given by a pair $(\Delta, g)$ where:

- $\Delta$ is a **totally nonholonomic distribution** of rank $m \leq n$ on $M$ which is defined locally by
  \[
  \Delta(x) = \operatorname{Span}\left\{X^1(x), \ldots, X^m(x)\right\} \subset T_x M,
  \]
  where $X^1, \ldots, X^m$ is a family of $m$ linearly independent smooth vector fields satisfying the **Hörmander condition**.

- $g_x$ is a **scalar product** over $\Delta(x)$.

The sub-Riemannian geodesic distance on $M$ is denoted by $d_{SR}$ and the metric space $(M, d_{SR})$ is always assumed to be complete.

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Some open problems in SR geometry

- The Sard Conjecture.
- The minimizing Sard Conjecture.
- Validity of Measure Contractions Properties.
The End-Point mapping

Assume that $\Delta$ is globally spanned by $k$ smooth vector fields $X^1, \ldots, X^k$, that is $\Delta(x) = \text{Span} \{X^1(x), \ldots, X^k(x)\}$ for all $x \in M$. For every $x \in M$ and every $u \in L^2([0, 1]; \mathbb{R}^k)$, denote by $\gamma_{x,u}: [0, 1] \rightarrow M$ the solution to the Cauchy problem

$$\dot{\gamma}(t) = \sum_{i=1}^{k} u_i(t)X^i(\gamma(t)) \text{ for a.e. } t \in [0, 1], \quad \gamma(0) = x.$$
The End-Point mapping

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$$\dot{\gamma}(t) = \sum_{i=1}^{k} u_i(t)X^i(\gamma(t)) \text{ for a.e. } t \in [0, 1], \quad \gamma(0) = x.$$

Definition

Given a point $x \in M$, the **End-Point mapping**

$$E^{x,1} : L^2 \in ([0, 1]; \mathbb{R}^k) \rightarrow M$$

is defined by

$$E^{x,1}(u) := \gamma_{x,u}(1) \quad \forall u \in L^2 \in ([0, 1]; \mathbb{R}^k).$$
Singular horizontal paths and Examples

**Definition**

An horizontal path is called **singular** if it is, through the correspondence $\gamma \leftrightarrow u$, a critical point of the End-Point mapping $E^{x,1} : L^2 \rightarrow M$ (with $x = \gamma(0)$).

---

Example 1: Riemannian case

Let $\Delta(x) = T_xM$, any path in $W^{1,2}$ is horizontal. There are no singular curves.

Example 2: Heisenberg, fat distributions

In $\mathbb{R}^3$, $\Delta$ given by $X_1 = \partial x, X_2 = \partial y + x \partial z$ does not admit nontrivial singular horizontal paths.
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Example 3: Martinet distributions
In $\mathbb{R}^3$, let $\Delta = \text{Vect}\{X^1, X^2\}$ with $X^1, X^2$ given by

$$X^1 = \partial_{x_1} \quad \text{and} \quad X^2 = \partial_{x_2} + x_1^2 \partial_{x_3}.$$ 

The singular horizontal paths are the horizontal paths which are contained in the set $\{x_1 = 0\}$. 

Example 4: Rank-2 distributions in dimension 3
In this case, the singular horizontal paths are those horizontal paths which are contained in the Martinet surface $\Sigma_{\Delta} = \{x \in M | \Delta(x) + [\Delta, \Delta](x) \neq T_x M\}$. 

Example 5: Rank-2 distributions in dimension 4
In this case, at least for a generic germ, the singular horizontal paths are given by the orbits of a smooth vector field in $\Delta$. 

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Sard conjectures and MCP in SR geometry
**Example 3:** Martinet distributions

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Sard conjectures and MCP in SR geometry
The Sard Conjecture

Let \((\Delta, g)\) be a SR structure on \(M\) and \(x \in M\) be fixed. Set

\[
S_x^\Delta = \{ \gamma(1) | \gamma : [0, 1] \to M, \gamma(0) = x, \gamma \text{ hor., sing.} \}.
\]

Conjecture (Sard Conjecture)

The set \(S_x^\Delta\) has Lebesgue measure zero.
The Sard Conjecture

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Conjecture (Sard Conjecture)

The set $S^x_\Delta$ has Lebesgue measure zero.

Remark

By the Brown-Morse-Sard Theorem (1935-42), we know that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function of class $C^k$ with

$$k \geq \max\{1, n - m + 1\},$$

then the set $f(C_f)$ of critical values of $f$ has Lebesgue measure zero.
The Sard Theorem is false in infinite dimension. Let $f : \ell^2 \rightarrow \mathbb{R}$ be defined by

$$f \left( \sum_{n=1}^{\infty} x_n e_n \right) = \sum_{n=1}^{\infty} \left( 3 \cdot 2^{-n/3} x_n^2 - 2x_n^3 \right).$$

The function $f$ is polynomial ($f^{(4)} \equiv 0$) with critical set

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and critical values

$$f(C_f) = \left\{ \sum_{n=1}^{\infty} \delta_n 2^{-n} \mid \delta_n \in \{0, 1\} \right\} = [0, 1].$$
Let $M$ be a smooth manifold of dimension 3 and $\Delta$ be a totally nonholonomic distribution of rank 2 on $M$. We define the **Martinet surface** by

$$
\Sigma_\Delta = \left\{ x \in M \mid \Delta(x) + [\Delta, \Delta](x) \neq T_x M \right\}
$$

If $\Delta$ is generic, $\Sigma_\Delta$ is a surface in $M$. If $\Delta$ is analytic then $\Sigma_\Delta$ is analytic of dimension $\leq 2$. If $\Delta$ is smooth, $\Sigma_\Delta$ is countably 2-rectifiable.
The case of Martinet surfaces

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**Proposition**

*The singular horizontal paths are the orbits of the trace of $\Delta$ on $\Sigma_\Delta$.***
Let $M$ be a smooth manifold of dimension 3 and $\Delta$ be a totally nonholonomic distribution of rank 2 on $M$. We define the **Martinet surface** by

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**Proposition**

*The singular horizontal paths are the orbits of the trace of $\Delta$ on $\Sigma_\Delta$.***

As a consequence, the Sard conjecture holds!!! In fact, we expect a stronger result...
The Sard Conjecture on Martinet surfaces

Transverse case

$\Sigma_\Delta$
Generic tangent case
(Zelenko-Zhitomirskii, 1995)
Let $M$ be of dimension 3 and $\Delta$ of rank 2.

$$S^x_\Delta = \{ \gamma(1) | \gamma : [0, 1] \to M, \gamma(0) = x, \gamma \text{ hor., sing.} \}.$$ 

Conjecture (Strong Sard Conjecture)

The set $S^x_\Delta$ has vanishing $H^2$-measure.

Theorem (Belotto-R, 2016)

The above conjecture holds true under one of the following assumptions:

- The Martinet surface is smooth;
- All datas are analytic and

$$\Delta(x) \cap T_x \text{Sing}(\Sigma_\Delta) = T_x \text{Sing}(\Sigma_\Delta) \quad \forall x \in \text{Sing}(\Sigma_\Delta).$$
Ingredients of the proof

- Control of the divergence of vector fields which generates the trace of $\Delta$ over $\Sigma_\Delta$ of the form

$$|\text{div} \mathcal{Z}| \leq C |\mathcal{Z}|.$$
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- Resolution of singularities.
In $\mathbb{R}^3$, 

$$X = \partial_y \quad \text{and} \quad Y = \partial_x + \left[ \frac{y^3}{3} - x^2y(x + z) \right] \partial_z.$$ 

Martinet Surface: $\Sigma_\Delta = \left\{ y^2 - x^2(x + z) = 0 \right\}$. 

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Sard conjectures and MCP in SR geometry
The minimizing Sard Conjecture

Let \((\Delta, g)\) be a SR structure on \(M\) and \(x \in M\) fixed.

\[
S_{\Delta, \text{min}}^x = \left\{ \gamma(1) | \gamma : [0, 1] \to M, \gamma(0) = x, \gamma \text{ hor., sing., min.} \right\}.
\]

Conjecture (SR or Minimizing Sard Conjecture)
The set \(S_{\Delta, \text{min}}^x\) has Lebesgue measure zero.
Let \((\Delta, g)\) be a SR structure on \(M\) and \(x \in M\) fixed.

\[ S^x_{\Delta, \text{min}} = \left\{ \gamma(1) | \gamma : [0, 1] \to M, \gamma(0) = x, \gamma \text{ hor., sing., min.} \right\}. \]

Conjecture (SR or Minimizing Sard Conjecture)

The set \(S^x_{\Delta, \text{min}}\) has Lebesgue measure zero.

Remark

We know since the 90’s that there are examples of sub-Riemannian structures with (strictly) singular minimizing curves (cf. Montgomery '94, Liu-Sussmann '95).
Theorem (Agrachev, 2009)

The set $S^x_{\Delta, \min}$ is closed with empty interior.

Proposition

The Sard minimizing Conjecture holds true in the following cases:

- Medium-fat distributions, that is for every $x \in M$ and all smooth section $X$ of $\Delta$ with $X(x) \neq 0$,

$$T_x M = \Delta(x) + [\Delta, \Delta](x) + [X, [\Delta, \Delta]](x).$$
The minimizing Sard Conjecture: State of the art

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- Generic distributions of rank $m \geq 3$. 

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Sard conjectures and MCP in SR geometry
Following Agrachev, we introduce the following definition:

**Definition**

We call **smooth point** of the function $y \mapsto d_{SR}(x, y)$ any $y \in M$ for which there is $p \in T^*_x M$ which is not a critical point of $\exp_x$ and such that the projection of the normal extremal starting at $(x, p)$ is the unique minimizing geodesic from $x$ to $y$. 
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**Proposition**

Let \( x \in M \) be fixed, the following properties are equivalent:

- The Minimizing Sard conjecture is satisfied at \( x \).
- The function \( y \mapsto d_{SR}(x, y) \) is differentiable almost everywhere in \( M \).
- The set of smooth points \( \mathcal{O}_x \) is an open set with full measure in \( M \) (\( d_{SR}(x, \cdot) \) is smooth on \( \mathcal{O}_x \)).
Geodesic interpolation

Let \((\Delta, g)\) be a SR structure on \(M\) and \(x \in M\) such that there is a measurable set \(C(x) \subset M\) with Lebesgue measure zero and a measurable map \(\gamma_x : (M \setminus C(x)) \times [0, 1] \to M\) such that for every \(y \in M \setminus C(x)\), the curve

\[
\gamma_x(s, y) \quad \text{for } s \in [0, 1]
\]

is the unique minimizing horizontal path from \(x\) to \(y\).

**Definition**

Let \(A \subset M\) be a measurable set, for every \(s \in [0, 1]\), the \(s\)-interpolation of \(A\) from \(x\) is defined by

\[
A_s := \left\{ \gamma_x(s, y) \mid y \in A \setminus C(x) \right\} \quad \forall s \in [0, 1].
\]
Let $\mu$ a measure absolutely continuous with respect to $\mathcal{L}^n$ and $K \in \mathbb{R}, N > 1$ be fixed. The measure contraction property MCP($K, N$) at $x$ consists in comparing the contraction of volumes along minimizing geodesics from $x$ with respect to the classical model space of Riemannian geometry of curvature $K$ in dimension $N$. 

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Sard conjectures and MCP in SR geometry
Let $\mu$ a measure absolutely continuous with respect to $\mathcal{L}^n$ and $K \in \mathbb{R}$, $N > 1$ be fixed. The measure contraction property MCP($K$, $N$) at $x$ consists in comparing the contraction of volumes along minimizing geodesics from $x$ with respect to the classical model space of Riemannian geometry of curvature $K$ in dimension $N$.

**Definition**

The property MCP($K$, $N$) holds at $x$ if for every measurable set $A \subset M \setminus \mathcal{C}(x)$ (provided that $A \subset B_{SR}(x, \pi \sqrt{N - 1}/K)$ if $K > 0$) with $0 < \mu(A) < \infty$,

$$
\mu(A_s) \geq \int_A s \left[ \frac{s_K (sd_{SR}(x, z)/\sqrt{N - 1})}{s_K (d_{SR}(x, z)/\sqrt{N - 1})} \right]^{N-1} ds,
$$

for all $s \in [0, 1]$. 
In particular, we have:

**Definition**

The sub-Riemannian structure \((\Delta, g)\) equipped with \(\mu\) satisfies MCP\((0, N)\) at \(x\) if for every measurable set \(A \subset M \setminus C(x)\) with \(0 < \mu(A) < \infty\),

\[
\mu(A_s) \geq s^N \mu(A) \quad \forall s \in [0, 1].
\]
Two qualitative results

Recall that a distribution $\Delta$ is two-step if

$$[\Delta, \Delta](x) = T_x M \quad \forall x \in M.$$ 

**Theorem (Badreddine-R, 2017)**

Every two-step sub-Riemannian structure on a compact manifold equipped with a smooth measure satisfies $\text{MCP}(0, N)$ for some $N > 0$. 

**Theorem (Badreddine-R, 2017)**

Every medium-fat Carnot group with the Haar measure satisfies $\text{MCP}(0, N)$ for some $N > 0$. 

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Theorem (Badreddine-R, 2017)

Every medium-fat Carnot group with the Haar measure satisfies $\text{MCP}(0, N)$ for some $N > 0$. 
Ingredients of the proof

Let $f := d_{SR}(x, \cdot)^2/2$ and $\nabla^h f$ its horizontal gradient. The control of $\mu(A_s)$ from below, $\mu(A_s) \geq s^N \mu(A)$, is equivalent to a control on the divergence of $\nabla^h f$ from above:

$$\text{div}^\mu (\nabla^h f) \leq N$$
Ingredients of the proof

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- The function $f$ is nearly horizontally semiconcave.
Ingredients of the proof

- Let \( f := d_{SR}(x, \cdot)^2/2 \) and \( \nabla^hf \) its horizontal gradient. The control of \( \mu(A_s) \) from below, \( \mu(A_s) \geq s^N \mu(A) \), is equivalent to a control on the divergence of \( \nabla^hf \) from above:
  \[
  \text{div}^\mu (\nabla^hf) \leq N
  \]
- The function \( f \) is nearly horizontally semiconcave.
- \( f \) is globally Lipschitz (Agrachev-Lee, 2009).
Recall that a function $f : U \to \mathbb{R}$ is $C$-semiconcave in an open set $U \subset \mathbb{R}^n$ if for every $x \in U$ there is a function $\psi : U \to \mathbb{R}$ of class $C^2$ with $\|\psi\|_{C^2} \leq C$ such that

$$f(y) \leq \psi(y) \quad \forall y \in U.$$
Nearly horizontally semiconcave functions

**Definition**

A function $f : U \to \mathbb{R}$ in an open set $U \subset M$ is said to be $C$-nearly horizontally semiconcave with respect to $(\Delta, g)$ if for every $y \in U$, there are an open neighborhood $V^y$ of 0 in $\mathbb{R}^m$, a function $\varphi^y : V^y \subset \mathbb{R}^m \to U$ of class $C^2$ and a function $\psi^y : V^y \subset \mathbb{R}^m \to \mathbb{R}$ of class $C^2$ such that

$$
\varphi^y(0) = y, \quad \psi^y(0) = f(y), \quad f(\varphi^y(v)) \leq \psi^y(v) \quad \forall v \in V^y,
$$

$$
\left\{ d_0 \varphi^y(e_1), \ldots, d_0 \varphi^y(e_m) \right\} \text{ is orthonormal in } \Delta(y),
$$

and

$$
\| \varphi^y \|_{C^2}, \| \psi^y \|_{C^2} \leq C.
$$
Proposition (Badreddine-R, 2017)

If $M$ is compact and $\Delta$ is two-step then there is $C > 0$ such that all functions $f^x = d_{SR}(x, \cdot)^2/2$ are $C$-nearly horizontally semiconcave in $M$. 

Lemma

There is $B > 0$ such that for every $x \in M$ the following property holds: there is locally an orthonormal family of smooth vector fields $X_1, \ldots, X_m$ which parametrize $\Delta$ such that $\|X_i\| \leq C_1 \leq B$ for $i = 1, \ldots, m$ and $X_i \cdot (X_i \cdot f^x) \leq B |\nabla z f^x| + B \forall i = 1, \ldots, m$.

We note that $\text{div} \, \mu(y) (\nabla h f^x) = \sum_{i=1}^m (X_i \cdot f^x)(y) \text{div} \, \mu(y) (X_i) + \sum_{i=1}^m [(X_i \cdot (X_i \cdot f^x))(y)]$.
End of the proof

Proposition (Badreddine-R, 2017)

If $M$ is compact and $\Delta$ is two-step then there is $C > 0$ such that all functions $f^x = d_{SR}(x, \cdot)^2 / 2$ are $C$-nearly horizontally semiconcave in $M$.

Lemma

There is $B > 0$ such that for every $x \in M$ the following property holds: there is locally a orthonormal family of smooth vector fields $X^1, \ldots, X^m$ which parametrize $\Delta$ such that $\|X^i\|_{C^1} \leq B$ for $i = 1, \ldots, m$ and

$$X^i \cdot (X^i \cdot f^x) \leq B |\nabla_z f^x| + B \quad \forall i = 1, \ldots, m.$$
Proposition (Badreddine-R, 2017)

If $M$ is compact and $\Delta$ is two-step then there is $C > 0$ such that all functions $f^x = d_{SR}(x, \cdot)^2/2$ are $C$-nearly horizontally semiconcave in $M$.

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There is $B > 0$ such that for every $x \in M$ the following property holds: there is locally a orthonormal family of smooth vector fields $X^1, \ldots, X^m$ which parametrize $\Delta$ such that $\|X^i\|_{C^1} \leq B$ for $i = 1, \ldots, m$ and

$$X^i \cdot (X^i \cdot f^x) \leq B |\nabla_z f^x| + B \quad \forall i = 1, \ldots, m.$$ 

We note that $\operatorname{div}^{\mu} y (\nabla^h f^x) = \sum_{i=1}^m (X^i \cdot f^x)(y) \operatorname{div}^{\mu} y (X^i) + \sum_{i=1}^m [X^i \cdot (X^i \cdot f)](y)$. 

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Thank you for your attention!!