Closing geodesics in $C^1$ topology

L. Rifford

July 8, 2010

Abstract

Given a closed Riemannian manifold, we show how to close a nonwandering orbit of the geodesic flow by a small perturbation of the metric in $C^1$ topology.

1 Introduction

Let $M$ be a smooth compact manifold without boundary of dimension $n \geq 2$ (throughout the paper, smooth always means of class $C^\infty$). For every Riemannian metric $g$ on $M$ of class $C^k$ with $k \geq 2$, denote by $|v|^2_g$ the norm of a vector $v \in T_xM$, by $U^gM$ the unit tangent bundle, and by $\phi^t_g$ the geodesic flow on $U^gM$. Moreover, for every $(x, v) \in U^gM$, denote by $\gamma^g_{x,v} : \mathbb{R} \to M$ the unit geodesic starting at $x$ with initial velocity $v$. A pair $(x, v) \in U^gM$ is called $g$-nonwandering if it is nonwandering with respect to the geodesic flow $\phi^t_g$, that is if for every neighborhood $V$ of $(x, v)$ in $U^gM$, there exist $t \geq 1$ and $(x', v') \in V$ such that $\phi^t_g(x', v') \in V$.

The aim of the present paper is to show how to close $g$-nonwandering orbits with a small conformal perturbation of the metric in $C^1$ topology. Pick a Riemannian distance on $TM$ and denote by $d_{TM}(\cdot, \cdot)$ the geodesic distance associated to it on $TM$. Note that since all Riemannian distances are Lipschitz equivalent on compact subsets, the choice of the metric on $TM$ is not important. Our main result is the following:

Theorem 1. Let $g$ be a Riemannian metric on $M$ of class $C^k$ with $k \geq 3$ (resp. $k = \infty$) and $\epsilon > 0$ be fixed. Let $(x, v) \in U^gM$ be a $g$-nonwandering point. Then there exist a metric $\tilde{g} = e^f g$ with $f : M \to \mathbb{R}$ of class $C^{k-1}$ (resp. $C^\infty$) satisfying $\|f\|_{C^1} < \epsilon$, and $(\tilde{x}, \tilde{v}) \in U^{\tilde{g}}M$ with $d_{TM}(x, v), (\tilde{x}, \tilde{v})) < \epsilon$, such that the geodesic $\gamma^\tilde{g}_{(\tilde{x}, \tilde{v})}$ is periodic.

The author has been supported by the program “Project ANR-07-BLAN-0361, Hamilton-Jacobi and théorie KAM faible”.

*Université de Nice-Sophia Antipolis, Labo. J.-A. Dieudonné, UMR CNRS 6621, Parc Valrose, 06108 Nice Cedex 02, France (Ludovic.Rifford@math.cnrs.fr)
There is a constant $C > 0$ such that if $(x, v), (\tilde{x}, \tilde{v}) \in TM$ satisfy $(x, v) \in U^g M$ and $d_{TM}(x, v), (\tilde{x}, \tilde{v})) < \epsilon$ with $\epsilon > 0$ small enough, then there is a smooth diffeomorphism $\Phi : M \rightarrow M$ such that

$$\Phi(x) = \Phi(\tilde{x}), \quad d\Phi(x, v) = (\tilde{x}, \tilde{v}), \quad \text{and} \quad \| \Phi - Id\|_{C^2} < C\epsilon.$$ 

Therefore, the following result is an easy consequence of Theorem 1:

**Corollary 2.** Let $g$ be a Riemannian metric on $M$ of class $C^k$ with $k \geq 3$ (resp. $k = \infty$) and $\epsilon > 0$ be fixed. Let $(x, v) \in U^g M$ be a $g$-nonwandering point. Then there exists a metric $\tilde{g}$ of class $C^{k-1}$ (resp. $C^\infty$) with $\| \tilde{g} - g\|_{C^1} < \epsilon$ such that the geodesic $\tilde{\gamma}_{(x, v)}$ is periodic.

In 1951, Lyusternik and Fet proved that at least one closed geodesic exists on every smooth compact Riemannian manifold (see [4, 5]). Corollary 2 shows that any non-wandering pair $(x, v) \in U^g M$ of the geodesic flow $\phi^g_t$ may indeed be seen as a pair $(\gamma_k(0), \dot{\gamma}_k(0))$ for some sequence of closed orbits $\{\gamma_k\}$ with respect to smooth Riemannian metrics $\{g_k\}$ converging to $g$ in $C^1$ topology.

The paper is organized as follows: In Section 2, we state and prove a result which is crucial to prove Theorem 1. This result, Proposition 3, shows how to connect two close geodesics while preserving a finite set of transverse geodesics, by a conformal perturbation of the initial metric with a control on the support of the conformal factor and its $C^1$ norm. Then, the proof of Theorem 1 is given in Section 3 and the proofs of some technical results are postponed to the appendix.

Notations: Throughout this paper, we denote by $| \cdot |$ the Euclidean norm in $\mathbb{R}^k$ and for any $x \in \mathbb{R}^k$ and any $r \geq 0$, we set $B^k(x, r) := \{y \in \mathbb{R}^k : |y - x| < r\}$.

## 2 Connecting geodesics with obstacles

### 2.1 Statement of the result

Let $n \geq 2$ be an integer, $\tau > 0$ be fixed, and let $\tilde{g}$ be a complete Riemannian metric of class $C^k$ with $k \geq 3$ or $k = \infty$ on $\mathbb{R}^n$. Denote by $|v|_{\tilde{g}}$ the norm with respect to $\tilde{g}$ of a vector $(x, v) \in T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, denote by $\phi^g_t$ the geodesic flow of $\tilde{g}$ on $\mathbb{R}^n \times \mathbb{R}^n$ and for every $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$, denote by $\tilde{\gamma}_{x,v}$ the geodesic with respect to $\tilde{g}$ which starts at $x$ with velocity $v$. Assume that the curve $\tilde{\gamma} : [0, \tau] \rightarrow \mathbb{R}^n$ is a geodesic with respect to $\tilde{g}$ satisfying the following property ($e_1$ denotes the first vector in the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$):

(A) $|\tilde{\gamma}(t) - e_1| \leq 1/10$, for every $t \in [0, \tau]$.  

2
Set 
\[ \begin{align*}
\tilde{x}^0 &= (\tilde{x}_0^0, \ldots, \tilde{x}_n^0) := \tilde{\gamma}(0), \\
\tilde{x}^\tau &= (\tilde{x}_1^\tau, \ldots, \tilde{x}_n^\tau) := \hat{\gamma}(\tau), \\
\tilde{v}^0 &= (\tilde{v}_0^0, \ldots, \tilde{v}_n^0) := \hat{\gamma}(0), \\
\tilde{v}^\tau &= (\tilde{v}_1^\tau, \ldots, \tilde{v}_n^\tau) := \hat{\gamma}(\tau).
\end{align*} \]

Our aim is to show that, given \((x, v), (y, w) \in \mathbb{R}^n \times \mathbb{R}^n\) with \(|v|^2 = |w|^2 = 1\) sufficiently close to \((\tilde{x}^0, \tilde{v}^0)\), there exists a Riemannian metric \(\tilde{g}\) of class \(C^{k-1}\) which is conformal to \(\tilde{g}\) and whose the support and the \(C^1\)-norm are controlled, which connects \((x, v)\) to \((\gamma_{y,w}(\tau), \phi^\tau_{y,w}(\tau)) = \phi^\tau_{y,w}(y, w)\) and which preserves finitely many transverse geodesics.

Set 
\[ \mathcal{R}(\rho) := \left\{ (t, z) \mid t \in [\tilde{x}_1^0, \tilde{x}_1^0], z \in B^{n-1}(0, \rho) \right\} \forall \rho > 0. \]

Let us state our result \((\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) denotes the canonical projection on the state variable):

**Proposition 3.** Let \(\tau > 0\) and \(\tilde{\gamma} : [0, \tau] \to \mathbb{R}^n\) satisfying assumption (A) be fixed. Let \(\rho > 0\) be such that \(\tilde{\gamma}([0, \tau]) \subset \mathcal{R}(\rho/2)\) be fixed. There are \(\delta = \delta(\tau, \rho) \in (0, \tau/3)\) and \(C = C(\tau, \rho) > 0\) such that the following property is satisfied: For every \((x, v), (y, w) \in U^g\mathbb{R}^n\) satisfying
\[ |x - \tilde{x}^0|, |y - \tilde{x}^0|, |v - \tilde{v}^0|, |w - \tilde{v}^0| < \delta, \]
and for every finite set of unit geodesics
\[ \tilde{c}_1 : I_1 = [a_1, b_1] \to \mathbb{R}^n, \ldots, \tilde{c}_L : I_L = [a_L, b_L] \to \mathbb{R}^n \]

satisfying
\[ \tilde{c}_l(a_l), \tilde{c}_l(b_l) \notin \mathcal{R}(\rho) \quad \forall l \in \{1, \ldots, L\}, \quad (2.2) \]
\[ (\tilde{c}_l(s), \tilde{c}_l(s)) \neq \phi^\delta_{l}(x, v), \phi^\delta_{l}(y, w), \quad \forall l \in \{1, \ldots, L\}, \forall s \in I_l, \forall t \in [0, \tau], \quad (2.3) \]
there are \(\bar{\tau} > 0\) and a Riemannian metric \(\bar{g} = e^f \tilde{g}\) on \(\mathbb{R}^n\) with \(f : \mathbb{R}^n \to \mathbb{R}\) of class \(C^{k-1}\) (or \(f\) of class \(C^\infty\) if \(\tilde{g}\) is itself \(C^\infty\)) satisfying the following properties:

(i) \(\text{Supp } (f) \subset \mathcal{R}(\rho)\);
(ii) \(\|f\|_{C^1} < C \| (x, v) - (y, w) \|\);
(iii) \(|\bar{\tau} - \tau| < C \| (x, v) - (y, w) \|\);
(iv) \(\phi^\delta_{l}(x, v) = \phi^\delta_{l}(y, w)\);
(v) for every \(l \in \{1, \ldots, L\}\) \(\tilde{c}_l\) is, up to reparametrization, a geodesic with respect to \(\bar{g}\).
The proof of Proposition 3 occupies Sections 2.2 to 2.4. First, in Section 2.2, we restrict our attention to assertions (i)-(iv) by showing how two connect two unit geodesics in a constructive way (compare [2, Proposition 3.1] and [3, Proposition 2.1]). Then, in Section 2.3, we provide a lemma (Lemma 5) which explains how a conformal factor may preserve geodesic curves. Finally, in Section 2.4, we invoke transversality arguments together with Lemma 5 to conclude the proof of Proposition 3.

2.2 Connecting geodesics without obstacles

Let us first forget about assertion (v). For every \( x \in \mathbb{R}^n \), denote by \( \bar{G}(x) \) the \( n \times n \) matrix whose coefficients are the \((\bar{g})_{i,j}\), set \( \bar{Q} := \bar{G}^{-1} \) and define the Hamiltonian \( \bar{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) of class \( C^k \) by

\[
\bar{H}(x,p) := \frac{1}{2} \langle p, \bar{Q}(p) \rangle \quad \forall x \in \mathbb{R}^n, \forall p \in \mathbb{R}^n.
\]

There is a one-to-one correspondence between the geodesics associated with \( \bar{g} \) and the Hamiltonian trajectories of \( \bar{H} \). For every \((x,v) \in \mathbb{R}^n \times \mathbb{R}^n\), the trajectory \((x(\cdot),p(\cdot)) : [0,\infty) \to \mathbb{R}^n \times \mathbb{R}^n\) defined by

\[
(x(t),p(t)) := \left( \bar{g}_{x,v}(t), \bar{G}(\bar{g}_{x,v}(t)) \bar{\gamma}_{x,v}(t) \right) \quad \forall t \geq 0,
\]

is the solution of the Hamiltonian system

\[
\begin{align*}
\dot{x}(t) &= \frac{\partial \bar{H}}{\partial p}(x(t),p(t)) \\
\dot{p}(t) &= -\frac{\partial \bar{H}}{\partial x}(x(t),p(t))
\end{align*}
\]

such that \((x(0),p(0)) = (x, \bar{G}(x) v)\). Let \((x,v), (y,w) \in U^{\bar{g}}\mathbb{R}^n\) be fixed, set

\[
x^0 := x, \quad p^0 := \bar{G}(x) v, \quad x^\tau := \bar{g}_{y,w}(\tau), \quad v^\tau := \bar{\gamma}_{y,w}(\tau), \quad p^\tau := \bar{G}(x^\tau) v^\tau.
\]

Our aim is first to find a metric \( \tilde{g} \) whose the associated matrices \( \tilde{G}, \tilde{Q} \) have the form

\[\tilde{G}(x)^{-1} = \tilde{Q}(x) = e^{-f(x)} \bar{Q}(x) \quad \forall x \in \mathbb{R}^n,\]

in such a way that the trajectory \((x(\cdot),p(\cdot)) : [0,\infty) \to \mathbb{R}^n \times \mathbb{R}^n\) of the Hamiltonian system

\[
\begin{align*}
\dot{x}(t) &= \frac{\partial \tilde{H}}{\partial p}(x(t),p(t)) \\
\dot{p}(t) &= -\frac{\partial \tilde{H}}{\partial x}(x(t),p(t))
\end{align*}
\]

associated with the new Hamiltonian \( \tilde{H} = \tilde{H}_f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) defined by

\[\tilde{H}(x,p) = H_f(x,p) := \frac{1}{2} \langle p, \tilde{Q}(x)p \rangle = \frac{e^{-f(x)}}{2} \langle p, \bar{Q}(x)p \rangle \quad \forall x \in \mathbb{R}^n, \forall p \in \mathbb{R}^n,\]

\[4\]
and starting at \((x^0, p^0)\) satisfies \((x(\tau), p(\tau)) = (x^\tau, p^\tau)\). Note that there holds for any \(x, p \in \mathbb{R}^n\),

\[
\frac{\partial H_f}{\partial p}(x, p) = \bar{Q}(x)p = e^{-f(x)}\bar{Q}(x)p
\]  

(2.8)

and for every \(i = 1, \ldots, n\),

\[
\frac{\partial H_f}{\partial x_i}(x, p) = \frac{1}{2} \left\langle p, \frac{\partial \bar{Q}}{\partial x_i}(x) p \right\rangle = \frac{e^{-f(x)}}{2} \left\langle p, \frac{\partial \bar{Q}}{\partial x_i}(x) p \right\rangle - \frac{1}{2} \left\langle p, \bar{Q}(x) p \right\rangle \frac{\partial f}{\partial x_i}(x). \tag{2.9}
\]

Let us fix a smooth function \(\psi : [0, \tau] \rightarrow [0, 1]\) satisfying

\[
\psi(t) = 0 \quad \forall t \in [0, \tau/3] \quad \text{and} \quad \psi(t) = 1 \quad \forall t \in [2\tau/3, \tau].
\]

Given \((x, v), (y, w) \in U^\beta \mathbb{R}^n\), we define a trajectory

\[
\mathcal{X}(\cdot; (x, v), (y, w)) : [0, \tau] \rightarrow \mathbb{R}^n
\]

of class \(C^{k+1}\) by

\[
\mathcal{X}(t; (x, v), (y, w)) := (1 - \psi(t)) \bar{\gamma}_{x,v}(t) + \psi(t) \bar{\gamma}_{y,w}(t) \quad \forall t \in [0, \tau]. \tag{2.10}
\]

We note that the mapping \((t, (x, v), (y, w)) \mapsto \mathcal{X}(t; (x, v), (y, w))\) is \(C^{k+1}\) in the \(t\) variable but only \(C^{k-1}\) in the variables \(x, v, y, w\). Let \(\alpha(\cdot; (x, v), (y, w)) : [0, \tau] \rightarrow [0, +\infty)\) be the function defined as

\[
\alpha(t; (x, v), (y, w))
\]

\[
:= \int_0^t \sqrt{\left\langle \dot{\mathcal{X}}(s; (x, v), (y, w)), G\left(\mathcal{X}(s; (x, v), (y, w))\right) \dot{\mathcal{X}}(s; (x, v), (y, w)) \right\rangle} \, ds,
\]

for every \(t \in [0, \tau]\). We observe that \(\alpha(\cdot; (x, v), (y, w))\) is strictly increasing, of class \(C^{k+1}\) in the \(t\) variable, and of class \(C^{k-1}\) in the variables \(x, v, y, w\). Let

\[
\theta(\cdot; (x, v), (y, w)) : [0, \bar{\tau}] = \bar{\tau}(x, v), (y, w)) := \alpha(\tau; (x, v), (y, w)) \rightarrow [0, \tau]
\]

denote its inverse, which is of class \(C^{k+1}\) in \(t\), \(C^{k-1}\) in \(x, v, y, w\), and satisfies (we set \(\theta(\cdot) = \theta((\cdot; (x, v), (y, w))\) and \(\mathcal{X}(\cdot) = \mathcal{X}((\cdot; (x, v), (y, w))\))

\[
\dot{\theta}(s) = \frac{1}{\sqrt{\left\langle \dot{\mathcal{X}}(\theta(s)), G\left(\mathcal{X}(\theta(s))\right) \dot{\mathcal{X}}(\theta(s)) \right\rangle}} \quad \forall s \in [0, \bar{\tau}].
\]
Then, we define a new trajectory
\[ \tilde{x}(\cdot) = \tilde{x}(\cdot; (x,v),(y,w)) : [0, \tilde{\tau}((x,v),(y,w))] \rightarrow \mathbb{R}^n \]
of class \( C^{k+1} \) by
\[
\tilde{x}(t; (x,v),(y,w)) := \mathcal{X}(\theta(t)) \quad \forall t \in [0, \tilde{\tau}].
\]

By construction, there holds
\[
\left\{ \begin{array}{l}
\tilde{x}(t) = \mathcal{X}(t,(x,v),(y,w)) = \tilde{\gamma}_{x,v}(t) \quad \forall t \in [0, \tau/3], \\
\tilde{x}(t) = \mathcal{X}(t; (x,v),(y,w)) = \tilde{\gamma}_{y,w}(t) \quad \forall t \in [\tilde{\tau} - \tau/3, \tilde{\tau}],
\end{array} \right.
\]
\tag{2.11}
and
\[
\langle \dot{\tilde{x}}(t), \tilde{G}(\tilde{x}(t)) \dot{\tilde{x}}(t) \rangle = 1 \quad \forall t \in [0, \tilde{\tau}].
\]

This means that the adjoint trajectory
\[ \tilde{p}(\cdot) = \tilde{p}(\cdot; (x,v),(y,w)) : [0, \tilde{\tau}((x,v),(y,w))] \rightarrow \mathbb{R}^n \]
defined by
\[
\tilde{p}(t; (x,v),(y,w)) := \tilde{G}(\tilde{x}(t)) \dot{\tilde{x}}(t) \quad \forall t \in [0, \tilde{\tau}],
\]
\tag{2.12}
satisfies
\[
\dot{\tilde{x}}(t) = \frac{\partial H}{\partial \tilde{p}}(\tilde{x}(t), \tilde{p}(t)) \quad \forall t \in [0, \tilde{\tau}]
\]
\tag{2.13}
and
\[
\tilde{H}(\tilde{x}(t), \tilde{p}(t)) = \frac{1}{2} \quad \forall \in [0, \tilde{\tau}].
\]
\tag{2.14}

We now define the function
\[
\tilde{u}(\cdot) = \left( \tilde{u}_1(\cdot; (x,v),(y,w)), \ldots, \tilde{u}_n(\cdot; (x,v),(y,w)) \right) : [0, \tilde{\tau}] \rightarrow \mathbb{R}^n
\]
by
\[
\tilde{u}_i(t) := 2\tilde{p}_i(t) + \left\langle \tilde{p}(t), \frac{\partial \tilde{Q}}{\partial x_i}(\tilde{x}(t)) \tilde{p}(t) \right\rangle \quad \forall i = 1, \ldots, n, \forall t \in [0, \tilde{\tau}].
\]
\tag{2.15}

By construction, the function \( \tilde{p} \) is of class \( C^k \) in the \( t \) variable, \( \tilde{u} \) is \( C^{k-1} \) in the \( t \) variable, and all the functions \( \tilde{\tau}, \tilde{p}, \tilde{u} \) are \( C^{k-1} \) in the \( x,y,v,w \) variables. Furthermore, there holds
\[
\dot{\tilde{p}}(t) = -\frac{\partial H}{\partial \tilde{x}}(\tilde{x}(t), \tilde{p}(t)) + \frac{1}{2} \tilde{u}(t) \quad \forall t \in [0, \tilde{\tau}],
\]
\[
\begin{align*}
\begin{cases}
(\tilde{x}(0), \tilde{p}(0)) = (x^0, p^0), \\
(\tilde{x}(\tau), \tilde{p}(\tau)) = (x^\tau, p^\tau),
\end{cases}
\end{align*}
\]
(using the notations (2.5) and remembering (2.11)), and
\[
\bar{u}(t; (x, v), (y, w)) = 0_n \quad \forall t \in [0, \tau/3] \cup [\tilde{\tau} - \tau/3, \tilde{\tau}]
\] (2.16)
(by (2.11), (2.12), and (2.15)). Since \( \bar{H} \) is of class \( C^k \) with \( k \geq 3 \), the mapping
\[
Q : ((x, v), (y, w), s) \in (\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n) \times [0, 1]
\]
\[\longmapsto (\tilde{\tau}((x, v), (y, w)), \bar{u}(s\tilde{\tau}((x, v), (y, w)); (x, v), (y, w)))\]
is of class at least \( C^1 \). Therefore, since for all \( (x, v) \in \mathcal{U}^\beta \mathbb{R}^n \) with \( |x - \bar{x}^0| \leq 1 \), there holds
\[
\begin{align*}
Q((x, v), (x, v), s) &= (\tau, 0) \quad \forall s \in [0, 1], \\
|\tilde{\tau}((x, v), (y, w)) - \tau| &\leq |Q((x, v), (y, w), 0) - Q((x, v), (x, v), 0)| \\
&\leq K |(x, v) - (y, w)|,
\end{align*}
\] (2.17)
and analogously
\[
\|\bar{u}(\cdot; (x, v), (y, w))\|_{C^0} \leq K |(x, v) - (y, w)|. \tag{2.18}
\]
Furthermore, we notice that differentiating (2.14) yields
\[
\left\langle \frac{\partial \bar{H}}{\partial x}(\tilde{x}(t), \tilde{p}(t)), \dot{x}(t) \right\rangle + \left\langle \frac{\partial \bar{H}}{\partial p}(\tilde{x}(t), \tilde{p}(t)), \dot{p}(t) \right\rangle = 0 \quad \forall t \in [0, \tilde{\tau}],
\]
which together with (2.13) and (2.15) gives
\[
\left\langle \bar{u}(t), \dot{x}(t) \right\rangle = 0 \quad \forall t \in [0, \tilde{\tau}]. \tag{2.19}
\]
In conclusion, for every \( (x, v), (y, w) \in \mathcal{U}^\beta \mathbb{R}^n \) verifying \( |x - \bar{x}^0|, |y - \bar{x}^0| \leq 1 \), the function
\[
t \in [0, \tilde{\tau}((x, v), (y, w))] \longmapsto \left(\tilde{x}(t; (x, v), (y, w)), \tilde{p}(t; (x, v), (y, w)), \bar{u}(t; (x, v), (y, w))\right)
\]
satisfies for every \( t \in [0, \tilde{\tau}((x, v), (y, w))] \) and every \( i = 1, \ldots, n, \)
\[
\begin{align*}
\dot{x}(t) &= Q(\tilde{x}(t)) \tilde{p}(t) \\
\dot{p}_i(t) &= -\frac{1}{2} \left\langle \tilde{p}(t), \frac{\partial Q}{\partial x_i}(\tilde{x}(t)) \tilde{p}(t) \right\rangle - \frac{1}{2} \left\langle \tilde{p}(t), Q(\tilde{x}(t)) \tilde{p}(t) \right\rangle \bar{u}_i(t).
\end{align*}
\] (2.20)
and properties (2.17)-(2.19) hold. The proof of the following lemma (taken from [2]) is postponed to Section A.1.
Lemma 4. Let $T, \beta, \mu \in (0, 1)$ with $3\mu \leq \beta < T$, and let $y(\cdot), w(\cdot) : [0, T] \to \mathbb{R}^n$ be two functions of class respectively $C^k$ and $C^{k-1}$ satisfying
\[
|y(t) - e_1| \leq 1/5 \quad \forall t \in [0, T],
\]
\[
w(t) = 0_n \quad \forall t \in [0, \beta] \cup [T - \beta, T],
\]
\[
\langle \dot{y}(t), w(t) \rangle = 0 \quad \forall t \in [0, T].
\]
Then, there exist a constant $K$ depending only on the dimension and $T$, and a function $W : \mathbb{R}^n \to \mathbb{R}$ of class $C^k$ such that the following properties hold:
\begin{enumerate}
  \item $\text{Supp}(W) \subset \{y(t) + (0, z) | t \in [\beta/2, T - \beta/2], z \in B^{n-1}(0, \mu)\}$;
  \item $\|W\|_{C^1} \leq \frac{K}{\mu} \|w(\cdot)\|_{C^0}$;
  \item $\nabla W(y(t)) = w(t)$ for every $t \in [0, T]$;
  \item $W(y(t)) = 0$ for every $t \in [0, T]$.
\end{enumerate}

Therefore taking $\bar{\delta} \in (0, \tau/3)$ small enough, applying the above Lemma with $y(\cdot) = \bar{x}(\cdot), w(\cdot) = \bar{u}(\cdot), T = \bar{\tau}, \beta = \tau/3$, and $\mu > 0$ small enough, and remembering assumption (A), that $\bar{\gamma}([0, \tau]) \subset \mathcal{R}(\rho/2)$, (2.16), and (2.18)-(2.19) yields a universal constant $C = C(\tau, \rho) > 0$ and a function $f : \mathbb{R}^n \to \mathbb{R}$ of class $C^k$ satisfying the following properties:
\begin{enumerate}
  \item $\text{Supp}(f) \subset \mathcal{R}(\rho)$;
  \item $\|f\|_{C^1} < C \|x - v - (y, w)\|$;
  \item for every $t \in [0, \bar{\tau}], \nabla f(\bar{x}(t)) = \bar{u}(t)$;
  \item for every $t \in [0, \bar{\tau}], f(\bar{x}(t)) = 0$.
\end{enumerate}

Then, there is a one-to-one correspondence between the geodesics of $\bar{\gamma} := e^f \bar{g}$ and the solutions of the Hamiltonian system (2.6) associated with $\bar{H} = H_f$ given by (2.7). By construction of $f$, the function $(\bar{x}(\cdot), \bar{p}(\cdot)) : [0, \tau] \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ satisfies for every $t \in [0, \tau],$
\[
\dot{\bar{x}}(t) = e^{-f(\bar{x}(s))} \bar{Q}(\bar{x}(t)) \bar{p}(t)
\]
and for every $i = 1, \ldots, n$ and every $t \in [0, \tau],$
\[
\dot{\bar{p}}_i(t) = -\frac{e^{-f(\bar{x}(s))}}{2} \left( \bar{p}(t), \frac{\partial \bar{Q}}{\partial x_i}(\bar{x}(t)) \bar{p}(t) \right) - \frac{e^{-f(\bar{x}(s))}}{2} \left( \bar{p}(t), \bar{Q}(\bar{x}(t)) \bar{p}(t) \right) \frac{\partial f}{\partial x_i}(\bar{x}(t)).
\]
This means that $\bar{x}(\cdot)$ is a geodesic on $[0, \bar{\tau}]$ with respect to $\bar{g}$ starting from $\bar{x}(0) = x^0 = x$ with initial velocity $\dot{v} = \bar{G}(x^0)^{-1} p^0 = \bar{G}(x^0)^{-1} \bar{p}(0)$ and ending at $\bar{x}(\bar{\tau}) = x^\tau$ with final velocity $\dot{v}^\tau = \bar{G}(x^\tau)^{-1} p^\tau = \bar{G}(x^\tau)^{-1} \bar{p}(\bar{\tau})$. This proves of Proposition 3 (i)-(iv).
2.3 One remark about reparametrization

The following result will be useful to insure that the geodesic curves \( \tilde{c}_I(I) \) are preserved.

**Lemma 5.** Let \( \tilde{c} : I = [a, b] \to \mathbb{R}^n \) be a unit geodesic with respect to \( \bar{g} \), \( \bar{f} : \mathbb{R}^n \to \mathbb{R} \) be a function of class at least \( C^2 \), and \( \lambda : \mathbb{R}^n \to \mathbb{R} \) be such that

\[
\nabla \bar{f} (\tilde{c}(t)) = \dot{\lambda}(t) \bar{p}(t) := \dot{\lambda}(t) \bar{G}(\tilde{c}(t)) \dot{\tilde{c}}(t) \quad \forall t \in I, \tag{2.24}
\]

where \( \nabla \bar{f} \) denotes the gradient of \( \bar{f} \) with respect to the Euclidean metric. Then up to reparametrization, \( c \) is a unit geodesic with respect to the metric \( e^{\bar{f}} \bar{g} \).

Of course, Lemma 5 is a consequence of the fact that the gradient of \( f \) with respect to \( \bar{g} \) at \( \tilde{c}(t) \) is always colinear with the velocity \( \dot{\tilde{c}}(t) \). For sake of completeness, we prove Lemma 5 with Hamiltonian point of view.

**Proof of Lemma 5.** Define the function \( \beta : I \to \mathbb{R} \) by

\[
\beta(t) := \int_0^t e^{-\frac{\bar{f}(c(s))}{2}} ds \quad \forall t \in I. \tag{2.25}
\]

It is a strictly increasing function of class at least \( C^3 \) from \( I \) to \( \tilde{I} = [0, \tilde{\tau}] := \beta(I) \). Denote by \( \theta : \tilde{I} \to I \) its inverse. Note that \( \theta \) is at least \( C^3 \) and satisfies

\[
\dot{\theta}(s) = e^{-\frac{\bar{f}(c(s))}{2}} \quad \forall s \in [0, \tilde{\tau}]. \tag{2.26}
\]

Define \( \tilde{c}, \tilde{p} : \tilde{I} \to \mathbb{R}^n \) by

\[
\tilde{c}(s) := \tilde{c}(\theta(s)) \quad \text{and} \quad \tilde{p}(s) := e^\frac{\bar{f}(c(s))}{2} \tilde{p}(\theta(s)) \quad \forall s \in \tilde{I}.
\]

The metric \( \tilde{g} := e^{\bar{f}} \bar{g} \) is associated with matrices \( \tilde{G}, \tilde{Q} \) given by

\[
\tilde{G}(x)^{-1} = \tilde{Q}(x) = e^{-\bar{f}(x)} \bar{Q}(x) \quad \forall x \in \mathbb{R}^n.
\]

Then, for every \( s \in \tilde{I} \), \( \dot{\tilde{c}}(s) \) and \( \tilde{p}(s) \) are given by

\[
\dot{\tilde{c}}(s) = \dot{\theta}(s) \tilde{c}(\theta(s)) = \dot{\theta}(s) \tilde{Q}(\tilde{c}(\theta(s))) \tilde{p}(\theta(s)) = \dot{\tilde{Q}}(\tilde{c}(s)) \tilde{p}(s)
\]

and (using (2.26))

\[
(\tilde{p})_i(s) = \frac{d}{ds} \left( e^\frac{\bar{f}(c(s))}{2} \right) (\tilde{p})_i(\theta(s)) + e^\frac{\bar{f}(c(s))}{2} \dot{\theta}(s) (\tilde{p})_i(\theta(s))
\]

\[
= \frac{d}{ds} \left( e^\frac{\bar{f}(c(s))}{2} \right) (\tilde{p})_i(\theta(s)) - \frac{1}{2} \left( \tilde{p}(s), \frac{\partial \bar{Q}_i}{\partial x_j}(\tilde{c}(\theta(s))) \tilde{p}(\theta(s)) \right)
\]

\[
= \frac{d}{ds} \left( e^\frac{\bar{f}(c(s))}{2} \right) (\tilde{p})_i(\theta(s)) - \frac{e^{-\bar{f}(\tilde{c}(s))}}{2} \left( \tilde{p}(s), \frac{\partial \bar{Q}_i}{\partial x_j} (\tilde{c}(s) \tilde{p}(s)) \right),
\]

\[9\]
where the first term is equal to (using (2.24))

\[
\frac{d}{ds} \left( e^{\frac{f(c(s))}{2}} \right) (\tilde{p})_i(\theta(s)) = \frac{e^{\frac{f(c(s))}{2}}}{2} \left\langle \nabla \bar{f}(\tilde{c}(s)), \tilde{c}(s) \right\rangle (\tilde{p})_i(\theta(s)) = \frac{1}{2} e^{\frac{f(c(s))}{2}} \left( \bar{\lambda}(\theta(s)) \tilde{p}(\theta(s)), \tilde{Q}(\tilde{c}(s)) \tilde{p}(s) \right) (\tilde{p})_i(\theta(s)) = \frac{1}{2} \left( \tilde{p}(s), \tilde{Q}(\tilde{c}(s)) \tilde{p}(s) \right) \left( \tilde{c}(s) \right).
\]

Remembering (2.8)-(2.9) with \( f = \bar{f} \) and \( \tilde{Q} = \tilde{Q} \), this proves that \( (\tilde{c}(\cdot), \tilde{p}(\cdot)) : \tilde{I} \to \mathbb{R}^n \times \mathbb{R}^n \) is a trajectory of the Hamiltonian system associated with \( \tilde{H} = H_{\bar{f}} \) and in turn concludes the proof of the lemma. \( \square \)

### 2.4 Dealing with obstacles

We now proceed to explain how to modify our construction in order to get assertion (v) of Proposition 3. We fix \( (x, v), (y, w) \in U^g \mathbb{R}^n \) verifying (2.1) and consider a finite set of unit geodesics

\[
\bar{c}_1 : I_1 \to \mathbb{R}^n, \quad \cdots, \quad \bar{c}_L : I_L \to \mathbb{R}^n
\]

satisfying assumptions (2.2)-(2.3). We set

\[
\bar{\Gamma} := \bigcup_{l=1}^L \bar{c}_l(I_l).
\]

The construction that we performed in the previous section together with transversality arguments yield the following result.

**Lemma 6.** Taking \( \bar{\delta} > 0 \) small enough, there is a positive constant \( C = C(\tau, \rho) \), \( \bar{\tau} = \bar{\tau}((x, v), (y, w)) > 0 \), a function

\[
(\bar{x}(\cdot), \bar{p}(\cdot)) = (\bar{x}(\cdot; (x, v), (y, w)), \bar{p}(\cdot; (x, v), (y, w))) : [0, \bar{\tau}] \to \mathbb{R}^n
\]

of class \( C^k \), and a function

\[
\bar{u}(\cdot) = \bar{u}(\cdot; (x, v), (y, w)) : [0, \bar{\tau}] \to \mathbb{R}^n
\]

of class \( C^{k-1} \) satisfying (2.19), (2.20),

\[
|\bar{\tau} - \tau| < C \left| (x, v) - (y, w) \right|,
\]

(2.27)
\[ \text{Supp}(\tilde{u}(\cdot)) \subset [\tau/5, 4\tau/5], \quad (2.28) \]

\[ \|\tilde{u}\|_{C^0} \leq C \right| (x,v) - (y,w)\right|, \quad (2.29) \]

\[ (\tilde{x}(0), \tilde{p}(0)) = (x^0, p^0), \quad (\tilde{x}(\tau), \tilde{p}(\tau)) = (x^\tau, p^\tau), \quad (2.30) \]

such that the following properties are satisfied:

(i) the curve \( \tilde{x}(\text{Supp}(\tilde{u}(\cdot))) \) is transverse to \( \bar{\Gamma} \);

(ii) if \( n \geq 3 \), the set \( T_{\tilde{u}} \subset \text{Supp}(\tilde{u}(\cdot)) \) defined by

\[ T_{\tilde{u}} := \left\{ t \in \text{Supp}(\tilde{u}(\cdot)) \mid \tilde{x}(t) \in \bar{\Gamma} \right\} \]

is empty;

(iii) if \( n = 2 \), the set \( T_{\tilde{u}} \) is finite and for every \( t \in T_{\tilde{u}} \), there is a unique \( l \in [0, L] \) such that \( \tilde{x}(t) = \bar{c}_l(s) \) for some \( s \in I_l \) and there are \( a < b \in [0, \tau] \) with \( a < t < b \) such that \( \tilde{u}(s) = 0 \) for every \( s \in [a, b] \).

**Proof of Lemma 6.** Let us consider the trajectory

\[ \mathcal{X}(\cdot) = \mathcal{X}(\cdot; (x,v), (y,w)) : [0, \tau] \rightarrow \mathbb{R}^n \]

of class \( C^{k+1} \) defined by (2.10). Since \( \mathcal{X}(\cdot) \) coincides respectively with \( \bar{\gamma}_{x,v} \) and \( \bar{\gamma}_{y,w} \) on the intervals \([0, \tau/3]\) and \([2\tau/3, \tau]\) and since the \( \bar{c}_l \)'s are unit geodesics satisfying (2.3), there are \( t_1 \in (0, \tau/3), t_2 \in (2\tau/3, \tau) \) and \( \nu \in (0, \tau/100) \) such that

\[ \mathcal{X}(t) \notin \bar{\Gamma} \quad \forall t \in [t_1 - \nu, t_1 + \nu] \cup [t_2 - \nu, t_2 + \nu]. \quad (2.31) \]

Taking \( \tilde{\delta} > 0 \) small enough, it is sufficient to show that we can perturb the curve \( \mathcal{X}([0, \tau]) \) to make it transverse to all the geodesic curves \( \bar{c}(I_l) \) verifying

\[ |\bar{c}_l(s) - e_1| < 1/2 \quad \forall s \in I_l = [a_l, b_l]. \]

Without loss of generality, we may assume that for each such curve (denote by \( \mathcal{L} \) the set of such \( l \)), there holds \( (\bar{c}_l(a_l))_1 \leq \bar{x}^0 \) and \( (\bar{c}_l(b_l))_1 \geq \bar{x}^\tau \) (remember (2.2)). Let us parametrize both curves \( \mathcal{X}(\cdot) \) and \( \bar{c}_l(\cdot) \) by their first coordinates. Namely, there are two functions \( \theta_1 : J_1 = [\alpha, \beta] \rightarrow [0, \tau], \theta_2 : J_2 \rightarrow I_l \) of class \( C^{k+1} \) such that

\[ ((\mathcal{X} \circ \theta_1)(s))_1 = s \quad \forall s \in J_1 \quad \text{and} \quad ((\bar{c}_l \circ \theta_2)(s))_1 = s \quad \forall s \in J_2. \quad (2.32) \]
Extending \( I_1 \) if necessary, we may indeed assume that \( J_1 \subset J_2 \). Define the function 
\[ h_l : I \to \mathbb{R}^{n-1} \]
by 
\[ h_l(s) := (X \circ \theta_1)(s) - (\tilde{c}_l \circ \theta_2)(s) \quad \forall s \in J_1 = [\alpha, \beta]. \]

Set 
\[ s_1^- := \theta_1(t_1 - \nu), \quad s_1^+ := \theta_1(t_1 + \nu), \quad s_2^- := \theta_2(t_2 - \nu), \quad s_2^+ := \theta_2(t_2 + \nu), \]
and fix a smooth function \( \psi : J_1 \to [0, 1] \) satisfying 
\[ \psi(s) = 0 \quad \forall s \in [\alpha, s_1^-] \cup [s_1^+, \beta] \quad \text{and} \quad \psi(s) = 1 \quad \forall s \in [s_2^+, s_2^+] \quad (2.33) \]

For every \( \chi \in \mathbb{R}^n \) with \( \chi_1 = 0 \), define the curve \( X_\chi : [0, \tau] \to \mathbb{R}^n \) by 
\[ X_\chi(t) := X(t) + \psi(t)\chi \quad \forall t \in [0, \tau], \]
If \( X_\chi([s_1^-, s_2^+]) \) intersects \( \tilde{c}_l(I_l) \) for some \( l \in \mathcal{L} \), there holds 
\[ 0_n = X_\chi(t) - \tilde{c}_l(s) = X(t) - \tilde{c}_l(s) + \psi(t)\chi = (X \circ \theta_1)(\theta_1^{-1}(t)) - (\tilde{c}_l(s) \circ \theta_2)(\theta_2^{-1}(s)) + \psi(t)\chi, \]
for some \( t \in [s_1^-, s_2^+] \) and \( s \in J_1 \). Since \( \chi_1 = 0 \) and (2.32) is satisfied, we have necessarily \( \theta_1^{-1}(t) = \theta_2^{-1}(s) \), then we obtain 
\[ 0_n = (X \circ \theta_1)(\theta_1^{-1}(t)) - (\tilde{c}_l(s) \circ \theta_2)(\theta_1^{-1}(t)) + \psi(t)\chi = h_l(\theta_1^{-1}(t)) + \psi(t)\chi. \]

Furthermore, thanks to (2.31), if \( \chi \) is small enough, the restriction of \( X_\chi(\cdot) \) to the two intervals \([t_1 - \nu, t_1 + \nu]\) and \([t_2 - \nu, t_2 + \nu]\) cannot intersect \( \tilde{\Gamma} \). By (2.33), we infer that 
\[ h_l(\theta_1^{-1}(t)) + \chi = 0_n \quad \text{for some} \ t \in [s_1^+, s_2^+]. \]

By Sard’s Theorem (see for instance [1]), almost every value of \( h_l \) is regular. In addition, if \(-\chi \) is a regular value of \( h_l \), then \( h_l(s) \neq 0_n \) for all \( s \) such that \( h_l(s) = -\chi \). This shows that if \(-\chi \) is a small enough regular value of \( h_l \), then \( X_\chi([s_1^-, s_2^+]) \) is transverse to \( \tilde{c}_l(I_l) \). Finally, we observe that 
\[ \begin{cases} \dot{X}_\chi(t) = \dot{X}(t) + \psi(t)\chi \\ \ddot{X}_\chi(t) = \ddot{X}(t) + \ddot{X}(t)\chi \end{cases} \quad \forall t \in [0, \tau]. \quad (2.34) \]

Then taking a small enough \( \chi \in \mathbb{R}^n \) with \( \chi_1 = 0 \) and proceeding as in Section 2.2 provides \( \tilde{\tau} = \tilde{\tau}(\langle x, v \rangle, \langle y, w \rangle) > 0 \) and a triple 
\[ (\tilde{x}(\cdot), \tilde{p}(\cdot), \tilde{u}(\cdot)) = (\tilde{x}(\cdot; \langle x, v \rangle, \langle y, w \rangle), \tilde{p}(\cdot; \langle x, v \rangle, \langle y, w \rangle), \tilde{u}(\cdot; \langle x, v \rangle, \langle y, w \rangle)) : [0, \tilde{\tau}] \to \mathbb{R}^n \]
satisfying (2.19), (2.20), and (2.30). Moreover, \( \tilde{\tau} \) is given by

\[
\tilde{\tau} := \int_0^\tau \sqrt{\langle \dot{X}(s), \bar{G}(X(s)) \dot{X}(s) \rangle} \, ds
\]

and for every \( t \in [0, \tilde{\tau}] \),

\[
\tilde{u}(t) = 2\dot{\bar{p}}(t) + 2 \frac{\partial H}{\partial x}(\tilde{x}(t), \tilde{p}(t))
\]

\[
= 2 \frac{d}{dt} \{ \bar{G}(\tilde{x}(t)) \dot{\tilde{x}}(t) \} + 2 \frac{\partial H}{\partial x}(\tilde{x}(t), \tilde{p}(t)).
\]

Thanks to (2.34) and (2.17)-(2.18), we deduce that taking \( \chi \) small enough yield (2.27) and (2.29) for some universal constant \( C = C(\tau, \rho) > 0 \). All in all, this shows assertion (i).

To show assertion (ii) and the first part (uniqueness) of assert (iii), just move a bit the curve \( X_\chi \) to get a new curve \( Y : [0, \tau] \to \mathbb{R}^n \) such that the new curve \( \tilde{Y} : [0, \tilde{\tau}] \to \mathbb{R}^n \) constructed as a reparametrization of \( Y \) (as \( \tilde{x} \) was obtained from \( X \) in Section 2.2) satisfies the result. Finally, to get the second part of assertion (iii), replace the curve \( \tilde{Y} \) by a piece of unit geodesic (with respect to \( \bar{g} \)) in a neighborhood of each \( t \in [0, \tilde{\tau}] \) such that \( \tilde{Y}(t) \in \bar{\Gamma} \) and reparametrize it as in Section 2.2. Let us explain briefly how to proceed. Given \( \bar{t} \in (0, \tilde{\tau}) \) such that \( \tilde{Y}(\bar{t}) \in \bar{\Gamma} \) and \( \lambda > 0 \), define \( \tilde{Y}_\lambda : [0, \tilde{\tau}] \to \mathbb{R}^n \) a small perturbation of \( \tilde{Y} \) by

\[
\tilde{Y}_\lambda(t) := \psi \left( \frac{t-\bar{t}}{\lambda} \right) \tilde{Y}(t) + \left[ 1 - \psi \left( \frac{t-\bar{t}}{\lambda} \right) \right] \tilde{r}_{Y(t), \bar{\Gamma}}(t-\bar{t}) \quad \forall t \in [0, \tilde{\tau}],
\]

where \( \psi : \mathbb{R} \to [0, 1] \) is a smooth function verifying

\[
\psi(t) = 0 \quad \forall t \in (-\infty) \cup [1, +\infty) \quad \text{and} \quad \psi(t) = 1 \quad \forall t \in [-1/2, 1/2].
\]

We leave the reader to check that taking \( \lambda > 0 \) small enough yields the desired result.

Proposition 3 follows easily from the following result whose the technical proof is postponed to Appendix A.2.

**Lemma 7.** There are \( C = C(\tau, \rho) > 0 \) and a function \( f : \mathbb{R}^n \to \mathbb{R} \) of class \( C^{k-1} \) such that the following properties are satisfied:

(i) \( \text{Supp} \ (f) \subset \mathcal{R}(\rho) \);

(ii) \( \|f\|_{C^1} < C \| (x, v) - (y, w) \| \);

(iii) for every \( t \in [0, \tau] \), \( \nabla f(\tilde{x}(t)) = \tilde{u}(t) \);

(iv) for every \( l \in \{1, \ldots, L\} \) and every \( s \in I_l \), there is \( \lambda_l(s) \) such that

\[
\nabla f(\tilde{c}_l(s)) = \lambda_l(s) \bar{p}_l(s) = \lambda_l(s) \bar{G}(\tilde{c}_l(s)) \dot{\tilde{c}}_l(s).
\]
3 Proof of Theorem 1

Let \( \gamma = \gamma_{x,v} : \mathbb{R} \to M \) be the geodesic starting from \( x \) with velocity \( v \in U^g_x M \) and \( \epsilon > 0 \) be fixed. Let \( \tau \in (0, 1/20) \) be a small enough time such that the curve \( \gamma_{x,v}([-10\tau, 10\tau]) \) has no self-intersection. There exist an open neighborhood \( U_x \) of \( x \) and a smooth diffeomorphism

\[ \theta_x : U_x \longrightarrow B^n(0, 1) \quad \text{with} \quad \theta_x(x) = 0_n \]

such that

\[ \left| \frac{d}{dt} (\theta_x \circ \gamma_{x,v})(t) - e_1 \right| \leq \frac{1}{10} \quad \forall t \in [-10\tau, 10\tau]. \tag{3.1} \]

Set

\[ \bar{\gamma}(t) := \theta_x (\gamma_{x,v}(t)) \quad \forall t \in [-10\tau, 10\tau] \]

and

\[ \bar{x}^0 := \bar{\gamma}(0) = 0_n, \quad \bar{v}^0 := \dot{\bar{\gamma}}(0), \quad \bar{x}^\tau := \bar{\gamma}(\tau), \quad \bar{v}^\tau := \dot{\bar{\gamma}}(\tau). \]

The metric \( g \) is sent, via the smooth diffeomorphism \( \theta_x \), onto a Riemannian metric \( \bar{g} \) of class \( C^k \) on \( B^n(0, 1) \). Without loss of generality, we may assume that \( \bar{g} \) is the restriction to \( B^n(0, 1) \) of a complete Riemannian metric of class \( C^k \) defined on \( \mathbb{R}^n \). Denote by \( \phi_t^\bar{g} \) the geodesic flow on \( \mathbb{R}^n \times \mathbb{R}^n \). Set

\[ \mathcal{H}_0 := \left\{ y = (y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_1 = 0 \right\}. \]

Since \( \tau \leq 1/2 \) and \( \bar{\gamma}(0) = 0_n \), (3.1) implies

\[ \bar{\gamma}(t) \in \mathcal{R}(1/4) := \left\{ (t, z) \mid t \in [0, \bar{x}^\tau], \ z \in B^{n-1}(0, 1/4) \right\} \subset B^n(0, 1). \]

Then, we can apply Proposition 3 to the curve \( \bar{\gamma} : [0, \tau] \to \mathbb{R}^n \) and \( \rho = 1/2 \). Consequently, there are \( \bar{\delta} = \bar{\delta}(\tau, \rho) \in (0, \tau/3) \) and \( C = C(\tau, \rho) > 0 \) such that the property stated in Proposition 3 is satisfied. Define the section \( S \subset TM \) by

\[ S := d\theta_x^{-1} (\mathcal{H}_0 \times \mathbb{R}^n). \]

Since \((x, v)\) is \( g\)-nonwandering and \( \gamma_{x,v} \) is transverse to \( S \) at time zero, for every \( r > 0 \) small, there exist \( (x^r, v^r), (x^r_s, v^r_s) \in S \cap U^g M \), \( T^r > 0 \) and \( y^r, y^r_s, w^r, w^r_s \in B^n(0, 1) \) such that

(a) \( (x^r_s, v^r_s) = \phi_{T^r}^g (x^r, v^r) \).

(b) \( (y^r, w^r) = d\theta_x (x^r, v^r), (y^r_s, w^r_s) = d\theta_x (x^r_s, v^r_s) \);
(c) \( (y^r, w^r), (y^*_r, w^*_r) \in U^g \mathbb{R}^n; \)

(d) \( y^r, y^*_r \in \mathcal{H}_0; \)

(e) \(|x - \bar{x}^0|, |y - \bar{y}^0|, |v - \bar{v}^0| < \delta;\)

(f) \(|(y^r, w^r) - (y^*_r, w^*_r)| < r.\)

Recall that the cylinder \( R(1/2) \) is defined by

\[ R(1/2) := \left\{ (t, z) \mid t \in [0, \bar{x}^1], z \in B^{n-1}(0, 1/2) \right\} \subset B^n(0, 1). \]

The intersection of the curve \( \gamma_{x^r, v^r}([5\tau, T^r - 5\tau]) \) with the open set \( \theta_x^{-1}(R(1/2)) \) can be covered by a finite number of connected curves. More precisely, there are a finite number of unit geodesic arcs

\[ \bar{c}_1 : I_1 = [a_1, b_1] \longrightarrow B^n(0, 1), \quad \cdots, \quad \bar{c}_L : I_L = [a_L, b_L] \longrightarrow B^n(0, 1) \]

such that the following properties are satisfied:

(g) For every \( l \in \{1, \ldots, L\}, \bar{c}_l(a_l), \bar{c}_l(b_l) \in B^n(0, 1) \setminus R(1/2); \)

(h) there are disjoint closed intervals \( \mathcal{J}_1, \ldots, \mathcal{J}_L \subset [-5\tau, T_r - 5\tau] \) such that

\[ \gamma_{x^r, v^r}(\mathcal{J}_l) \subset \mathcal{U}_x, \quad \bar{c}_l(I_l) = \theta_x(\gamma_{x^r, v^r}(\mathcal{J}_l)) \quad \forall l = 1, \ldots, L, \]

and

\[ \left( \theta_x(\gamma_{x^r, v^r}([5\tau, T_r - 5\tau]) \cap \mathcal{U}_x) \cap R(1/2) \right) \subset \bigcup_{l=1}^L \bar{c}_l(I_l). \]

We connect \( (y^*_r, w^*_r) \) to \( \phi^g_y(y^r, w^r) \) by preserving the curves \( \bar{c}_1(I_1), \ldots, \bar{c}_L(I_L) \). We define the metric \( \tilde{g} \) on \( M \) by

\[ \tilde{g} = \left\{ \begin{array}{ll} \tilde{g} \text{ on } M \setminus \mathcal{U}_x & \\
\theta_x^*(e^f \tilde{g}) \text{ on } \mathcal{U}_x. & \end{array} \right. \]

We leave the reader to check that by construction the geodesic starting from \( x^r \) with initial velocity \( v^r \) is periodic. Taking \( r > 0 \) small enough yields \( d_{TM}((x, v),(x^r, v^r)) < \epsilon \) and \( \|f\|_{C^1} < \epsilon. \)
A Proof of Lemmas 4 and 7

A.1 Proof of Lemma 4

Define the function $\Phi : [0,T] \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ by

$$\Phi(t,z) := y(t) + (0,z) \quad \forall (t,z) \in [0,T] \times \mathbb{R}^{n-1}. $$

We can easily check that, thanks to (2.21), $\Phi$ is a diffeomorphism of class $C^k$ from $[0,T] \times \mathbb{R}^{n-1}$ into $[y_1(0), y_1(\tau)] \times \mathbb{R}^{n-1}$ which sends the cylinder $[\beta/2, T - \beta/2] \times B^{n-1}(0, \mu)$ into the “cylinder”

$$C_y(\mu) := \left\{ y(t) + (0,z) | t \in [\beta/2, T - \beta/2], z \in B^{n-1}(0, \mu) \right\},$$

and which satisfies

$$\|\Phi\|_{C^1}, \|\Phi^{-1}\|_{C^1} \leq K_0,$$

for some positive constant $K_0$ depending on $T$ only. Define the function $\tilde{w}(\cdot) : [0,T] \to \mathbb{R}^n$ by

$$\tilde{w}(t) := (d\Phi(t,0_{n-1}))^*(w(t)) \quad \forall t \in [0,T].$$

The function $\tilde{w}$ is $C^{k-1}$; in addition, by (2.23), there holds

$$\tilde{w}(t) = 0_n \quad \forall t \in [0,\beta] \cup [T - \beta, T] \quad \text{and} \quad \tilde{w}_1(t) = 0 \quad \forall t \in [0,T].$$

Let $\psi : \mathbb{R} \to [0,1]$ be an even function of class $C^\infty$ satisfying the following properties:

- $\psi(s) = 1$ for $s \in [0,1/3]$;
- $\psi(s) = 0$ for $s \geq 2/3$;
- $|\psi(s)|, |\psi'(s)| \leq 10$ for any $s \in [0, +\infty)$. 

Extend the function $\tilde{w}(\cdot)$ on $\mathbb{R}$ by $\tilde{w}(t) := 0$ for $t \leq 0$ and $t \geq T$, and define the function $\tilde{W} : [0,T] \times \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$\tilde{W}(t,z) = \psi \left( \frac{|z|}{\mu} \right) \left[ \sum_{i=2}^n \int_0^{x_i} \tilde{w}_i(t+s) ds \right] \quad \forall (t,z) \in [0,T] \times \mathbb{R}^{n-1}.$$ 

Since $\tilde{w}$ and $\phi$ are both at least $C^k$, it is easy to check that $\tilde{W}$ is of class $C^k$ as well. Moreover, (using that $3\mu \leq \beta < T$) we check easily that

$$\text{Supp } (\tilde{W}) \subset [\beta/2, T - \beta/2] \times B^{n-1}(0, 2\mu/3),$$

16
∇\tilde{W}(t,0) = \tilde{w}(t), \ W(t,0) = 0 \ \ \forall t \in [0,\tau],

and that (see the proof of [2, Lemma 3.3])

$$\left\| \tilde{W} \right\|_{C^1} \leq \frac{K_1}{\mu} \left\| \tilde{w}(\cdot) \right\|_{C^0},$$

for some constant $K_1 > 0$. Finally, define the function $W : \mathbb{R}^n \to \mathbb{R}$ by

$$W(x) := \begin{cases} \tilde{W}(\Phi^{-1}(x)) & \text{if } x \in C_y(\mu) \\ 0 & \text{otherwise.} \end{cases}$$

We conclude easily.

A.2 Proof of Lemma 7

We proceed in several steps.

Step 1: Applying Lemma 4, we get a universal constant $C_1 = C_1(\tau,\rho) > 0$ and a function $f_1 : \mathbb{R}^n \to \mathbb{R}$ of class $C^k$ such that the following properties are satisfied:

(i)_1 \quad \text{Supp } (f_1) \subset \mathcal{R}(2\rho/3);

(ii)_1 \quad \left\| f_1 \right\|_{C^1} < C_1 \left| (x,v) - (y,w) \right|;

(iii)_1 \quad \nabla f_1(\tilde{x}(t)) = \tilde{u}(t), \text{ for every } t \in [0,\tilde{\tau}];

(iv)_1 \quad f_1(\tilde{x}(t)) = 0, \text{ for every } t \in [0,\tau].

Step 2: Let $x_1, \ldots, x_N$ be a set of points in $\mathcal{R}(2\rho/3)$ such that

$$\left( \bigcup_{k,l=1,k \neq l}^{L} \left( \bar{c}_k(I_k) \cap \bar{c}_l(I_l) \right) \right) \cap \mathcal{R}(2\rho/3) = \{x_1, \ldots, x_N\}.$$ 

Note that by Lemma 6 (ii)-(iii), the set $\{x_1, \ldots, x_N\}$ does not intersect the curve $\tilde{x}(\text{Supp } (\tilde{u}(\cdot)))$. Let $\mu > 0$ be such that the $N$ balls $B^n(x_1,2\mu), \ldots, B^n(x_N,2\mu)$ are disjoint and do not intersect neither the curve $\tilde{x}(\text{Supp } (\tilde{u}(\cdot)))$ nor the boundary of $\mathcal{R}(2\rho/3)$. Define the $C^k$ function $f_2 : \mathbb{R}^n \to \mathbb{R}$ by

$$f_2(x) := f_1 \left( \sum_{k=1}^{N} \psi \left( \frac{|x-x_k|}{3\mu} \right) x_k + \left( 1 - \psi \left( \frac{|x-x_k|}{3\mu} \right) \right) x \right) \quad \forall x \in \mathbb{R}^n.$$ 

By contruction, there is a universal constant $C_2 = C_2(\tau,\rho) > 0$ such that $f_2$ satisfies the following properties:
(i) $\text{Supp } (f_2) \subset \mathcal{R}(2\rho/3)$;

(ii) $||f_2||_{C^1} < C_2 |(x, v) - (y, w)|$;

(iii) $\nabla f_2(\tilde{x}(t)) = \tilde{u}(t)$, for every $t \in [0, \tilde{\tau}]$;

(iv) $f_2(\tilde{x}(t)) = 0$, for every $t \in [0, \tilde{\tau}]$;

(v) $f_2(x) = f_1(x)$ for every $x \in \mathbb{R}^n \setminus \left(\bigcup_{k=1}^{N} B^n(x_k, 2\mu)\right)$;

(vi) $\nabla f_2(x) = 0$ for every $x \in \bigcup_{k=1}^{N} B^n(x_k, \mu)$.

Step 3: Let $t_1, \ldots, t_K \in [0, \tau]$ be the set of times such that

$$\tilde{x}(\text{Supp } (\tilde{u}(\cdot))) \cap \left(\bigcup_{l=1}^{L} \tilde{e}_l(I_l)\right) = \{\tilde{x}(t_k) \mid k = 1, \ldots, K\}.$$

Taking $\mu > 0$ smaller if necessary, we may assume that the balls $B^n(\tilde{x}(t_1), 5\mu)$, $\ldots, B^n(\tilde{x}(t_K), 5\mu)$ are disjoint, do not intersect the boundary of $\mathcal{R}(\rho/2)$, and such that $\tilde{u}(t) = 0$ for every $t \in [0, \tilde{\tau}]$ with $\tilde{x}(t) \in \bigcup_{k=1}^{Q} B^n(\tilde{x}(t_k), 5\mu)$ (remember Lemma 6 (iii)). Set

$$\Omega := \bigcup_{k=1}^{Q} B^n(\tilde{x}(t_k), 2\mu).$$

Taking $\mu > 0$ smaller if necessary again, the projection (with respect to the Euclidean metric) $P_0 : \Omega \rightarrow \mathbb{R}^n$ to the set

$$S := \bigcup_{k=1}^{K} \left( B^n(\tilde{x}(t_k), 2\mu) \cap \tilde{x}([0, \tilde{\tau}] ) \right),$$

is of class $C^{k-1}$, has a $C^1$ norm $||P_0||_{C^1}$ which is bounded by a universal constant, and satisfies

$$P_0(x) = x \quad \forall x \in S,$$

$$P_0(x) \in S \quad \forall x \in \Omega,$$

$$|x - P_0(x)| < \frac{\mu}{2} \quad \forall x \in \bigcup_{k=1}^{K} \left( B^n(\tilde{x}(t_k), \mu/2) \right).$$
Define the $C^{k-1}$ function $f_3 : \mathbb{R}^n \to \mathbb{R}$ by

$$f_3(x) := \begin{cases} f_2(h(x)\mathcal{P}_0(x) + (1 - h(x))x) & \text{if } x \in \Omega \\ f_2(x) & \text{otherwise,} \end{cases}$$

where $h : \Omega \to \mathbb{R}$ is defined by

$$h(x) := \psi \left( \sum_{q=1}^{Q} \frac{2|x - \tilde{x}(t_q)|}{3\mu} \right) \quad \forall x \in \Omega.$$ 

We note that $h(x) = 1$ for every $x \in \bigcup_{k=1}^{K} \left( B^n(\tilde{x}(t_k), \mu/2) \right)$ and $h(x) = 0$ for every $x \in \Omega$ which does not belong to the set $\bigcup_{k=1}^{K} \left( B^n(\tilde{x}(t_k), \mu) \right)$. Consequently, by contraction, there is a universal constant $C_3 = C_3(\tau, \rho) > 0$ such that $f_3$ satisfies the following properties:

(i) $\text{Supp } (f_3) \subset \mathcal{R}(2\rho/3)$;

(ii) $\|f_3\|_{C^1} \leq C_3 \|(x, v) - (y, w)\|$;

(iii) $\nabla f_3(\tilde{x}(t)) = \tilde{u}(t)$, for every $t \in [0, \tilde{\tau}]$;

(iv) $f_3(\tilde{x}(t)) = 0$, for every $t \in [0, \tilde{\tau}]$;

(v) $f_3(x) = f_2(x)$ for every $x \in \mathbb{R}^n \setminus \Omega$;

(vi) $\nabla f_3(x) = 0$ for every $x \in \bigcup_{k=1}^{K} B^n(\tilde{x}(t_k), \mu/2)$.

Step 4: Denote by $d_\tilde{g} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the Riemannian distance with respect to the Riemannian metric $\tilde{g}$. Denote by $\text{dist}_\tilde{g}^n(\cdot)$ the distance function (with respect to $\tilde{g}$) to the set $\bar{\Gamma}$. For every $\delta > 0$, let $S_{\delta} \subset \mathcal{R}(2\rho/3 + \delta)$ be the subset of $\bar{\Gamma}$ defined by

$$S_{\delta} := \left( \bar{\Gamma} \cap \mathcal{R}(\tau, 2\rho/3 + \delta) \right) \setminus \left( \bigcup_{k=1}^{N} B^n(x_k, \mu/2) \cup \bigcup_{k=1}^{K} B^n(\tilde{x}(t_k), \mu/4) \right).$$

For every $\delta, \mu > 0$, we denote by $S_{\delta}^{\mu}$ the open set of points whose distance (with respect to $\tilde{g}$) to $S_{\delta}$ is strictly less than $\mu$. There are $\delta, \mu > 0$ such that the function $\text{dist}_\tilde{g}^n(\cdot)$ is of class $C^k$ on $S_{\delta}^{\mu}$, the projection $\mathcal{P}_{\tilde{g}}^\Gamma$ to $\bar{\Gamma}$ with respect to $\tilde{g}$ is $C^{k-1}$ on $S_{\delta}^{\mu}$, and both $\|\text{dist}_\tilde{g}^n(\cdot)\|_{C^1(S_{\delta}^{\mu})}, \|\mathcal{P}_{\tilde{g}}^\Gamma(\cdot)\|_{C^1(S_{\delta}^{\mu})}$ are bounded by a universal constant. Define the function $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) := \begin{cases} f_3(P(x)) & \text{if } x \in S_{\delta}^{\mu} \\ f_3(x) & \text{otherwise,} \end{cases}$$

19
where the mapping $P : S^\mu_\delta \to \mathbb{R}^n$ is defined by

$$P(x) := \psi \left( \frac{2\text{dist}_{\bar{g}}^F(x)}{3\mu} \right) P^F_{\bar{g}}(x) + \left( 1 - \psi \left( \frac{2\text{dist}_{\bar{g}}^F(x)}{3\mu} \right) \right) x \quad \forall x \in S^\mu_\delta.$$ 

We leave the reader to check that if $\mu > 0$ is small enough, the function $f$ is of class $C^{k-1}$ and satisfies assertions (i)-(iv) of Lemma 7 for some universal constant $C = C(\tau, \rho) > 0$. For that, it is worth noticing that assertion (iv) means that both vectors $\nabla f(\bar{c}_l(s))$ and $\dot{\bar{c}}_l(s)$ are colinear with respect to $\bar{g}$.

References


